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# THE BOUNDARY LAYER FOR THE REISSNER–MINDLIN PLATE MODEL*

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**Abstract.** The structure of the solution of the Reissner–Mindlin plate equations is investigated, emphasizing its dependence on the plate thickness. For the transverse displacement, rotation, and shear stress, asymptotic expansions in powers of the plate thickness are developed. These expansions are uniform up to the boundary for the transverse displacement, but for the other variables there is a boundary layer. Rigorous error bounds are given for the errors in the expansions in Sobolev norms. As applications, new regularity results for the solutions and new estimates for the difference between the Reissner–Mindlin solution and the solution to the biharmonic equation are derived. Boundary conditions for a clamped edge are considered for most of the paper, and the very similar case of a hard simply-supported plate is discussed briefly at the end. Other boundary conditions will be treated in a forthcoming paper.

**Key words.** Reissner, Mindlin, plate, boundary layer

**AMS(MOS) subject classifications.** 73K10, 35B25

**1. Introduction.** The Reissner–Mindlin model describes the deformation of a plate subject to a transverse loading in terms of the transverse displacement of the midplane and the rotation of fibers normal to the midplane [9], [10]. This linear model, as well as its generalization to shells, is frequently used for plates and shells of small to moderate thickness. Specifically, let $\Omega$ denote the region in $\mathbb{R}^2$ occupied by the midsection of the plate and $\omega$ and $\boldsymbol{\phi}$ the transverse displacement of $\Omega$ and the rotation of the fibers normal to $\Omega$, respectively. The Reissner–Mindlin model for the bending of a clamped isotropic elastic plate in equilibrium determines $\omega$ and $\boldsymbol{\phi}$ as the solution of the partial differential equations

$$-\,\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}) - \lambda\bar{t}^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = 0, \tag{1.1}$$

$$-\lambda\bar{t}^{-2}\,\mathrm{div}(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = g, \tag{1.2}$$

in $\Omega$ and the boundary conditions

$$\boldsymbol{\phi} = 0,\ \omega = 0, \tag{1.3}$$

on $\partial\Omega$. Here $g\bar{t}^3$ is the transverse load force density per unit area, $\bar{t}$ is the plate thickness, $\lambda = Ek/2(1+\nu)$ with $E$ the Young's modulus, $\nu$ the Poisson ratio, and $k$ the shear correction factor, $\mathcal{E}(\boldsymbol{\phi})$ is the symmetric part of the gradient of $\boldsymbol{\phi}$, and the fourth-order tensor $\mathcal{C}$ is defined by

$$\mathcal{C}\mathcal{T} = D\left[(1-\nu)\mathcal{T} + \nu\,\mathrm{tr}(\mathcal{T})\mathcal{I}\right], \quad D = \frac{E}{12(1-\nu^2)},$$

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for any $2 \times 2$ matrix $\mathcal{T}$ ($\mathcal{I}$ denotes the $2 \times 2$ identity matrix). Note that the load has been scaled so that the solution tends to a nonzero limit as $\bar{t}$ tends to zero. The Dirichlet boundary conditions (1.3) model a plate which experiences no displacement along its lateral edge. This is commonly referred to as a clamped edge (which is the terminology we adopt here), although the terms welded or built-in are perhaps more descriptive.

The Reissner–Mindlin model is an alternative to the biharmonic model for plate bending. The biharmonic model gives the transverse displacement as the solution to the boundary value problem

$$D\,\Delta^2\,\omega_0 = g \quad \text{in } \Omega, \quad \omega_0 = \partial\omega_0/\partial n = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

With our scaling of the load function, the solution $\omega_0$ is independent of the plate thickness. By contrast, the solution of the Reissner–Mindlin model depends in a complex way on the plate thickness. It is the purpose of this paper to investigate the structure of solution in its dependence on $\bar{t}$.

We shall develop asymptotic expansions with respect to $\bar{t}$ for $\omega$ and $\boldsymbol{\phi}$ (as well as other quantities associated with the solution such as the shear stress). The expansions are of the following forms[1]

$$\omega \sim \omega_0 + \bar{t}^2\omega_2 + \bar{t}^3\omega_3 + \cdots,$$
$$\boldsymbol{\phi} \sim \boldsymbol{\phi}_0 + \bar{t}^2(\boldsymbol{\phi}_2 + \chi\boldsymbol{\Phi}_0) + \bar{t}^3(\boldsymbol{\phi}_3 + \chi\boldsymbol{\Phi}_1) + \cdots.$$

Here the functions $\omega_i$ and $\boldsymbol{\phi}_i$, the interior expansion functions, are independent of $\bar{t}$. The functions $\boldsymbol{\Phi}_i$ are boundary correctors. They depend on $\bar{t}$ only through the quantity $\rho/\bar{t}$, where $\rho$ is the distance of a point of $\Omega$ from the boundary. More specifically,

$$\boldsymbol{\Phi}_i = \hat{\boldsymbol{\Phi}}_i(\rho/\bar{t}, \theta)$$

where $\theta$ is a coordinate which roughly gives arclength along the boundary (see § 2), and the function $\hat{\boldsymbol{\Phi}}_i(\eta, \theta)$ has the form of a polynomial with respect to $\eta$ times $\exp(-\sqrt{12k}\eta)$. Thus $\boldsymbol{\Phi}_i$ represents a boundary layer function, which essentially lives in a strip of width $\bar{t}$ around the boundary. Finally, $\chi$ is a cutoff function which is independent of $\bar{t}$ and identically equal to unity in a neighborhood of $\partial\Omega$.

In §§ 3 and 6 we construct all terms of these expansions. Here we summarize the results for the principal terms. The function $\omega_0$ is the solution to the biharmonic problem above, $\omega_2$ solves

$$D\,\Delta^2\,\omega_2 = \frac{-1}{6k(1-\nu)}\,\Delta\, g \quad \text{in } \Omega, \quad \omega_2 = 0, \quad \frac{\partial\omega_2}{\partial n} = \frac{-1}{6k(1-\nu)}\frac{\partial}{\partial n}\,\Delta\,\omega_0 \quad \text{on } \partial\Omega,$$

and $\omega_3$ solves

$$D\,\Delta^2\,\omega_3 = 0 \quad \text{in } \Omega, \quad \omega_3 = 0, \quad \frac{\partial\omega_3}{\partial n} = \frac{-1}{12\sqrt{3k^3}(1-\nu)}\frac{\partial^2}{\partial s^2}\,\Delta\,\omega_0 \quad \text{on } \partial\Omega.$$

[1] In order not to introduce unnecessary distractions, in this introduction we use a slightly different notation than in the following sections. The $\omega_i$ and $\boldsymbol{\phi}_i$ of this section are $\lambda^{-i/2}$ times the corresponding quantities used in the remaining sections, and $\hat{\boldsymbol{\Phi}}_{i-2}(\eta, \theta)$ here is $\lambda^{-i/2}$ times $\hat{\boldsymbol{\Phi}}_{i-2}(\sqrt{\lambda}\eta, \theta)$ used later.

For the expansion of $\phi$, we have $\boldsymbol{\phi}_0 = \mathbf{grad}\,\omega_0$, $\boldsymbol{\phi}_2 = \mathbf{grad}\,\psi$, where

$$\Delta^2\,\psi = 0 \quad \text{in } \Omega, \qquad \psi = \frac{1}{6k(1-\nu)}\,\Delta\,\omega_0, \quad \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

and

$$\hat{\boldsymbol{\Phi}}_0(\eta,\theta) = -\frac{\exp(-\sqrt{12k}\eta)}{6k(1-\nu)}\frac{\partial}{\partial s}\,\Delta\,\omega_0(0,\theta)\boldsymbol{s},$$

where $\boldsymbol{s} = \boldsymbol{s}(\theta)$ is the unit tangent vector to $\partial\Omega$.

We prove a priori estimates for all terms of the expansions in § 4, and establish error bounds for the remainders in § 5. With these results, we may easily investigate the regularity of solutions of the Reissner–Mindlin system and their limit as $\bar{t} \to 0$. Supposing that $g$ is sufficiently smooth, we have the following estimates, in which the constant $C$ depends on $g$, $\Omega$, and the elastic constants, but is independent of $\bar{t}$. Here $\|\cdot\|_s$ and $|\cdot|_s$ denote the norms in the Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$ (see § 2).

The transverse displacement $\omega$ is regular uniformly in $\bar{t}$, but the regularity of the rotation $\boldsymbol{\phi}$ is limited by the boundary layer:

$$\|\omega\|_s \le C, \qquad \|\boldsymbol{\phi}\|_s \le C\bar{t}^{\min(0,5/2-s)}, \qquad s \in \mathbb{R}.$$

Thus all derivatives of $\omega$ remain bounded uniformly in $L^2$ as $\bar{t} \to 0$, while for $\boldsymbol{\phi}$, the second derivatives remain bounded in $L^2$, but higher derivatives will in general blow up as $\bar{t} \to 0$.

The quantity $\boldsymbol{\zeta} := \lambda\bar{t}^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi})$, which is related to the shear stress, is often of interest. From the above expansions we get

$$\lambda^{-1}\boldsymbol{\zeta} \sim \mathbf{grad}\,\omega_2 - \boldsymbol{\phi}_2 - \chi\boldsymbol{\Phi}_0 + t(\mathbf{grad}\,\omega_3 - \boldsymbol{\phi}_3 - \chi\boldsymbol{\Phi}_1) + \cdots,$$

so it has a stronger boundary layer. Indeed, $\boldsymbol{\zeta}$ is not uniformly bounded in $H^s$ for $s > \frac{1}{2}$:

$$\|\boldsymbol{\zeta}\|_s \le C\bar{t}^{\min(0,1/2-s)}, \ s \in \mathbb{R}.$$

Of course, the boundary layer does not limit the regularity of $\boldsymbol{\phi}$ or $\boldsymbol{\zeta}$ at a positive distance from $\partial\Omega$ nor does it affect the smoothness of their restrictions to $\partial\Omega$. Thus

$$\|\boldsymbol{\phi}\|_{H^s(\Omega_c)} + |\boldsymbol{\phi}|_s + \|\boldsymbol{\zeta}\|_{H^s(\Omega_c)} + |\boldsymbol{\zeta}|_s \le C, \ s \in \mathbb{R},$$

for any compact subdomain $\Omega_c$ of $\Omega$.

In the limit as $\bar{t} \to 0$, each of the variables $\omega$, $\boldsymbol{\phi}$, and $\boldsymbol{\zeta}$ tends in $L^2$ to the leading terms of its asymptotic expansions. The number of derivatives which converge and the rate of convergence may be determined by examining the first neglected interior and boundary terms of the expansions. We get, for each $s \in \mathbb{R}$, that

$$\begin{aligned}
&\|\omega - \omega_0\|_s \le C\bar{t}^2,\\
&\|\boldsymbol{\phi} - \boldsymbol{\phi}_0\|_s \le C\bar{t}^{\min(2,5/2-s)},\\
&\|\boldsymbol{\zeta} - \lambda(\mathbf{grad}\,\omega_2 - \boldsymbol{\phi}_2)\|_s \le C\bar{t}^{\min(1/2,1/2-s)}.
\end{aligned}$$

Note that for $\boldsymbol{\phi}$ and $\boldsymbol{\zeta}$, the rate of convergence depends on the Sobolev norm under consideration. For each of the variables, taking more terms from the expansions increases the rates of convergence. For example,

$$\|\omega - \omega_0 - \bar{t}^2\omega_2\|_s \le C\bar{t}^3, \qquad \|\boldsymbol{\phi} - \boldsymbol{\phi}_0 - \bar{t}^2(\boldsymbol{\phi}_2 + \chi\boldsymbol{\Phi}_0)\|_s \le C\bar{t}^{\min(3,7/2-s)}.$$

Taking sufficiently many terms in the expansions gives approximations of any desired algebraic order of convergence in $\bar{t}$ in any desired Sobolev space (provided $g$ is sufficiently regular).

It is also possible to use the asymptotic expansion to derive estimates in function spaces other than $H^s$. Thus for example, we show at the end of § 5 that

$$\|\boldsymbol{\phi}\|_{W^s_\infty} \le C\bar{t}^{\min(0,2-s)},$$

and, in particular, that $\|\boldsymbol{\phi}\|_{W^2_\infty}$ is uniformly bounded. Note that this is a better estimate than we would get applying the Sobolev Embedding Theorem directly to the estimates for $\boldsymbol{\phi}$ in $H^s$. It is also easy to show that

$$\|\omega - \omega_0\|_{L^\infty} \le C\bar{t}^2, \qquad \|\boldsymbol{\phi} - \boldsymbol{\phi}_0\|_{L^\infty} \le C\bar{t}^2,$$

but $\boldsymbol{\zeta}$ does not in general converge in $L^\infty(\Omega)$.

The Reissner–Mindlin model is discussed in many places (under various names), although not very much attention has been devoted to the boundary layer behavior. The existence of a boundary layer is noted in [6, Chaps. 8.9–8.10] and [11, Chap. 3.5]. Assiff and Yen [2] also note the existence of a boundary layer, and use separation of variable techniques to compute the exact solution to the equations on a circular plate with a special load. This calculation exhibits the boundary layer, and may be taken as an example of our theory. Häggblad and Bathe recently studied the boundary layer in more general situations via formal techniques and numerical experiments in [7]. They also consider the effect of corners, which is not treated here. In [6], [11], and [7], the authors emphasize a reformulation of the Reissner–Mindlin system consisting of a biharmonic equation for $\omega$ (with different right-hand side than (1.4)), and a singularly perturbed Laplacian for $\operatorname{rot}\boldsymbol{\phi}$. These equations are coupled through somewhat complicated boundary conditions, however, and we have preferred not to use them. As far as we know, the explicit form of the asymptotic expansions and error bounds for them are new.

**2. Notation and preliminaries.** The letter $C$ denotes a generic constant, not necessarily the same in each occurrence. We assume that $\Omega$ is a smooth, bounded, and simply-connected domain in $\mathbb{R}^2$. The $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products are denoted by $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$ respectively. We shall use the usual $L^2$-based Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$, $s \in \mathbb{R}$, with norms denoted by $\|\cdot\|_s$ and $|\cdot|_s$. The reader is referred to [8] for precise definitions of these spaces and their properties, of which we recall only a few here. For $s \ge 0$, $H^{-s}$ may be identified with the dual of $\mathring{H}^s$, the closure of $C_0^\infty$ in $H^s$. If $s \ge 0$, $n \ge i \ge 0$ are real numbers, then the interpolation inequality

$$\|g\|_{s+i}^n \le C\|g\|_s^{n-i}\|g\|_{s+n}^i \tag{2.1}$$

holds. If $g \in L^2(\Omega)$ and $\Delta^{-1} g$ denotes the unique function in $H^2(\Omega) \cap \mathring{H}^1(\Omega)$ whose Laplacian is equal to $g$, then

$$C^{-1}\|\,\Delta^{-1} g\|_{s+2} \le \|g\|_s \le C\|\,\Delta^{-1} g\|_{s+2}, \qquad s \ge 0,$$

where the constant $C$ may depend on $s$ and $\Omega$, but not on $g$. In other words, $g \mapsto \|\,\Delta^{-1} g\|_{s+2}$ defines an equivalent norm on $H^s(\Omega)$ for $s \ge 0$. This is also true for $s = -1$, but slightly different negative norms are needed to extend this shift theorem to other negative values. We define $|||g|||_s = \|\,\Delta^{-1}\,g\|_{s+2}$ for $g \in L^2(\Omega)$ and all real $s$. Then $|||\cdot|||_s$ is equivalent to the ordinary Sobolev norm $\|\cdot\|_s$ for $s \ge 0$ and $s = -1$. For $s = -2$, $|||\cdot|||_s$ is equivalent to the norm in the dual space of $H^2(\Omega) \cap \mathring{H}^1(\Omega)$. The norm $|||\cdot|||$ can be identified for other values of $s$ as well, but this is not necessary for our purposes. From (2.1) we have

$$|||g|||_{s+i}^n \le C|||g|||_s^{n-i}|||g|||_{s+n}^i,$$

valid for all real $s \ge -2$, $n \ge i \ge 0$. We shall make frequent use of this fact to bound sums of the form $\sum_{i=0}^n t^i|||g|||_{s+i}$ by a multiple of the sum of the first and last terms.

We also require the quotient space $H^s(\Omega)/\mathbb{R}$. An element $p \in H^s(\Omega)/\mathbb{R}$ is a coset consisting of all functions in $H^s(\Omega)$ differing from a fixed function by a constant. The quotient norm is given by

$$\|p\|_{s/\mathbb{R}} = \min_{q \in p} \|q\|_s.$$

In fact, $\|p\|_{s/\mathbb{R}} = \|p_0\|_s$ where $p_0$ is the unique function in the coset $p$ having mean value zero.

We use boldface type to denote 2-vector-valued functions, operators whose values are vector-valued functions, and spaces of vector-valued functions. Script type is used in a similar way for 2 × 2-matrix objects. Thus, for example, $\operatorname{div}\boldsymbol{\psi} \in L^2(\Omega)$ for $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$, while $\mathbf{div}\,\mathcal{T} \in \boldsymbol{L}^2(\Omega)$ for $\mathcal{T} \in \mathcal{H}^1(\Omega)$. Finally, we use various standard differential operators:

$$\mathbf{grad}\,r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \qquad \operatorname{div}\boldsymbol{\psi} = \frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y},$$

$$\mathbf{div}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} \partial t_{11}/\partial x + \partial t_{12}/\partial y \\ \partial t_{21}/\partial x + \partial t_{22}/\partial y \end{pmatrix},$$

$$\mathbf{curl}\,p = \begin{pmatrix} -\partial p/\partial y \\ \partial p/\partial x \end{pmatrix}, \qquad \operatorname{rot}\boldsymbol{\psi} = \frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial x}.$$

Note that these differential operators annihilate constants, and consequently induce operators on the quotient space $H^s(\Omega)/\mathbb{R}$ for each $s$. We denote the induced operator in the same way as the original. Thus, for example, if $p \in H^1(\Omega)/\mathbb{R}$, $\mathbf{curl}\,p$ denotes the element of $\boldsymbol{L}^2$ obtained by applying the curl to any element in the coset $p$.

In our analysis, we rely on an equivalent formulation of the Reissner–Mindlin plate equations, suggested by Brezzi and Fortin [3]. This formulation is derived by using the Helmholtz theorem to decompose the scaled transverse shear stress vector:

$$\boldsymbol{\zeta} := \lambda\bar{t}^{\,-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = \mathbf{grad}\,r + \mathbf{curl}\,p, \qquad r \in \mathring{H}^1(\Omega), \quad p \in H^1(\Omega)/\mathbb{R}. \tag{2.2}$$

Setting $t^2 = \bar{t}^{\,2}/\lambda$, one finds

$$-\Delta\,r = g, \tag{2.3}$$

$$-\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}) - \mathbf{curl}\,p = \mathbf{grad}\,r, \tag{2.4}$$

$$-\operatorname{rot}\boldsymbol{\phi} + t^2\,\Delta\,p = 0, \tag{2.5}$$

$$-\Delta\,\omega = -\operatorname{div}\boldsymbol{\phi} - t^2\,\Delta\,r, \tag{2.6}$$

with the boundary conditions

$$r = 0, \qquad \boldsymbol{\phi} = 0, \qquad \frac{\partial p}{\partial n} = 0, \qquad \omega = 0. \tag{2.7}$$

Note that $r$ satisfies a Dirichlet problem for Poisson's equation, which decouples from the other three equations. Once $r$ has been determined, $\boldsymbol{\phi}$ and $p$ may be computed from (2.4) and (2.5) and their boundary conditions, and then $\omega$ is determined by a second Dirichlet problem for Poisson's equation. Thus all the difficulties of the problem have been concentrated in the system (2.4)–(2.5) for $\boldsymbol{\phi}$ and $p$. When $t = 0$, this system of partial differential equations is very similar to the Stokes equations. For positive $t$, these two equations represent a singularly perturbed Stokes-like system.

It is easy to check that this reformulation is equivalent to the usual Reissner–Mindlin formulation (cf. [3] or [1]). That is, if $(\omega, \boldsymbol{\phi}) \in H^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ solves (1.1)–(1.3) and $(r, p) \in \mathring{H}^1(\Omega) \times H^1(\Omega)/\mathbb{R}$ are defined (uniquely) by (2.2), then (2.3)–(2.7) are satisfied, and, conversely, if $(\omega, \boldsymbol{\phi}, p, r) \in H^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times H^1(\Omega)/\mathbb{R} \times H^1(\Omega)$ solves (2.3)–(2.7) then (1.1)–(1.3) hold.

To describe the boundary layer for the Reissner–Mindlin plate, we shall employ the standard technique of making a change of variable in a neighborhood of the boundary. Let $(X(\theta), Y(\theta))$, $\theta \in [0, L)$, be a parametrization of $\partial\Omega$ by arclength, and let $\Omega_0$ be a normal tubular neighborhood of $\partial\Omega$ in $\Omega$. Then, for each point $z = (x, y) \in \Omega_0$ there is a unique nearest point $z_0 \in \partial\Omega$. Let $\theta$ denote the arclength parameter, with counterclockwise orientation, corresponding to $z_0$ and $\rho = |z - z_0|$ the distance from the point $z$ to the boundary. Since $\Omega_0$ is a tubular neighborhood of $\partial\Omega$, the correspondence $(x, y) \mapsto (\rho, \theta)$ is a diffeomorphism between $\Omega_0$ and $(0, \rho_0) \times \mathbb{R}/L$ for some $\rho_0 > 0$. Explicitly, $x = X(\theta) - \rho Y'(\theta)$, $y = Y(\theta) + \rho X'(\theta)$. A simple computation shows that the Jacobian of the transformation from $(x, y)$ coordinates to $(\rho, \theta)$ coordinates on $\Omega_0$ is given by $1 - \kappa(\theta)\rho$, where $\kappa$ denotes the curvature of $\partial\Omega$. With these definitions, the unit outward normal and counterclockwise unit tangent vectors are given by

$$\boldsymbol{n} = -\,\mathbf{grad}\,\rho = -\,\mathbf{curl}\,\theta, \quad \boldsymbol{s} = \mathbf{grad}\,\theta = -\,\mathbf{curl}\,\rho, \quad \text{on } \partial\Omega.$$

We use tildes to denote the corresponding change of variables for functions, i.e.,

$$\tilde{f}(\rho, \theta) := f(x, y).$$

We shall also use the stretched variable $\hat{\rho} = \rho/t$. Circumflexes denote the corresponding change of variables

$$\hat{f}(\hat{\rho}, \theta) := \tilde{f}(\rho, \theta) = f(x, y).$$

**3. An asymptotic expansion of the solution.** We now turn to the construction of an asymptotic expansion with respect to the scaled thickness $t = \bar{t}/\sqrt{\lambda}$ of the solution of the Reissner–Mindlin clamped plate model using the formulation given in (2.3)–(2.7). Clearly $r$ does not depend on $t$, so we begin, in this section, with the expansion of $\boldsymbol{\phi}$ and $p$. In § 6 we consider the expansion of $\omega$ and the shear stress.

Our immediate goal is to develop approximations of $\boldsymbol{\phi}$ and $p$ by sums of the form

$$\boldsymbol{\phi}(x,y) \sim \boldsymbol{\phi}^I(x,y) + \boldsymbol{\phi}^B(x,y) := \sum_{i=0}^{\infty} t^i \boldsymbol{\phi}_i(x,y) + t^2 \tilde{\chi}(\rho) \sum_{i=0}^{\infty} t^i \hat{\boldsymbol{\Phi}}_i(\hat{\rho},\theta),$$

$$p(x,y) \sim p^I(x,y) + p^B(x,y) := \sum_{i=0}^{\infty} t^i p_i(x,y) + t\tilde{\chi}(\rho) \sum_{i=0}^{\infty} t^i \hat{P}_i(\hat{\rho},\theta),$$

where $\tilde{\chi}(\rho)$ is a smooth cutoff function which is identically one for $0 \le \rho \le \rho_0/3$ and which is identically zero for $\rho > 2\rho_0/3$. (The power of $t$ multiplying the second sum in each expansion was chosen in anticipation of the results that follow.) In this section we shall calculate formally in order to motivate appropriate definitions of the interior expansion functions $\boldsymbol{\phi}_i$ and $p_i$, and the boundary correctors $\hat{\boldsymbol{\Phi}}_i$ and $\hat{P}_i$. In the next section we derive some estimates for these functions, and in § 5 we give rigorous bounds for the errors in the asymptotic expansions.

Now $\boldsymbol{\phi} \in \boldsymbol{H}^1(\Omega)$ and $p \in H^1(\Omega)/\mathbb{R}$ are uniquely determined by the equations

$$
\begin{gathered}
-\operatorname{\mathbf{div}} C\,\mathcal{E}(\boldsymbol{\phi}) - \operatorname{\mathbf{curl}} p = \operatorname{\mathbf{grad}} r, \quad \text{in } \Omega,\\
-\operatorname{rot}\boldsymbol{\phi} + t^2\,\Delta p = 0 \pmod{\mathbb{R}}, \quad \text{in } \Omega,\\
\boldsymbol{\phi} = 0, \quad \frac{\partial p}{\partial n} = 0, \quad \text{on } \partial\Omega.
\end{gathered}
$$

In writing the second equation modulo $\mathbb{R}$ we mean that $\boldsymbol{\phi}$ and $p$ are to be determined with $-\operatorname{rot}\boldsymbol{\phi} + t^2\,\Delta p$ equal to an unspecified constant function. In fact if we integrate this equation using the divergence theorem, it follows that if $\boldsymbol{\phi}$ and $p$ also satisfy the boundary conditions, then the constant must vanish. Thus, although the equation modulo $\mathbb{R}$ is formally weaker than (2.5), in fact together with the other equation and the boundary conditions, we have an equivalent problem to (2.4), (2.5), (2.7). The reason for introducing this complication is that it is more convenient to define an asymptotic expansion that satisfies the second equation only up to an additive constant.

Formally, $(\boldsymbol{\phi}^I, p^I)$ will be determined such that

$$
-\operatorname{\mathbf{div}} C\,\mathcal{E}(\boldsymbol{\phi}^I) - \operatorname{\mathbf{curl}} p^I = \operatorname{\mathbf{grad}} r, \quad \text{in } \Omega, \tag{3.1}
$$

$$
-\operatorname{rot}\boldsymbol{\phi}^I + t^2\,\Delta p^I = 0 \pmod{\mathbb{R}}, \quad \text{in } \Omega, \tag{3.2}
$$

$$
\boldsymbol{\phi}^I = -\boldsymbol{\phi}^B, \quad \text{on } \partial\Omega, \tag{3.3}
$$

and $(\boldsymbol{\phi}^B, p^B)$ will be determined such that

$$
-\operatorname{\mathbf{div}} C\,\mathcal{E}(\boldsymbol{\phi}^B) - \operatorname{\mathbf{curl}} p^B = 0, \quad \text{in } \Omega, \tag{3.4}
$$

$$
-\operatorname{rot}\boldsymbol{\phi}^B + t^2\,\Delta p^B = 0, \quad \text{in } \Omega, \tag{3.5}
$$

$$
\frac{\partial p^B}{\partial n} = -\frac{\partial p^I}{\partial n}, \quad \text{on } \partial\Omega. \tag{3.6}
$$

Inserting the series expansions for $\boldsymbol{\phi}^I$, $p^I$, and $\boldsymbol{\phi}^B$ in the first boundary value problem and equating coefficients of corresponding powers of $t$, we obtain the boundary value problems defining the interior expansion functions $\boldsymbol{\phi}_i$ and $p_i$:

$$
-\operatorname{\mathbf{div}} C\,\mathcal{E}(\boldsymbol{\phi}_i) - \operatorname{\mathbf{curl}} p_i = \begin{cases} \operatorname{\mathbf{grad}} r, \text{for } i = 0,\\ 0, \text{for } i > 0, \end{cases} \tag{3.7}
$$

$$
-\operatorname{rot}\boldsymbol{\phi}_i = \begin{cases} 0 \pmod{\mathbb{R}}, & \text{for } i = 0, 1,\\ -\Delta\, p_{i-2} \pmod{\mathbb{R}}, & \text{for } i \ge 2, \end{cases} \tag{3.8}
$$

and the boundary conditions

$$\boldsymbol{\phi}_i = \begin{cases} 0, & \text{for } i = 0, 1, \\ -\boldsymbol{\Phi}_{i-2}, & \text{for } i \geq 2. \end{cases} \tag{3.9}$$

In fact, (3.8) can be replaced by the simpler equation

$$\operatorname{rot} \boldsymbol{\phi}_i = 0 \quad (\text{mod } \mathbb{R}). \tag{3.10}$$

To see this, apply rot to (3.7). Using simple calculus identities, we get that

$$-\frac{E}{24(1+\nu)} \Delta \operatorname{rot} \boldsymbol{\phi}_i + \Delta p_i = 0.$$

It then follows from (3.8) that $\Delta p_i = 0$ for $i = 0, 1$ and, for $i \geq 2$,

$$\Delta p_i = \frac{E}{24(1+\nu)} \Delta^2 p_{i-2}.$$

By induction, $\Delta p_i = 0$ for all $i$. We thus use the system (3.7), (3.10), (3.9) to define the interior expansion functions. This system is essentially the Stokes equations and admits a unique solution in $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ (see Lemma ? below). Note that the implied constant in (3.10) is uniquely determined by compatibility between this equation and the boundary conditions in (3.9). We also remark that the right-hand side of all three equations vanishes for $i = 1$, so $\boldsymbol{\phi}_1 = 0$ and $p_1 = 0$.

To obtain the defining equations for the boundary correctors, we transform the system (3.4)–(3.6) to $\rho$-$\theta$ coordinates. The equation for $\tilde{\boldsymbol{\phi}}^B$ and $\tilde{p}^B$ corresponding to (3.4) is

$$\tilde{\mathcal{A}}_0 \frac{\partial^2 \tilde{\boldsymbol{\phi}}^B}{\partial \rho^2} + \tilde{\mathcal{A}}_1 \frac{\partial^2 \tilde{\boldsymbol{\phi}}^B}{\partial \rho \partial \theta} + \tilde{\mathcal{A}}_2 \frac{\partial \tilde{\boldsymbol{\phi}}^B}{\partial \rho} + \tilde{\mathcal{A}}_3 \frac{\partial^2 \tilde{\boldsymbol{\phi}}^B}{\partial \theta^2} + \tilde{\mathcal{A}}_4 \frac{\partial \tilde{\boldsymbol{\phi}}^B}{\partial \theta} + \tilde{\boldsymbol{A}}_5 \frac{\partial \tilde{p}^B}{\partial \rho} + \tilde{\boldsymbol{A}}_6 \frac{\partial \tilde{p}^B}{\partial \theta} = 0, \tag{3.11}$$

where

$$\mathcal{A}_0 = -D \begin{pmatrix} (\rho_x)^2 + (1-\nu)(\rho_y)^2/2 & (1+\nu)\rho_x\rho_y/2 \\ (1+\nu)\rho_x\rho_y/2 & (\rho_y)^2 + (1-\nu)(\rho_x)^2/2 \end{pmatrix},$$

$$\mathcal{A}_1 = -D \begin{pmatrix} 2\theta_x\rho_x + (1-\nu)\theta_y\rho_y & (1+\nu)(\theta_y\rho_x + \theta_x\rho_y)/2 \\ (1+\nu)(\theta_y\rho_x + \theta_x\rho_y)/2 & 2\theta_y\rho_y + (1-\nu)\theta_x\rho_x \end{pmatrix},$$

$$\mathcal{A}_2 = -D \begin{pmatrix} \rho_{xx} + (1-\nu)\rho_{yy}/2 & (1+\nu)\rho_{xy}/2 \\ (1+\nu)\rho_{xy}/2 & \rho_{yy} + (1-\nu)\rho_{xx}/2 \end{pmatrix},$$

$$\mathcal{A}_3 = -D \begin{pmatrix} (\theta_x)^2 + (1-\nu)(\theta_y)^2/2 & (1+\nu)\theta_x\theta_y/2 \\ (1+\nu)\theta_x\theta_y/2 & (\theta_y)^2 + (1-\nu)(\theta_x)^2/2 \end{pmatrix},$$

$$\mathcal{A}_4 = -D \begin{pmatrix} \theta_{xx} + (1-\nu)\theta_{yy}/2 & (1+\nu)\theta_{xy}/2 \\ (1+\nu)\theta_{xy}/2 & \theta_{yy} + (1-\nu)\theta_{xx}/2 \end{pmatrix},$$

$$\boldsymbol{A}_5 = -\mathbf{curl}\,\rho, \quad \boldsymbol{A}_6 = -\mathbf{curl}\,\theta.$$

Note that

$$\boldsymbol{A}_5 = \boldsymbol{s}, \quad \boldsymbol{A}_6 = \boldsymbol{n}, \quad \text{on } \partial\Omega.$$

In $\rho$-$\theta$ coordinates (3.5) becomes

$$-\tilde{\boldsymbol{A}}_5 \cdot \frac{\partial \tilde{\boldsymbol{\phi}}^B}{\partial \rho} - \tilde{\boldsymbol{A}}_6 \cdot \frac{\partial \tilde{\boldsymbol{\phi}}^B}{\partial \theta} + t^2 \left( \frac{\partial^2 \tilde{p}^B}{\partial \rho^2} + \tilde{A}_7 \frac{\partial \tilde{p}^B}{\partial \rho} + \tilde{A}_8 \frac{\partial^2 \tilde{p}^B}{\partial \theta^2} + \tilde{A}_9 \frac{\partial \tilde{p}^B}{\partial \theta} \right) = 0, \tag{3.12}$$

where

$$A_7 = \Delta \rho, \quad A_8 = |\,\mathbf{grad}\, \theta|^2, \quad A_9 = \Delta \theta.$$

In deriving (3.12), we use the facts that $|\,\mathbf{grad}\, \rho| = 1$ and $\mathbf{grad}\, \rho \cdot \mathbf{grad}\, \theta = 0$ . The boundary condition (3.6) becomes

$$\frac{\partial \tilde{p}^B}{\partial \rho} = \widetilde{\frac{\partial p^I}{\partial n}} \quad \text{on } \partial\Omega. \tag{3.13}$$

The exact form of the coefficient functions in (3.11) and (3.12) is not essential. However, the coefficients $\tilde{\mathcal{A}}_0$ and $\tilde{\boldsymbol{A}}_5$ have some properties which will prove important.

LEMMA 3.1. *The matrix-valued function $\tilde{\mathcal{A}}_0(\rho, \theta)$ and the vector-valued function $\tilde{\boldsymbol{A}}_5(\rho, \theta)$ are independent of $\rho$. Moreover for each fixed $\theta$, $\tilde{\mathcal{A}}_0$ is symmetric negative definite and $\tilde{\boldsymbol{A}}_5$ is a unit eigenvector of $\tilde{\mathcal{A}}_0$ with eigenvalue $-D(1-\nu)/2$.*

*Proof.* That these coefficients are independent of $\rho$ follows from the observation that $\widetilde{\partial \rho/\partial x}$ and $\widetilde{\partial \rho/\partial y}$ depend on $\theta$ but not on $\rho$. The second sentence is easily verified using the fact that $|\,\mathbf{grad}\, \rho| = 1$. ∎

The remaining coefficients are in general functions of both $\rho$ and $\theta$ and to obtain the boundary layer equations, we expand them in Taylor series about $\rho = 0$. That is, we define operators $\mathcal{A}_i^j(\theta)$ by the formal Taylor series expansions:

$$\tilde{\mathcal{A}}_i(\rho, \theta) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \tilde{\mathcal{A}}_i^j(\theta), \qquad i = 1, 2, 3, 4,$$

and define $\tilde{\boldsymbol{A}}_6^j$ and $\tilde{A}_i^j$, $i = 7, 8, 9$, similarly. Formally inserting these expansions in (3.11) and at the same time making the change of variable $\rho = t\hat{\rho}$ gives

$$\begin{aligned} &t^{-2} \hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\phi}}^B}{\partial \hat{\rho}^2} + t^{-1} \left[ \hat{\mathcal{A}}_1^0 \frac{\partial^2 \hat{\boldsymbol{\phi}}^B}{\partial \hat{\rho} \partial \theta} + \hat{\mathcal{A}}_2^0 \frac{\partial \hat{\boldsymbol{\phi}}^B}{\partial \hat{\rho}} + \hat{\boldsymbol{A}}_5 \frac{\partial \hat{p}^B}{\partial \hat{\rho}} \right] + \\ &\sum_{j=0}^{\infty} t^j \left[ \frac{\hat{\rho}^{j+1}}{(j+1)!} \left( \hat{\mathcal{A}}_1^{j+1} \frac{\partial^2 \hat{\boldsymbol{\phi}}^B}{\partial \hat{\rho} \partial \theta} + \hat{\mathcal{A}}_2^{j+1} \frac{\partial \hat{\boldsymbol{\phi}}^B}{\partial \hat{\rho}} \right) \right. \\ &\qquad \left. + \frac{\hat{\rho}^j}{j!} \left( \hat{\mathcal{A}}_3^j \frac{\partial^2 \hat{\boldsymbol{\phi}}^B}{\partial \theta^2} + \hat{\mathcal{A}}_4^j \frac{\partial \hat{\boldsymbol{\phi}}^B}{\partial \theta} + \hat{\boldsymbol{A}}_6^j \cdot \frac{\partial \hat{p}^B}{\partial \theta} \right) \right] = 0. \end{aligned} \tag{3.14}$$

Similarly (3.12) becomes:

(3.15)

$$\begin{aligned} &-t^{-1} \hat{\boldsymbol{A}}_5 \cdot \frac{\partial \hat{\boldsymbol{\phi}}^B}{\partial \hat{\rho}} - \hat{\boldsymbol{A}}_6^0 \cdot \frac{\partial \hat{\boldsymbol{\phi}}^B}{\partial \theta} + \frac{\partial^2 \hat{p}^B}{\partial \hat{\rho}^2} \\ &\quad + \sum_{j=0}^{\infty} t^{j+1} \left[ -\frac{\hat{\rho}^{j+1}}{(j+1)!} \hat{\boldsymbol{A}}_6^{j+1} \cdot \frac{\partial \hat{\boldsymbol{\phi}}^B}{\partial \theta} + \frac{\hat{\rho}^j}{j!} \left( \hat{A}_7^j \frac{\partial \hat{p}^B}{\partial \hat{\rho}} + t \hat{A}_8^j \frac{\partial^2 \hat{p}^B}{\partial \theta^2} + t \hat{A}_9^j \frac{\partial \hat{p}^B}{\partial \theta} \right) \right] = 0. \end{aligned}$$

We now calculate the differential equations determining the boundary correctors by inserting the series expansions for $\boldsymbol{\phi}^B$ and $p^B$ (defined at the beginning of this section) in (3.14) and (3.15) and equating coefficients of corresponding powers of $t$. Neglecting the cutoff function $\chi$, we obtain from (3.14) the equations

$$\hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_0}{\partial \hat{\rho}^2} + \hat{\boldsymbol{A}}_5 \frac{\partial \hat{P}_0}{\partial \hat{\rho}} = 0,$$

$$\hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_1}{\partial \hat{\rho}^2} + \hat{\boldsymbol{A}}_5 \frac{\partial \hat{P}_1}{\partial \hat{\rho}} + \hat{\mathcal{A}}_1^0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_0}{\partial \hat{\rho} \partial \theta} + \hat{\mathcal{A}}_2^0 \frac{\partial \hat{\boldsymbol{\Phi}}_0}{\partial \hat{\rho}} + \hat{\boldsymbol{A}}_6^0 \frac{\partial \hat{P}_0}{\partial \theta} = 0,$$

and, for $i = 2, 3, \cdots$,

$$\hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_i}{\partial \hat{\rho}^2} + \hat{\boldsymbol{A}}_5 \frac{\partial \hat{P}_i}{\partial \hat{\rho}} + \hat{\mathcal{A}}_1^0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-1}}{\partial \hat{\rho} \partial \theta} + \hat{\mathcal{A}}_2^0 \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1}}{\partial \hat{\rho}} + \hat{\boldsymbol{A}}_6^0 \frac{\partial \hat{P}_{i-1}}{\partial \theta}$$
$$+ \sum_{j=0}^{i-2} \left[ \frac{\hat{\rho}^{j+1}}{(j+1)!} \left( \hat{\mathcal{A}}_1^{j+1} \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-2-j}}{\partial \hat{\rho} \partial \theta} + \hat{\mathcal{A}}_2^{j+1} \frac{\partial \hat{\boldsymbol{\Phi}}_{i-2-j}}{\partial \hat{\rho}} + \hat{\boldsymbol{A}}_6^{j+1} \frac{\partial \hat{P}_{i-2-j}}{\partial \theta} \right) \right.$$
$$\left. + \frac{\hat{\rho}^j}{j!} \left( \hat{\mathcal{A}}_3^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-2-j}}{\partial \theta^2} + \hat{\mathcal{A}}_4^j \frac{\partial \hat{\boldsymbol{\Phi}}_{i-2-j}}{\partial \theta} \right) \right] = 0.$$

Introducing the convention $\hat{\boldsymbol{\Phi}}_i = 0$, $\hat{P}_i = 0$ for $i < 0$, we may write these three equations as

$$\hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_i}{\partial \hat{\rho}^2} + \hat{\boldsymbol{A}}_5 \frac{\partial \hat{P}_i}{\partial \hat{\rho}} = -\hat{\boldsymbol{F}}_i(\hat{\rho}, \theta), \qquad i \in \mathbb{N}, \tag{3.16}$$

where

$$\hat{\boldsymbol{F}}_i(\hat{\rho}, \theta) = \sum_{j=0}^{i-1} \frac{\hat{\rho}^j}{j!} \left( \hat{\mathcal{A}}_1^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-1-j}}{\partial \hat{\rho} \partial \theta} + \hat{\mathcal{A}}_2^j \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1-j}}{\partial \hat{\rho}} + \hat{\mathcal{A}}_3^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-2-j}}{\partial \theta^2} \right.$$
$$\left. + \hat{\mathcal{A}}_4^j \frac{\partial \hat{\boldsymbol{\Phi}}_{i-2-j}}{\partial \theta} + \hat{\boldsymbol{A}}_6^j \frac{\partial \hat{P}_{i-1-j}}{\partial \theta} \right).$$

Similarly, from (3.15), we obtain

$$- \hat{\boldsymbol{A}}_5 \cdot \frac{\partial \hat{\boldsymbol{\Phi}}_i}{\partial \hat{\rho}} + \frac{\partial^2 \hat{P}_i}{\partial \hat{\rho}^2} = \hat{G}_i(\hat{\rho}, \theta) :=$$
$$- \sum_{j=0}^{i-1} \frac{\hat{\rho}^j}{j!} \left( -\hat{\boldsymbol{A}}_6^j \cdot \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1-j}}{\partial \theta} + \hat{A}_7^j \frac{\partial \hat{P}_{i-1-j}}{\partial \hat{\rho}} + \hat{A}_8^j \frac{\partial^2 \hat{P}_{i-2-j}}{\partial \theta^2} + \hat{A}_9^j \frac{\partial \hat{P}_{i-2-j}}{\partial \theta} \right),$$
$$i \in \mathbb{N}. \tag{3.17}$$

Inserting the asymptotic expansions for $p^I$ and $p^B$ in (3.13), changing variables from $\rho$ to $\hat{\rho}$, and matching powers, we obtain the boundary conditions

$$\frac{\partial \hat{P}_i}{\partial \hat{\rho}}(0, \theta) = \widetilde{\frac{\partial p_i}{\partial n}}(0, \theta), \qquad i \in \mathbb{N}. \tag{3.18}$$

Finally, in order to determine the boundary correctors uniquely, we also impose the conditions at infinity

$$\lim_{\hat\rho\to\infty} \hat{\boldsymbol{\Phi}}_i(\hat\rho,\theta) = 0, \quad \lim_{\hat\rho\to\infty} \hat P_i(\hat\rho,\theta) = 0. \tag{3.19}$$

We remark that (3.17) is to be satisfied exactly, rather than up to an additive constant (as was (3.10)). Similarly, because of the boundary condition at infinity, we have specified $\hat P_i$ completely, not just up to an additive constant as was the case for $p_i$. Once the boundary correctors $\hat{\boldsymbol{\Phi}}_k$ and $\hat P_k$ for $k < i$ and the interior expansion function $p_i$ are known, we may view (3.16)–(3.19) as a boundary value problem in ordinary differential equations in which the independent variable is $\hat\rho$, the unknowns are $\hat{\boldsymbol{\Phi}}_i$ and $\hat P_i$, and $\theta$ plays the role of a parameter. As we shall see (in Theorem ?), this problem has a unique solution. Therefore we can recursively determine all the interior expansion functions and boundary correctors as follows. First we determine $(\boldsymbol{\phi}_0, p_0)$ by (3.7), (3.10), and (3.9). Then we determine $(\hat{\boldsymbol{\Phi}}_0, \hat P_0)$ by (3.16)–(3.19) (the right-hand sides of (3.16) and (3.17) being zero and the right-hand side of (3.18) being known). Then $(\boldsymbol{\phi}_1, p_1)$ is uniquely determined by (3.7), (3.10), and (3.9), and so forth. Thus we have proved the following theorem.

THEOREM 3.2. *There exist functions $\boldsymbol{\phi}_i(x,y)$, $p_i(x,y)$ on $\Omega$ and $\hat{\boldsymbol{\Phi}}_i(\hat\rho,\theta)$, $\hat P_i(\hat\rho,\theta)$ on $\hat\Omega_0$, $i \in \mathbb{N}$, unique except that $p_i$ is determined only up to an additive constant, which satisfy the boundary value problems* (3.7), (3.10), (3.9) *and* (3.16)–(3.19).

The Stokes-like boundary value problem (3.7), (3.10), (3.9) is well posed, but, of course, we cannot in general determine its solution in closed form, even if $r$ were known in closed form. (However, the regularity of solutions to this problem is well understood—cf. Lemma ?.) The system (3.16)–(3.19) can, in principle, be solved in closed form. For example, the solution for $i = 0$ is

$$\hat P_0(\hat\rho,\theta) = -\frac{1}{c}\widehat{\frac{\partial p_0}{\partial n}}(0,\theta)e^{-c\hat\rho}, \qquad \hat{\boldsymbol{\Phi}}_0(\hat\rho,\theta) = \hat{\boldsymbol{A}}_5(\theta)\widehat{\frac{\partial p_0}{\partial n}}(0,\theta)e^{-c\hat\rho}, \tag{3.20}$$

where $c = [24(1+\nu)/E]^{1/2}$. (We show that this is the only solution in the proof of Theorem ?.)

The following theorem gives the form of the solution for general $i$. In particular, it states that $\hat{\boldsymbol{\Phi}}_i$ and $\hat P_i$ are polynomials in $\hat\rho$ times the decaying exponential $e^{-c\hat\rho}$.

THEOREM 3.3. *For each $i \in \mathbb{N}$, the system* (3.16)–(3.19) *has a unique solution $(\hat{\boldsymbol{\Phi}}_i, \hat P_i)$. Moreover there exist smooth functions $\boldsymbol{\alpha}_{ijkl}(\theta)$ and $\alpha_{ijkl}(\theta)$ depending only on $i$, the domain $\Omega$, and the plate constants $E$ and $\nu$ such that*

$$\hat{\boldsymbol{\Phi}}_i(\hat\rho,\theta) = e^{-c\hat\rho}\sum_{k=0}^{i}\sum_{j=0}^{i}\sum_{l=0}^{i-j}\boldsymbol{\alpha}_{ijkl}(\theta)\hat\rho^k\frac{\partial^l}{\partial\theta^l}\widetilde{\frac{\partial p_j}{\partial n}}(0,\theta),$$

$$\hat P_i(\hat\rho,\theta) = e^{-c\hat\rho}\sum_{k=0}^{i}\sum_{j=0}^{i}\sum_{l=0}^{i-j}\alpha_{ijkl}(\theta)\hat\rho^k\frac{\partial^l}{\partial\theta^l}\widetilde{\frac{\partial p_j}{\partial n}}(0,\theta).$$

The proof, an exercise in ordinary differential equations based on the form of the coefficients of (3.16)–(3.17) as given in Lemma 3.1, is given in the Appendix.

This completes the construction of the interior and boundary layer asymptotic expansions. In the next section we bound the individual terms of the series and determine how nearly the finite sums satisfy the differential systems which motivated their definitions. Then, in § 5, we prove error bounds for the finite sums of the expansions.

**4. A priori estimates.** We begin this section by deriving a priori bounds on the boundary correctors using Theorem 3.3.

THEOREM 4.1 (A PRIORI ESTIMATES FOR BOUNDARY CORRECTORS). *Let $i$ be a nonnegative integer. There exists a constant $C$ depending only on the domain $\Omega$, the elastic constants $E$ and $\nu$, and $s$ and $i$, such that*

$$|\boldsymbol{\Phi}_i|_s + |P_i|_s \le C \sum_{j=0}^{i} \left| \frac{\partial p_j}{\partial n} \right|_{s+i-j}, \qquad s \in \mathbb{R},$$

$$\|\boldsymbol{\Phi}_i\|_{s,\Omega_0} + \|P_i\|_{s,\Omega_0} \le C t^{1/2-s} \sum_{j=0}^{i} \sum_{m=0}^{s} t^m \left| \frac{\partial p_j}{\partial n} \right|_{m+i-j}, \qquad s \in \mathbb{N}.$$

*Proof.* The first estimate follows from Theorem 3.3 by setting $\hat\rho = 0$ and using the triangle inequality. We now consider the second inequality. To establish the bound for $\boldsymbol{\Phi}_i$, we change to $(\rho, \theta)$ coordinates and seek bounds on the integrals

(4.1)
$$\left[ \int_0^L \int_0^{\rho_0} |\partial^{s-m+k} \tilde{\boldsymbol{\Phi}}_i / \partial\rho^{s-m} \partial\theta^k|^2 |1 - \kappa(\theta)\rho| \, d\rho \, d\theta \right]^{1/2}, \qquad 0 \le m \le s, \quad 0 \le k \le m.$$

Now $\tilde{\boldsymbol{\Phi}}_i$ is a sum of terms of the form

(4.2)
$$\boldsymbol{\alpha}(\theta) \exp(-c\rho/t) f(\rho/t) \frac{\partial^l}{\partial\theta^l} \widetilde{\frac{\partial p_j}{\partial n}}(0,\theta), \qquad \boldsymbol{\alpha} \text{ smooth, } f \text{ polynomial, } j \le i,\ l \le i-j.$$

The $L^2(\Omega_0)$ norm of (4.2) is bounded by $Ct^{1/2}|\partial p_j/\partial n|_l$, since

$$\int_0^{\rho_0} |\exp(-c\rho/t) f(\rho/t)|^2 \, d\rho \le t \int_0^\infty |\exp(-c\hat\rho) f(\hat\rho)|^2 \, d\hat\rho.$$

Applying $\partial^{s-m+k}/\partial\rho^{s-m}\partial\theta^k$ to (4.2) gives $t^{m-s}$ times a sum of terms of the same form except that $l$ may be as large as $i - j + k \le m + i - j$. Thus (4.1) is bounded by $Ct^{1/2} \sum_{j=0}^{i} t^{m-s} |\partial p_j/\partial n|_{m+i-j}$. Summing over $m = 0, 1, \cdots, s$ gives the desired bound for $\boldsymbol{\Phi}_i$, and that for $P_i$ is proved identically. ∎

We next summarize the basic regularity properties of the Stokes-like system which defines the interior expansion functions.

LEMMA 4.2. *Let $s \in \mathbb{N}$, $f \in H^s(\Omega) \cap \mathring{H}^1(\Omega)$, $g \in H^s(\Omega)/\mathbb{R}$, and $\boldsymbol{l} \in \boldsymbol{H}^{s+1/2}(\partial\Omega)$ be given. Then there exist unique $\boldsymbol{\psi} \in \boldsymbol{H}^{s+1}(\Omega)$, $q \in H^s(\Omega)/\mathbb{R}$ satisfying the partial differential equations*

$$-\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\psi}) - \mathbf{curl}\, q = \mathbf{grad}\, f, \tag{4.3}$$

$$-\operatorname{rot} \boldsymbol{\psi} = g \pmod{\mathbb{R}}, \tag{4.4}$$

*and the boundary conditions*

$$\boldsymbol{\psi} = \boldsymbol{l}.$$

*Moreover, there exists a constant $C$ depending only on $s$, $E$, $\nu$, and $\Omega$ such that*

$$\|\boldsymbol{\psi}\|_{s+1} + \|q\|_{s/\mathbb{R}} \le C(\|f\|_s + \|g\|_{s/\mathbb{R}} + |\boldsymbol{l}|_{s+1/2}).$$

*Remarks.* 1. The restriction that the forcing function in (4.3) be the gradient of an $\mathring{H}^1(\Omega)$ function is sufficient for our purposes and allows us to avoid some technical points concerning duals of Sobolev spaces and trace operators with values in negative order spaces. 2. If we replace $\mathbf{div}\, \mathcal{C}\, \mathcal{E}(\boldsymbol{\psi})$ with $\Delta$ then the simple change of variables $(\psi_1, \psi_2) \to (\psi_2, -\psi_1)$ converts (4.3), (4.4) to a generalized Stokes system, and this result is well known [5]. Here we give a proof which works for general $\mathcal{C}$ based on regularity results for the biharmonic.

*Proof.* Written in weak form, the boundary value problem is to find $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{\psi} = \boldsymbol{l}$ on $\partial\Omega$ and $q \in L^2(\Omega)/\mathbb{R}$ satisifying

$$\begin{aligned}(\mathcal{C}\, \mathcal{E}(\boldsymbol{\psi}), \mathcal{E}(\boldsymbol{\mu})) - (q, \operatorname{rot} \boldsymbol{\mu}) &= -(f, \operatorname{div} \boldsymbol{\mu}) \\ -(\operatorname{rot} \boldsymbol{\psi}, v) &= (g, v),\end{aligned}$$

for all $\boldsymbol{\mu} \in \mathring{H}^1(\Omega)$ and all $v \in L^2(\Omega)$ of mean value zero. Existence and uniqueness of a solution in $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ is proved just as for the generalized Stokes equations, e.g., by applying Brezzi's theorem [4]. The estimate for $s = 0$ follows from the same argument. To establish the estimate with $s \ge 1$, we apply the Helmholtz decomposition to $\boldsymbol{\psi}$ to get $\boldsymbol{\psi} = \mathbf{grad}\, z + \mathbf{curl}\, b$, with $z \in \mathring{H}^1(\Omega)$, $b \in H^1(\Omega)/\mathbb{R}$. From (4.4) we get

$$\Delta\, b = g \pmod{\mathbb{R}} \quad \text{in } \Omega,$$

with boundary conditions $\partial b/\partial n = \boldsymbol{l} \cdot \boldsymbol{s}$. Taking the divergence of equation (4.3) gives

$$-D\, \Delta^2 z = \Delta\, f \quad \text{in } \Omega,$$

with boundary conditions $z = 0$, $\partial z/\partial n = \boldsymbol{l} \cdot \boldsymbol{n} + \partial b/\partial s$. Applying regularity results for the biharmonic problem and the Laplacian, we obtain

$$\begin{aligned}\|b\|_{s+2/\mathbb{R}} &\le C(\|g\|_{s/\mathbb{R}} + +|\boldsymbol{l}|_{s+1/2}), \\ \|z\|_{s+2} &\le C(\|\Delta\, f\|_{s-2} + |\boldsymbol{l}|_{s+1/2} + |\partial b/\partial s|_{s+1/2}) \\ &\le C(\|f\|_s + |\boldsymbol{l}|_{s+1/2} + \|b\|_{s+2/\mathbb{R}}) \\ &\le C(\|f\|_s + |\boldsymbol{l}|_{s+1/2} + \|g\|_{s/\mathbb{R}}).\end{aligned}$$

The bound for $\boldsymbol{\psi}$ now follows directly by the triangle inequality and the bound for $q$ then follows from (4.3). ∎

We now use the previous two theorems to obtain estimates for the interior expansion functions.

THEOREM 4.3 (A PRIORI ESTIMATES FOR INTERIOR EXPANSION FUNCTIONS). *Let $\boldsymbol{\phi}_i$ and $p_i$ be the interior expansion functions. Then for all $s \ge 0$ and $i \in \mathbb{N}$, there exists a constant $C$ such that*

$$\|\boldsymbol{\phi}_i\|_{s+1} + \|p_i\|_{s/\mathbb{R}} \le C|\!|\!|g|\!|\!|_{s+i-2}.$$

*Proof.* Since $-\Delta\, r = g$ and $r$ vanishes on $\partial\Omega$, we have

$$\|r\|_s \le C|\!|\!|g|\!|\!|_{s-2}. \tag{4.5}$$

Thus, it suffices to prove that

$$\|\boldsymbol{\phi}_i\|_{s+1} + \|p_i\|_{s/\mathbb{R}} \le C\|r\|_{s+i}. \tag{4.6}$$

We prove this first for $s \in \mathbb{N}$ by induction on $i$. For $i = 0$, we get this immediately from the defining equations (3.7), (3.10), (3.9), and Lemma 4.2. As already noted $\boldsymbol{\phi}_1 = p_1 = 0$, so (4.6) holds for $i = 1$ also. For $i \ge 2$, we apply Lemma 4.2, Theorem 4.1, and the trace theorem to obtain

$$\|\boldsymbol{\phi}_i\|_{s+1} + \|p_i\|_{s/\mathbb{R}} \le C|\boldsymbol{\Phi}_{i-2}|_{s+1/2} \le C\sum_{j=0}^{i-2}\left|\frac{\partial p_j}{\partial n}\right|_{s-3/2+i-j} \le C\sum_{j=0}^{i-2}\|p_j\|_{s+i-j}.$$

Application of the inductive hypothesis completes the proof of (4.6) for integer $s$. The proof for noninteger $s$ now follows by a standard interpolation argument. ∎

COROLLARY 4.4. *For $s \ge -\frac{3}{2}$ and $i \in \mathbb{N}$, there exists a constant $C$ such that*

$$\left|\frac{\partial p_i}{\partial s}\right|_s + \left|\frac{\partial p_i}{\partial n}\right|_s + |\boldsymbol{\Phi}_i|_s + |P_i|_s \le C|||g|||_{s+i-1/2}.$$

*Proof.* As remarked in the previous section, $p_i$ is harmonic for all $i$. Consequently, the trace inequality

$$\left|\frac{\partial p_i}{\partial s}\right|_s + \left|\frac{\partial p_i}{\partial n}\right|_s \le C\|p_i\|_{s+3/2/\mathbb{R}}$$

holds for all $s$, so the bounds on $p_i$ follow easily from the theorem. The bounds on $\boldsymbol{\Phi}_i$ and $P_i$ then follow from Theorem 4.1. ∎

We now combine Theorems 4.1 and 4.3 to obtain an essential result for the derivation of error bounds for the boundary layer expansion.

THEOREM 4.5 (A PRIORI ESTIMATES FOR BOUNDARY CORRECTORS). *Let $i$, $k$, $l$, $n$, and $s$ be nonnegative integers and define functions $\boldsymbol{f}$ and $f$ on $\Omega_0$ by*

$$\tilde{\boldsymbol{f}} = \left(\frac{\rho}{t}\right)^k t^l \frac{\partial^{l+n}}{\partial\rho^l\partial\theta^n}\tilde{\boldsymbol{\Phi}}_i, \qquad \tilde{f} = \left(\frac{\rho}{t}\right)^k t^l \frac{\partial^{l+n}}{\partial\rho^l\partial\theta^n}\tilde{P}_i.$$

*Then there exists a constant $C$ depending only on the domain $\Omega$, the elastic constants $E$ and $\nu$, and $i$, $k$, $l$, $n$, and $s$, such that*

$$\|\boldsymbol{f}\|_{s,\Omega_0} + \|f\|_{s,\Omega_0} \le C(t^{1/2-s}|||g|||_{i+n-1/2} + t^{1/2}|||g|||_{s+i+n-1/2}).$$

*Proof.* Note that, if in (4.2) we differentiate with respect to $\hat{\rho}$ (i.e., differentiate with respect to $\rho$ and multiply by $t$), we obtain something of the same form. The same is true if we multiply by $\hat{\rho}$. If we differentiate with respect to $\theta$ we obtain a sum of two terms of the same form, with one higher order of differentiation on $\widetilde{\partial p_j/\partial n}$. Hence, reasoning just as in the proof of Theorem 4.1, we obtain

$$\|f\|_{s,\Omega_0} + \|\boldsymbol{f}\|_{s,\Omega_0} \le Ct^{1/2-s}\sum_{j=0}^{i}\sum_{m=0}^{s} t^m \left|\frac{\partial p_j}{\partial n}\right|_{m+i+n-j}.$$

An application of Corollary 4.4 completes the proof. ∎

We now consider the partial sums given by

$$\boldsymbol{\phi}_m^I(x,y) = \sum_{i=0}^m t^i \boldsymbol{\phi}_i(x,y), \qquad p_m^I(x,y) = \sum_{i=0}^m t^i p_i(x,y),$$
$$\boldsymbol{\phi}_m^B(x,y) = t^2 \tilde{\chi}(\rho) \sum_{i=0}^m t^i \hat{\boldsymbol{\Phi}}_i(\hat{\rho},\theta), \quad p_m^B(x,y) = t\tilde{\chi}(\rho) \sum_{i=0}^m t^i \hat{P}_i(\hat{\rho},\theta).$$

Note that while $\boldsymbol{\Phi}_i$ and $P_i$ are only defined on the tubular neighborhood $\Omega_0$ of $\partial\Omega$, $\boldsymbol{\phi}_m^B$ and $p_m^B$ are defined on all of $\Omega$ because of the cutoff function $\chi$. By construction, $(\boldsymbol{\phi}_m^I, p_m^I)$ and $(\boldsymbol{\phi}_m^B, p_m^B)$ should almost satisfy the boundary value problems (3.1)–(3.3) and (3.4)–(3.6), respectively. We now make precise to what extent this is true.

For the interior expansion this is easy. The following theorem follows directly from (3.7), (3.10), (3.9).

THEOREM 4.6 (BOUNDARY VALUE PROBLEM FOR THE INTERIOR EXPANSION). *Let $m \in \mathbb{N}$. The finite interior expansion $(\boldsymbol{\phi}_m^I, p_m^I) \in \boldsymbol{H}^1(\Omega) \times H^1(\Omega)/\mathbb{R}$ satisfies the boundary value problem*

$$-\operatorname{\mathbf{div}} C\,\mathcal{E}(\boldsymbol{\phi}_m^I) - \operatorname{\mathbf{curl}} p_m^I = \operatorname{\mathbf{grad}} r \quad \text{in } \Omega,$$
$$-\operatorname{rot} \boldsymbol{\phi}_m^I + t^2\,\Delta\, p_m^I = 0 \pmod{\mathbb{R}}, \quad \text{in } \Omega,$$
$$\boldsymbol{\phi}_m^I = -\boldsymbol{\phi}_{m-2}^B, \quad \text{on } \partial\Omega.$$

For the boundary expansion, it follows from (3.18) that the boundary condition

$$\frac{\partial p_m^B}{\partial n} = -\frac{\partial p_m^I}{\partial n} \quad \text{on } \partial\Omega, \tag{4.7}$$

is satisfied exactly, but for the differential equations the situation is more complicated. Define the residuals $\boldsymbol{R}_m$ and $R_m$ by the equations:

$$-\operatorname{\mathbf{div}} C\,\mathcal{E}(\boldsymbol{\phi}_m^B) - \operatorname{\mathbf{curl}} p_m^B = \boldsymbol{R}_m \quad \text{in } \Omega, \tag{4.8}$$
$$-\operatorname{rot} \boldsymbol{\phi}_m^B + t^2\,\Delta\, p_m^B = R_m \quad \text{in } \Omega. \tag{4.9}$$

The following theorem shows that these residuals are indeed of high order with respect to $t$.

THEOREM 4.7 (BOUNDARY VALUE PROBLEM FOR THE BOUNDARY LAYER EXPANSION). *Let $m \in \mathbb{N}$. The finite boundary layer expansion $(\boldsymbol{\phi}_m^B, p_m^B) \in \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ satisfies the boundary value problem (4.7)–(4.9), with the following bounds valid for the forcing functions $\boldsymbol{R}_m$, $R_m$:*

$$\|\boldsymbol{R}_m\|_s \le C(t^{m+3/2-s}\|g\|_{m+1/2} + t^{m+5/2}\|g\|_{m+s+3/2}), \qquad s = -1, 0, \cdots,$$
$$\|R_m\|_s \le C(t^{m+5/2-s}\|g\|_{m+1/2} + t^{m+7/2}\|g\|_{m+s+3/2}), \qquad s \in \mathbb{N}.$$

*Proof.* It suffices to prove the theorem with the right-hand sides replaced by

$$Ct^{m+3/2-s}\sum_{i=0}^{s+1} t^i \|g\|_{m+1/2+i} \text{ and } Ct^{m+5/2-s}\sum_{i=0}^{s+1} t^i \|g\|_{m+1/2+i},$$

respectively. In addition to the portion of the residual due to truncating the series after finitely many terms, we must consider the contributions from two other sources, namely the replacement of the coefficients by Taylor polynomial approximations and the suppression of the cutoff function $\chi$. Because of the presence of the cutoff function $\chi$ in the definitions of $\boldsymbol{\phi}_m^B$ and $p_m^B$, it is easy to see that the residuals $\boldsymbol{R}_m$ and $R_m$ will vanish for $\rho \geq \rho_0$, i.e., in $\Omega \setminus \Omega_0$. In $\Omega_0$, after changing to $(\hat{\rho}, \theta)$ variables, we have (cf. (3.11))

$$\hat{\boldsymbol{R}}_m = t^{-2}\hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\phi}}_m^B}{\partial \hat{\rho}^2} + t^{-1}\left(\hat{\mathcal{A}}_1 \frac{\partial^2 \hat{\boldsymbol{\phi}}_m^B}{\partial \hat{\rho}\partial\theta} + \hat{\mathcal{A}}_2 \frac{\partial \hat{\boldsymbol{\phi}}_m^B}{\partial \hat{\rho}}\right) + \hat{\mathcal{A}}_3 \frac{\partial^2 \hat{\boldsymbol{\phi}}_m^B}{\partial \theta^2} + \hat{\mathcal{A}}_4 \frac{\partial \hat{\boldsymbol{\phi}}_m^B}{\partial \theta}$$
$$+ t^{-1}\hat{\boldsymbol{A}}_5 \frac{\partial \hat{p}_m^B}{\partial \hat{\rho}} + \hat{\boldsymbol{A}}_6 \frac{\partial \hat{p}_m^B}{\partial \theta} = \tilde{\chi}(\rho)\hat{\boldsymbol{R}}_m^1 + \hat{\boldsymbol{R}}_m^2,$$

where

$$\begin{aligned}
\hat{\boldsymbol{R}}_m^1 &= \sum_{i=0}^{m} t^i \left(\hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_i}{\partial \hat{\rho}^2} + \hat{\boldsymbol{A}}_5 \frac{\partial \hat{P}_i}{\partial \hat{\rho}}\right) \\
&+ \sum_{i=0}^{m+1} t^i \left(\hat{\mathcal{A}}_1 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-1}}{\partial \hat{\rho}\partial\theta} + \hat{\mathcal{A}}_2 \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1}}{\partial \hat{\rho}} + \hat{\mathcal{A}}_3 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-2}}{\partial \theta^2} + \hat{\mathcal{A}}_4 \frac{\partial \hat{\boldsymbol{\Phi}}_{i-2}}{\partial \theta} + \hat{\boldsymbol{A}}_6 \frac{\partial \hat{P}_{i-1}}{\partial \theta}\right) \\
&+ t^{m+2}\left(\hat{\mathcal{A}}_3 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_m}{\partial \theta^2} + \hat{\mathcal{A}}_4 \frac{\partial \hat{\boldsymbol{\Phi}}_m}{\partial \theta}\right), \\
\hat{\boldsymbol{R}}_m^2 &= \tilde{\chi}'(\rho)\left[\sum_{i=0}^{m} t^i \left(2\hat{\mathcal{A}}_0 \frac{\partial \hat{\boldsymbol{\Phi}}_i}{\partial \hat{\rho}} + \hat{\boldsymbol{A}}_5 \hat{P}_i\right) + \sum_{i=0}^{m+1} t^i \left(\hat{\mathcal{A}}_1 \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1}}{\partial \theta} + \hat{\mathcal{A}}_2 \hat{\boldsymbol{\Phi}}_{i-1}\right)\right] \\
&+ \tilde{\chi}''(\rho)\sum_{i=0}^{m} t^i \hat{\mathcal{A}}_0 \hat{\boldsymbol{\Phi}}_i,
\end{aligned}$$

and we have again used the convention that terms with negative indices vanish. Now for any $k \geq -1$

$$\tilde{\mathcal{A}}_1(\rho,\theta) = \sum_{j=0}^{k} \frac{\rho^j}{j!}\tilde{\mathcal{A}}_1^j(\theta) + \rho^{k+1}\bar{\mathcal{A}}_1^{k+1}(\rho,\theta)$$

where

$$\bar{\mathcal{A}}_1^{k+1}(\rho,\theta) = \begin{cases} \displaystyle\int_0^1 \frac{\partial^{k+1}\tilde{\mathcal{A}}_1}{\partial \rho^{k+1}}(s\rho,\theta)\frac{(1-s)^k}{k!}\,ds, & k \geq 0, \\ \tilde{\mathcal{A}}_1(\rho,\theta), & k = -1. \end{cases}$$

The other coefficients admit similar Taylor expansions (except for $\tilde{\mathcal{A}}_0$ and $\tilde{\boldsymbol{A}}_5$ which are functions of $\theta$ only). Substituting these expansions for $k = m - i$ and using $\rho = t\hat{\rho}$, we get

$$\hat{\boldsymbol{R}}_m^1 = \sum_{i=0}^{m} t^i \left(\hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_i}{\partial \hat{\rho}^2} + \hat{\boldsymbol{A}}_5 \frac{\partial \hat{P}_i}{\partial \hat{\rho}}\right) + \sum_{i=0}^{m+1} t^i \sum_{j=0}^{m-i} \frac{(t\hat{\rho})^j}{j!}\left(\hat{\mathcal{A}}_1^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-1}}{\partial \hat{\rho}\partial\theta} + \hat{\mathcal{A}}_2^j \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1}}{\partial \hat{\rho}}\right.$$
$$\left. + \hat{\mathcal{A}}_3^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-2}}{\partial \theta^2} + \hat{\mathcal{A}}_4^j \frac{\partial \hat{\boldsymbol{\Phi}}_{i-2}}{\partial \theta} + \hat{\boldsymbol{A}}_6^j \frac{\partial \hat{P}_{i-1}}{\partial \theta}\right)$$

$$
\begin{aligned}
&\quad + \sum_{i=0}^{m+1} t^i (t\hat\rho)^{m-i+1} \Bigg( \hat{\bar{\mathcal{A}}}_1^{m-i+1} \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-1}}{\partial\hat\rho\partial\theta} + \hat{\bar{\mathcal{A}}}_2^{m-i+1} \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1}}{\partial\hat\rho} \\
&\qquad + \hat{\bar{\mathcal{A}}}_3^{m-i+1} \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-2}}{\partial\theta^2} + \hat{\bar{\mathcal{A}}}_4^{m-i+1} \frac{\partial \hat{\boldsymbol{\Phi}}_{i-2}}{\partial\theta} + \hat{\bar{\boldsymbol{A}}}_6^{m-i+1} \frac{\partial \hat P_{i-1}}{\partial\theta} \Bigg) \\
&\quad + t^{m+2} \left( \hat{\mathcal{A}}_3 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_m}{\partial\theta^2} + \hat{\mathcal{A}}_4 \frac{\partial \hat{\boldsymbol{\Phi}}_m}{\partial\theta} \right) \\
&= \sum_{i=0}^{m} t^i \Bigg[ \hat{\mathcal{A}}_0 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_i}{\partial\hat\rho^2} + \hat{\boldsymbol{A}}_5 \frac{\partial \hat P_i}{\partial\hat\rho} + \sum_{j=0}^{i} \frac{\hat\rho^j}{j!} \Bigg( \hat{\mathcal{A}}_1^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-j-1}}{\partial\hat\rho\partial\theta} + \hat{\mathcal{A}}_2^j \frac{\partial \hat{\boldsymbol{\Phi}}_{i-j-1}}{\partial\hat\rho} \\
&\qquad + \hat{\mathcal{A}}_3^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-j-2}}{\partial\theta^2} + \hat{\mathcal{A}}_4^j \frac{\partial \hat{\boldsymbol{\Phi}}_{i-j-2}}{\partial\theta} + \hat{\boldsymbol{A}}_6^j \frac{\partial \hat P_{i-j-1}}{\partial\theta} \Bigg) \Bigg] \\
&\quad + t^{m+1} \sum_{i=0}^{m+1} \hat\rho^{m-i+1} \Bigg( \hat{\bar{\mathcal{A}}}_1^{m-i+1} \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-1}}{\partial\hat\rho\partial\theta} + \hat{\bar{\mathcal{A}}}_2^{m-i+1} \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1}}{\partial\hat\rho} \\
&\qquad + \hat{\bar{\mathcal{A}}}_3^{m-i+1} \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-2}}{\partial\theta^2} + \hat{\bar{\mathcal{A}}}_4^{m-i+1} \frac{\partial \hat{\boldsymbol{\Phi}}_{i-2}}{\partial\theta} + \hat{\bar{\boldsymbol{A}}}_6^{m-i+1} \frac{\partial \hat P_{i-1}}{\partial\theta} \Bigg) \\
&\quad + t^{m+2} \left( \hat{\mathcal{A}}_3 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_m}{\partial\theta^2} + \hat{\mathcal{A}}_4 \frac{\partial \hat{\boldsymbol{\Phi}}_m}{\partial\theta} \right),
\end{aligned}
$$

where we have used the identity $\sum_{i=0}^{m}\sum_{j=0}^{m-i} F(i,j) = \sum_{i=0}^{m}\sum_{j=0}^{i} F(i-j,j)$ to obtain the second equality. Now the term in brackets vanishes by construction (cf. (3.16)). Thus

$$
\begin{aligned}
\hat{\boldsymbol{R}}_m^1 &= t^{m+1} \sum_{i=0}^{m+1} \hat\rho^{m-i+1} \Bigg( \hat{\bar{\mathcal{A}}}_1^{m-i+1} \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-1}}{\partial\hat\rho\partial\theta} + \hat{\bar{\mathcal{A}}}_2^{m-i+1} \frac{\partial \hat{\boldsymbol{\Phi}}_{i-1}}{\partial\hat\rho} \\
&\qquad + \hat{\bar{\mathcal{A}}}_3^{m-i+1} \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{i-2}}{\partial\theta^2} + \hat{\bar{\mathcal{A}}}_4^{m-i+1} \frac{\partial \hat{\boldsymbol{\Phi}}_{i-2}}{\partial\theta} + \hat{\bar{\boldsymbol{A}}}_6^{m-i+1} \frac{\partial \hat P_{i-1}}{\partial\theta} \Bigg) \\
&\quad + t^{m+2} \left( \hat{\mathcal{A}}_3 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_m}{\partial\theta^2} + \hat{\mathcal{A}}_4 \frac{\partial \hat{\boldsymbol{\Phi}}_m}{\partial\theta} \right),
\end{aligned}
$$

or

$$
\begin{aligned}
\hat{\boldsymbol{R}}_m^1 &= t^{m+1} \Bigg[ \sum_{j=0}^{m} \hat\rho^j \Bigg( \hat{\bar{\mathcal{A}}}_1^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{m-j}}{\partial\hat\rho\partial\theta} + \hat{\bar{\mathcal{A}}}_2^j \frac{\partial \hat{\boldsymbol{\Phi}}_{m-j}}{\partial\hat\rho} + \hat{\bar{\mathcal{A}}}_3^j \frac{\partial^2 \hat{\boldsymbol{\Phi}}_{m-j-1}}{\partial\theta^2} \\
&\qquad + \hat{\bar{\mathcal{A}}}_4^j \frac{\partial \hat{\boldsymbol{\Phi}}_{m-j-1}}{\partial\theta} + \hat{\bar{\boldsymbol{A}}}_6^j \frac{\partial \hat P_{m-j-1}}{\partial\theta} \Bigg) + t \left( \hat{\mathcal{A}}_3 \frac{\partial^2 \hat{\boldsymbol{\Phi}}_m}{\partial\theta^2} + \hat{\mathcal{A}}_4 \frac{\partial \hat{\boldsymbol{\Phi}}_m}{\partial\theta} \right) \Bigg].
\end{aligned}
\tag{4.10}
$$

A similar computation gives $\hat R_m = \tilde\chi(\rho)\hat R_m^1 + \hat R_m^2$, where

$$
\begin{aligned}
\hat R_m^1 &= t^{m+2} \Bigg[ \sum_{j=0}^{m} \hat\rho^j \Bigg( -\hat{\bar{\boldsymbol{A}}}_6^j \cdot \frac{\partial \hat{\boldsymbol{\Phi}}_{m-j}}{\partial\theta} + \hat{\bar{A}}_7^j \frac{\partial \hat P_{m-j}}{\partial\hat\rho} + \hat{\bar{A}}_8^j \frac{\partial^2 \hat P_{m-j-1}}{\partial\theta^2} \\
&\qquad + \hat{\bar{A}}_9^j \frac{\partial \hat P_{m-j-1}}{\partial\theta} \Bigg) + t \left( \hat A_8 \frac{\partial^2 \hat P_m}{\partial\theta^2} + \hat A_9 \frac{\partial \hat P_m}{\partial\theta} \right) \Bigg]
\end{aligned}
$$

and

$$\hat{R}_m^2 = \tilde{\chi}'(\rho)\left[t\sum_{i=0}^{m} t^i\left(-\hat{A}_5\hat{\boldsymbol{\Phi}}_i + 2\frac{\partial\hat{P}_i}{\partial\hat{\rho}}\right) + t\sum_{i=0}^{m+1} t^i\hat{P}_{i-1}\right] + \tilde{\chi}''(\rho)t\sum_{i=0}^{m} t^i\hat{P}_i.$$

It suffices to show that the desired bounds are satisfied by each of the terms $\boldsymbol{R}_m^1$, $\boldsymbol{R}_m^2$, $R_m^1$, and $R_m^2$. The bounds on $\|\boldsymbol{R}_m^1\|_s$ and $\|R_m^1\|_s$, $s \geq 0$, follow from the expressions for the residuals just computed and Theorem 4.5.

We next bound $\|\boldsymbol{R}_m^2\|_s$ and $\|R_m^2\|_s$, $s \geq 0$. Using the expressions for $\hat{\boldsymbol{\Phi}}_i$ and $\hat{P}_i$ given in Theorem 3.3, we can write $\boldsymbol{R}_m^2$ and $R_m^2$ as a sum of terms all with a common factor of $e^{-c\hat{\rho}}$. Now, because of the presence of the factors $\tilde{\chi}'$ and $\tilde{\chi}''$ in the definitions of $\boldsymbol{R}_m^2$ and $R_m^2$, each of these terms vanishes for $\rho \leq \rho_0/3$. On the region where $\rho \geq \rho_0/3$

$$e^{-c\hat{\rho}} \leq K_j\left(\frac{c\rho_0}{3}\right)^{-j} t^j =: C_j t^j$$

with $K_j := \max_{x\geq 0} x^j e^{-x} < \infty$, for *any* desired power $j$. Using this result, referring to the expressions given in Theorem 3.3, and applying Corollary 4.4, it is not difficult to show that for any $j$ and suitable $C$

$$\|\boldsymbol{R}_m^2\|_s \leq Ct^j\|g\|_{m+s+1/2}, \qquad \|R_m^2\|_s \leq Ct^j\|g\|_{m+s-1/2}. \tag{4.11}$$

Finally, we establish the first estimate when $s = -1$. First we note that

$$\hat{\mathcal{A}}_1^j = \frac{1}{j!}\hat{\mathcal{A}}_1^j + t\hat{\rho}\hat{\tilde{\mathcal{A}}}_1^{j+1}.$$

Substituting this and analogous expressions for $\hat{\mathcal{A}}_2^j$, $\hat{\mathcal{A}}_3^j$, $\hat{\mathcal{A}}_4^j$, and $\hat{\boldsymbol{A}}_6^j$ in (4.10) we get

$$\hat{\boldsymbol{R}}_m^1 = \hat{\boldsymbol{R}}_m^{11} + \hat{\boldsymbol{R}}_m^{12} + \hat{\boldsymbol{R}}_m^{13},$$

where

$$\hat{\boldsymbol{R}}_m^{11} = t^{m+1}\left[\sum_{j=0}^{m}\frac{\hat{\rho}^j}{j!}\left(\hat{\mathcal{A}}_1^j\frac{\partial^2\hat{\boldsymbol{\Phi}}_{m-j}}{\partial\hat{\rho}\partial\theta} + \hat{\mathcal{A}}_2^j\frac{\partial\hat{\boldsymbol{\Phi}}_{m-j}}{\partial\hat{\rho}} + \hat{\mathcal{A}}_3^j\frac{\partial^2\hat{\boldsymbol{\Phi}}_{m-j-1}}{\partial\theta^2} + \hat{\mathcal{A}}_4^j\frac{\partial\hat{\boldsymbol{\Phi}}_{m-j-1}}{\partial\theta} + \hat{\boldsymbol{A}}_6^j\frac{\partial\hat{P}_{m-j-1}}{\partial\theta}\right)\right],$$

$$\hat{\boldsymbol{R}}_m^{12} = t^{m+2}\hat{\mathcal{A}}_3\frac{\partial^2\hat{\boldsymbol{\Phi}}_m}{\partial\theta^2},$$

and

$$\hat{\boldsymbol{R}}_m^{13} = t^{m+2}\left[\sum_{j=0}^{m}\hat{\rho}^{j+1}\left(\hat{\tilde{\mathcal{A}}}_1^{j+1}\frac{\partial^2\hat{\boldsymbol{\Phi}}_{m-j}}{\partial\hat{\rho}\partial\theta} + \hat{\tilde{\mathcal{A}}}_2^{j+1}\frac{\partial\hat{\boldsymbol{\Phi}}_{m-j}}{\partial\hat{\rho}} + \hat{\tilde{\mathcal{A}}}_3^{j+1}\frac{\partial^2\hat{\boldsymbol{\Phi}}_{m-j-1}}{\partial\theta^2} + \hat{\tilde{\mathcal{A}}}_4^{j+1}\frac{\partial\hat{\boldsymbol{\Phi}}_{m-j-1}}{\partial\theta} + \hat{\tilde{\boldsymbol{A}}}_6^{j+1}\frac{\partial\hat{P}_{m-j-1}}{\partial\theta}\right) + \hat{\mathcal{A}}_4\frac{\partial\hat{\boldsymbol{\Phi}}_m}{\partial\theta}\right]. \tag{4.12}$$

By (3.16)

$$\hat{\boldsymbol{R}}_m^{11} = -t^{m+1}\left(\hat{\mathcal{A}}_0\frac{\partial^2\hat{\boldsymbol{\Phi}}_{m+1}}{\partial\hat{\rho}^2} + \hat{\boldsymbol{A}}_5\frac{\partial\hat{P}_{m+1}}{\partial\hat{\rho}}\right),$$

or

$$\tilde{\boldsymbol{R}}_m^{11} = -t^{m+2}\left(\hat{\mathcal{A}}_0 t\frac{\partial^2\tilde{\boldsymbol{\Phi}}_{m+1}}{\partial\rho^2} + \hat{\boldsymbol{A}}_5\frac{\partial\tilde{P}_{m+1}}{\partial\rho}\right).$$

Therefore, for any $\boldsymbol{\psi} \in \mathring{H}^1(\Omega)$,

(4.13)

$$\begin{aligned}(\chi\boldsymbol{R}_m^{11}, \boldsymbol{\psi}) &= -t^{m+2}\int_0^L\int_0^{\rho_0}\tilde{\chi}(\rho)\left(\hat{\mathcal{A}}_0 t\frac{\partial^2\tilde{\boldsymbol{\Phi}}_{m+1}}{\partial\rho^2} + \hat{\boldsymbol{A}}_5\frac{\partial\tilde{P}_{m+1}}{\partial\rho}\right)\tilde{\boldsymbol{\psi}}(\rho,\theta)[1-\kappa(\theta)\rho]\,d\rho d\theta \\ &= t^{m+2}\int_0^L\int_0^{\rho_0}\left(\hat{\mathcal{A}}_0 t\frac{\partial\tilde{\boldsymbol{\Phi}}_{m+1}}{\partial\rho} + \hat{\boldsymbol{A}}_5\tilde{P}_{m+1}\right)\frac{\partial}{\partial\rho}\left\{\tilde{\chi}(\rho)\tilde{\boldsymbol{\psi}}(\rho,\theta)[1-\kappa(\theta)\rho]\right\}\,d\rho d\theta.\end{aligned}$$

Applying the Schwarz inequality and Theorem 4.5 gives

$$(\chi\boldsymbol{R}_m^{11}, \boldsymbol{\psi}) \le Ct^{m+5/2}\|g\|_{m+1/2}\|\boldsymbol{\psi}\|_1, \tag{4.14}$$

or, since $\boldsymbol{\psi}$ was arbitrary,

$$\|\chi\boldsymbol{R}_m^{11}\|_{-1} \le Ct^{m+5/2}\|g\|_{m+1/2}.$$

Similarly,

$$\begin{aligned}(\chi\boldsymbol{R}_m^{12}, \boldsymbol{\psi}) &= t^{m+2}\int_0^L\int_0^{\rho_0}\tilde{\chi}(\rho)\hat{\mathcal{A}}_3\frac{\partial^2\tilde{\boldsymbol{\Phi}}_m}{\partial\theta^2}\tilde{\boldsymbol{\psi}}(\rho,\theta)[1-\kappa(\theta)\rho]\,d\rho d\theta \\ &= -t^{m+2}\int_0^L\int_0^{\rho_0}\tilde{\chi}(\rho)\hat{\mathcal{A}}_3\frac{\partial\tilde{\boldsymbol{\Phi}}_m}{\partial\theta}\frac{\partial}{\partial\theta}\left\{\tilde{\boldsymbol{\psi}}(\rho,\theta)[1-\kappa(\theta)\rho]\right\}\,d\rho d\theta,\end{aligned}$$

whence

$$\|\chi\boldsymbol{R}_m^{12}\|_{-1} \le Ct^{m+5/2}\|g\|_{m+1/2}.$$

Finally, applying Theorem 4.5 directly to (4.12) and (4.11), respectively, we get

$$\|\chi\boldsymbol{R}_m^{13}\|_{-1} \le C\|\boldsymbol{R}_m^{13}\|_0 \le Ct^{m+5/2}\|g\|_{m+1/2},$$

and

$$\|\boldsymbol{R}_m^2\|_{-1} \le \|\boldsymbol{R}_m^2\|_0 \le Ct^{m+5/2}\|g\|_{m+1/2}.$$

Since $\boldsymbol{R}_m = \chi\boldsymbol{R}_m^{11} + \chi\boldsymbol{R}_m^{12} + \chi\boldsymbol{R}_m^{13} + \boldsymbol{R}_m^2$, the last three equations imply

$$\|\boldsymbol{R}_m\|_{-1} \le Ct^{m+5/2}\|g\|_{m+1/2},$$

as desired. ∎

**5. Error estimates.** Let

$$\begin{aligned}\boldsymbol{\phi}_n^E &= \boldsymbol{\phi} - \boldsymbol{\phi}_n^I - \boldsymbol{\phi}_{n-2}^B \\ &= \boldsymbol{\phi} - [\boldsymbol{\phi}_0 + t\boldsymbol{\phi}_1 + \cdots + t^n\boldsymbol{\phi}_n + \chi(t^2\boldsymbol{\Phi}_0 + t^3\boldsymbol{\Phi}_1 + \cdots + t^n\boldsymbol{\Phi}_{n-2})], \\ p_n^E &= p - p_n^I - p_{n-2}^B \\ &= p - [p_0 + tp_1 + \cdots + t^n p_n + \chi(tP_0 + t^2P_1 + \cdots + t^{n-1}P_{n-2})].\end{aligned}$$

Thus $\boldsymbol{\phi}_n^E$ and $p_n^E$ denote the errors in the asymptotic expansions up to order roughly $n$. Since $\boldsymbol{\phi}_1$ and $p_1$ vanish,

$$\boldsymbol{\phi}_0^E = \boldsymbol{\phi}_1^E = \boldsymbol{\phi} - \boldsymbol{\phi}_0, \qquad p_0^E = p_1^E = p - p_0.$$

In this section we derive rigorous error bounds for $\boldsymbol{\phi}_n^E$ and $p_n^E$. In Theorem ? we bound the error in $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ and in Theorem ? we bound the error in higher order Sobolev norms.

THEOREM 5.1 (ERROR ESTIMATES FOR $\boldsymbol{\phi}$ AND $p$ IN ENERGY NORM). *There exists a constant $C$ independent of $t$ such that*

$$\|\boldsymbol{\phi}_1^E\|_1 + \|p_1^E\|_{0/\mathbb{R}} + t\|\,\mathbf{grad}\, p_1^E\|_0 \le Ct^{3/2}\|g\|_{-1/2}$$

*and for $n \ge 2$*

$$\|\boldsymbol{\phi}_n^E\|_1 + \|p_n^E\|_{0/\mathbb{R}} + t\|\,\mathbf{grad}\, p_n^E\|_0 \le C(t^{n+1/2}\|g\|_{n-3/2} + t^{n+3/2}\|g\|_{n-1/2}).$$

*Proof.* It follows easily from (2.4), (2.5), (2.7), Theorem 4.6, and (4.7)–(4.9) that $(\boldsymbol{\phi}_n^E, p_n^E)$ satisfy the partial differential equations

$$-\,\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_n^E) - \mathbf{curl}\, p_n^E = -\boldsymbol{R}_{n-2}, \tag{5.1}$$

$$-\operatorname{rot} \boldsymbol{\phi}_n^E + t^2\,\Delta\, p_n^E = -R_{n-2} \pmod{\mathbb{R}}, \tag{5.2}$$

and the boundary conditions

$$\boldsymbol{\phi}_n^E = 0, \qquad \frac{\partial p_n^E}{\partial n} = -t^{n-1}\frac{\partial p_{n-1}}{\partial n} - t^n\frac{\partial p_n}{\partial n}. \tag{5.3}$$

Writing these equations variationally, we get for all $\boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(\Omega)$ and $q \in L^2(\Omega)$ with mean value zero,

$$\begin{gathered}(\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_n^E), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}\, p_n^E, \boldsymbol{\psi}) = -(\boldsymbol{R}_{n-2}, \boldsymbol{\psi}), \\ (\boldsymbol{\phi}_n^E, \mathbf{curl}\, q) + t^2(\mathbf{grad}\, p_n^E, \mathbf{grad}\, q) = (R_{n-2}, q) - t^{n+1}\langle \partial p_{n-1}/\partial n + t\partial p_n/\partial n, q\rangle.\end{gathered} \tag{5.4}$$

Now let

$$\bar{p}_n^E = p_n^E - \frac{1}{\operatorname{meas}\Omega}\int_\Omega p_n^E\,dx$$

denote the difference between $p_n^E$ and its mean value. Choosing $\boldsymbol{\psi} = \boldsymbol{\phi}_n^E$ and $q = \bar{p}_n^E$ and adding the equations, we obtain

$$\begin{aligned}&(\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_n^E), \mathcal{E}(\boldsymbol{\phi}_n^E)) + t^2(\mathbf{grad}\, p_n^E, \mathbf{grad}\, p_n^E) \\ &\qquad = -(\boldsymbol{R}_{n-2}, \boldsymbol{\phi}_n^E) + (R_{n-2}, \bar{p}_n^E) - t^{n+1}\langle \partial p_{n-1}/\partial n + t\partial p_n/\partial n, \bar{p}_n^E\rangle.\end{aligned}$$

Applying Korn's inequality and standard estimates, we thus obtain

$$\|\boldsymbol{\phi}_n^E\|_1^2 + t^2\|\mathbf{grad}\, p_n^E\|_0^2 \le C(\|\boldsymbol{R}_{n-2}\|_{-1}\|\boldsymbol{\phi}_n^E\|_1 + \|R_{n-2}\|_0\|p_n^E\|_{0/\mathbb{R}}. \\ + t^{n+1}(|\partial p_{n-1}/\partial n|_0 + t|\partial p_n/\partial n|_0)|\bar{p}_n^E|_0).$$

Now

$$|\bar{p}_n^E|_0 \le C\|\bar{p}_n^E\|_0^{1/2}\|\bar{p}_n^E\|_1^{1/2} \le C(t^{-1/2}\|p_n^E\|_{0/\mathbb{R}} + t^{1/2}\|\mathbf{grad}\, p_n^E\|_0),$$

so the last term in the previous estimate may be bounded by

$$Ct^{n+1/2}(|\partial p_{n-1}/\partial n|_0 + t|\partial p_{n-1}/\partial n|_0)(\|p_n^E\|_{0/\mathbb{R}} + t\|\mathbf{grad}\, p_n^E\|_0).$$

Now choose $\boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(\Omega)$ satisfying

$$\operatorname{rot}\boldsymbol{\psi} = \bar{p}_n^E, \qquad \|\boldsymbol{\psi}\|_1 \le C\|p_n^E\|_{0/\mathbb{R}}.$$

(The existence of $\boldsymbol{\psi}$ follows from Lemma 4.2.) From the first variational equation, we obtain

$$\begin{aligned}\|p_n^E\|_{0/\mathbb{R}}^2 &= (p_n^E, \bar{p}_n^E) \\ &= (\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_n^E), \mathcal{E}(\boldsymbol{\psi})) + (\boldsymbol{R}_{n-2}, \boldsymbol{\psi}) \\ &\le C\|\boldsymbol{\psi}\|_1(\|\boldsymbol{\phi}_n^E\|_1 + \|\boldsymbol{R}_{n-2}\|_{-1}),\end{aligned}$$

and so

$$\|p_n^E\|_{0/\mathbb{R}} \le C(\|\boldsymbol{\phi}_n^E\|_1 + \|\boldsymbol{R}_{n-2}\|_{-1}).$$

Combining all these results and using the arithmetic-geometric mean inequality, we obtain

$$\|\boldsymbol{\phi}_n^E\|_1 + \|p_n^E\|_{0/\mathbb{R}} + t\|\mathbf{grad}\, p_n^E\|_0 \\ \le C\left[\|\boldsymbol{R}_{n-2}\|_{-1} + \|R_{n-2}\|_0 + t^{n+1/2}|\partial p_{n-1}/\partial n|_0 + t^{n+3/2}|\partial p_n/\partial n|_0\right].$$

Note that if $n = 1$, the right-hand side reduces to $Ct^{3/2}|\partial p_0/\partial n|_0$. The theorem follows immediately from this estimate, Corollary 4.4, and Theorem 4.7. ∎

We now turn to the derivation of error estimates in higher norms.

THEOREM 5.2 (ERROR ESTIMATES FOR $\boldsymbol{\phi}$ AND $p$ IN HIGHER NORMS). *Let $s \ge 2$ be an integer. Then*

$$\|\boldsymbol{\phi}_1^E\|_s + t\|p_1^E\|_{s/\mathbb{R}} \le C(t^{5/2-s}\|g\|_{-1/2} + t\|g\|_{s-2})$$

*and for $n \ge 2$*

$$\|\boldsymbol{\phi}_n^E\|_s + t\|p_n^E\|_{s/\mathbb{R}} \le C(t^{n+3/2-s}\|g\|_{n-3/2} + t^{n+1}\|g\|_{n+s-2}).$$

*Proof.* By standard regularity results for the Dirichlet problem for plane elasticity and (5.1),

$$\|\boldsymbol{\phi}_n^E\|_s \le C\|\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_n^E)\|_{s-2} \le C(\|\mathbf{grad}\, p_n^E\|_{s-2} + \|\boldsymbol{R}_{n-2}\|_{s-2}). \tag{5.5}$$

Using regularity for the Neumann problem for the Laplacian and (5.2) and (5.3), we similarly obtain

$$\|p_n^E\|_{s/\mathbb{R}} \le C\left(\|\,\Delta\, p_n^E\|_{s-2/\mathbb{R}} + \left|\frac{\partial p_n^E}{\partial n}\right|_{s-3/2}\right)$$
$$\le C\left(t^{-2}\|\operatorname{rot}\boldsymbol{\phi}_n^E\|_{s-2} + t^{-2}\|R_{n-2}\|_{s-2} + t^{n-1}\left|\frac{\partial p_{n-1}}{\partial n}\right|_{s-3/2} + t^n\left|\frac{\partial p_n}{\partial n}\right|_{s-3/2}\right).$$

Combining these results and using Corollary 4.4 and Theorem 4.7, we get for $n \ge 1$, $s \ge 2$,

$$\|\boldsymbol{\phi}_n^E\|_s + t\|p_n^E\|_{s/\mathbb{R}} \le C\Bigg(\|\,\mathbf{grad}\, p_n^E\|_{s-2} + \|\boldsymbol{R}_{n-2}\|_{s-2} + t^{-1}\|\operatorname{rot}\boldsymbol{\phi}_n^E\|_{s-2}$$
$$+t^{-1}\|R_{n-2}\|_{s-2} + t^n\left|\frac{\partial p_{n-1}}{\partial n}\right|_{s-3/2} + t^{n+1}\left|\frac{\partial p_n}{\partial n}\right|_{s-3/2}\Bigg)$$
$$\le C(\|p_n^E\|_{s-1/\mathbb{R}} + t^{-1}\|\boldsymbol{\phi}_n^E\|_{s-1} + t^{n+3/2-s}\|g\|_{n-3/2}$$
$$+ t^{n+1/2}\|g\|_{n+s-5/2} + t^n\|g\|_{n+s-3} + t^{n+1}\|g\|_{n+s-2}).$$

Since $\boldsymbol{R}_{-1}$, $R_{-1}$, and $p_1$ vanish, for $n = 1$ we can simplify this result to

$$\|\boldsymbol{\phi}_1^E\|_s + t\|p_1^E\|_{s/\mathbb{R}} \le C(\|p_1^E\|_{s-1/\mathbb{R}} + t^{-1}\|\boldsymbol{\phi}_1^E\|_{s-1} + t\|g\|_{s-2}).$$

Thus

$$\|\boldsymbol{\phi}_n^E\|_s + t\|p_n^E\|_{s/\mathbb{R}}$$
$$\le \begin{cases} C(\|p_1^E\|_{s-1/\mathbb{R}} + t^{-1}\|\boldsymbol{\phi}_1^E\|_{s-1} + t\|g\|_{s-2}), & n = 1, \\ C(\|p_n^E\|_{s-1/\mathbb{R}} + t^{-1}\|\boldsymbol{\phi}_n^E\|_{s-1} + t^{n+3/2-s}\|g\|_{n-3/2} + t^{n+1}\|g\|_{n+s-2}), & n \ge 2. \end{cases}$$

For $s = 2$, the theorem follows from this relation and Theorem 5.1. We can complete the proof using this relation and a simple induction on $s$. ∎

As a consequence of Theorems 5.1, 5.2, and 4.3, we easily obtain bounds on $\boldsymbol{\phi}$ and $p$.

THEOREM 5.3 (BOUNDS ON $\boldsymbol{\phi}$ AND $p$).

$$\|\boldsymbol{\phi}\|_s \le C(t^{5/2-s}\|g\|_{-1/2} + t\|g\|_{s-2} + |||g|||_{s-3}), \qquad s = 1, 2, \cdots,$$
$$\|p\|_{s/\mathbb{R}} \le C(t^{3/2-s}\|g\|_{-1/2} + |||g|||_{s-2}), \qquad s \in \mathbb{N}.$$

*Proof.* From Theorems 5.1 and 5.2 we have

$$\|p_1^E\|_{s/\mathbb{R}} \le C(t^{3/2-s}\|g\|_{-1/2} + |||g|||_{s-2}), \qquad s \in \mathbb{N}.$$

By Theorem 4.3,

$$\|p_0\|_{s/\mathbb{R}} \le C|||g|||_{s-2}.$$

Applying the triangle inequality, we obtain

$$\|p\|_{s/\mathbb{R}} \le \|p_1^E\|_{s/\mathbb{R}} + \|p_0\|_{s/\mathbb{R}} \le C(t^{3/2-s}\|g\|_{-1/2} + |||g|||_{s-2}).$$

A similar argument gives the estimate on $\boldsymbol{\phi}$. ∎

We may use the interpolation property of the Sobolev norms to obtain bounds on $\|\boldsymbol{\phi}_n^E\|_s$ and $\|p_n^E\|_s$ for noninteger $s$ similar to those given in Theorems 5.1 and 5.2 for integer $s$. In particular we have

$$\begin{aligned}\|\boldsymbol{\phi}_1^E\|_{5/2} &\le C(\|\boldsymbol{\phi}_1^E\|_2\|\boldsymbol{\phi}_1^E\|_3)^{1/2}\\ &\le C[(t^{1/2}\|g\|_{-1/2}+t\|g\|_0)(t^{-1/2}\|g\|_{-1/2}+t\|g\|_1)]^{1/2}\\ &\le C(\|g\|_{-1/2}+t^{3/2}\|g\|_1)\end{aligned}$$

and, similarly,

$$\begin{aligned}\|p_1^E\|_{3/2/\mathbb{R}} &\le C(\|p_1^E\|_{1/\mathbb{R}}\|p_1^E\|_{2/\mathbb{R}})^{1/2}\\ &\le C[t^{1/2}\|g\|_{-1/2}(t^{-1/2}\|g\|_{-1/2}+\|g\|_0)]^{1/2}\\ &\le C(\|g\|_{-1/2}+t^{1/2}\|g\|_0).\end{aligned}$$

Combining with Theorem 4.3 as above, we get

$$\|\boldsymbol{\phi}\|_{5/2} \le C(\|g\|_{-1/2}+t^{3/2}\|g\|_1), \tag{5.6}$$

$$\|p\|_{3/2/\mathbb{R}} \le C(\|g\|_{-1/2}+t^{1/2}\|g\|_0). \tag{5.7}$$

In general, however, higher norms of $\boldsymbol{\phi}$ and $p$ do not remain bounded as $t \to 0$.

Thus far our estimates have all been in the $L^2$-based Sobolev spaces $H^s$. In closing this section, we note that our asymptotic expansions and error estimates can be used to study the dependence of the solution on $t$ in many other function spaces as well, for example in the $L^p$-based Sobolev spaces $W_p^s$ or the Hölder spaces $C^{m,\alpha}$. To determine the behavior of the norm $\|\boldsymbol{\phi}\|_{W_\infty^s}$ with respect to $t$, for example, we may write $\boldsymbol{\phi} = \boldsymbol{\phi}_n^E + \boldsymbol{\phi}_n^I + \boldsymbol{\phi}_{n-2}^B$. Now, assuming $g$ is sufficiently smooth, $\|\boldsymbol{\phi}_n^E\|_{n+3/2}$ is bounded uniformly in $t$. Hence, if $n$ is sufficiently large ($n > s - \frac{1}{2}$ in this case), then the Sobolev Embedding Theorem implies that $\|\boldsymbol{\phi}_n^E\|_{W_\infty^s}$ is bounded uniformly. Each of the interior expansion functions is bounded in all the $H^s$ spaces, so $\|\boldsymbol{\phi}_n^I\|_{W_\infty^s}$ is also bounded uniformly. Thus the behavior of $\boldsymbol{\phi}$ is determined by that of $\boldsymbol{\phi}_{n-2}^B = \chi(t^2\boldsymbol{\Phi}_0+t^3\boldsymbol{\Phi}_1+\cdots+t^n\boldsymbol{\Phi}_{n-2})$. Since we have quite explicit expressions for the boundary correctors (Theorem 3.3), it is not difficult to determine the behavior of $\boldsymbol{\phi}_{n-2}^B$. We see that $\|\boldsymbol{\phi}_{n-2}^B\|_{W_\infty^s} = O(t^{2-s})$. Thus

$$\|\boldsymbol{\phi}\|_{W_\infty^s} = O(t^{\min(2-s,0)}).$$

Estimates of other quantities, including the errors in the partial sums of the asymptotic expansions can be derived similarly. With a little effort we can get a bound which indicates explicitly the dependence of the norm on the load function $g$ as well. However, we do not expect that the required regularity on $g$ in these estimates (and in some of the previous ones as well) is optimal.

**6. Asymptotic expansion of the transverse displacement and shear.** In the previous sections we obtained and justified an asymptotic expansion for the rotation variable $\boldsymbol{\phi}$. We now turn to the other primitive variable, $\omega$, and obtain an expansion for it. In contrast to $\boldsymbol{\phi}$, we will see that $\omega$ has no boundary layer.

Define the auxiliary variable $v = \omega - t^2 r$. Clearly $v = 0$ on $\partial\Omega$ and, from (2.6), $\Delta v = \operatorname{div} \boldsymbol{\phi}$. Then, taking the divergence of (2.4) and substituting (2.3), we easily compute that $D\,\Delta^2 v = D\,\Delta \operatorname{div} \boldsymbol{\phi} = g$. Next, note that $\mathbf{grad}\, v = \mathbf{grad}\, w - t^2\, \mathbf{grad}\, r = \boldsymbol{\phi} + t^2\, \mathbf{curl}\, p$. Since $\boldsymbol{\phi}$ vanishes on $\partial\Omega$,

$$\frac{\partial v}{\partial n} = -t^2 \frac{\partial p}{\partial s}, \quad \text{on } \partial\Omega.$$

Thus $v$ is completely characterized as the solution of a certain Dirichlet problem for the biharmonic operator, and it is easy to see how to expand it in powers of $t$. For $i \in \mathbb{N}$, define $v_i$ by the biharmonic problem

$$D\,\Delta^2 v_i = \begin{cases} g, & i = 0, \\ 0, & i \geq 1, \end{cases} \quad \text{in } \Omega,$$

$$v_i = 0, \quad \partial v_i/\partial n = \begin{cases} 0, & i = 0, 1, \\ -\partial p_0/\partial s, & i = 2, \\ -\partial p_{i-2}/\partial s - \partial P_{i-3}/\partial s, & i \geq 3, \end{cases} \quad \text{on } \partial\Omega.$$

The coefficients in the asymptotic expansion of $\omega$ are then given by

$$\omega_i = \begin{cases} v_i, & i \neq 2, \\ v_2 + r, & i = 2. \end{cases}$$

Note that $\omega_0$ satisfies the boundary value problem

$$D\,\Delta^2 \omega_0 = g \quad \text{in } \Omega, \qquad \omega_0 = \frac{\partial \omega_0}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

It is useful to express the first terms of the expansions for $w$ and $\boldsymbol{\phi}$ in terms of $\omega_0$. First of all, there is a simple relation between $\omega_0$ and $\boldsymbol{\phi}_0$.

THEOREM 6.1.

$$\boldsymbol{\phi}_0 = \mathbf{grad}\, \omega_0.$$

*Proof.* From (3.10) and (3.9), it follows that $\boldsymbol{\phi}_0 = \mathbf{grad}\, \mu$ for some $\mu \in \mathring{H}^2(\Omega)$. Inserting in (3.7) and taking the divergence gives

$$D\,\Delta^2 \mu = -\Delta r = g.$$

Comparing with the defining equations for $\omega_0$, we see that $\mu = \omega_0$. ∎

Clearly $\omega_1 = v_1 = 0$ and $\omega_2 = v_2 + r$, where

$$\Delta^2 v_2 = 0 \quad \text{in } \Omega, \qquad v_2 = 0, \quad \partial v_2/\partial n = -\partial p_0/\partial s \quad \text{on } \partial\Omega, \tag{6.1}$$

$$\Delta r = -g \quad \text{in } \Omega, \qquad r = 0 \quad \text{on } \partial\Omega. \tag{6.2}$$

Now, from (3.7),

$$\frac{\partial p_0}{\partial s} = \mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_0) \cdot \boldsymbol{n} + \frac{\partial r}{\partial n} = D \frac{\partial\, \Delta\, \omega_0}{\partial n} + \frac{\partial r}{\partial n}. \tag{6.3}$$

$$\frac{\partial p_0}{\partial n} = -\,\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_0) \cdot \boldsymbol{s} = -D \frac{\partial\, \Delta\, \omega_0}{\partial s}. \tag{6.4}$$

Using (6.3) in (6.1) and combining with (6.2), we get

$$\Delta^2 \omega_2 = -\Delta g \quad \text{on } \Omega, \qquad \omega_2 = 0, \quad \frac{\partial \omega_2}{\partial n} = -D \frac{\partial \Delta \omega_0}{\partial n},$$

which is a biharmonic problem for $\omega_2$. From the definitions, $\omega_3 = v_3$ is a biharmonic function vanishing on $\partial\Omega$ with $\partial\omega_3/\partial n = -\partial P_0/\partial s$ (since $p_1 = 0$). Using (3.20) and (6.4) to simplify the latter boundary condition gives the following biharmonic problem for $\omega_3$:

$$\Delta^2 \omega_3 = 0 \quad \text{in } \Omega, \qquad \omega_3 = 0, \quad \frac{\partial \omega_3}{\partial n} = -\frac{D}{c} \frac{\partial^2}{\partial s^2} \Delta \omega_0 \quad \text{on } \partial\Omega.$$

Turning to the expansion for $\boldsymbol{\phi}$, the expression for $\boldsymbol{\Phi}_0$ in (3.20) becomes, in light of (6.4),

$$\hat{\boldsymbol{\Phi}}_0(\hat\rho, \theta) = -D \widehat{\frac{\partial \Delta \omega_0}{\partial s}}(0, \theta) e^{-c\hat\rho} \boldsymbol{s}. \tag{6.5}$$

To determine $\boldsymbol{\phi}_2$, we note from (3.10) that $\operatorname{rot} \boldsymbol{\phi}_2$ is constant. Since

$$\int_\Omega \operatorname{rot} \boldsymbol{\phi}_2 = -\int_{\partial\Omega} \boldsymbol{\phi}_2 \cdot \boldsymbol{s} = \int_{\partial\Omega} \boldsymbol{\Phi}_0 \cdot \boldsymbol{s} = 0,$$

$\operatorname{rot} \boldsymbol{\phi}_2 = 0$ and $\boldsymbol{\phi}_2 = \mathbf{grad}\, \psi$ for some function $\psi$. Substituting in (3.7) and taking the divergence shows that $\psi$ is biharmonic. Then the boundary conditions $\mathbf{grad}\, \psi = -\boldsymbol{\Phi}_0$ on $\partial\Omega$ determine $\psi$ modulo $\mathbb{R}$ and $\boldsymbol{\phi}_2$ completely. In light of (6.5), the boundary conditions on $\psi$ become

$$\psi = D \Delta \omega_0 \quad (\text{mod } \mathbb{R}), \quad \partial\psi/\partial n = 0 \quad \text{on } \partial\Omega.$$

We now obtain a priori estimates for the $\omega_i$ and error estimates for the finite sums of the expansion.

Theorem 6.2 (A priori estimates for the $\omega_i$). *Let $i \in \mathbb{N}$, $s \ge 2$. Then*

$$\|\omega_i\|_s \le C |\!|\!| g |\!|\!|_{s+i-4}.$$

*Proof.* This follows easily from regularity for the biharmonic equation, Corollary 4.4, and (4.5). ∎

Let $\omega_n^E = \omega - \sum_{i=0}^n t^i \omega_i$ denote the error in the partial sums of the asymptotic expansion

$$\omega \sim \sum_{i=0}^\infty t^i \omega_i.$$

The next theorem bounds the error in expansion. Note that the order of the error is the same in all Sobolev norms, reflecting the fact that $\omega$ does not involve a boundary layer.

Theorem 6.3 (error estimates for $\omega$). *For $n = 1, 2, \cdots$ and $s = 1, 2, \cdots$*

$$\|\omega_n^E\|_s \le C(t^{n+1} \|g\|_{n+s-3} + t^{n+s+1} \|g\|_{n+2s-3}).$$

*Proof.* Set $v_n^E = v - \sum_{i=0}^n t^i v_i$. Note that $\omega_n^E = v_n^E$ for $n > 1$, and $\omega_1^E = v_1^E + t^2 r$, so it suffices to prove the theorem with $\omega_n^E$ replaced by $v_n^E$.

Now

$$D\,\Delta^2 v_n^E = 0, \quad \text{in } \Omega,$$
$$v_n^E = 0, \quad \text{on } \partial\Omega,$$

and

$$\begin{aligned}\frac{\partial v_n^E}{\partial n} &= -t^2 \frac{\partial}{\partial s}(p - p_{n-2}^I - p_{n-3}^B) \\ &= -t^2 \frac{\partial}{\partial s}\left(p_{n+s-1}^E + \sum_{i=n-1}^{n+s-1} t^i p_i + \sum_{i=n-1}^{n+s-2} t^i P_{i-1}\right).\end{aligned}$$

Thus, using regularity results for the biharmonic problem,

$$\begin{aligned}\|v_n^E\|_s &\le C \left|\frac{\partial v_n^E}{\partial n}\right|_{s-3/2} \\ &\le Ct^2 \left(\|p_{n+s-1}^E\|_{s/\mathbb{R}} + \sum_{i=n-1}^{n+s-1} t^i \|p_i\|_{s/\mathbb{R}} + \sum_{i=n-1}^{n+s-2} t^i |P_{i-1}|_{s-1/2}\right).\end{aligned}$$

Applying Theorem 5.1 or 5.2, Theorem 4.3, and Corollary 4.4, we get

$$\begin{aligned}\|v_n^E\|_s &\le Ct^2 \left(t^{n-1/2}\|g\|_{n+s-5/2} + t^{n+s-1}\|g\|_{n+2s-3} + \sum_{i=n-1}^{n+s-1} t^i \|g\|_{i+s-2}\right) \\ &\le C(t^{n+1}\|g\|_{n+s-3} + t^{n+s+1}\|g\|_{n+2s-3}),\end{aligned}$$

as desired. □

Using a similar argument, we can also obtain regularity estimates for $\omega$ in $H^s(\Omega)$ uniform with respect to $t$.

THEOREM 6.4. *For $s = 2, 3, \cdots$ there exists a constant $C$ independent of $t$ such that*

$$\|\omega\|_s \le C(\|g\|_{s-4} + t^2\|g\|_{s-2}), \qquad s = 2, 3,$$
$$\|\omega\|_s \le C(\|g\|_{s-4} + t^s\|g\|_{2s-4}), \quad s \ge 4.$$

*Proof.* Using standard regularity results for the biharmonic problem and (4.5), we get

$$\|\omega\|_s \le \|v\|_s + t^2\|r\|_s \le C(\|g\|_{s-4} + t^2|\partial p/\partial s|_{s-3/2} + t^2\|g\|_{s-2}). \tag{6.6}$$

When $s \ge 4$, we substitute $p = \sum_{i=0}^{s-2} t^i p_i + \chi \sum_{i=0}^{s-4} t^{i+1} P_i + p_{s-2}^E$, into (6.6) and apply Corollary 4.4 and Theorem 5.2 to estimate the right-hand side, obtaining

$$\|\omega\|_s \le C\left(\|g\|_{s-4} + \sum_{i=0}^{s-2} t^{i+2}\|g\|_{s+i-2} + \sum_{i=0}^{s-4} t^{i+3}\|g\|_{s+i-1} + t^{1/2}\|g\|_{s-7/2}\right),$$

which gives the desired result.

When $s = 2$ or $s = 3$, we substitute $p = p_0 + p_1^E$ into (6.6) and complete the proof with a similar argument. ∎

Recall that the scaled transverse shear stress is given by $\boldsymbol{\zeta} = t^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi})$, which we decomposed as $\mathbf{grad}\,r + \mathbf{curl}\,p$. We can obtain an asymptotic expansion for the shear stress from either of these expressions, in the former case noting a cancellation due to Theorem 6.1. Thus, formally,

$$\begin{aligned}\boldsymbol{\zeta} &\sim (\mathbf{grad}\,\omega_2 - \boldsymbol{\phi}_2) + t(\mathbf{grad}\,\omega_3 - \boldsymbol{\phi}_3) + \cdots - \chi(\boldsymbol{\Phi}_0 + t\boldsymbol{\Phi}_1 + \cdots)\\ &\sim (\mathbf{grad}\,r + \mathbf{curl}\,p_0) + t^2\,\mathbf{curl}\,p_2 + t^3\,\mathbf{curl}\,p_3 + \cdots + \chi(t\,\mathbf{curl}\,P_0 + t^2\,\mathbf{curl}\,P_1 + \cdots).\end{aligned}$$

In light of our previous results, it is straightforward to bound the individual terms in either of these expansions as well as the remainders when the expansions are terminated. Here we content ourselves with determining the regularity of the shear stress vector and its dependence on $t$.

THEOREM 6.5. *Let $s \ge -1$ be an integer. Then there exists a constant $C$ independent of $t$ such that*

$$\|\boldsymbol{\zeta}\|_s \le C(t^{1/2-s}\|g\|_{-1/2} + |||g|||_{s-1}).$$

*Proof.* This follows immediately from Theorem 5.3 and (4.5). ∎

Similar bounds hold in the noninteger order Sobolev spaces. In particular,

$$\|\boldsymbol{\zeta}\|_{1/2} \le C(\|g\|_{-1/2} + t^{1/2}\|g\|_0),$$

as follows immediately from (4.5) and (5.7). In general $\|\boldsymbol{\zeta}\|_s$ will blow up as $t \to 0$ if $s > 1/2$. Thus the shear stress evidences a rather strong boundary layer.

**7. Hard simply-supported boundary conditions.** Two sets of boundary conditions are commonly used with the Reissner–Mindlin equations to model a simply-supported plate. Boundary conditions for a hard simply-supported plate are

$$M_n\boldsymbol{\phi} = 0, \quad \boldsymbol{\phi}\cdot\boldsymbol{s} = 0, \quad w = 0,$$

where $M_n\boldsymbol{\phi} = \boldsymbol{n}^t\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n}$, or, in $(\rho,\theta)$ coordinates,

$$\widetilde{M_n\boldsymbol{\phi}} = D\left(-\frac{\partial\tilde{\boldsymbol{\phi}}}{\partial\rho}\cdot\boldsymbol{n} + \nu\frac{\partial\tilde{\boldsymbol{\phi}}}{\partial\theta}\cdot\boldsymbol{s}\right).$$

(For a soft simply-supported plate the condition $\boldsymbol{\phi}\cdot\boldsymbol{s} = 0$ is replaced by $\boldsymbol{s}^t\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n} = 0$. Thus, in both cases the lateral edge of the undisplaced plate is not permitted to displace vertically. In the soft case a vertical fiber on the lateral edge is permitted to rotate freely, while in the hard case it may only rotate in the plane normal to the edge. The soft conditions would seem to be easier to realize in practice.)

The boundary layer analysis for the hard simply-supported plate, which we consider in this section, is very similar to that for the clamped plate. The soft simply-supported plate has a significantly stronger boundary layer which will be investigated in a subsequent paper.

The only difference in the asymptotic expansions themselves for the hard simply-supported Reissner–Mindlin plate and the clamped plate is that the boundary conditions for the problems defining the interior expansion functions must be modified. All the major estimates for the expansion functions and all the error analysis carries over. However, at a few places in the analysis additional terms must be considered. In this section we indicate very briefly these additional considerations.

As in the case of the clamped plate, we use the decomposition of the shear stress vector, given by formula (2.2). We then obtain the reformulation of (2.3)–(2.6), where the boundary conditions (2.7) are replaced by:

$$r = 0, \quad \boldsymbol{\phi} \cdot \boldsymbol{s} = 0, \quad M_n \boldsymbol{\phi} = 0, \quad \partial p / \partial n = 0, \quad w = 0.$$

The forms of the asymptotic expansions for $\boldsymbol{\phi}$ and $p$ are the same as those given in § 3 for the clamped plate, and the interior approximations satisfy the same partial differential equations (3.7), (3.10), but the boundary conditions (3.9) are replaced by

$$\boldsymbol{\phi}_i \cdot \boldsymbol{s} = \begin{cases} 0, & i = 0, 1, \\ -\hat{\boldsymbol{\Phi}}_{i-2} \cdot \boldsymbol{s}, & i \geq 2, \end{cases}$$

and

$$M_n \boldsymbol{\phi}_i = \begin{cases} 0, & i = 0, \\ D(\partial \hat{\boldsymbol{\Phi}}_0 / \partial \hat{\rho}) \cdot \boldsymbol{n}, & i = 1, \\ D[(\partial \hat{\boldsymbol{\Phi}}_{i-1} / \partial \hat{\rho}) \cdot \boldsymbol{n} - \nu (\partial \hat{\boldsymbol{\Phi}}_{i-2} / \partial \theta) \cdot \boldsymbol{s}], & i \geq 2. \end{cases}$$

The boundary correctors are again defined by (3.16)–(3.19). Thus, the analysis in § 3 remains valid. In particular, Theorem 3.3 still holds and the formula for the first boundary corrector is again (3.20). It follows immediately that $M_n \boldsymbol{\phi}_1 = 0$ on $\partial\Omega$ and hence $\boldsymbol{\phi}_1 = 0$, $p_1 = 0$.

To bound the errors in the asymptotic expansions, we need analogues of the results proved in § 4. From the form of the boundary correctors (given in Theorem 3.3), we get immediately that

$$|\boldsymbol{\Phi}_i|_s + t \left| \frac{\partial \boldsymbol{\Phi}_i}{\partial n} \right|_s \leq C \sum_{j=0}^{i} \left| \frac{\partial p_j}{\partial n} \right|_{s+i-j}, \tag{7.1}$$

for all $s \in \mathbb{R}$, $i \in \mathbb{N}$. To estimate the interior expansion functions we use the following analogue of Lemma 4.2.

LEMMA 7.1. *Let $s \in \mathbb{N}$, $f \in H^s(\Omega) \cap \mathring{H}^1(\Omega)$, $g \in H^s(\Omega)/\mathbb{R}$, $k \in H^{s+1/2}(\partial\Omega)$, and $l \in H^{s-1/2}(\partial\Omega)$ be given. Then there exist unique $\boldsymbol{\psi} \in \boldsymbol{H}^{s+1}(\Omega)$, $q \in H^s(\Omega)/\mathbb{R}$ satisfying the partial differential equations*

$$-\mathbf{div}\, \mathcal{C}\, \mathcal{E}(\boldsymbol{\psi}) - \mathbf{curl}\, q = \mathbf{grad}\, f, \tag{7.2}$$

$$-\operatorname{rot} \boldsymbol{\psi} = g \pmod{\mathbb{R}}, \tag{7.3}$$

*and the boundary conditions*

$$\boldsymbol{\psi} \cdot \boldsymbol{s} = k, \qquad M_n \boldsymbol{\psi} = l.$$

*Moreover, there exists a constant $C$ depending only on $s$, $E$, $\nu$, and $\Omega$ such that*

$$\|\boldsymbol{\psi}\|_{s+1} + \|q\|_{s/\mathbb{R}} \leq C(\|f\|_s + \|g\|_{s/\mathbb{R}} + |k|_{s+1/2} + |l|_{s-1/2}).$$

*Proof.* The weak form of the boundary value problem is to find $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{\psi} \cdot \boldsymbol{s} = k$ on $\partial\Omega$ and $q \in L^2(\Omega)/\mathbb{R}$ satisifying

$$\begin{aligned}&(\mathcal{C}\,\mathcal{E}(\boldsymbol{\psi}), \mathcal{E}(\boldsymbol{\mu})) - (q, \operatorname{rot}\boldsymbol{\mu}) = -(f, \operatorname{div}\boldsymbol{\mu}) + \langle l, \boldsymbol{\mu}\cdot\boldsymbol{n}\rangle,\\ &-(\operatorname{rot}\boldsymbol{\psi}, v) = (g, v),\end{aligned}$$

for all $\boldsymbol{\mu} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{\mu}\cdot\boldsymbol{s} = 0$ on $\partial\Omega$ and all $v \in L^2(\Omega)$ of mean value zero. Existence and uniqueness of a solution in $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ is proved just as for the generalized Stokes equations, e.g., by applying Brezzi's theorem [4]. The estimate for $s = 0$ follows from the same argument. To establish the claimed regularity for $s \geq 1$, we apply the Helmholtz decomposition to $\boldsymbol{\psi}$ to get $\boldsymbol{\psi} = \mathbf{grad}\, z + \mathbf{curl}\, b$, with $z \in \mathring{H}^1(\Omega)$, $b \in H^1(\Omega)/\mathbb{R}$. Now, it suffices to show that

$$\|b\|_{s+2/\mathbb{R}} + \|z\|_{s+2} \leq C(\|f\|_s + \|g\|_{s/\mathbb{R}} + |k|_{s+1/2}), \quad s \in \mathbb{N},$$

since this gives the estimate on $\boldsymbol{\psi}$ immediately, and that on $q$ then follows from (7.2).

From (7.3) we have

$$\Delta\, b = g \pmod{\mathbb{R}} \quad \text{in } \Omega,$$

with boundary conditions $\partial b/\partial n = k$, so the desired bound on $b$ follows from regularity for the Neumann problem for Laplace's equation. We prove the desired estimate for $z$ by induction on $s$. The case $s = 0$ follows from the bound on $\|\boldsymbol{\psi}\|_1$ since $z = \operatorname{div}\boldsymbol{\psi}$. Thus we assume that $s$ is a positive integer.

Let $w = D\,\Delta\, z + f$. Since

$$\begin{aligned}M_n(\mathbf{grad}\, z) &= D\left[\Delta\, z - (1-\nu)\left(\frac{\partial^2 z}{\partial s^2} + \kappa\frac{\partial z}{\partial n}\right)\right],\\ M_n(\mathbf{curl}\, b) &= D(1-\nu)\left(-\frac{\partial}{\partial s}\frac{\partial b}{\partial n} + \kappa\frac{\partial b}{\partial s}\right),\end{aligned}$$

the boundary conditions for $\boldsymbol{\psi}$ imply that

$$w - f = D\,\Delta\, z = l + D(1-\nu)(\partial^2 z/\partial s^2 + \kappa\partial z/\partial n + \partial k/\partial s - \kappa\partial b/\partial s) \quad \text{on } \partial\Omega,$$

or, since $z$ and $f$ vanish on $\partial\Omega$,

$$w = l + D(1-\nu)(\kappa\partial z/\partial n + \partial k/\partial s - \kappa\partial b/\partial s) \quad \text{on } \partial\Omega.$$

Now, taking the divergence of equation (7.2) gives

$$-D\,\Delta^2\, z = \Delta\, f \quad \text{in } \Omega.$$

so $w$ is harmonic. Applying regularity for the Dirichlet problem for Laplace's equation then gives

$$\begin{aligned}\|w\|_s &\leq C(|\partial z/\partial n|_{s-1/2} + |\partial k/\partial s|_{s-1/2} + |\partial b/\partial s|_{s-1/2})\\ &\leq C(\|z\|_{s+1} + |k|_{s+1/2} + \|b\|_{s+1/\mathbb{R}})\\ &\leq C(\|z\|_{s+1} + |k|_{s+1/2} + \|g\|_{s/\mathbb{R}}).\end{aligned}$$

Finally $z$ satisfies

$$-\Delta z = D^{-1}(f - w) \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega,$$

so another application of regularity for the Dirichlet problem shows that

$$\|z\|_{s+2} \le C(\|f\|_s + \|w\|_s) \le C(\|f\|_s + \|g\|_{s/\mathbb{R}} + |k|_{s+1/2} + \|z\|_{s+1}),$$

and the proof is completed by induction. ∎

Using this result and (7.1), it follows that Theorem 4.3 holds also in the hard simply-supported case, and then that Theorem 4.5 also remains valid.

Turning to the finite interior and boundary expansions, Theorem 4.6 and Theorem 4.7 hold as before. However, in order to prove the analogue of Theorem 5.1, we need a slight refinement of the estimate of $\|\boldsymbol{R}_m\|_{-1}$.

THEOREM 7.2. *If* $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$ *satisfies* $\boldsymbol{\psi}\cdot\boldsymbol{s} = 0$ *on* $\partial\Omega$ *and* $m \in \mathbb{N}$, *then*

$$\left|(\boldsymbol{R}_m, \boldsymbol{\psi}) - t^{m+3}\left\langle D\frac{\partial \boldsymbol{\Phi}_{m+1}}{\partial n}\cdot\boldsymbol{n}, \boldsymbol{\psi}\cdot\boldsymbol{n}\right\rangle\right| \le Ct^{m+5/2}\|g\|_{m+1/2}\|\boldsymbol{\psi}\|_1.$$

*Proof.* The proof is very close to that of the $H^{-1}$ estimate in Theorem 4.7. The only difference is that instead of (4.14) we must show that

$$\left|(\chi \boldsymbol{R}_m^{11}, \boldsymbol{\psi}) - t^{m+3}\left\langle D\frac{\partial \boldsymbol{\Phi}_{m+1}}{\partial n}\cdot\boldsymbol{n}, \boldsymbol{\psi}\cdot\boldsymbol{n}\right\rangle\right| \le Ct^{m+5/2}\|g\|_{m+1/2}\|\boldsymbol{\psi}\|_1 \tag{7.4}$$

(which is the same as (4.14) for $\boldsymbol{\psi} \in \mathring{H}^1(\Omega)$). Since $\boldsymbol{\psi}$ does not vanish on the boundary, when we integrate by parts in (4.13) we get a boundary term:

$$\begin{aligned}(\chi \boldsymbol{R}_m^{11}, \boldsymbol{\psi}) &= t^{m+2}\int_0^L\int_0^{\rho_0}\left(\hat{\mathcal{A}}_0 t\frac{\partial \tilde{\boldsymbol{\Phi}}_{m+1}}{\partial \rho} + \hat{\boldsymbol{A}}_5\tilde{P}_{m+1}\right)\frac{\partial}{\partial\rho}\left\{\tilde{\chi}(\rho)\tilde{\boldsymbol{\psi}}(\rho,\theta)[1-\kappa(\theta)\rho]\right\}\,d\rho d\theta \\ &\qquad - t^{m+2}\left\langle \mathcal{A}_0 t\frac{\partial \boldsymbol{\Phi}_{m+1}}{\partial n} - \boldsymbol{A}_5 P_{m+1}, \boldsymbol{\psi}\right\rangle.\end{aligned}$$

Now $\boldsymbol{A}_5\cdot\boldsymbol{\psi} = \boldsymbol{s}\cdot\boldsymbol{\psi} \equiv 0$ and $\mathcal{A}_0\boldsymbol{\psi} = \mathcal{A}_0\boldsymbol{n}(\boldsymbol{\psi}\cdot\boldsymbol{n}) = -D\boldsymbol{n}(\boldsymbol{\psi}\cdot\boldsymbol{n})$, so

$$\left\langle \mathcal{A}_0 t\frac{\partial \boldsymbol{\Phi}_{m+1}}{\partial n} - \boldsymbol{A}_5 P_{m+1}, \boldsymbol{\psi}\right\rangle = -t\left\langle D\frac{\partial \boldsymbol{\Phi}_{m+1}}{\partial n}\cdot\boldsymbol{n}, \boldsymbol{\psi}\cdot\boldsymbol{n}\right\rangle.$$

The proof of inequality (7.4) and the remainder of the theorem now proceed just as in Theorem 4.7. ∎

Defining $\boldsymbol{\phi}_n^E$ and $p_n^E$ as in § 5, we see that they again satisfy the partial differential equations (5.1) and (5.2). The boundary conditions now become

$$\boldsymbol{\phi}_n^E\cdot\boldsymbol{s} = 0, \qquad M_n\boldsymbol{\phi}_n^E = t^{n+1}D\frac{\partial \boldsymbol{\Phi}_{n-1}}{\partial n}\cdot\boldsymbol{n}, \qquad \frac{\partial p_n^E}{\partial n} = -t^{n-1}\frac{\partial p_{n-1}}{\partial n} - t^n\frac{\partial p_n}{\partial n},$$

and the variational equation (5.4) which enters the proof of Theorem 5.1 thus becomes

$$(\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_n^E), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}\, p_n^E, \boldsymbol{\psi}) = -(\boldsymbol{R}_{n-2}, \boldsymbol{\psi}) + t^{n+1}D\left\langle \frac{\partial \boldsymbol{\Phi}_{n-1}}{\partial n}\cdot\boldsymbol{n}, \boldsymbol{\psi}\cdot\boldsymbol{n}\right\rangle,$$

valid for $\boldsymbol{\psi}$ with vanishing tangential component on $\partial\Omega$. We bounded the right-hand side of this equation in Theorem 7.2. This is the only additional consideration in establishing Theorem 5.1 in the hard simply-supported case.

The higher-order estimates in Theorem 5.2 also carry over to the present case, but again there is an additional term to be bounded because $\boldsymbol{\phi}_n^E$ does not vanish on $\partial\Omega$. The bound for $\|\boldsymbol{\phi}_n^E\|_s$ given in (5.5) must be modified to include the additional term

$$t^{n+1}\left|\frac{\partial \boldsymbol{\Phi}_{n-1}}{\partial n}\cdot \boldsymbol{n}\right|_{s-3/2}.$$

In view of (7.1) and Theorem 4.3, this term is easily bounded by $t^n\|g\|_{s+n-3}$, which is no larger than other terms which were treated in the proof of Theorem 5.2. Of course, once Theorems 4.3, 5.1, and 5.2 are established, the regularity results given in Theorem 5.3 follow.

An asymptotic expansion and regularity results for the transverse displacement and the shear stress can be developed as in § 6. Naturally the boundary conditions in the defining problems for the expansion functions are changed. The boundary conditions on $\boldsymbol{\phi}$ and $p$ imply that $v=\omega-t^2r$ satisfies, in addition to the differential equation $D\,\Delta^2\,v=g$, the boundary conditions

$$v=0,\qquad (1-\nu)\partial^2 v/\partial n^2+\nu\,\Delta\,v=t^2(1-\nu)\kappa\partial p/\partial s,\quad \text{on } \partial\Omega,$$

where $\kappa$ denotes the curvature of $\partial\Omega$. It is then clear how to define the regular expansion for $v$ and hence $\omega$, and all the analysis of § 6 carries over easily.

**Appendix.** In this appendix we give the proof of Theorem 3.3 concerning the existence, uniqueness, and form of the solution of the boundary value problems defining the boundary correctors.

*Proof.* Differentiating (3.17) with respect to $\hat\rho$, we obtain

$$-\hat{\boldsymbol{A}}_5\cdot\frac{\partial^2\hat{\boldsymbol{\Phi}}_i}{\partial\hat\rho^2}+\frac{\partial^3\hat P_i}{\partial\hat\rho^3}=\frac{\partial\hat G_i(\hat\rho,\theta)}{\partial\hat\rho}.$$

Multiplying (3.16) by $\hat{\mathcal{A}}_0^{-1}$ and taking the inner product with $\hat{\boldsymbol{A}}_5$, we obtain

$$\hat{\boldsymbol{A}}_5\cdot\frac{\partial^2\hat{\boldsymbol{\Phi}}_i}{\partial\hat\rho^2}+\hat{\boldsymbol{A}}_5^t\hat{\mathcal{A}}_0^{-1}\hat{\boldsymbol{A}}_5\frac{\partial\hat P_i}{\partial\hat\rho}=-\hat{\boldsymbol{A}}_5^t\hat{\mathcal{A}}_0^{-1}\hat{\boldsymbol{F}}_i(\hat\rho,\theta).$$

Adding these equations and observing from Lemma 3.1 that $\hat{\boldsymbol{A}}_5^t\hat{\mathcal{A}}_0^{-1}\hat{\boldsymbol{A}}_5=-c^2$, we get

$$\frac{\partial^3\hat P_i}{\partial\hat\rho^3}-c^2\frac{\partial\hat P_i}{\partial\hat\rho}=-\hat{\boldsymbol{A}}_5^t\hat{\mathcal{A}}_0^{-1}\hat{\boldsymbol{F}}_i(\hat\rho,\theta)+\frac{\partial\hat G_i(\hat\rho,\theta)}{\partial\hat\rho}=:\hat H_i. \tag{8.1}$$

The general solution of the associated homogeneous equation is $c_1(\theta)+c_2(\theta)e^{-c\hat\rho}+c_3(\theta)e^{c\hat\rho}$, with the functions $c_i$ arbitrary. Now if we have two solutions to (3.16)–(3.19), then the difference in the values of $\hat P_i$ must be of this form. Applying (3.19) implies that $c_1$ and $c_3$ vanish, and then the homogeneous form of (3.18) implies that $c_2$ vanishes. Thus there can be at most one function $\hat P_i$ satisfying (3.16)–(3.19). Once $\hat P_i$ is known, $\hat{\boldsymbol{\Phi}}_i$ is determined up to the addition of a function linear in $\hat\rho$ by (3.16).

In light of (3.19), $\hat{\boldsymbol{\Phi}}_i$ is uniquely determined. Thus we have shown that there can be at most one solution $(\hat{\boldsymbol{\Phi}}_i, \hat{P}_i)$ to (3.16)–(3.19).

Let us say that a scalar-valued function $\hat{Q}(\hat{\rho}, \theta)$ is of type $(m, i)$ if

$$\hat{Q}(\hat{\rho}, \theta) = e^{-c\hat{\rho}} \sum_{k=0}^{m} \sum_{j=0}^{i} \sum_{l=0}^{i-j} \alpha_{jkl}(\theta) \hat{\rho}^k \frac{\partial^l}{\partial \theta^l} \widetilde{\frac{\partial p_j}{\partial n}}(0, \theta)$$

for some smooth functions $\alpha_{jkl}(\theta)$. A vector-valued function is of type $(m, i)$ if all components are. We claim that there is a solution $(\hat{\boldsymbol{\Phi}}_i, \hat{P}_i)$ to (3.16)–(3.19) which is of of type $(i, i)$. We will establish the claim by induction on $i$, thereby completeing the proof of the theorem. The solution given in (3.20) verifies the claim for $i = 0$. Now suppose that $(\hat{\boldsymbol{\Phi}}_j, \hat{P}_j)$ is of type $(j, j)$ for $j = 0, 1, \cdots, i-1$. It follows easily from their respective definitions (just after (3.16) and in (3.17) and (8.1)) that $\hat{\boldsymbol{F}}_i$, $\hat{G}_i$, and $\hat{H}_i$ are of type $(i-1, i)$. It is then elementary to see that the differential equation (8.1) has a unique solution of type $(i, i)$ satisfying the boundary condition (3.18). Next, there is a unique function $\hat{\boldsymbol{\Phi}}_i$ of type $(i, i)$ satisfying (3.16). Together (3.16), (8.1), and the decay at infinity of $\hat{\boldsymbol{\Phi}}_i$, $\hat{G}_i$, and the $\hat{\rho}$-derivatives of $\hat{P}_i$ imply (3.17). Thus $(\hat{\boldsymbol{\Phi}}_i, \hat{P}_i)$ satisfy (3.16)–(3.19) and are of the desired form. This completes the induction.

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