Finite Elements for the Reissner-Mindlin Plate

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1 Introduction

In this paper, we consider the approximation of the equations of linear elasticity in the case when the body is an isotropic, homogeneous, linearly elastic plate. To describe the geometry of the plate, it will be convenient to consider the plate as occupying the region $P_t = \Omega \times (-t/2, t/2)$, where Ω is a bounded domain in $\mathbb{R}^2$ and $t \in (0, 1]$. We are interested in the case when the

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plate is thin, so that the thickness t will be small. We denote the union of the top and bottom surfaces of the plate by $\partial P_t^{\pm} = \Omega \times \{-t/2, t/2\}$ and the lateral boundary by $\partial P_t^{\mathrm{L}} = \partial\Omega \times (-t/2, t/2)$ (see Fig. 1). We suppose that the plate is loaded by a surface force density $\underline{g}\colon \partial P_t^{\pm} \to \mathbb{R}^3$ and a volume force density $\underline{f}\colon P_t \to \mathbb{R}^3$, and is clamped along its lateral boundary. The resulting stress $\underline{\underline{\sigma}}^*\colon P_t \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and displacement $\underline{u}^*\colon P_t \to \mathbb{R}^3$ then satisfy the boundary-value problem

$$\begin{aligned} &\mathcal{A}\underline{\underline{\sigma}}^* = \underline{\underline{\varepsilon}}(\underline{u}^*), \quad -\,\underline{\mathrm{div}}\,\underline{\underline{\sigma}}^* = \underline{f} \text{ in } P_t, \\ &\underline{\underline{\sigma}}^*\underline{n} = \underline{g} \text{ on } \partial P_t^{\pm}, \quad \underline{u}^* = 0 \text{ on } \partial P_t^{\mathrm{L}}. \end{aligned} \tag{1}$$

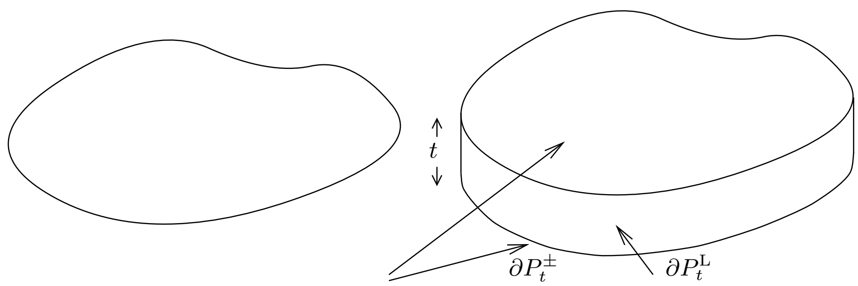


Fig. 1. The two-dimensional domain Ω and plate domain P_t

Here $\underline{\underline{\varepsilon}}(\underline{u}^*)$ denotes the infinitesimal strain tensor associated to the displacement vector $\underline{u}^*$, namely the symmetric part of its gradient, and $\underline{\mathrm{div}}\,\underline{\underline{\sigma}}$ denotes the vector divergence of the symmetric matrix $\underline{\underline{\sigma}}$ taken by rows. The compliance tensor $\mathcal{A}$ is given by $\mathcal{A}\underline{\underline{\tau}} = (1+\nu)\underline{\underline{\tau}}/E - \nu\,\mathrm{tr}(\underline{\underline{\tau}})\underline{\underline{\delta}}/E$, with $E > 0$ Young's modulus, $\nu \in [0, 1/2)$ Poisson's ratio, and $\underline{\underline{\delta}}$ the 3×3 identity matrix.

A plate model seeks to approximate the solution of the elasticity problem (1) in terms of the solution of a system of partial differential equations on the two-dimensional domain Ω without requiring the solution of a three-dimensional problem. The passage from the 3-D problem to a plate model is known as *dimensional reduction*.

By taking odd and even parts with respect to the variable x_3, the three-dimensional plate problem splits into two decoupled problems which correspond to *stretching* and *bending* of the plate. The most common plate stretching models are variants of the equations of generalized plane stress. The most common plate bending models are variants of the Kirchhoff-Love biharmonic plate model or of the Reissner-Mindlin plate model. We speak of variants here, because the specification of the forcing functions for the 2-D differential equations in terms of the 3-D loads $\underline{g}$ and $\underline{f}$ differs for different models to be found in the literature, as does the specification of the approximate 3-D stresses and displacements in terms of the solutions of the 2-D boundary-value problems. Moreover, there is a coefficient in the Reissner-Mindlin model, the so-called shear correction factor, which is given different values in the literature. So there is no universally accepted basic two-dimensional model of plate stretching or bending.

2 A variational approach to dimensional reduction

The Hellinger-Reissner principle gives a variational characterization of the solution to the three-dimensional problem (1). We will consider two forms of this principle.

2.1 The first variational approach

To state the first form of the Hellinger-Reissner principle, which we label HR, we define

$$\underline{\underline{\Sigma}}^\bullet = \underline{\underline{L}}^2(P_t), \quad \underline{V}^\bullet = \{\underline{v} \in \underline{H}^1(P_t) \colon \underline{v} = 0 \text{ on } \partial P_t^{\mathrm{L}}\}.$$

Then HR characterizes $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ as the unique critical point (namely a saddle point) of the HR functional

$$J(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2}\int_{P_t} \mathcal{A}\underline{\underline{\tau}} \colon \underline{\underline{\tau}}\, d\underline{x} - \int_{P_t} \underline{\underline{\tau}} \colon \underline{\underline{\varepsilon}}(\underline{v})\, d\underline{x} + \int_{P_t} \underline{f} \cdot \underline{v}\, d\underline{x} + \int_{\partial P_t^\pm} \underline{g} \cdot \underline{v}\, d\underset{\sim}{x}$$

on $\underline{\underline{\Sigma}}^\bullet \times \underline{V}^\bullet$. Equivalently, $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ is the unique element of $\underline{\underline{\Sigma}}^\bullet \times \underline{V}^\bullet$ satisfying the weak equations

$$\int_{P_t} \mathcal{A}\underline{\underline{\sigma}}^* \colon \underline{\underline{\tau}}\, d\underline{x} - \int_{P_t} \underline{\underline{\varepsilon}}(\underline{u}) \colon \underline{\underline{\tau}}\, d\underline{x} = 0 \quad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{\Sigma}}^\bullet, \tag{2}$$

$$\int_{P_t} \underline{\underline{\sigma}} \colon \underline{\underline{\varepsilon}}(\underline{v})\, d\underline{x} = \int_{P_t} \underline{f} \cdot \underline{v}\, d\underline{x} + \int_{\partial P_t^\pm} \underline{g} \cdot \underline{v}\, d\underset{\sim}{x} \quad \text{for all } \underline{v} \in \underline{V}^\bullet. \tag{3}$$

Plate models may be derived by replacing $\underline{\underline{\Sigma}}^\bullet$ and $\underline{V}^\bullet$ in HR with subspaces $\underline{\underline{\Sigma}}$ and $\underline{V}$ which admit only a specified polynomial dependence on x_3 and then defining $(\underline{\underline{\sigma}}, \underline{u})$ as the unique critical point of J over $\underline{\underline{\Sigma}} \times \underline{V}$. This is equivalent to restricting the trial and test spaces in the weak formulation to $\underline{\underline{\Sigma}} \times \underline{V}$. We insure a unique solution by requiring that $\underline{\underline{\varepsilon}}(\underline{V}) \subset \underline{\underline{\Sigma}}$. Here we shall consider only one of these models, which we denote HR(1). Define the two-dimensional analogue of the compliance tensor by $A\underset{\approx}{\tau} = (1+\nu)\underset{\approx}{\tau}/E - \nu\,\mathrm{tr}(\underset{\approx}{\tau})\underset{\approx}{\delta}/E$. It can be shown that the HR(1) solution is given by

$$\underline{u}(\underline{x}) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}) \\ 0 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\phi}(\underset{\sim}{x})x_3 \\ \omega(\underset{\sim}{x}) \end{pmatrix},$$

$$\underline{\underline{\sigma}}(\underline{x}) = \begin{pmatrix} A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\eta}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\phi})x_3 & \frac{E}{2(1+\nu)}(\underset{\sim}{\nabla}\omega - \underset{\sim}{\phi}) \\ \frac{E}{2(1+\nu)}(\underset{\sim}{\nabla}\omega - \underset{\sim}{\phi})^T & 0 \end{pmatrix},$$

where $\underset{\sim}{\eta}$ is determined by a classical generalized plane stress problem and $\underset{\sim}{\phi}$ and ω by a Reissner-Mindlin problem. Specifically,

$$-t \underset{\sim}{\operatorname{div}} A^{-1} \underset{\approx}{\varepsilon}(\underset{\sim}{\eta}) = 2\underset{\sim}{g}^0 + \underset{\sim}{f}^0 \text{ in } \Omega, \qquad \underset{\sim}{\eta} = 0 \text{ on } \partial\Omega, \tag{4}$$

$$-\frac{t^3}{12} \underset{\sim}{\operatorname{div}} A^{-1} \underset{\approx}{\varepsilon}(\underset{\sim}{\phi}) + t\frac{E}{2(1+\nu)}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\omega) = -t(\underset{\sim}{g}^1 + \underset{\sim}{f}^1) \text{ in } \Omega, \tag{5}$$

$$t\frac{E}{2(1+\nu)} \operatorname{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\omega) = 2g_3^0 + f_3^0 \text{ in } \Omega, \tag{6}$$

$$\underset{\sim}{\phi} = 0, \quad \omega = 0 \text{ on } \partial\Omega. \tag{7}$$

In the above, (and in this section only for clarity), we use $\sim$ and $\approx$ to denote two-dimensional vectors and 2×2 matrices and $\_$ and $=$ to denote three-dimensional vectors and 3×3 matrices, respectively. We also define

$$g_3^0(\underset{\sim}{x}) = \frac{1}{2}\big[g_3(\underset{\sim}{x}, t/2) + g_3(\underset{\sim}{x}, -t/2)\big], \quad g_3^1(\underset{\sim}{x}) = \frac{1}{2}\big[g_3(\underset{\sim}{x}, t/2) - g_3(\underset{\sim}{x}, -t/2)\big],$$

$$f_3^0(\underset{\sim}{x}) = \int_{-t/2}^{t/2} f_3(\underset{\sim}{x}, x_3)\, dx_3, \quad f_3^1(\underset{\sim}{x}) = \int_{-t/2}^{t/2} f_3(\underset{\sim}{x}, x_3)\frac{x_3}{t}\, dx_3,$$

with $\underset{\sim}{g}^0$, $\underset{\sim}{g}^1$, $\underset{\sim}{f}^0$, and $\underset{\sim}{f}^1$ defined analogously. The verification of these equations is straightforward, but tedious.

In the case of a purely transverse bending load, the system (5)–(7) is the classical Reissner-Mindlin system with shear correction factor 1. When the bending is also affected by nonzero $\underset{\sim}{g}^1$ or $\underset{\sim}{f}^1$, then these appear as an applied couple in the Reissner-Mindlin system. Thus we see that the HR(1) method is a simple approach to deriving generalized plane stress and Reissner-Mindlin type models. There is an alternative approach, however, that produces models that are both more accurate and more amenable to rigorous justification than the methods based on HR discussed above. We discuss this approach below.

2.2 An alternative variational approach

A second form of the Hellinger-Reissner principle, which we shall call HR′, leads to somewhat different plate models. For HR′ we define

$$\underline{\underline{\Sigma}}_g^* = \big\{\underline{\underline{\sigma}} \in \underline{\underline{H}}(\underline{\operatorname{div}}, P_t) \mid \underline{\underline{\sigma}}\underline{n} = \underline{g} \text{ on } \partial P_t^{\pm}\big\}, \quad \underline{V}^* = \underline{L}^2(P).$$

Then HR′ characterizes $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ as the unique critical point (again a saddle point) of the HR′ functional

$$J'(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2}\int_{P_t} \mathcal{A}\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x} + \int_{P_t} \underline{\operatorname{div}}\, \underline{\underline{\tau}} \cdot \underline{v}\, d\underline{x} + \int_{P_t} \underline{f} \cdot \underline{v}\, d\underline{x}$$

on $\underline{\underline{\Sigma}}_g^* \times \underline{V}^*$. Equivalently, $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ is the unique element of $\underline{\underline{\Sigma}}_g^* \times \underline{V}^*$ satisfying the weak equations

$$\int_{P_t} \mathcal{A}\underline{\underline{\sigma}}^*: \underline{\underline{\tau}}\, d\underline{x} + \int_{P_t} \underline{u} \cdot \underline{\operatorname{div}}\, \underline{\underline{\tau}}\, d\underline{x} = 0 \quad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{\Sigma}}_0^*,$$
$$\int_{P_t} \underline{\operatorname{div}}\, \underline{\underline{\sigma}} \cdot \underline{v}\, d\underline{x} = -\int_{P_t} \underline{f} \cdot \underline{v}\, d\underline{x} \quad \text{for all } \underline{v} \in \underline{V}^*.$$

Here $\underline{\underline{\Sigma}}_0^* = \{\underline{\underline{\sigma}} \in \underline{\underline{H}}(\underline{\operatorname{div}}, P_t) \mid \underline{\underline{\sigma}}\underline{n} = 0 \text{ on } \partial P_t^{\pm}\}$. Note that the displacement boundary conditions, which were essential to the first form of the Hellinger-Reissner principle, are natural in this setting, while the reverse situation holds for the traction boundary conditions.

By restricting J' to subspaces of $\underline{\underline{\Sigma}}_g^*$ and $\underline{V}^*$ with a specified polynomial dependence on x_3, we also obtain a variety of plate models. Here we shall consider only one of these, which we denote HR$'$(1). The HR$'$(1) solution is:

$$\underline{u}(\underline{x}) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}) \\ \varrho(\underset{\sim}{x})x_3 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\phi}(\underset{\sim}{x})x_3 \\ \omega(\underset{\sim}{x}) + \omega_2(\underset{\sim}{x})r(x_3) \end{pmatrix},$$
$$\underline{\underline{\sigma}}(\underline{x}) = \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}) & \frac{2x_3}{t}\underset{\sim}{g}^0(\underset{\sim}{x}) \\ \frac{2x_3}{t}\underset{\sim}{g}^0(\underset{\sim}{x})^T & g_3^1(\underset{\sim}{x}) + \sigma_{33}^0(\underset{\sim}{x})q(x_3) \end{pmatrix}$$
$$+ \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x})\frac{x_3}{t} & \underset{\sim}{g}^1(\underset{\sim}{x}) + \underset{\sim}{\sigma}^0(\underset{\sim}{x})q(x_3) \\ \underset{\sim}{g}^1(\underset{\sim}{x})^T + \underset{\sim}{\sigma}^0(\underset{\sim}{x})^T q(x_3) & g_3^0(\underset{\sim}{x})\frac{2x_3}{t} + \sigma_{33}^1(x_3)s(x_3) \end{pmatrix},$$

where the coefficient functions $\underset{\sim}{\eta}$, ϱ, $\underset{\sim}{\phi}$, ω, ω_2, $\underset{\approx}{\sigma}^0$, σ_{33}^0, $\underset{\approx}{\sigma}^1$, $\underset{\sim}{\sigma}^0$, and σ_{33}^1 are functions of $\underset{\sim}{x}$ which we shall describe, and the polynomials q, r, and s are given by $q(z) = 3/2 - 6z^2/t^2$, $r(z) = 6z^2/t^2 - 3/10$, and $s(z) = (5/2)z/t - 10z^3/t^3$.

The stretching portion of the solution is determined by the solution to the boundary-value problem

$$-t\, \underset{\sim}{\operatorname{div}}\, A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\eta}) = \underset{\sim}{l}_1 + t\frac{\nu}{1-\nu}\underset{\sim}{\nabla} l_2 \text{ in } \Omega, \qquad \underset{\sim}{\eta} = 0 \text{ on } \partial\Omega, \tag{8}$$
$$\text{where} \quad \underset{\sim}{l}_1 = 2\underset{\sim}{g}^0 + \underset{\sim}{f}^0, \quad l_2 = g_3^1 + \frac{t}{6}\operatorname{div}\underset{\sim}{g}^0 + f_3^1.$$

With $\underset{\sim}{\eta}$ uniquely determined by (8), the remaining solution quantities are

$$\underset{\approx}{\sigma}^0 = A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\eta}) + \frac{\nu}{1-\nu}l_2\underset{\approx}{\delta}, \qquad \sigma_{33}^0 = \frac{t}{6}\operatorname{div}\underset{\sim}{g}^0 + f_3^1,$$
$$\varrho = \frac{1}{E}\left[-\nu\operatorname{tr}(\underset{\approx}{\sigma}^0) + \frac{6}{5}\sigma_{33}^0 + g_3^1\right].$$

The bending portion of the solution is determined by the solution to the boundary-value problem

$$-\frac{t^3}{12}\underset{\sim}{\operatorname{div}}\, A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\phi}) + t\frac{5}{6}\frac{E}{2(1+\nu)}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\omega) = t\underset{\sim}{k}_1 - \frac{t^2}{12}\underset{\sim}{\nabla} k_2 \text{ in } \Omega,$$
$$t\frac{5}{6}\frac{E}{2(1+\nu)}\operatorname{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\omega) = k_3 \text{ in } \Omega, \qquad \underset{\sim}{\phi} = 0, \quad \omega = 0 \text{ on } \partial\Omega, \tag{9}$$

$$\text{where} \qquad \underset{\sim}{k}_1 = -\frac{5}{6}\underset{\sim}{g}^1 - \underset{\sim}{f}^1, \qquad k_2 = \frac{\nu}{1-\nu}\left[\frac{t}{5}\operatorname{div}\underset{\sim}{g}^1 + \frac{12}{5}g_3^0 + f_3^2\right],$$
$$f_3^2(\underset{\sim}{x}) = \int_{-t/2}^{t/2} f_3(\underline{x}) r(x_3)\,\mathrm{d}x_3, \qquad k_3 = \frac{t}{6}\operatorname{div}\underset{\sim}{g}^1 + 2g_3^0 + f_3^0.$$

The boundary value problem (9) determining the bending solution is a somewhat different version of the Reissner-Mindlin equations than (5)–(7), which arose from the HR(1) model. Not only are the formulas for the applied load and couple more involved, but a shear correction factor of 5/6 has been introduced. With $\underset{\sim}{\phi}$ and ω determined by (9), we find

$$\underset{\approx}{\sigma}^1 = -tA^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\phi}) + k_2\underset{\approx}{\delta}, \qquad \underset{\sim}{\sigma}^0 = \frac{5}{6}\left[\frac{E}{2(1+\nu)}(-\underset{\sim}{\phi} + \underset{\sim}{\nabla}\omega) - \underset{\sim}{g}^1\right],$$
$$\sigma_{33}^1 = \frac{t}{5}\operatorname{div}\underset{\sim}{g}^1 + \frac{2}{5}g_3^0 + f_3^2, \qquad \omega_2 = \frac{t}{E}\left[\frac{1}{6}g_3^0 + \frac{5}{42}\sigma_{33}^1 - \frac{\nu}{12}\operatorname{tr}(\underset{\approx}{\sigma}^1)\right].$$

For this model, it is possible to use the "two-energies principle" to derive rigorous error estimates between the solution of the three-dimensional model and the two-dimensional reduced model as a function of the plate thickness (see [1] for details).

3 The Reissner–Mindlin model

From the previous section, we see that if we introduce the tensor $\mathsf{C} = A^{-1}$ and scale the right hand side, then the Reissner-Mindlin equations may be written in the form

$$-\operatorname{\mathbf{div}} \mathsf{C}\,\mathcal{E}(\boldsymbol{\theta}) - \lambda t^{-2}(\operatorname{\mathbf{grad}} w - \boldsymbol{\theta}) = -\boldsymbol{f},$$
$$-\operatorname{div}(\operatorname{\mathbf{grad}} w - \boldsymbol{\theta}) = \lambda^{-1}t^2 g,$$

with λ a constant depending on the particular version of the model that is chosen. We also have a Reissner-Mindlin energy

$$J(\boldsymbol{\theta}, w) = \frac{1}{2}\int_\Omega \mathsf{C}\,\mathcal{E}(\boldsymbol{\theta}) : \mathcal{E}(\boldsymbol{\theta}) + \frac{1}{2}\lambda t^{-2}\int_\Omega |\operatorname{\mathbf{grad}} w - \boldsymbol{\theta}|^2 - \int_\Omega gw + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\theta}, \tag{10}$$

for which the above equations are the Euler equations. As both a theoretical and computational tool, it is useful to introduce the shear stress $\boldsymbol{\gamma} = \lambda t^{-2}(\operatorname{\mathbf{grad}} w - \boldsymbol{\theta})$. Then we have the equivalent Reissner-Mindlin system

$$-\operatorname{\mathbf{div}} \mathsf{C}\,\mathcal{E}(\boldsymbol{\theta}) - \boldsymbol{\gamma} = -\boldsymbol{f}, \tag{11}$$
$$-\operatorname{div}\boldsymbol{\gamma} = g, \tag{12}$$
$$\operatorname{\mathbf{grad}} w - \boldsymbol{\theta} - \lambda^{-1}t^2\boldsymbol{\gamma} = 0, \tag{13}$$

For simplicity we restrict our attention to the case of a clamped plate, i.e., we consider the boundary conditions $\boldsymbol{\theta} = 0$ and $w = 0$ on the boundary $\partial\Omega$. A weak formulation of the Reissner–Mindlin model is then given by:

Find $\boldsymbol{\theta} \in \mathring{\boldsymbol{H}}^1(\Omega)$, $w \in \mathring{H}^1(\Omega)$, $\boldsymbol{\gamma} \in \boldsymbol{L}^2(\Omega)$ such that

$$a(\boldsymbol{\theta}, \boldsymbol{\phi}) + (\boldsymbol{\gamma}, \mathbf{grad}\, v - \boldsymbol{\phi}) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \quad \boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega), v \in \mathring{H}^1(\Omega), \tag{14}$$

$$(\mathbf{grad}\, w - \boldsymbol{\theta}, \boldsymbol{\eta}) - \lambda^{-1} t^2 (\boldsymbol{\gamma}, \boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega), \tag{15}$$

where $a(\boldsymbol{\theta}, \boldsymbol{\phi}) = (\mathcal{C}\, \mathcal{E}(\boldsymbol{\theta}), \mathcal{E}(\boldsymbol{\phi}))$.

4 Properties of the solution

As $t \to 0$, $\boldsymbol{\theta} \to \boldsymbol{\theta}^0$ and $w \to w^0$, where $\boldsymbol{\theta}^0 = \mathbf{grad}\, w^0$. One can then show that w^0 satisfies the limit problem: Find $w^0 \in \mathring{H}^2(\Omega) = \{v \in H^2(\Omega) : v = \partial v/\partial n = 0 \text{ on } \partial\Omega\}$ such that

$$a(\mathbf{grad}\, w^0, \mathbf{grad}\, v) = (g, v) - (\boldsymbol{f}, \mathbf{grad}\, v), \qquad v \in \mathring{H}^2(\Omega).$$

This is the weak form of the equation: $\operatorname{div} \mathbf{div}\, \mathcal{C}\, \mathcal{E}(\mathbf{grad}\, w^0) = g + \operatorname{div} \boldsymbol{f}$, which after the application of some calculus identities becomes:

$$D\, \Delta^2\, w^0 = g + \operatorname{div} \boldsymbol{f}, \qquad D = \frac{E}{12(1-\nu^2)}. \tag{16}$$

Hence, the limiting problem is the biharmonic problem.

To understand this limiting behavior and also to derive the regularity results presented in the next section, it is useful to introduce the Helmholtz decomposition, and rewrite the Reissner-Mindlin system as a perturbed Stokes equation. For some $r \in \mathring{H}^1(\Omega)$ and $p \in \hat{H}^1(\Omega)$, we can write

$$\boldsymbol{\gamma} = \lambda t^{-2}(\mathbf{grad}\, w - \boldsymbol{\theta}) = \mathbf{grad}\, r + \mathbf{curl}\, p.$$

Then it is easy to check that problem (14)-(15) is equivalent to the system:

Find $(r, \boldsymbol{\theta}, p, w) \in \mathring{H}^1(\Omega) \times \mathring{\boldsymbol{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ such that

$$(\mathbf{grad}\, r, \mathbf{grad}\, \mu) = (g, \mu), \quad \mu \in \mathring{H}^1(\Omega), \tag{17}$$

$$a(\boldsymbol{\theta}, \boldsymbol{\phi}) - (\mathbf{curl}\, p, \boldsymbol{\phi}) = (\mathbf{grad}\, r, \boldsymbol{\phi}) - (\boldsymbol{f}, \boldsymbol{\phi}), \quad \boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega), \tag{18}$$

$$-(\boldsymbol{\theta}, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p, \mathbf{curl}\, q) = 0, \quad q \in \hat{H}^1(\Omega), \tag{19}$$

$$(\mathbf{grad}\, w, \mathbf{grad}\, s) = (\boldsymbol{\theta} + \lambda^{-1} t^2\, \mathbf{grad}\, r, \mathbf{grad}\, s), \quad s \in \mathring{H}^1(\Omega). \tag{20}$$

We then define $(\boldsymbol{\theta}^0, p^0, w^0) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ as the solution of (17)-(20) with $t = 0$. Note that for r known and $t = 0$, (18)-(19) is the ordinary Stokes system for $(\theta_2^0, -\theta_1^0, p^0)$.

5 Regularity results

A key issue in the approximation of the Reissner-Mindlin plate problem is the regularity of the solution and especially its dependence on the plate thickness t. For this problem, there is a boundary layer, whose strength depends on the particular boundary condition. There are a number of physically interesting boundary conditions:

$$\begin{aligned}
\boldsymbol{\theta}\cdot\boldsymbol{n} = \boldsymbol{\theta}\cdot\boldsymbol{s} = w = 0 &\qquad \text{hard clamped (hc)},\\
\boldsymbol{\theta}\cdot\boldsymbol{n} = M_s(\boldsymbol{\theta})\cdot\boldsymbol{s} = w = 0 &\qquad \text{soft clamped (sc)},\\
M_n(\boldsymbol{\theta}) = \boldsymbol{\theta}\cdot\boldsymbol{s} = w = 0 &\qquad \text{hard simply supported (hss)},\\
M_n(\boldsymbol{\theta}) = M_s(\boldsymbol{\theta}) = w = 0 &\qquad \text{soft simply supported (sss)},\\
M_n(\boldsymbol{\theta}) = M_s(\boldsymbol{\theta}) = \partial w/\partial n - \boldsymbol{\theta}\cdot\boldsymbol{n} = 0 &\qquad \text{free (f)},
\end{aligned}$$

where $\boldsymbol{n}$ and $\boldsymbol{s}$ denote the unit normal and counterclockwise unit tangent vectors, respectively, and $M_n(\boldsymbol{\theta}) = \boldsymbol{n}\cdot \mathcal{C}\,\mathcal{E}(\boldsymbol{\theta})\boldsymbol{n}$, $M_s(\boldsymbol{\theta}) = \boldsymbol{s}\cdot \mathcal{C}\,\mathcal{E}(\boldsymbol{\theta})\boldsymbol{n}$. In the case of a domain with smooth boundary, it is shown in [12] and [13] that for all boundary conditions, the transverse displacement and all its derivatives are bounded uniformly in t, i.e., $\|w\|_s \le C$, $s \in \mathbb{R}$. Estimates showing the boundary layers, ordered from weakest to strongest, are given below.

$$\begin{aligned}
&\|\boldsymbol{\theta}\|_s \le Ct^{\min(0,7/2-s)}, &&\|\boldsymbol{\gamma}\|_s \le Ct^{\min(0,3/2-s)}, \quad s\in\mathbb{R}, \quad \text{(sc)}\\
&\|\boldsymbol{\theta}\|_s \le Ct^{\min(0,5/2-s)}, &&\|\boldsymbol{\gamma}\|_s \le Ct^{\min(0,1/2-s)}, \quad s\in\mathbb{R}, \quad \text{(hc), (hss)},\\
&\|\boldsymbol{\theta}\|_s \le Ct^{\min(0,3/2-s)}, &&\|\boldsymbol{\gamma}\|_s \le Ct^{\min(0,-1/2-s)}, \quad s\in\mathbb{R}, \quad \text{(sss), (f)}.
\end{aligned}$$

Additional results can be found in [10].

We will also need estimates that show the precise dependence on the data of the problem and which are valid when Ω is a convex polygon, the case we consider in the derivation of error estimates for finite element approximation schemes. We establish such estimates below for the case of the clamped plate.

Theorem 5.1 *Let Ω be a convex polygon or a smoothly bounded domain in the plane. For any $t \in (0,1]$, $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, and $g \in H^{-1}(\Omega)$, there exists a unique solution $(r,\boldsymbol{\theta},p,w) \in \mathring{H}^1(\Omega)\times\mathring{\boldsymbol{H}}^1(\Omega)\times\hat{H}^1(\Omega)\times\mathring{H}^1(\Omega)$ satisfying (17)-(20). Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, then $\boldsymbol{\theta} \in \boldsymbol{H}^2(\Omega)$ and there exists a constant C independent of t, $\boldsymbol{f}$, and g, such that*

$$\|\boldsymbol{\theta}\|_2 + \|r\|_1 + \|p\|_1 + t\|p\|_2 + \|w\|_1 + \|\boldsymbol{\gamma}\|_0 \le C(\|\boldsymbol{f}\|_0 + \|g\|_{-1}), \qquad (21)$$

If, in addition, $g \in L^2(\Omega)$, then r and $w \in H^2(\Omega)$ and

$$\|r\|_2 + \|w\|_2 + t\|\boldsymbol{\gamma}\|_1 + \|\operatorname{div}\boldsymbol{\gamma}\|_0 \le C(\|g\|_0 + \|\boldsymbol{f}\|_0). \qquad (22)$$

Finally, if $(\boldsymbol{\theta}^0, w^0)$ denotes the solution of (17)-(20) with $t = 0$, then

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}^0\|_1 \le Ct(\|\boldsymbol{f}\|_0 + \|g\|_{-1}), \qquad \|w - w^0\|_2 \le Ct(\|\boldsymbol{f}\|_0 + \|g\|_{-1} + t\|g\|_0),$$
$$\|w^0\|_3 \le C(\|\boldsymbol{f}\|_0 + \|g\|_{-1}). \qquad (23)$$

Proof. Existence and uniqueness are easy to establish using the equivalence of this system to (14)-(15) and standard results, so we concentrate on the regularity estimates. We first observe that standard regularity results for Poisson's equation gives

$$\|r\|_1 \le C\|g\|_{-1}, \qquad \|r\|_2 \le \|g\|_0.$$

We next recall a regularity result for the Stokes system, valid both for the case of a domain with smooth boundary and for a convex polygon.

$$\|\boldsymbol{\theta}^0\|_2 + \|p^0\|_1 \le C(\|\boldsymbol{f}\|_0 + \|r\|_1) \le C(\|\boldsymbol{f}\|_0 + \|g\|_{-1}).$$

Now from (18) and (19), and the corresponding equations for $\boldsymbol{\theta}^0$ and p^0, we get

$$a(\boldsymbol{\theta}-\boldsymbol{\theta}^0, \boldsymbol{\phi}) - (\mathbf{curl}(p-p^0), \boldsymbol{\phi}) + (\boldsymbol{\theta}-\boldsymbol{\theta}^0, \mathbf{curl}\, q) + \lambda^{-1}t^2(\mathbf{curl}(p-p^0), \mathbf{curl}\, q)$$
$$= -\lambda^{-1}t^2(\mathbf{curl}\, p^0, \mathbf{curl}\, q), \quad (\boldsymbol{\phi}, q) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \hat{H}^1(\Omega).$$

Choosing $\boldsymbol{\phi} = \boldsymbol{\theta} - \boldsymbol{\theta}^0$ and $q = p - p^0$, we obtain

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^0\|_1^2 + t^2\|\,\mathbf{curl}(p-p^0)\|_0^2 \le Ct^2\|p^0\|_1\|\,\mathbf{curl}(p-p^0)\|_0.$$

It easily follows that

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^0\|_1 + t\|p - p^0\|_1 \le Ct\|p^0\|_1 \le Ct(\|\boldsymbol{f}\|_0 + \|g\|_{-1}), \tag{24}$$

which establishes the first estimate in (23). We also get that

$$\|p\|_1 \le C(\|\boldsymbol{f}\|_0 + \|g\|_{-1}).$$

Applying standard estimates for second order elliptic problems to (18), we further obtain

$$\|\boldsymbol{\theta}\|_2 \le C(\|p\|_1 + \|r\|_1 + \|\boldsymbol{f}\|_0) \le C(\|\boldsymbol{f}\|_0 + \|g\|_{-1}).$$

Now from (19) and the definition of $\boldsymbol{\theta}^0$, we get

$$\lambda^{-1}t^2(\mathbf{curl}\, p, \mathbf{curl}\, q) = -(\boldsymbol{\theta}, \mathbf{curl}\, q) = (\boldsymbol{\theta}^0 - \boldsymbol{\theta}, \mathbf{curl}\, q), \quad q \in \hat{H}^1(\Omega).$$

Thus p is the weak solution of the boundary value problem

$$-\Delta p = \lambda t^{-2} \operatorname{rot}(\boldsymbol{\theta}^0 - \boldsymbol{\theta}) \quad \text{in } \Omega, \qquad \partial p/\partial n = 0 \text{ on } \partial\Omega.$$

Applying elliptic regularity and (24), we get

$$\|p\|_2 \le Ct^{-2}\|\boldsymbol{\theta}^0 - \boldsymbol{\theta}\|_1 \le Ct^{-1}(\|\boldsymbol{f}\|_0 + \|g\|_{-1}).$$

The estimate for w in (21) now follows directly from (20) and the estimate for $\boldsymbol{\gamma}$ in (21) follows immediately from its definition and the estimates for r and p. The estimate (22) follows directly from the regularity result for r, the definition of $\boldsymbol{\gamma}$, elliptic regularity of w, and the previous results. Finally, it remains to establish the last two estimates in (23). Subtracting the analogue of (20) from (20), we get that

$$(\mathbf{grad}(w - w^0), \mathbf{grad}\, s) = (\boldsymbol{\theta} - \boldsymbol{\theta}^0 + \lambda^{-1} t^2\, \mathbf{grad}\, r, \mathbf{grad}\, s), \quad s \in \mathring{H}^1(\Omega).$$

This is the weak form of the equation

$$-\Delta(w - w^0) = -\operatorname{div}(\boldsymbol{\theta} - \boldsymbol{\theta}^0) - \lambda^{-1} t^2\, \Delta\, r.$$

Combining standard regularity estimates for Poisson's equation with our previous results, we get

$$\|w - w^0\|_2 \le C(\|\boldsymbol{\theta} - \boldsymbol{\theta}^0\|_1 + t^2 \|r\|_2) \le Ct(\|\boldsymbol{f}\| + \|g\|_{-1} + t\|g\|_0).$$

Finally, using the fact that w^0 satisfies the biharmonic equation (16), together with the boundary conditions $w^0 = \partial w^0/\partial n = 0$, we get the estimate

$$\|w^0\|_3 \le C\|g + \operatorname{div} \boldsymbol{f}\|_{-1} \le C(\|g\|_{-1} + \|\boldsymbol{f}\|_0).$$

6 Finite element discretizations

The challenge in devising finite element approximation schemes for the Reissner-Mindlin plate model is to find schemes whose approximation accuracy does not deteriorate as the plate thickness becomes very small. For example, if one minimizes the Reissner-Mindlin energy over subspaces consisting of low order finite elements, then the resulting approximation suffers from the problem of "locking." This problem is most easily described by recalling that as $t \to 0$, the minimizer $(\boldsymbol{\theta}, w)$ of (10) approaches $(\boldsymbol{\theta}^0, w^0)$, where $\boldsymbol{\theta}^0 = \mathbf{grad}\, w^0$. If we discretize the problem directly by seeking $\boldsymbol{\theta}_h \in \boldsymbol{\Theta}_h$ and $w_h \in W_h$ minimizing $J(\boldsymbol{\theta}, w)$ over $\boldsymbol{\Theta}_h \times W_h$, then as $t \to 0$ we will have $(\boldsymbol{\theta}_h, w_h) \to (\boldsymbol{\theta}_h^0, w_h^0)$ where, again, $\boldsymbol{\theta}_h^0 = \mathbf{grad}\, w_h^0$. The locking problem occurs because, for low order finite element spaces, this last condition is too restrictive to allow for good approximations of smooth functions. In particular, if continuous piecewise linear functions are chosen to approximate both variables, then $\boldsymbol{\theta}_h^0 \equiv \mathbf{grad}\, w_h^0$ would be continuous *and* piecewise constant, with zero boundary conditions: Only the choice $\boldsymbol{\theta}_h^0 = 0$ can satisfy all these conditions. Hence, unless the combination of finite element spaces is chosen carefully, this problem is likely to occur.

Many of the successful locking-free finite element schemes have taken the following approach. Let $\boldsymbol{\Theta}_h \subset \mathring{\boldsymbol{H}}^1(\Omega)$, $W_h \subset \mathring{H}^1(\Omega)$, $\boldsymbol{\Gamma}_h \subset \boldsymbol{L}^2(\Omega)$, where $\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h$. Let $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ be an interpolation operator mapping $\mathring{\boldsymbol{H}}^1(\Omega)$ to $\boldsymbol{\Gamma}_h$. Then consider finite element approximation schemes of the form:

Find $\boldsymbol{\theta}_h \in \boldsymbol{\Theta}_h$, $w_h \in W_h$, $\boldsymbol{\gamma}_h \in \boldsymbol{\Gamma}_h$ such that

$$a(\boldsymbol{\theta}_h, \boldsymbol{\phi}) + (\boldsymbol{\gamma}_h, \mathbf{grad}\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi}) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \qquad \boldsymbol{\phi} \in \boldsymbol{\Theta}_h, v \in W_h,$$
$$(\mathbf{grad}\, w_h - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}_h, \boldsymbol{\eta}) - \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in \boldsymbol{\Gamma}_h. \tag{25}$$

The point of introducing the operator $\boldsymbol{\Pi}^{\Gamma}$ is that now as $t \to 0$, we will get that $\mathbf{grad}\, w_{h,0} \to \boldsymbol{\Pi}^{\Gamma} \boldsymbol{\theta}_{h,0}$. If $\boldsymbol{\Pi}^{\Gamma}$ is chosen properly, this condition may be much easier to satisfy, while still maintaining good approximation properties of each subspace.

We will also consider some nonconforming discretizations in which the either the space $\boldsymbol{\Theta}_h$ or W_h consists of functions which belong to H^1 on each triangle, but not globally. In the first case, the operator $\mathcal{E}$ entering into the definition of the bilinear form a must be replaced with $\mathcal{E}_h$, the operator obtained by applying $\mathcal{E}$ piecewise on each triangle. Similarly, in the second case, the operator **grad** must be replaced by its piecewise counterpart, $\mathbf{grad}_h$.

7 Abstract error analysis

In order to analyze approximation schemes using a common framework, we first prove several abstract approximation results. These results will make use of the following assumptions about the approximation properties of the finite dimensional subspaces and the operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ that define the various methods.

$$\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h, \tag{26}$$
$$\|\boldsymbol{\eta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \eta\| \le ch \|\boldsymbol{\eta}\|_1, \qquad \boldsymbol{\eta} \in \boldsymbol{H}^1(\Omega), \tag{27}$$

for some constant c independent of h. Letting $\boldsymbol{M}_r$ denote the space of discontinuous piecewise polynomials of degree $\le r$, we also define $r_0 \ge -1$ as the greatest integer r for which

$$(\boldsymbol{\eta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\eta}, \boldsymbol{\zeta}) = 0, \qquad \boldsymbol{\zeta} \in \boldsymbol{M}_r. \tag{28}$$

Of course this relation trivially holds for $r = -1$. We then let $\boldsymbol{\Pi}^0$ denote the $\boldsymbol{L}^2$ projection into $\boldsymbol{M}_{r_0}$.

The following basic result is close to Lemma 3.1 of Durán and Liberman [33].

Theorem 7.1 *Let* $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$, $w^I \in W_h$ *be arbitrary, and define* $\boldsymbol{\gamma}^I = \lambda t^{-2} (\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}^I) \in \boldsymbol{\Gamma}_h$. *Then*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \le C(\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_0 + h\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0 \boldsymbol{\gamma}\|_0).$$

Proof. Clearly

$$a(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\phi}) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \mathbf{grad}\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi}) = (\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}] \boldsymbol{\phi}), \tag{29}$$

for all $\boldsymbol{\phi} \in \boldsymbol{\Theta}_h$ and $v \in W_h$, so

$$a(\boldsymbol{\theta}^I - \boldsymbol{\theta}_h, \boldsymbol{\phi}) + (\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h, \mathbf{grad}\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi}) = a(\boldsymbol{\theta}^I - \boldsymbol{\theta}, \boldsymbol{\phi}) \\ + (\boldsymbol{\gamma}^I - \boldsymbol{\gamma}, \mathbf{grad}\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi}) + (\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\phi}).$$

Taking $\boldsymbol{\phi} = \boldsymbol{\phi}^I - \boldsymbol{\phi}_h$ and $v = w^I - w_h$, noting that $\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}^I = \lambda^{-1} t^2 \boldsymbol{\gamma}^I$ and $\mathbf{grad}\, w_h - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}_h = \lambda^{-1} t^2 \boldsymbol{\gamma}_h$, and using (28), we get the identity

$$a(\boldsymbol{\theta}^I - \boldsymbol{\theta}_h, \boldsymbol{\theta}^I - \boldsymbol{\theta}_h) + \lambda^{-1} t^2 (\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h, \boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h) = a(\boldsymbol{\theta}^I - \boldsymbol{\theta}, \boldsymbol{\theta}^I - \boldsymbol{\theta}_h) \\ + \lambda^{-1} t^2 (\boldsymbol{\gamma}^I - \boldsymbol{\gamma}, \boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h) + (\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}^I - \boldsymbol{\theta}_h]).$$

Using Schwarz's inequality, and (27) and (28), we can bound the last term:

$$|(\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}^I - \boldsymbol{\theta}_h])| \le Ch \|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0 \boldsymbol{\gamma}\|_0 \|\boldsymbol{\theta}^I - \boldsymbol{\theta}_h\|_1.$$

The theorem then follows easily.

Note that if we apply this theorem in a naive way, then the error estimates we obtain will blow up as $t \to 0$. More specifically, if we use the simple estimate

$$t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\| = \lambda t^{-1} \| \mathbf{grad}(w - w^I) - (\boldsymbol{\theta} - \Pi^{\boldsymbol{\Gamma}} \boldsymbol{\theta}^I)\| \\ \le \lambda t^{-1} (\| \mathbf{grad}(w - w^I)\| + \|\boldsymbol{\theta} - \Pi^{\boldsymbol{\Gamma}} \boldsymbol{\theta}^I\|),$$

and use approximation theory to bound each of the terms on the right separately, then the bound will contain the term t^{-1}.

The key idea to using this theorem to obtain error estimates that are independent of the plate thickness t is to find functions $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$ and $w^I \in W_h$ that satisfy

$$\boldsymbol{\gamma}^I = \lambda t^{-2} (\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}^I) = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\gamma}. \tag{30}$$

We then have the following corollary.

Corollary 7.2 *If $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$ and $w^I \in W_h$ satisfy* (30), *then*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \le C(\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\gamma}\|_0 + h\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0 \boldsymbol{\gamma}\|_0).$$

If we also make assumptions about the approximation properties of the functions $\boldsymbol{\theta}^I$, w^I, and $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\gamma}$, we immediately obtain order of convergence estimates. One such result is the following.

Theorem 7.3 *Let $n \ge 1$ and assume for each $\boldsymbol{\theta} \in \boldsymbol{H}^{n+1}(\Omega) \cap \mathring{\boldsymbol{H}}^1(\Omega)$ and $w \in H^{n+2}(\Omega) \cap \mathring{H}^1(\Omega)$, there exists $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$ and $w^I \in W_h$ satisfying* (30). *If for $1 \le r \le n$,*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \le Ch^r \|\boldsymbol{\theta}\|_{r+1}, \tag{31}$$

$$\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\gamma}\|_0 \le Ch^r \|\boldsymbol{\gamma}\|_r, \tag{32}$$

then

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \le C\left(h^r \|\boldsymbol{\theta}\|_{r+1} + h^r t \|\boldsymbol{\gamma}\|_r + h^{r_0+2} \|\boldsymbol{\gamma}\|_{r_0+1}\right).$$

Proof. The proof follows immediately from the hypotheses of the theorem and standard approximation properties of $\boldsymbol{\Pi}^0$.

We now state and prove an abstract estimate for the L^2 errors for the rotation and the transverse displacement. To do so, we first define an appropriate dual problem. Given $\boldsymbol{F} \in \boldsymbol{L}^2(\Omega)$ and $G \in L^2(\Omega)$, define $\boldsymbol{\psi}$, u, and $\boldsymbol{\zeta}$ to be the solution to the auxiliary problem

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\mathbf{grad}\, v - \boldsymbol{\phi}, \boldsymbol{\zeta}) = (\boldsymbol{\phi}, \boldsymbol{F}) + (v, G), \quad \boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1, v \in \mathring{H}^1(\Omega), \tag{33}$$

$$(\boldsymbol{\eta}, \mathbf{grad}\, u - \boldsymbol{\psi}) - \lambda^{-1} t^2 (\boldsymbol{\eta}, \boldsymbol{\zeta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega). \tag{34}$$

Then by the regularity results (21) and (22),

$$\|\boldsymbol{\psi}\|_2 + \|u\|_2 + \|\boldsymbol{\zeta}\| + t\|\boldsymbol{\zeta}\|_1 + \|\operatorname{div} \boldsymbol{\zeta}\|_0 \le c(\|\boldsymbol{F}\|_0 + \|G\|_0). \tag{35}$$

With these definitions we have the following estimate.

Theorem 7.4 *If the hypotheses of Theorems 7.1 and 7.3 are satisfied, then*

$$\begin{aligned} \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|^2/2 + \|w - w_h\|_0^2/2 \le Ch^2(\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0^2) \\ + ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}_h, \boldsymbol{\zeta}) + (\gamma, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\psi}^I). \end{aligned} \tag{36}$$

Proof. Let $\boldsymbol{F} = \boldsymbol{\theta} - \boldsymbol{\theta}_h$ and $G = (w - w_h)$. Then, setting $\boldsymbol{\phi} = \boldsymbol{\theta} - \boldsymbol{\theta}_h$, $v = w - w_h$ in (34) and using the definitions of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_h$ we get

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0^2 + \|w - w_h\|_0^2 = a(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\psi}) + \lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\zeta}) + ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}_h, \boldsymbol{\zeta}). \tag{37}$$

Now, the error equation (29) gives

$$a(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\psi}^I) + \lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\zeta}^I) = (\gamma, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\bar{\boldsymbol{\psi}}).$$

where $\boldsymbol{\zeta}^I = \lambda t^{-2}(\mathbf{grad}\, u^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\psi}^I)$, so (37) becomes

$$\begin{aligned} \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|^2 + \|w - w_h\|_0^2 = a(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\psi} - \boldsymbol{\psi}^I) \\ + \lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\zeta} - \boldsymbol{\zeta}^I) + ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}_h, \boldsymbol{\zeta}) + (\gamma, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\psi}^I). \end{aligned} \tag{38}$$

The first two terms on the right side of (38) are easily bounded by

$$\begin{aligned} C(\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 \|\boldsymbol{\psi} - \boldsymbol{\psi}^I\|_1 + t^2 \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \|\boldsymbol{\zeta} - \boldsymbol{\zeta}^I\|_0) \\ \le Ch(\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0)(\|\boldsymbol{\psi}\|_2 + t\|\boldsymbol{\zeta}\|_1) \\ \le Ch(\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0)(\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0). \end{aligned} \tag{39}$$

Application of the arithmetic-geometric mean inequality establishes the result.

Remark 1. Bounds on the last two terms will depend on the particular method being analyzed.

Next, we establish an abstract estimate for the approximation of the derivatives of the transverse displacement.

Theorem 7.5 *For all $w_I \in W_h$, we have*

$$\begin{aligned}&\|\,\mathbf{grad}[w - w_h]\|_0 \\ &\qquad \le C(\|\,\mathbf{grad}[w - w_I]\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}\|_0 + h\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0).\end{aligned}$$

Proof. Choosing $\eta = \mathbf{grad}\, v_h$, $v_h \in W_h$, we get for all $w_I \in W_h$,

$$(\mathbf{grad}[w_I - w_h], \mathbf{grad}\, v_h) = (\mathbf{grad}[w_I - w], \mathbf{grad}\, v_h) + (\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}_h, \mathbf{grad}\, v_h).$$

Then choosing $v_h = w_h - w_I$, it easily follows that

$$\begin{aligned}&\|\,\mathbf{grad}[w_I - w_h]\|_0 \le \|\,\mathbf{grad}[w_I - w]\|_0 + \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}_h\|_0 \\ &\le \|\,\mathbf{grad}[w_I - w]\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}]\|_0 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 \\ &\qquad \le \|\,\mathbf{grad}[w_I - w]\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}\|_0 + Ch\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0.\end{aligned}$$

The result follows from the triangle inequality.

In some cases, it is also possible to establish improved estimates for the shear stress $\boldsymbol{\gamma}$ in negative norms. We will not derive such estimates here, but will state known results in some cases.

8 Applications of the abstract error estimates

Most of our discussion will be centered on triangular elements. We will henceforth assume that Ω is a convex polygonal domain in the plane, and we let $\mathcal{T}_h$ denote a triangulation of Ω. Let $\mathbf{V}$ and $\mathbf{E}$ denote the set of vertices and edges, respectively in the mesh $\mathcal{T}_h$. We will use the following finite element spaces based on the mesh $\mathcal{T}_h$.

$M_k(\mathcal{T}_h)$:	arbitrary piecewise polynomials of degree $\le k$,
$M_k^l(\mathcal{T}_h)$:	$M_k \cap C^l(\Omega)$,
$M_k^*(\mathcal{T}_h)$:	elements of M_k continuous at k Gauss-points of each interelement edge,
$B_k(\mathcal{T}_h)$:	elements of M_k^0 which vanish on interelement edges,
$\boldsymbol{RT}_k^\perp(\mathcal{T}_h)$:	Raviart–Thomas discretization of order k to $\boldsymbol{H}(\mathrm{rot})$,
$\boldsymbol{BDM}_k^\perp(\mathcal{T}_h)$:	Brezzi-Douglas-Marini discretization of order k to $\boldsymbol{H}(\mathrm{rot})$,
$\boldsymbol{BDFM}_k^\perp(\mathcal{T}_h)$:	Brezzi-Douglas-Fortin-Marini discretization of order k to $\boldsymbol{H}(\mathrm{rot})$.

When there is no risk of confusion, we write M_k for $M_k(\mathcal{T}_h)$, etc. For the scalar-valued function spaces in this list, we have vector-valued analogues in the obvious way. For example, $\boldsymbol{M}_k := M_k \times M_k$. Note that $B_k = 0$ for $k < 3$. For convenience, we interpret M_{-1} as the zero space.

The degrees of freedom for each space determine an interpolation operator from $C^\infty(\Omega)$ or $\boldsymbol{C}^\infty(\Omega)$ into the corresponding space. We denote these operators Π^{M_k}, etc. The operators Π^{M_k} and Π^{B_k} extend boundedly to L^2; the operators $\Pi^{M_k^0}$ extend boundedly to $W_p^1(\Omega)$ for any $p > 2$; the other interpolation operators extend boundedly to H^1 or $\boldsymbol{H}^1$. (These are not the largest possible domain spaces.) With each space we have a corresponding space in which all degrees of freedom associated with edges or vertices contained in the boundary are set equal to zero. Thus $\mathring{M}_k = M_k \cap \mathring{H}^1$.

We will now consider some specific choices of the subspaces in the general method (25).

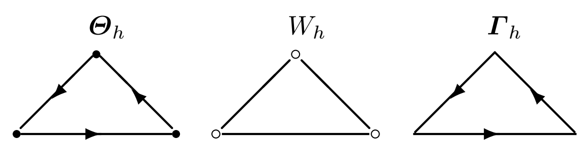
8.1 The Durán–Liberman element

[33] (see also [25], p.145). This element corresponds to the choices

$$\boldsymbol{\Theta}_h = \{\, \boldsymbol{\phi} \in \mathring{\boldsymbol{M}}_2^0 \,|\, \boldsymbol{\phi}\cdot\boldsymbol{n} \in P_1(e), e \in \mathbf{E}\,\}, \quad W_h = \mathring{M}_1^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{RT}_0^\perp,$$

depicted in the element diagram below. We then take $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ to the usual interpolant into $\boldsymbol{RT}_0^\perp$ defined for $\boldsymbol{\gamma} \in \boldsymbol{H}^1(\Omega)$ by

$$\int_e \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}\cdot\boldsymbol{s}\,\mathrm{d}s = \int_e \boldsymbol{\gamma}\cdot\boldsymbol{s}\,\mathrm{d}s, \qquad e \in \mathbf{E}.$$



Durán–Liberman

We then get the following error estimate.

Theorem 8.1

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 + \|w-w_h\|_1 \le Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

Proof. Using standard approximation properties of the space $\boldsymbol{\Theta}_h$, we may find a function $\boldsymbol{\theta}^I$ satisfying (31) with $n = 1$ and the condition $\int_e \boldsymbol{\theta}^I\cdot\boldsymbol{s}\,\mathrm{d}s = \int_e \boldsymbol{\theta}\cdot\boldsymbol{s}\,\mathrm{d}s$. on each edge e. Then

$$\int_e \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I\cdot\boldsymbol{s}\,\mathrm{d}s = \int_e \boldsymbol{\theta}^I\cdot\boldsymbol{s}\,\mathrm{d}s = \int_e \boldsymbol{\theta}\cdot\boldsymbol{s}\,\mathrm{d}s = \int_e \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}\cdot\boldsymbol{s}\,\mathrm{d}s,$$

so

$$\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}.$$

Next observe that if $\Pi^W w$ is the standard piecewise linear interpolant of w, and e is the edge joining vertices v_a and v_b, then

$$\int_e \mathbf{grad}\, \Pi^W w \cdot \boldsymbol{s} \, \mathrm{d}s = \int_e \partial \Pi^W w / \partial s \, \mathrm{d}s = \Pi^W w(v_b) - \Pi^W w(v_a)$$
$$= w(v_b) - w(v_a) = \int_e \partial w / \partial s \, \mathrm{d}s = \int_e \mathbf{grad}\, w \cdot \boldsymbol{s} \, \mathrm{d}s, \quad (40)$$

so

$$\mathbf{grad}\, \Pi^W w = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \, \mathbf{grad}\, w.$$

If we choose $w^I = \Pi^W w$, then $\boldsymbol{\gamma}^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\gamma}$, so (30) is satisfied and Theorem 7.3 is satisfied with $n = 1$. Since (28) is satisfied with $r_0 = -1$, the first two estimates of the theorem follow directly from Theorem 7.3 and the a priori estimate (21). The final estimate is an easy consequence of Theorem 7.5.

To obtain L^2 estimates, we apply Theorem 7.4. In this regard, the following technical lemma will be useful.

Lemma 8.2 *(cf. [32]) For $\boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(\Omega)$, denote by $\boldsymbol{\psi}_c$ a piecewise linear approximation to $\boldsymbol{\psi}$ satisfying*

$$\|\boldsymbol{\psi}_c\|_1 \le C \|\boldsymbol{\psi}\|_1, \qquad \|\boldsymbol{\psi} - \boldsymbol{\psi}_c\|_1 \le Ch \|\boldsymbol{\psi}\|_2.$$

Then for all $\boldsymbol{\zeta} \in \boldsymbol{H}(\mathrm{div}, \Omega)$

$$|(\boldsymbol{\zeta}, \boldsymbol{\psi}_c - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\psi}_c)| \le Ch^2 \| \operatorname{div} \boldsymbol{\zeta} \|_0 \|\boldsymbol{\psi}\|_1.$$

Theorem 8.3

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0 \le Ch^2 (\|\boldsymbol{f}\|_0 + \|g\|_0).$$

Proof. Estimates for the first two terms on the right side of (36) are given by Theorem 8.1. For the third term in (36), let $\boldsymbol{\theta}_c$ be an approximation to $\boldsymbol{\theta}$ satisfying the hypotheses of Lemma 8.2 and write

$$([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}] \boldsymbol{\theta}_h, \boldsymbol{\zeta}) = ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}_c], \boldsymbol{\zeta}) + ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}] \boldsymbol{\theta}_c, \boldsymbol{\zeta}).$$

From Lemma 8.2 we have

$$([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}] \boldsymbol{\theta}_c, \boldsymbol{\zeta}) \le Ch^2 \| \operatorname{div} \boldsymbol{\zeta} \|_0 \|\boldsymbol{\theta}\|_1,$$

and using Lemma 8.2 and Theorem 8.1, we have

$$([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}_c], \boldsymbol{\zeta}) \le Ch \|\boldsymbol{\zeta}\|_0 \|\boldsymbol{\theta}_h - \boldsymbol{\theta}_c\|_1$$
$$\le Ch \|\boldsymbol{\zeta}\|_0 (\|\boldsymbol{\theta}_h - \boldsymbol{\theta}\|_1 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_c\|_1) \le Ch^2 (\|\boldsymbol{\theta}\|_2 + \|\boldsymbol{f}\|_0 + \|g\|_0) \|\boldsymbol{\zeta}\|_0.$$

Combining these results and applying (21) and (35), we get

$$|([\boldsymbol{I}-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}_h, \boldsymbol{\zeta})| \leq Ch^2(\|\boldsymbol{f}\|_0 + \|g\|_0)\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0.$$

We bound the last term in (36) in an analogous manner, obtaining

$$|(\boldsymbol{\gamma}, [\boldsymbol{I}-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\psi}^I)| \leq Ch^2(\|\boldsymbol{f}\|_0 + \|g\|_0)\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0.$$

The theorem follows directly by combining these results.

We note that it is also possible to show that

$$\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{-1} \leq Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

8.2 The MITC triangular families

See [23], [25], and [44] for analysis of these methods and [19] for some experimental results. There are three triangular families considered in [25], defined for integer $k \geq 2$. For each of these families, the space $\boldsymbol{\Theta}_h$ is chosen to be

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_k^0 + \boldsymbol{B}_{k+1}, \quad k = 2, 3, \qquad \boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_k^0, \quad k \geq 4.$$

We then define

$$\begin{aligned} \text{Family I:} &\qquad W_h = \mathring{M}_k^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{RT}_{k-1}^{\perp}, \\ \text{Family II:} &\qquad W_h = \mathring{M}_k^0 + B_{k+1}, \quad \boldsymbol{\Gamma}_h = \boldsymbol{BDFM}_k^{\perp}, \\ \text{Family III:} &\qquad W_h = \mathring{M}_{k+1}^0 \quad \boldsymbol{\Gamma}_h = \boldsymbol{BDM}_k^{\perp}, \end{aligned}$$

and choose $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ to be the usual interpolant into each $\boldsymbol{\Gamma}_h$ space.

The MITC elements are based on a common idea expressed in [23], i.e., "to combine in a proper way some known results on the approximation of Stokes problems with other known results on the approximation of linear elliptic problems." This combination is summarized in a list of five properties relating the spaces $\boldsymbol{\Theta}_h$, W_h, $\boldsymbol{\Gamma}_h$, and an auxiliary space Q_h (not part of the method). These properties are:

P1 $\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h$.
P2 $\operatorname{rot} \boldsymbol{\Gamma}_h \subset Q_h$.
P3 $\operatorname{rot} \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi} = \Pi^0 \operatorname{rot} \boldsymbol{\phi}$, for $\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega)$, with $\Pi^0 : L_0^2(\Omega) \mapsto Q_h$ denoting the L^2-projection ($L_0^2(\Omega)$ denotes functions in $L^2(\Omega)$ with mean value zero.)
P4 If $\eta \in \boldsymbol{\Gamma}_h$ satisfies $\operatorname{rot} \boldsymbol{\eta} = 0$, then $\boldsymbol{\eta} = \mathbf{grad}\, v$ for some $v \in W_h$.
P5 $(\boldsymbol{\Theta}_h^{\perp}, Q_h)$ is a stable pair for the Stokes problem, i.e.,

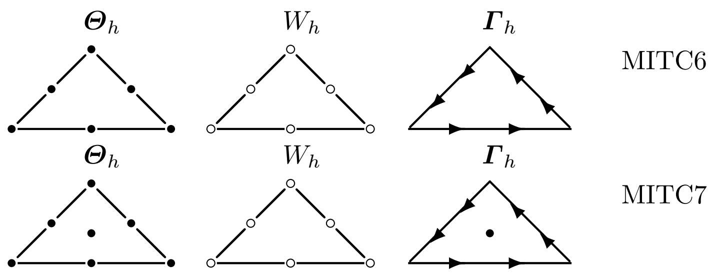
$$\sup_{\mathbf{0} \neq \boldsymbol{\phi} \in \boldsymbol{\Theta}_h} \frac{(\operatorname{rot} \boldsymbol{\phi}, q)}{\|\boldsymbol{\phi}\|_1} \geq C\|q\|_0, \qquad q \in Q_h.$$

For each of the three families described above, the space

$$Q_h = \{q \in L_0^2(\Omega) : q_T \in P_{k-1}(T),\ T \in \mathcal{T}_h\}.$$

For this choice, the fact that the pair of spaces $(\boldsymbol{\Theta}_h, Q_h)$ satisfies **P5** follows from the corresponding results known for the Stokes equation.

Although these families are only defined for $k \geq 2$, it is interesting to see what the difficulties are in extending them to the case $k = 1$. Most obvious is that $\boldsymbol{B}_{k+1}$ is only defined for $k \geq 2$, so this space must be replaced. A suitable replacement space for $\boldsymbol{\Theta}_h$ in Family I is the one chosen in the Durán–Liberman element. With this choice, the Durán–Liberman element also fits this general framework, with $k = 1$. For Family II, a similar problem occurs for the choice of W_h and in addition $\boldsymbol{BDFM}_1^{\perp} = \boldsymbol{RT}_0^{\perp}$, so the method needs substantial change and does not give anything new. For Family III, the choices $W_h = \mathring{M}_2^0$ and $\boldsymbol{\Gamma}_h = \boldsymbol{BDM}_1^{\perp}$ make sense, and one can choose $\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_2^0$. This would correspond to the choice of piecewise constants for Q_h and the $\boldsymbol{P}_2 - P_0$ Stokes element. An element of this type is mentioned in [23] (page 1798). This element, which we label MITC6 is depicted below along with MITC7, the $k = 2$ element of Family II.



We give an analysis in this section only for Family I:

$$\boldsymbol{\Theta}_h = \begin{cases} \mathring{\boldsymbol{M}}_k^0 + \boldsymbol{B}_{k+1} & k = 2, 3 \\ \mathring{\boldsymbol{M}}_k^0 & k \geq 4 \end{cases}, \quad W_h = \mathring{M}_k^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{RT}_{k-1}^{\perp}.$$

The analysis of the other two families can be done in a similar manner.

Theorem 8.4 *For the MITC family of index $k \geq 2$, we have for $1 \leq r \leq k$*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|w - w_h\|_1 \leq Ch^r \left(\|\boldsymbol{\theta}\|_{r+1} + t\|\boldsymbol{\gamma}\|_r + \|\boldsymbol{\gamma}\|_{r-1}\right).$$

Proof. Using standard results about stable Stokes elements, we can find an interpolant $\boldsymbol{\theta}^I$ of $\boldsymbol{\theta} \in \boldsymbol{\Theta}_h$ satisfying (31) with $n = k$ and

$$\int_\Omega \operatorname{rot}(\boldsymbol{\theta} - \boldsymbol{\theta}^I)\, q \,\mathrm{d}x = 0, \quad \forall q \in M_{k-1}^{-1}.$$

By the definition of $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$, we have $\forall q \in M_{k-1}^{-1}$

$$0 = \int_\Omega \operatorname{rot}(\boldsymbol{\theta} - \boldsymbol{\theta}^I)\, q \,\mathrm{d}x = \int_\Omega \operatorname{rot} \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\theta}^I)\, q \,\mathrm{d}x.$$

Choosing $q = \operatorname{rot} \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\theta}^I)$ implies $\operatorname{rot} \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\theta}^I) = 0$. Hence,

$$\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\theta}^I) = \mathbf{grad}\, v^I, \qquad \text{for some } v^I \in W_h.$$

Let $\Pi^W w \in M_0^k$ be the interpolant of w defined for each vertex x, edge e and triangle T by

$$\Pi^W w(x) = w(x), \qquad \int_e \Pi^W w\, p \,\mathrm{d}s = \int_e w\, p \,\mathrm{d}s, \quad \text{for all } p \in P_{k-2}(e), \tag{41}$$

$$\int_T \Pi^W wp \,\mathrm{d}x = \int_T wp \,\mathrm{d}x, \quad \text{for all } p \in P_{k-3}(T). \tag{42}$$

It is easy to check that $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\mathbf{grad}\, w) = \mathbf{grad}\, \Pi^W w$. Hence, (30) is satisfied with $w^I = \Pi^W w - v^I$. By the definition of the space $\boldsymbol{\Gamma}_h$, (32) is satisfied with $n = k$ and (28) is satisfied with $r_0 = k - 2$. The estimate for the first two terms follows directly from Theorem 7.3. The final estimate is an easy consequence of Theorem 7.5.

Theorem 8.5 *For the MITC family of index $k \geq 2$, we have for $1 \leq r \leq k$*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0 \leq C h^{r+1} \left(\|\boldsymbol{\theta}\|_{r+1} + t\|\boldsymbol{\gamma}\|_r + \|\boldsymbol{\gamma}\|_{r-1}\right).$$

Proof. Estimates for the first two terms on the right side of (36) are given by Theorem 8.4. To estimate the third term, we write

$$\begin{aligned}
([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}_h, \boldsymbol{\zeta}) &= ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}], \boldsymbol{\zeta}) + ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}, \boldsymbol{\zeta}). \\
&= ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}], \boldsymbol{\zeta}) - \lambda^{-1}t^2(\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}})\gamma, \boldsymbol{\zeta}) + ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\, \mathbf{grad}\, w, \zeta) \\
&= ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}], \boldsymbol{\zeta}) - \lambda^{-1}t^2([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\gamma, \boldsymbol{\zeta}) \\
&\qquad + (\mathbf{grad}\, w - \mathbf{grad}\, \Pi^W w, \boldsymbol{\zeta}) \\
&= ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}], \boldsymbol{\zeta}) - \lambda^{-1}t^2([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\gamma, [\boldsymbol{I} - \boldsymbol{\Pi}^{M^0}]\boldsymbol{\zeta}) \\
&\qquad - (w - \Pi^W w, \operatorname{div} \boldsymbol{\zeta}).
\end{aligned}$$

Hence,

$$\begin{aligned}
|([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}_h, \boldsymbol{\zeta})| &\leq \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}]\|_0 \|\boldsymbol{\zeta}\|_0 \\
&\qquad + \lambda^{-1}t^2\|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\gamma\|_0 \|[\boldsymbol{I} - \boldsymbol{\Pi}^{M^0}]\boldsymbol{\zeta}\|_0 + \|w - \Pi^W w\|_0 \|\operatorname{div} \boldsymbol{\zeta}\|_0 \\
&\leq Ch \Big(\|\boldsymbol{\theta}_h - \boldsymbol{\theta}\|_1 \|\boldsymbol{\zeta}\|_0 + t\|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\gamma\|_0 t \|\boldsymbol{\zeta}\|_1 \\
&\qquad + h^{-1}\|w - \Pi^W w\|_0 \|\operatorname{div} \boldsymbol{\zeta}\|_0 \Big).
\end{aligned}$$

To estimate the final term, we write

$$(\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\psi}^I) = ([\boldsymbol{I} - \boldsymbol{\Pi}^0]\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\psi}^I).$$
$$= ([\boldsymbol{I} - \boldsymbol{\Pi}^0]\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\psi}^I - \boldsymbol{\psi}]) + ([\boldsymbol{I} - \boldsymbol{\Pi}^0]\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\psi}]).$$

Hence,

$$|(\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\psi}^I)| \le C\|[\boldsymbol{I} - \boldsymbol{\Pi}^0]\boldsymbol{\gamma}\|_0(\|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\psi}^I - \boldsymbol{\psi}]\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\psi}]\|_0) \le Ch^2\|[\boldsymbol{I} - \boldsymbol{\Pi}^0]\boldsymbol{\gamma}\|_0\|\boldsymbol{\psi}\|_2.$$

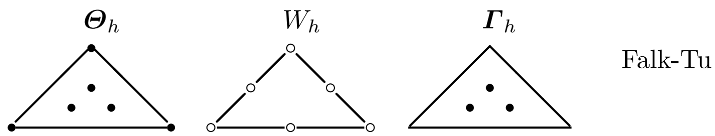
The theorem now follows by combining these results and applying (35) and standard estimates.

8.3 The Falk-Tu elements with discontinuous shear stresses [35]

For $k = 2, 3, \dots$ we choose

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}^0_{k-1} + \boldsymbol{B}_{k+2}, \quad W_h = \mathring{M}^0_k, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_{k-1},$$

and $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ to be the $\boldsymbol{L}^2$ projection into $\boldsymbol{\Gamma}_h$. See also the related element of Zienkiewicz–Lefebvre [52]. The element diagram for the lowest order Falk–Tu element ($k = 2$) is depicted below.



Theorem 8.6 *For the discontinuous shear stress family of index $k \ge 2$, we have for $1 \le r \le k-1$*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \le Ch^r \left(\|\boldsymbol{\theta}\|_{r+1} + \|w\|_{r+2} + t\|\boldsymbol{\gamma}\|_r + \|\boldsymbol{\gamma}\|_{r-1}\right).$$

For $k = 2$ and $r = 1$, we also have the estimate

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \le Ch\left(\|\boldsymbol{\theta}\|_2 + \|w^0\|_3 + \|\boldsymbol{\gamma}\|_0 + t\|\boldsymbol{\gamma}\|_1 + t^{-1}\|w - w^0\|_2\right) \le Ch\left(\|\boldsymbol{f}\|_0 + \|g\|_0\right).$$

Proof. For $1 \le r \le k-1$, let $\Pi^W w$ be a standard interpolant of w satisfying

$$\|w - \Pi^W w\|_0 + h\|w - \Pi^W w\|_1 \le Ch^{r+2}\|w\|_{r+2}$$

and $\boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta} \in \mathring{\boldsymbol{M}}^{k-1}_0$ a standard interpolant of $\boldsymbol{\theta}$ satisfying

$$\|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_0 + h\|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_1 \le Ch^{r+1}\|\boldsymbol{\theta}\|_{r+1}.$$

Define $\boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*) \in \boldsymbol{B}^{k+3}$ by

$$\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*) = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}\, w^* + \mathbf{grad}\,\Pi^W w^*,$$

where w^* shall be chosen as either w or w^0, the limiting transverse displacement obtained from the Reissner-Mindlin system when $t = 0$. We then set $w^I = \Pi^W w$ and $\boldsymbol{\theta}^I = \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta} + \boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*)$. In this case, $\boldsymbol{\theta}^I$ is not an interpolant of $\boldsymbol{\theta}$, since it depends on w^* also. Hence, (31) does not hold. However, we will show (the proof is postponed until after the completion of the proof of the theorem) that for $1 \le r \le k-1$,

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \le Ch^r\big(\|\boldsymbol{\theta}\|_{r+1} + \|w^*\|_{r+2}\big). \tag{43}$$

Using the definitions given above, we also get

$$\begin{aligned}
\boldsymbol{\gamma}^I &= \lambda t^{-2}(\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I)\\
&= \lambda t^{-2}(\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*))\\
&= \lambda t^{-2}(\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta} + \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}\, w^* - \mathbf{grad}\,\Pi^W w^*)\\
&= \lambda t^{-2}[\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\mathbf{grad}\, w - \boldsymbol{\theta}) + \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}(w^* - w) - \mathbf{grad}\,\Pi^W(w^* - w)]\\
&= \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} + \lambda t^{-2}\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}([I - \Pi^W][w^* - w]).
\end{aligned}$$

Note that if we choose $w^* = w$, then (30) will be satisfied, while the choice $w^* = w^0$ does not satisfy (30). The need for the second choice is a technical one, namely the fact that on a convex polygon, we do not have an a priori bound for $\|w\|_3$, but do have a bound for $\|w^0\|_3$. If we had been working on a domain with smooth boundary, the simpler choice $w^* = w$ would be sufficient. By the definition of the space $\boldsymbol{\Gamma}_h$, (32) is satisfied with $n = k-1$ and (28) is satisfied with $r_0 = k-2$. Choosing $w^* = w$, the first estimate of the theorem now follows from a simple modification of Theorem 7.3, in which we replace (31) by (43).

To establish the second estimate in the theorem, we choose $k = 2$, $r = 1$, and $w^* = w^0$, and first apply Theorem 5.1 to obtain

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \le Ch\big(\|\boldsymbol{\theta}\|_2 + \|w^0\|_3\big) \le Ch(\|\boldsymbol{f}\|_0 + \|g\|_0). \tag{44}$$

Since (30) does not hold in this case, we cannot obtain an error estimate by the same simple modification of Theorem 7.3 used above. Instead, we return to Theorem 7.1 and estimate each of the terms. From our approximability assumption on the space W_h and Theorem 5.1, we get that

$$\begin{aligned}
t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\| &\le t\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}\| + \lambda t^{-1}\|\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}([I - \Pi^W][w^0 - w]\|\\
&\le t\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}\| + Ct^{-1}\|\,\mathbf{grad}([I - \Pi^W][w^0 - w]\|\\
&\le Ch(t\|\boldsymbol{\gamma}\|_1 + t^{-1}\|w^0 - w\|_2) \le Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).
\end{aligned}$$

The estimate of the final term is straightforward, i.e.,

$$\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0\boldsymbol{\gamma}\| \le \|\boldsymbol{\gamma}\| \le C(\|\boldsymbol{f}\|_0 + \|g\|_{-1}).$$

Inserting the above estimates into Theorem 7.1, we obtain the second estimate of the theorem.

Finally, it remains to prove (43).

Lemma 8.7 *For $1 \le r \le k-1$,*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \le Ch^r\big(\|\boldsymbol{\theta}\|_{r+1} + \|w^*\|_{r+2}\big).$$

Proof. We first note that it is easy to show that if $\boldsymbol{\psi} \in \boldsymbol{B}_{k+2}$ and $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ denotes the $\boldsymbol{L}^2$ projection into $\boldsymbol{M}_{k-1}$, then

$$\|\boldsymbol{\psi}\|_0 \le C\|\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\psi}\|_0. \tag{45}$$

Hence, we have

$$\begin{aligned}
&\|\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*)\|_0 = \|\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}\,w^* + \mathbf{grad}\,\Pi^W w^*\|_0 \\
&\le \|\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta})\|_0 + \|(\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} - \boldsymbol{I})\,\mathbf{grad}\,w^*\|_0 + \|\,\mathbf{grad}(w^* - \Pi^W w^*)\|_0 \\
&\quad \le C\big(\|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_0 + \|(\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} - \boldsymbol{I})\,\mathbf{grad}\,w^*\|_0 + \|\,\mathbf{grad}(w^* - \Pi^W w^*)\|_0.
\end{aligned} \tag{46}$$

Now by the triangle inequality, standard approximation theory, (45), and (46):

$$\begin{aligned}
\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 &= \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*)\|_1 \le \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_1 + \|\boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*)\|_1 \\
&\le \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_1 + Ch^{-1}\|\boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*)\|_0 \\
&\le \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_1 + Ch^{-1}\|\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*)\|_0 \\
&\le C\Big[\|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_1 + h^{-1}\big(\|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_0 \\
&\qquad + \|(\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} - \boldsymbol{I})\,\mathbf{grad}\,w^*\|_0 + \|\,\mathbf{grad}(w^* - \Pi^W w^*)\|_0\big)\Big].
\end{aligned}$$

Applying our approximation theory results, we get for $1 \le r \le k-1$

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \le Ch^r\big(\|\boldsymbol{\theta}\|_{r+1} + \|w^*\|_{r+2}\big).$$

Using a slightly modified version of Theorem 7.4, (due to the fact that $\boldsymbol{\theta}^I$ depends on both $\boldsymbol{\theta}$ and w^*), one can derive $\boldsymbol{L}^2$ error estimates for $\boldsymbol{\theta} - \boldsymbol{\theta}_h$ and then error estimates for $w - w_h$. We state the results below.

Theorem 8.8 *For the discontinuous shear stress family of index $k \ge 2$, we have for $1 \le r \le k-1$*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_1 \le Ch^{r+1}\left(\|\boldsymbol{\theta}\|_{r+1} + \|w\|_{r+2} + t\|\boldsymbol{\gamma}\|_r + \|\boldsymbol{\gamma}\|_{r-1}\right).$$

For $k = 2$ and $r = 1$, we also have the estimate

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_1 \le Ch^2\left(\|\boldsymbol{f}\|_0 + \|g\|_0\right).$$

We note that we do not obtain a higher order of convergence for $\|w - w_h\|_0$.

8.4 Linked interpolation methods

There are a number of formulations of the linked interpolation method. One approach is to use the mixed formulation (25), but replace the space $\boldsymbol{\Theta}_h \times W_h$ by a space $\boldsymbol{V}_h$ in which the two spaces are linked by a constraint. The simplest example of such a method is the one introduced by Xu [51] and Auricchio and Taylor [16, 49], and analyzed in [41, 39, 15]. In this method, we choose

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^0 + \boldsymbol{B}_3, \quad W_h = \mathring{M}_1^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0,$$
$$\boldsymbol{V}_h = \{(\boldsymbol{\phi}, v + L\boldsymbol{\phi}) : \boldsymbol{\phi} \in \boldsymbol{\Theta}_h, v \in W_h\},$$

where following [41], we may define $L_T = L|_T$ as a mapping from $\boldsymbol{H}^1(T)$ onto $P_{2,-}(T)$ by

$$\int_e [(\mathbf{grad}\, L_T\boldsymbol{\phi} - \boldsymbol{\phi}) \cdot \boldsymbol{s}] \frac{\partial v}{\partial s} \, \mathrm{d}s = 0, \quad v \in P_{2,-}(T), \tag{47}$$

for every edge e of T, where $P_{2,-}(T)$ is the space of piecewise quadratics which vanish at the vertices of T.

We then seek an approximation $(\boldsymbol{\theta}_h, w_h^*; \boldsymbol{\gamma}_h) \in \boldsymbol{V}_h \times \boldsymbol{\Gamma}_h$ such that (25) holds for all $(\boldsymbol{\phi}, v^*;) \in \boldsymbol{V}_h \times \boldsymbol{\Gamma}_h$. Equivalently, we can write this method in terms of the usual spaces, but with a modified bilinear form, i.e., we seek $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in \boldsymbol{\Theta}_h \times W_h \times \boldsymbol{\Gamma}_h$ such that

$$a(\boldsymbol{\theta}_h, \boldsymbol{\phi}) + \lambda^{-1}t^2(\boldsymbol{\gamma}_h, \mathbf{grad}(v + L\boldsymbol{\phi}) - \boldsymbol{\phi}) = (g, v + L\boldsymbol{\phi}) - (\boldsymbol{f}, \boldsymbol{\phi}),$$
$$\boldsymbol{\phi} \in \boldsymbol{\Theta}_h, v \in W_h,$$
$$(\mathbf{grad}(w_h + L\boldsymbol{\theta}_h) - \boldsymbol{\theta}_h, \boldsymbol{\eta}) - \lambda^{-1}t^2(\boldsymbol{\gamma}_h, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{\Gamma}_h.$$

Note that we can write this discrete variational formulation as a slight perturbation of the formulation (25), by defining $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} = \boldsymbol{\Pi}^0(I - \mathbf{grad}\, L)$ (where $\boldsymbol{\Pi}^0$ denotes the $\boldsymbol{L}^2$ projection onto $\boldsymbol{\Gamma}_h$), and replacing the term (g, v) by $(g, v + L\boldsymbol{\phi})$. We omit the element diagram for this method, since depicting only the three basic spaces, without the additional space $P_{2,-}(T)$, is somewhat misleading.

We shall analyze this method using the usual spaces and the interpolation operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ defined above. We first observe that from [41],

$$|(g, L\boldsymbol{\phi})|_T \le \|g\|_{0,T}\|L_T\boldsymbol{\phi}\|_{0,T} \le Ch_T\|g\|_{0,T}\|\nabla L_T\boldsymbol{\phi}\|_{0,T} \le Ch_T^2\|g\|_{0,T}\|\boldsymbol{\phi}\|_{1,T},$$

so this term is a high order perturbation and may be dropped. To apply our previous error estimates, we first define $w^I = \Pi^W w$, the continuous piecewise linear interpolant of w, and $\boldsymbol{\theta}^I = \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta} + \boldsymbol{\Pi}^{\boldsymbol{B}}\boldsymbol{\theta}$, where $\boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}$ denotes an interpolant of $\boldsymbol{\theta}$ satisfying

$$\|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_0 + \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta}\|_1 \le Ch^s\|\boldsymbol{\theta}\|_s, \quad s = 1, 2,$$

and $\boldsymbol{\Pi}^{\boldsymbol{B}}\boldsymbol{\theta} \in \boldsymbol{B}_3$ is defined by:

$$\boldsymbol{\Pi}^0 \boldsymbol{\Pi}^{\boldsymbol{B}} \boldsymbol{\theta} = \boldsymbol{\Pi}^0 [(\boldsymbol{I} - \mathbf{grad}\, L)(\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta})]. \tag{48}$$

We note that

$$\begin{aligned}\|\boldsymbol{\Pi}^0 \boldsymbol{\Pi}^{\boldsymbol{B}} \boldsymbol{\theta}\|_0 &\le \|(\boldsymbol{I} - \mathbf{grad}\, L)(\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta})\|_0 \le \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta}\|_0 \\ &+ \|\mathbf{grad}\, L(\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta})\|_0 \le \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta}\|_0 + Ch \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta}\|_1.\end{aligned}$$

Since $\|\boldsymbol{\Pi}^{\boldsymbol{B}} \boldsymbol{\theta}\|_0 \le C \|\boldsymbol{\Pi}^0 \boldsymbol{\Pi}^{\boldsymbol{B}} \boldsymbol{\theta}\|_0$, we easily obtain for $s = 1, 2$ that

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_0 \le C(\|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta}\|_0 + h \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta}\|_1 \le C h^s \|\boldsymbol{\theta}\|_s.$$

Using the inverse inequality $\|\boldsymbol{\Pi}^{\boldsymbol{B}} \boldsymbol{\theta}\|_1 \le C h^{-1} \|\boldsymbol{\Pi}^{\boldsymbol{B}} \boldsymbol{\theta}\|_0$, we then obtain

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \le C(\|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta}\|_1 + h^{-1} \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta}\|_0) \le Ch \|\boldsymbol{\theta}\|_2.$$

Hence, hypotheses (31) and (32) of Theorem 7.3 are satisfied with $r = 1$ and $r_0 = -1$. Thus, it only remains to show that (30) is satisfied. Applying (47) with $\phi = \mathbf{grad}(w - w^I)$, and noting that $(L_T\, \mathbf{grad} - I)(w - w^I) = 0$ at the vertices of T, we get

$$\begin{aligned}0 &= \int_e [(\mathbf{grad}\, L_T - \boldsymbol{I})\, \mathbf{grad}(w - w^I)] \cdot \boldsymbol{s} \frac{\mathrm{d}v}{\mathrm{d}s}\, \mathrm{d}s \\ &= \int_e \frac{d}{ds} \left[(L_T\, \mathbf{grad} - I)(w - w^I)\right] \frac{\mathrm{d}v}{\mathrm{d}s}\, \mathrm{d}s \\ &= -\int_e (L_T\, \mathbf{grad} - I)(w - w^I) \frac{\mathrm{d}^2 v}{\mathrm{d}s^2}\, \mathrm{d}s, \quad v \in P_{2,-}(T).\end{aligned}$$

Since $\mathrm{d}^2 v / \mathrm{d}s^2$ is a constant on the edge e, we get for all $\boldsymbol{q} \in \boldsymbol{P}_0(T)$,

$$\begin{aligned}\int_T (\mathbf{grad}\, L - I)\, \mathbf{grad}(w - w^I) \cdot \boldsymbol{q}\, \mathrm{d}x &= \int_T \mathbf{grad}(L\, \mathbf{grad} - I)(w - w^I) \cdot \boldsymbol{q}\, \mathrm{d}x \\ &= -\int_{\partial T} (L\, \mathbf{grad} - I)(w - w^I) \boldsymbol{q} \cdot \boldsymbol{n}\, \mathrm{d}s = 0,\end{aligned}$$

and so

$$\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\, \mathbf{grad}(w - w^I) = \boldsymbol{\Pi}^0 (\mathbf{grad}\, L_T - \boldsymbol{I})\, \mathbf{grad}(w - w^I) = 0.$$

Finally, from (48) and the fact that $L \boldsymbol{\Pi}^{\boldsymbol{B}} \boldsymbol{\theta} = 0$, we get

$$\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} (\boldsymbol{\theta} - \boldsymbol{\theta}^I) = \boldsymbol{\Pi}^0 (\mathbf{grad}\, L_T - \boldsymbol{I})(\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{M}} \boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{B}} \boldsymbol{\theta}) = 0.$$

If we drop the term $(g, L\boldsymbol{\phi})$ from the right hand side of the method, then we get immediately from Theorems 7.3 and 7.5 the following estimate:

$$\begin{aligned}&\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|w - w_h\|_1 \\ &\qquad \le Ch(\|\boldsymbol{\theta}\|_2 + t \|\boldsymbol{\gamma}\|_1 + \|\boldsymbol{\gamma}_0 + \|w\|_2) \le Ch(\|g\|_0 + \|\boldsymbol{f}\|_0).\end{aligned}$$

A simple extension of this argument gives the same final result with this term included (the term $h^2\|g\|_0$ would need to be added to the intermediate result).

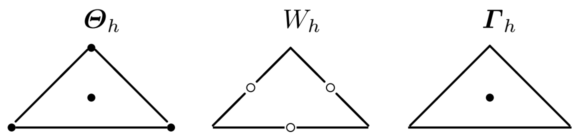
We note that the method of [53] analyzed in [34] is also of this type. The analysis given in [34] proceeds by comparing the method to the Durán–Liberman element described above. The two methods have the same choices for the spaces $\boldsymbol{\Theta}_h$ and $\boldsymbol{\Gamma}_h$,

8.5 The nonconforming element of Arnold and Falk [11]

See also [29].

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^0 + \boldsymbol{B}_3, \quad W_h = \mathring{M}_1^*, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0$$

where $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ is the $\boldsymbol{L}^2$ projection into $\boldsymbol{\Gamma}_h$.



Arnold-Falk

Since the space W_h is not contained in $\mathring{H}^1(\Omega)$, **grad** must be replaced by $\mathbf{grad}_h$ and some modifications need to be made in the basic error estimates proved earlier. Rather than prove a general abstract version of these results taking into account several types of nonconformity, we simply modify the proofs for the particular method being analyzed. We begin by first stating a standard result basic to the analysis of nonconforming methods.

Lemma 8.9 *(cf. [28]). Let $\boldsymbol{\phi} \in \boldsymbol{H}^1(\Omega)$ and $v \in W_h$. Then*

$$\left| \sum_{T\in\tau} \int_{\partial T} v\boldsymbol{\phi} \cdot \boldsymbol{n}_T \right| \leq Ch\|\boldsymbol{\phi}\|_1 \|\mathbf{grad}_h v\|_0.$$

Using this result, we can derive the following energy norm error estimate.

Theorem 8.10

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|\mathbf{grad}_h[w - w_h]\|_0 \leq Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

Proof. Since $W_h \notin H_0^1(\Omega)$, we cannot apply Theorems 7.1 and 7.3 directly. In particular, the error equation (29) must be replaced by a modified equation which contains an additional term for the consistency error.

$$a(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\phi}) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \mathbf{grad}_h v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\phi}) = (\boldsymbol{\gamma}, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\phi}) + \sum_{T\in\tau} \int_{\partial T} v\boldsymbol{\gamma} \cdot \boldsymbol{n}_T, \tag{49}$$

for all $\boldsymbol{\phi} \in \boldsymbol{\Theta}_h$ and $v \in W_h$. Following the proof of Theorem 7.1, we obtain

$$\|\boldsymbol{\theta}^I - \boldsymbol{\theta}_h\|_1^2 + t^2\|\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h\|_0^2 \le C\Big(\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_0^2 + h^2\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_0^2 + \Big|\sum_{T\in\tau}\int_{\partial T}(w^I - w_h)\boldsymbol{\gamma}\cdot\boldsymbol{n}_T\Big|\Big). \quad (50)$$

In this case, $\boldsymbol{\Pi}^0$ is $\boldsymbol{L}^2$ projection into piecewise constants, so we can use the trivial estimate $\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_0 \le \|\boldsymbol{\gamma}\|_0$. As in Theorem 7.3, we need to define $\boldsymbol{\theta}^I$ and w^I and hence $\boldsymbol{\gamma}^I$ to satisfy (30) and (31). The choice of $\boldsymbol{\theta}^I$ is the same as that used for the MINI element for the Stokes problem. This satisfies (31) with $n = 1$ (and the 1-norm replaced by the discrete 1-norm) and also the condition $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}$. Hence, to satisfy (30), we need only to find w^I such that

$$\mathbf{grad}_h\, w^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}\, w. \quad (51)$$

This is easily accomplished by choosing w^I to satisfy $\int_e w^I = \int_e w$ on each edge e. Then for all $\eta \in \boldsymbol{\Gamma}_h$

$$\int_T \mathbf{grad}\, w\cdot\boldsymbol{\eta}\,\mathrm{d}x = \int_{\partial T} w\eta\cdot\boldsymbol{n}_T\,\mathrm{d}s = \int_{\partial T} w^I\eta\cdot\boldsymbol{n}_T\,\mathrm{d}s = \int_T \mathbf{grad}\, w^I\cdot\boldsymbol{\eta}\,\mathrm{d}x,$$

which implies (51). Then (32) is satisfied with $n = 1$.

It only remains to estimate the term arising from the nonconforming approximation. Unfortunately, we cannot estimate this term by applying Lemma 8.9 directly, since the result would then contain the term $\|\boldsymbol{\gamma}\|_1$ which is not bounded independent of the thickness t. Instead, we use the Helmholtz decomposition to write $\boldsymbol{\gamma} = \mathbf{grad}\, r + \mathbf{curl}\, p$ with $r \in \mathring{H}^1(\Omega)$ and $p \in H^1(\Omega)$. Recalling that

$$\mathbf{grad}_h(w^I - w_h) = \lambda^{-1}t^2(\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h) + \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\boldsymbol{\theta}^I - \boldsymbol{\theta}_h),$$

we first use Lemma 8.9 to get

$$\begin{aligned}\Big|\sum_{T\in\tau}\int_{\partial T}(w^I - w_h)\,\mathbf{grad}\, r\cdot\boldsymbol{n}_T\,\mathrm{d}s\Big| &\le Ch\|r\|_2\|\,\mathbf{grad}_h(w^I - w_h)\|_0\\ &\le Ch\|r\|_2\left(t^2\|\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h\|_0 + \|\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\boldsymbol{\theta}^I - \boldsymbol{\theta}_h)\|_0\right)\\ &\le Ch\|r\|_2\left(t^2\|\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h\|_0 + \|\boldsymbol{\theta}^I - \boldsymbol{\theta}_h\|_0\right).\end{aligned}$$

Now for all $p^I \in M_1^0$,

$$\begin{aligned}\sum_{T\in\tau}\int_{\partial T}(w^I - w_h)\,\mathbf{curl}\, p\cdot\boldsymbol{n}_T\,\mathrm{d}s &= \sum_{T\in\tau}\int_T \mathbf{grad}(w^I - w_h)\cdot\mathbf{curl}\, p\,\mathrm{d}x\\ &= \sum_{T\in\tau}\int_T \mathbf{grad}(w^I - w_h)\cdot\mathbf{curl}(p - p^I)\,\mathrm{d}x\\ &= \lambda^{-1}t^2(\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h, \mathbf{curl}[p - p^I]) + (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}[\boldsymbol{\theta}^I - \boldsymbol{\theta}_h], \mathbf{curl}[p - p^I])\\ &= \lambda^{-1}t^2(\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h, \mathbf{curl}[p - p^I]) + ([\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} - I](\boldsymbol{\theta}^I - \boldsymbol{\theta}_h), \mathbf{curl}[p - p^I])\\ &\quad + (\mathrm{rot}[\boldsymbol{\theta}^I - \boldsymbol{\theta}_h], p - p^I).\end{aligned}$$

Choosing p^I to satisfy

$$\|p - p^I\|_0 + h\|p - p^I\|_1 \le Ch^s \|p^I\|_s, \quad s = 1, 2,$$

(e.g., the Clement interpolant), we have by standard estimates that

$$\Big|\sum_{T\in\tau} \int_{\partial T} (w^I - w_h)\, \mathbf{curl}\, p \cdot \boldsymbol{n}_T \, \mathrm{d}s\Big| \\ \le C\left(t^2 \|\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h\|_0 h \|p\|_2 + h\|\boldsymbol{\theta}^I - \boldsymbol{\theta}_h\|_1 \|p\|_1\right).$$

Combining these results, we obtain

$$\Big|\sum_{T\in\tau} \int_{\partial T} (w^I - w_h)\boldsymbol{\gamma} \cdot \boldsymbol{n}_T \, \mathrm{d}x\Big| \\ \le Ch\left(t\|\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h\|_0 + \|\boldsymbol{\theta}^I - \boldsymbol{\theta}_h\|_1\right)(\|r\|_2 + \|p\|_1 + t\|p\|_2).$$

The first two estimates of the theorem now follow by combining all these results and using the a priori estimate (21). To obtain an error estimate on the transverse displacement, we need a nonconforming version of Theorem 7.5.

Choosing $\eta = \mathbf{grad}_h\, v_h$, $v_h \in W_h$, we get for all $w_I \in W_h$,

$$(\mathbf{grad}_h [w_I - w_h], \mathbf{grad}_h\, v_h) = (\mathbf{grad}_h [w_I - w], \mathbf{grad}\, v_h) \\ + (\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}_h, \mathbf{grad}_h\, v_h) + \sum_T \int_{\partial T} v_h \frac{\partial w}{\partial n}\, \mathrm{d}s. \quad (52)$$

Then choosing $v_h = w_h - w_I$, it easily follows using Lemma 8.9 that

$$\|\,\mathbf{grad}_h [w_I - w_h]\|_0 \le \|\,\mathbf{grad}_h [w_I - w]\|_0 + \|\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}_h\|_0 + Ch\|w\|_2 \\ \le \|\,\mathbf{grad}_h [w_I - w]\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}][\boldsymbol{\theta}_h - \boldsymbol{\theta}]\|_0 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + Ch\|w\|_2 \\ \le \|\,\mathbf{grad}_h [w_I - w]\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}\|_0 + Ch\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + Ch\|w\|_2.$$

The desired result now follows from the triangle inequality and standard estimates.

Using a nonconforming version of Theorem 7.4, we can also establish the following L^2 error estimate.

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0 \le Ch^2(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

See also [36] and [30] for a modification of this element, and [2] for a relationship between these two approaches.

9 Some rectangular Reissner–Mindlin elements

Now let $\mathcal{T}_h$ denote a rectangular mesh of Ω and R an element of $\mathcal{T}_h$. We denote by Q_{k_1,k_2} the set of polynomials of separate degree $\leq k_1$ in x and $\leq k_2$ in y and set $Q_k = Q_{k,k}$. We also define the serendipity polynomials $Q_k^s = P_k \oplus x^k y \oplus xy^k$. Finally, we will also use the rotated versions of the rectangular Raviart-Thomas, Brezzi-Douglas-Marini, and Brezzi-Douglas-Fortin-Marini spaces, which we define locally for $k \geq 1$ as follows.

$$\boldsymbol{RT}_{k-1}^{\perp}(R) = \{\boldsymbol{\eta} : \boldsymbol{\eta} = (Q_{k-1,k}(R), Q_{k,k-1}(R))\},$$
$$\boldsymbol{BDM}_k^{\perp}(R) = \{\boldsymbol{\eta} : \boldsymbol{\eta} \in \boldsymbol{P}_k(R) \oplus \nabla(xy^{k+1}) \oplus \nabla(x^{k+1}y)\},$$
$$\boldsymbol{BDFM}_k^{\perp}(R) = \{\boldsymbol{\eta} : \boldsymbol{\eta} = (P_k(R) \setminus \{x^k\}, P_k(R) \setminus \{y^k\})\}.$$

9.1 Rectangular MITC elements and generalizations

[20, 17, 23, 48]. In the original MITC family, we choose for $k \geq 1$,

$$\boldsymbol{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in \boldsymbol{Q}_k(R)\}, \qquad W_h = \{v \in \mathring{H}^1(\Omega) : v|_R \in Q_k^s(R)\},$$
$$\boldsymbol{\Gamma}_h = \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\eta}|_R \in \boldsymbol{BDFM}_k^{\perp}(R)\}.$$

The auxiliary pressure space

$$Q_h = \{q \in L_0^2(\Omega) : q|_R \in P_{k-1}\}$$

and the reduction operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ is defined by

$$\int_e (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} - \boldsymbol{\gamma}) \cdot \boldsymbol{s}\, p_{k-1}(s)\, \mathrm{d}s = 0, \quad \forall e, \quad \forall p_{k-1} \in P_{k-1}(e),$$
$$\int_R (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} - \boldsymbol{\gamma}) \cdot \boldsymbol{p}_{k-2}\, \mathrm{d}x\, \mathrm{d}y = 0, \quad \forall R, \quad \forall \boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(R).$$

The lowest order element ($k = 1$) is called MITC4. In this case, the space $\boldsymbol{BDFM}_1^{\perp}(R)$ has the form $(a+by, c+dx)$ and coincides with the lowest order rotated rectangular Raviart-Thomas element $\boldsymbol{RT}_0^{\perp}(R)$. The space $Q_1^s(R) = Q_1(R)$. The MITC4 element was proposed in [20] and analyzed in [17], [18], [33], and most recently in [31], where the proof is extended to more general quadrilateral meshes using a macro-element technique and the results obtained under less regularity than previously required. For rectangular meshes, this method coincides with the T1 method of Hughes and Tezuyar [38]. The $k = 2$ method is known as MITC9 and has been analyzed in [23] and [33].

For $k \geq 3$, it is shown in [48] and [45] that it is possible to reduce the number of degrees of freedom in the rotation space $\boldsymbol{\Theta}_h$ without affecting the locking-free convergence. In particular, one can choose

$$\boldsymbol{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in [\boldsymbol{Q}_k(R) \cap \boldsymbol{P}_{k+2}(R)]\}.$$

Another possibility (cf. [48]) is to choose for $k \geq 2$

$$\begin{aligned}
\boldsymbol{\Theta}_h &= \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in [\boldsymbol{Q}_k(R) \cap \boldsymbol{P}_{k+2}(R)]\},\\
W_h &= \{v \in \mathring{H}^1(\Omega) : v|_R \in Q^s_{k+1}(R)\},\\
\boldsymbol{\Gamma}_h &= \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\eta}|_R \in \boldsymbol{BDM}^{\perp}_k(R)\}.
\end{aligned}$$

The auxiliary pressure space is again $Q_h = \{q \in L^2_0(\Omega) : q|_R \in P_{k-1}\}$ and the reduction operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ is defined by

$$\int_e (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} - \boldsymbol{\gamma}) \cdot \boldsymbol{s}\, p_k(s)\,\mathrm{d}s = 0, \quad \forall e, \quad \forall p_k \in P_k(e),$$

$$\int_R (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} - \boldsymbol{\gamma}) \cdot \boldsymbol{p}_{k-2}\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall R, \quad \forall \boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(R).$$

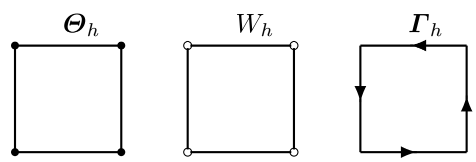
A fourth possibility discussed in [48] is to choose for $k \geq 2$

$$\begin{aligned}
\boldsymbol{\Theta}_h &= \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in [\boldsymbol{Q}_{k+1}(R), \boldsymbol{\phi}|_e \in \boldsymbol{P}_k(e)]\},\\
W_h &= \{v \in \mathring{H}^1(\Omega) : v|_R \in Q^s_k(R)\},\\
\boldsymbol{\Gamma}_h &= \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\eta}|_R \in \boldsymbol{RT}^{\perp}_{k-1}(R)\}.
\end{aligned}$$

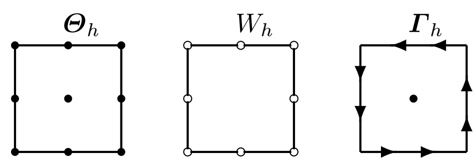
In this case, the auxiliary pressure space is now $Q_h = \{q \in L^2_0(\Omega) : q|_R \in Q_{k-1}\}$ and the reduction operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ is defined by

$$\int_e (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} - \boldsymbol{\gamma}) \cdot \boldsymbol{s}\, p_{k-1}(s)\,\mathrm{d}s = 0, \quad \forall e, \quad \forall p_{k-1} \in P_{k-1}(e),$$

$$\int_R (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} - \boldsymbol{\gamma}) \cdot \boldsymbol{r}_{k-2}\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall R,\ \forall \boldsymbol{r}_{k-2} \in Q_{k-1,k-2}(R) \times Q_{k-2,k-1}(R).$$



MITC4

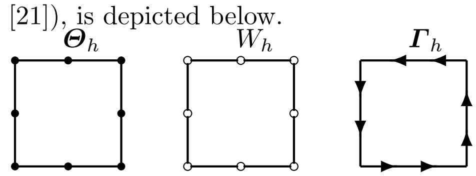


MITC9

One can also consider a low order element, associated with the choice

$$W_h = \{v \in \mathring{H}^1(\Omega) : v|_R \in Q^s_2(R)\},\ \boldsymbol{\Gamma}_h = \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\eta}|_R \in \boldsymbol{BDM}^{\perp}_1(R)\},$$

where we choose $\boldsymbol{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in \boldsymbol{Q}^s_2(R)\}$. This element, MITC8 (cf. [21]), is depicted below.



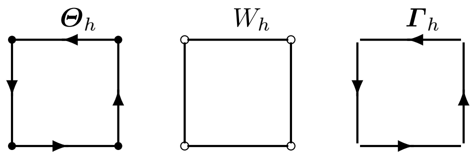
MITC8

9.2 DL4 method [31]

The DL4 method is the extension to rectangles of the Durán–Liberman triangular element defined previously. The spaces W_h and $\boldsymbol{\Gamma}_h$ are the same as those chosen for the MITC4 method, while the space of rotations is now chosen to be:

$$\boldsymbol{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_K \in \boldsymbol{Q}_1(K) \oplus \langle \boldsymbol{b}_1, , \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4 \rangle, \ \forall K \in \mathcal{T}_h\},$$

where $\boldsymbol{b}_i = b_i \boldsymbol{s}_i$, with $\boldsymbol{s}_i$ the counterclockwise unit tangent vector to the edge e_i of K and $b_i \in Q_2(K)$ vanishes on the edges e_j, $j \neq i$.



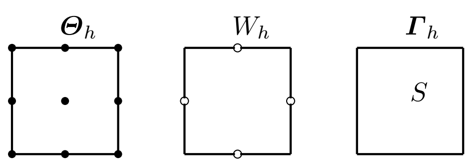
DL4

9.3 Ye's method

Ye's method is the extension to rectangles of the Arnold-Falk element. This is not completely straightforward, since the values at the midpoints of the edges of a rectangle are not a unisolvent set of degrees of freedom for a bilinear function (consider $(x-1/2)(y-1/2)$ on the unit square). Hence, the nonconforming space W_h must be chosen differently.

$$\begin{aligned}
\boldsymbol{\Theta}_h &= \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in \boldsymbol{Q}_2(R)\}, \\
\boldsymbol{\Gamma}_h &= \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\eta}|_R = (b + dx, c - dy) \equiv S\}. \\
W_h &= \{v \in \mathring{H}^1(\mathcal{T}_h) : v|_R = a + bx + cy + d(x^2 - y^2)/2\},
\end{aligned}$$

and $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ is the $\boldsymbol{L}^2$ projection.



Ye

10 Extension to quadrilaterals

Meshes of rectangular elements are very restrictive, so one would like to extend the elements defined above to quadrilaterals. To do so, we let $\boldsymbol{F}$ be an invertible bilinear mapping from the reference element $\hat{K} = [0,1] \times [0,1]$ to a convex quadrilateral K. For scalar functions, if $\hat{v}(\hat{\boldsymbol{x}})$ is function defined on

$\hat{K}$, we define $v(\boldsymbol{x})$ on K by $v = \hat{v} \circ \boldsymbol{F}^{-1}$. Then, for $\hat{V}$ a set of shape functions given on $\hat{K}$, we define

$$V_F(K) = \{v : v = \hat{v} \circ \boldsymbol{F}^{-1}, \hat{v} \in \hat{V}\}.$$

For all the examples given previously, the space W_h may be defined in this way, beginning with the shape functions denoted in the figures. This preserves the appropriate interelement continuity when the usual degrees of freedom are chosen. The same mapping, applied to each component, can be used with minor exceptions to define the space $\boldsymbol{\Theta}_h$. One exception occurs for the Durán-Liberman element, where one now defines the edge bubbles $\boldsymbol{b}_i = (\hat{b}_i \circ \boldsymbol{F}^{-1})\boldsymbol{s}_i$ where $\boldsymbol{s}_i$ denotes the unit tangent on the ith edge of K. There is also the possibility of using a different mapping to define the interior degrees of freedom for the space $\boldsymbol{\Theta}_h$, since this will not affect the interelement continuity.

To define the space $\boldsymbol{\Gamma}_h$, we use a rotated version of the Piola transform. Letting $D\boldsymbol{F}$ denote the Jacobian matrix of the transformation $\boldsymbol{F}$, if $\hat{\boldsymbol{\eta}}$ is a vector function defined on $\hat{K}$, we define $\boldsymbol{\eta}$ on K by

$$\boldsymbol{\eta}(\boldsymbol{x}) = \boldsymbol{\eta}(\boldsymbol{F}(\hat{\boldsymbol{x}})) = [D\boldsymbol{F}(\hat{\boldsymbol{x}})]^{-t}\hat{\boldsymbol{\eta}}(\hat{\boldsymbol{x}}),$$

where A^{-t} denotes the transpose of the inverse of the matrix A. Then if $\hat{\boldsymbol{V}}$ is a set of vector shape functions given on $\hat{K}$, we define

$$\boldsymbol{V}_F(K) = \{\boldsymbol{\eta} : \boldsymbol{\eta} = [D\boldsymbol{F}]^{-t}\hat{\boldsymbol{\eta}} \circ \boldsymbol{F}^{-1}, \hat{\boldsymbol{\eta}} \in \hat{\boldsymbol{V}}\}.$$

For $w \in W_h$, $\mathbf{grad}\, w = D\boldsymbol{F}^{-t}\hat{\mathbf{grad}}\, \hat{w}$. Hence, if on the reference square $\hat{\mathbf{grad}}\, \hat{w} \subseteq \hat{\boldsymbol{V}}$, we will also have $\mathbf{grad}\, w \subseteq \boldsymbol{\Gamma}_h$, a key condition in our analysis.

Although the extensions to quadrilaterals are in most cases straightforward to define, the question is whether the method retains the same order of approximation as in the rectangular case. The problem, as discussed in [3, 6, 5, 4], is that the approximation properties of some of the elements can deteriorate, depending on the way that the mesh is refined. Thus, much of the existing analysis for quadrilateral elements is restricted to the case of parallelograms (e.g., [48]), where the mapping $\boldsymbol{F}$ is affine, or to elements that are $O(h^2)$ perturbations of parallelograms. Another possibility is to restrict the refinement strategy to produce asymptotically affine meshes, so that the deterioration in approximation is also avoided. Error estimates are obtained for the DL4 method for shape-regular quadrilateral meshes and for the MITC4 method for asymptotically parallelogram meshes in [31]. However, numerical experiments do not indicate any deterioration of convergence rates for MITC4, even for more general shape regular meshes.

The MITC8 element approximates both $\boldsymbol{\theta}$ and w by spaces obtained from mappings of the quadratic serendipity space. Since this space does not contain all of Q_2 , (i.e, it is missing the basis function x^2y^2), we expect to see only $O(h)$ convergence. The space $\boldsymbol{\Gamma}_h$ is obtained by mapping the $\boldsymbol{BDM}_1^{\perp}$ space, which also degrades in convergence after a bilinear mapping. The MITC9 element uses the full Q_2 approximation for $\boldsymbol{\theta}$, but the use of the Q_2 serendipity space to approximate w and the $\boldsymbol{BDFM}_2^{\perp}$ space to approximate $\boldsymbol{\gamma}$ will cause degradation in the convergence rate on general quadrilateral meshes.

11 Other approaches

So far, all the finite element methods discussed have basically followed the common approach of modifying the original variational formulation only by the introduction of the reduction operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$. However, there are a number of other approaches that produce locking-free approximation schemes by modifying the variational formulation in other ways. Although we will not analyze these methods in detail, the main ideas are presented for a sampling of such methods in the following subsections.

11.1 Expanded mixed formulations

One of the first approaches to developing locking-free finite elements for the Reissner-Mindlin plate problem was the method proposed by Brezzi and Fortin [24], based on the expanded mixed formulation (17)-(20). There are now four variables to approximate and piecewise linear functions are used to approximate r, p, and w, while piecewise linears plus cubic bubble functions are used to approximate $\boldsymbol{\theta}$. The key idea was that equations (18)-(19) are perturbations of the stationary Stokes equations, and so a stable conforming approximation is obtained by Stokes elements with continuous pressures (note that (19) requires $p \in H^1(\Omega)$). The choice made for these two variables was the mini element. In fact, the Arnold-Falk method presented earlier was developed as a modification of this method that had the added feature that the finite element method was also equivalent to a method using only the primitive variables $\boldsymbol{\theta}$ and w. The new idea in [11] was to use a discrete Helmholtz decomposition of piecewise constant functions as the element-wise gradient of nonconforming piecewise linear functions plus the curl of continuous piecewise linear functions to reduce the discrete expanded mixed formulation back to a discrete formulation using only the primitive variables.

11.2 Simple modification of the Reissner-Mindlin energy

In this method by Arnold and Brezzi [7], the definition of the variable $\boldsymbol{\gamma}$ is modified to be

$$\boldsymbol{\gamma} = \lambda(t^{-2} - 1)(\boldsymbol{\theta} - \mathbf{grad}\, w)$$

and a new bilinear form is defined:

$$a(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) = (\mathcal{C}\, \mathcal{E}(\boldsymbol{\theta}), \mathcal{E}(\boldsymbol{\phi})) + \lambda(\boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\psi} - \mathbf{grad}\, v).$$

Then a modified weak formulation of the Reissner-Mindlin equations is to find $(\boldsymbol{\theta}, w, \boldsymbol{\gamma}) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ such that

$$a(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) + \lambda^{-1} t^2 (\boldsymbol{\gamma}, \boldsymbol{\phi} - \mathbf{grad}\, v) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}),$$
$$\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega), v \in \mathring{H}^1(\Omega),$$
$$(\mathbf{grad}\, w - \boldsymbol{\theta}, \boldsymbol{\eta}) - \frac{t^2}{\lambda(1-t^2)} (\boldsymbol{\gamma}, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega).$$

When this formulation is discretized by finite elements, we no longer need the condition that $\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h$, since the form $a(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v)$ is coercive over $\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)$. Hence, greater flexibility is allowed in the design of stable elements. Using this formulation, the choice

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^0 + \boldsymbol{B}_3, \quad W_h = \mathring{M}_2^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0.$$

gives a stable discretization and the error estimate

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|w - w_h\|_1 \le Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

11.3 Least-squares stabilization schemes

In this approach by Hughes-Franca [37] and Stenberg [47], the bilinear forms defining the method are modified by adding least-squares type stabilization terms. The approach of Stenberg is simpler and we present that here. A weak formulation of the Reissner-Mindlin equations without the introduction of the shear stress is to find $(\boldsymbol{\theta}, w) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)$ such that

$$B(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \quad \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(\Omega), v \in \mathring{H}^1(\Omega), \tag{53}$$

where

$$B(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) = a(\boldsymbol{\theta}, \boldsymbol{\phi}) + \lambda t^{-2} (\boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\phi} - \mathbf{grad}\, v).$$

In the stabilized scheme, we define

$$B_h(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) = a(\boldsymbol{\theta}, \boldsymbol{\phi}) - \alpha \sum_{T \in \mathcal{T}_h} h_T^2 (\boldsymbol{L}\boldsymbol{\theta}, \boldsymbol{L}\boldsymbol{\psi})_T$$
$$+ \sum_{T \in \mathcal{T}_h} (\lambda^{-1} t^2 + \alpha h_T^2)^{-1} (\boldsymbol{\theta} - \mathbf{grad}\, w + \alpha h_T^2 \boldsymbol{L}\boldsymbol{\theta}, \boldsymbol{\phi} - \mathbf{grad}\, v + \alpha h_T^2 \boldsymbol{L}\boldsymbol{\phi})_T,$$

where $\boldsymbol{L}\boldsymbol{\theta} = \operatorname{div} \mathcal{C}\, \mathcal{E}(\boldsymbol{\theta})$, and then seek an approximate solution $(\boldsymbol{\theta}_h, w_h) \in \boldsymbol{\Theta}_h \times W_h$ such that

$$B_h(\boldsymbol{\theta}_h, w_h; \boldsymbol{\phi}, v) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \quad \boldsymbol{\psi} \in \boldsymbol{\Theta}_h, v \in W_h,$$

The new bilinear form B_h is constructed so that the new formulation is both consistent and stable independent of the choice of finite element spaces. Dictated by approximation theory estimates with respect to the norms used, the choices $\boldsymbol{\Theta}_h = \boldsymbol{M}_k^0$, $W_h = M_{k+1}^0$ are considered for $k \geq 1$. In the lowest order case $k = 1$, $\boldsymbol{L\phi}|_T = 0$ for all $T \in \mathcal{T}_h$ and all $\boldsymbol{\phi} \in \boldsymbol{\Theta}_h$ and hence the bilinear form reduces to:

$$B_h(\boldsymbol{\theta}_h, w_h; \boldsymbol{\phi}, v) = a(\boldsymbol{\theta}, \boldsymbol{\phi}) + \sum_{T \in \mathcal{T}_h} (\lambda^{-1} t^2 + \alpha h_T^2)^{-1} (\boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\phi} - \mathbf{grad}\, v)_T,$$

a method proposed in Pitkäranta [46]. Under the hypothesis $0 < \alpha < C_I$ (for an appropriately chosen constant C_I), it is shown that

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + \|w - w_h\|_1 \leq C h^k (\|w\|_{k+2} + \|\boldsymbol{\theta}\|_{k+1}),$$

Estimates in other norms and for additional quantities are also obtained.

A modification of this method is also considered in [25]. In the modified method, $\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^0$, $W_h = \mathring{M}_1^0$, and the term $(\boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\phi} - \mathbf{grad}\, v)$ is modified to $(\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi} - \mathbf{grad}\, v)$ by adding the interpolation operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ into the space $\boldsymbol{RT}_0^{\perp}$. Thus, the method uses only linear elements. We also note that a stabilized version of the MITC4 element is proposed and analyzed in [42].

In Lyly [41], it is shown that the linked interpolation method discussed previously has close connections (and in some cases is equivalent) to the stabilized method of [25] and also to a stabilized linked method proposed by Tessler and Hughes [50]. The connection to the method of [25] is established by proving that for $\phi \in \mathring{\boldsymbol{M}}_0^1$, $\boldsymbol{\phi} - \mathbf{grad}\, L\boldsymbol{\phi} = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \phi$, where $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ denotes the usual interpolant in $\boldsymbol{RT}_{\perp}^0$. Connections to the stabilized methods are then established by using static condensation to eliminate the cubic bubble functions.

11.4 Discontinuous Galerkin methods [9], [8]

In this approach, the bilinear forms are modified to include terms that allow the use of totally discontinuous elements. We use the notation $H^s(\mathcal{T}_h)$ to denote functions whose restrictions to T belong to $H^s(T)$ for all $T \in \mathcal{T}_h$. To define the modified forms, we first define the jump and average of a function in $H^1(\mathcal{T}_h)$ as functions on the union of the edges of the triangulation. Let e be an internal edge of $\mathcal{T}_h$, shared by two elements T^+ and T^-, and let $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ denote the unit normals to e, pointing outward from T^+ and T^-, respectively. For a scalar function $\varphi \in H^1(\mathcal{T}_h)$, its average and jump on on e are defined respectively, by

$$\{\varphi\} = \frac{\varphi^+ + \varphi^-}{2}, \qquad [\![\varphi]\!] = \varphi^+ \boldsymbol{n}^+ + \varphi^- \boldsymbol{n}^-.$$

Note that the jump is a vector normal to e. The jump of a vector $\boldsymbol{\phi} \in \boldsymbol{H}^1(\mathcal{T}_h)$ is the symmetric matrix-valued function given on e by:

$$[\![\boldsymbol{\phi}]\!] = \boldsymbol{\phi}^+ \odot \boldsymbol{n}^+ + \boldsymbol{\phi}^- \odot \boldsymbol{n}^-,$$

where $\phi \odot n = (\phi \otimes n + n \otimes \phi)/2$ is the symmetric part of the tensor product of ϕ and n. On a boundary edge, the average $\{\varphi\}$ is defined simply as the trace of φ, while for a scalar-valued function, we define $[\![\varphi]\!]$ to be φn (with n the outward unit normal), and for a vector-valued function we define $[\![\phi]\!] = \phi \odot n$.

To obtain a DG discretization, we have to choose finite dimensional subspaces $\Theta_h \subset H^2(\mathcal{T}_h)$, $W_h \subset H^1(\mathcal{T}_h)$, and $\Gamma_h \subset H^1(\mathcal{T}_h)$. The method then takes the form:

Find $(\theta_h, w_h) \in \Theta_h \times W_h$ *and* $\gamma_h \in \Gamma_h$ *such that*

$$
\begin{aligned}
&(C\,\mathcal{E}_h(\theta_h), \mathcal{E}_h(\phi)) - \langle \{C\,\mathcal{E}_h(\theta_h)\}, [\![\phi]\!] \rangle - \langle [\![\theta_h]\!], \{C\,\mathcal{E}_h(\phi)\} \rangle \\
&\qquad + (\gamma_h, \mathbf{grad}_h\, v - \phi) - \langle \{\gamma_h\}, [\![v]\!] \rangle \\
&\qquad\qquad + p_\Theta(\theta_h, \phi) + p_W(w_h, v) = (g, v) - (f, \phi), \quad (\phi, v) \in \Theta_h \times W_h, \\
&\qquad (\mathbf{grad}_h\, w_h - \theta_h, \eta) - \langle [\![w_h]\!], \{\eta\} \rangle - t^2(\gamma_h, \eta) = 0, \quad \eta \in \Gamma_h.
\end{aligned}
$$

We make a standard choice for the interior penalty terms p_Θ and p_W:

$$
p_\Theta(\theta, \phi) = \sum_{e \in \mathcal{E}_h} \frac{\kappa^\Theta}{|e|} \int_e [\![\theta]\!] : [\![\phi]\!]\, ds, \quad p_W(w, v) = \sum_{e \in \mathcal{E}_h} \frac{\kappa^W}{|e|} \int_e [\![w]\!] \cdot [\![v]\!]\, ds,
$$

so that $p_\Theta(\phi, \phi)$, ($p_W(v, v)$, resp.) can be viewed as a measure of the deviation of ϕ ($v, resp.$) from being continuous. The parameters κ^Θ and κ^W are positive constants to be chosen; they must be sufficiently large to ensure stability. In the case when W_h consists of continuous elements, the penalty term p_W will not be needed.

In the simplest of such methods, one chooses for $k \geq 1$, $W_h = \mathring{M}^0_{k+1}$, i.e., continuous piecewise polynomials of degree $\leq k+1$. We then choose $w^I = \Pi^W w$, where Π^W is defined as for the MITC elements. Since the space Θ_h need not be continuous, we can now choose Θ_h so that condition (30) is satisfied without the need for a reduction operator Π^Γ. The simplest choice is $\Theta_h = BDM^\perp_{k-1}$. We note that $\mathbf{grad}\, W_h \subset \Theta_h$. We next define $\theta^I = \Pi^\Theta \theta$, where $\Pi^\Theta : H^1(\Omega) \mapsto \Theta_h$ is defined by the conditions:

$$
\begin{aligned}
\int_e (\phi - \Pi^\Theta \phi) \cdot s\, q\, \mathrm{d}s = 0, &\quad q \in P_{k-1}(e), \\
\int_T (\phi - \Pi^\Theta \phi) \cdot q\, \mathrm{d}x = 0, &\quad q \in RT_{k-3}(T),
\end{aligned}
$$

where RT_{k-3} is the usual (unrotated) Raviart-Thomas space of index $k-3$. We note that the interior degrees of freedom are not the original degrees of freedom defined for these spaces. However, the natural interpolant defined by these modified degrees of freedom satisfies the additional and key property that

$$
\Pi^\Theta\, \mathbf{grad}\, w = \mathbf{grad}\, \Pi^W w.
$$

From this condition, we get

$$\gamma^I = \lambda t^{-2}(\mathbf{grad}\, w^I - \boldsymbol{\theta}^I) = \lambda t^{-2}(\mathbf{grad}\, \Pi^W w - \boldsymbol{\Pi}^{\boldsymbol{\Theta}}\boldsymbol{\theta})$$
$$= \lambda t^{-2}\boldsymbol{\Pi}^{\boldsymbol{\Theta}}(\mathbf{grad}\, w - \boldsymbol{\theta}) = \boldsymbol{\Pi}^{\boldsymbol{\Theta}}\boldsymbol{\gamma}.$$

11.5 Methods using nonconforming finite elements

In the nonconforming element of Oñate, Zarate, and Flores [43], one chooses

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^*, \quad W_h = \mathring{M}_1^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{RT}_0^{\perp}.$$

In this case, $\boldsymbol{\Theta}_h$ is not contained in $\mathring{\boldsymbol{H}}^1(\Omega)$, and so $\mathcal{E}$ must be replaced by $\mathcal{E}_h$. The main problem with this method is that $\|\,\mathcal{E}_h(\boldsymbol{\theta}_h)\|_0^2$ is not a norm on $\boldsymbol{\Theta}_h$ because Korn's inequality fails for nonconforming piecewise linear functions. To partially compensate for this fact, one can use the following result, established in [14]. Define

$$\boldsymbol{Z}_h = \left\{(\boldsymbol{\psi}, \boldsymbol{\eta}) \in \mathring{\boldsymbol{M}}_1^* \times \boldsymbol{\Gamma}_h : \lambda^{-1} t^2 \operatorname{rot} \boldsymbol{\eta} = \operatorname{rot}_h \boldsymbol{\psi}\right\}. \tag{54}$$

Lemma 11.1 *There exists a constant c independent of h and t such that*

$$a_h(\boldsymbol{\psi}, \boldsymbol{\psi}) + \lambda^{-1}t^2(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq c[\min(1, h^2/t^2)\|\boldsymbol{\psi}\|_{1,h}^2 + \|\,\mathcal{E}_h\,\boldsymbol{\psi}\|_0^2$$
$$+ t^2\|\boldsymbol{\eta}\|_0^2 + h^2t^2\|\operatorname{rot}\boldsymbol{\eta}\|_0^2] \quad \textit{for all } (\boldsymbol{\psi}, \boldsymbol{\eta}) \in \boldsymbol{Z}_h.$$

Note that the bilinear form is not uniformly coercive. It is then possible to establish the following error estimates (cf. [14]).

Theorem 11.2 *There exists a constant C independent of h and t such that*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{1,h} + t^2\|\operatorname{rot}(\boldsymbol{\gamma} - \gamma_h)\|_0^2 \leq Ch\max(1, t^2/h^2)\|g\|_0,$$
$$\|\,\mathcal{E}(\boldsymbol{\theta} - \boldsymbol{\theta}_h)\|_0^2 + t\|\boldsymbol{\gamma} - \gamma_h\|_0 \leq Ch\max(1, t/h)\|g\|_0.$$
$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0 \leq C\max(h^2, t^2)\|g\|_0.$$

Note that this theorem does not imply convergence of the method. If $h \sim t$, however, the error will be small.

In the method proposed by Lovadina [40],

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^*, \quad W_h = \mathring{M}_1^*, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0,$$

so two of the spaces are nonconforming. Hence, both $\mathcal{E}$ and **grad** are replaced by their element-wise counterparts. In addition, the bilinear form $a(\boldsymbol{\theta}, \boldsymbol{\phi})$ is replaced by

$$a_h(\boldsymbol{\theta}, \boldsymbol{\phi}) = \sum_{T \in \mathcal{T}_h} a_T(\boldsymbol{\theta}, \boldsymbol{\phi}) + p_\Theta(\boldsymbol{\theta}, \boldsymbol{\phi}), \qquad a_T(\boldsymbol{\theta}, \boldsymbol{\phi}) = \int_T \mathsf{C}\,\mathcal{E}\,\boldsymbol{\theta}) : \mathcal{E}(\boldsymbol{\phi})\,\mathrm{d}x,$$

where p_Θ has the same definition as in the discontinuous Galerkin method. By adding the term p_Θ, one is able to establish a discrete Korn's inequality.

This method is a simplified version of a method proposed earlier by Brezzi-Marini [26]. Using a similar formulation, they made the choices

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^* + \boldsymbol{B}_2^*, \quad W_h = \mathring{M}_1^* + B_2^*, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0 + \mathbf{grad}_h\, B_2^*,$$

where B_2^* denotes the nonconforming quadratic bubble function that vanishes at the two Gauss points of each edge of a triangle. See also [27] for L^2 estimates for the method of [40].

11.6 A negative-norm least squares method

This method, proposed by Bramble-Sun [22], begins with the expanded mixed formulation used by Brezzi-Fortin. The problem is then reformulated as a least squares method using a special minus one norm developed previously by Bramble, Lazarov, and Pasciak. Only continuous finite elements are needed to approximate all the variables, and piecewise linears can be used. Optimal order error estimates are established uniformly in the thickness t. The stability result also gives a natural block diagonal preconditioner, using only standard preconditioners for second order elliptic problems, for the solution of the resulting least squares system.

12 Summary

We have treated in these notes only a selection of the finite element methods that have been developed for the approximation of the Reissner-Mindlin plate problem, concentrating on those for which there is a mathematical analysis. There are many other methods available in the engineering literature, and the list is too long to give proper citations.

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