SIAM J. NUMER. ANAL.
Vol. 29, No. 3, pp. 730–754, June 1992

007

LOCAL ERROR ESTIMATES FOR A FINITE ELEMENT METHOD FOR HYPERBOLIC AND CONVECTION-DIFFUSION EQUATIONS*

RICHARD S. FALK† AND GERARD R. RICHTER‡

Abstract. Local error estimates of near optimal order are derived for a finite element method for hyperbolic and convection dominated convection-diffusion equations in a domain $\Omega \subset R^2$. The method generates, in an explicit fashion, a continuous piecewise polynomial approximation of degree $n \geq 2$ over a triangulation of Ω. The scheme is shown to propagate disturbances a distance $O(\sqrt{h} \log \frac{1}{h})$ in the crosswind direction, where h is the meshsize. The analysis uses test functions which depend only on the crosswind variable. It is also shown to be applicable, in a parallel fashion, to the discontinuous Galerkin method, thus underscoring the close interrelationship of the two methods.

Key words. finite elements, hyperbolic equations, convection-diffusion

AMS(MOS) subject classifications. 65N30, 65M15

1. Introduction.

In this paper, we continue the analysis of a continuous finite element method for hyperbolic equations that was begun in [3] and later extended to convection dominated convection-diffusion equations in [15], [16]. Here we derive local error estimates for the method. We show how our analysis can be applied, in a parallel fashion, to the discontinuous Galerkin method, and develop a close relationship between the two methods. For ease of exposition, we shall deal only with constant coefficient problems; the extension to variable coefficients involves only minor but potentially diversionary technical details.

We first consider the case of a pure hyperbolic equation

$$\begin{cases} \boldsymbol{\alpha} \cdot \nabla u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_{\text{in}}(\Omega), \end{cases}$$

where $\Omega \subset R^2$ is a polygon with boundary Γ and $\boldsymbol{\alpha}$ is a unit vector. Here $\Gamma_{\text{in}}(\Omega)$ is the inflow part of Γ, defined by $\{x \in \Gamma | \boldsymbol{\alpha} \cdot n < 0\}$, where n denotes the unit outer normal. Let Ω be triangulated by a quasi-uniform mesh of size h, with minimum angle bounded away from zero, in such a way that $|\boldsymbol{\alpha} \cdot n| \neq 0$ for all triangle sides. The triangles then divide into two categories: those with one inflow side and two outflow sides (type I), and vice versa (type II). In addition, they can then be ordered explicitly with respect to domain of dependence (cf. [11]), creating the possibility that a finite element approximation can be generated in an explicit fashion, element by element. Two such methods originated in the neutron transport literature in an article by Reed and Hill [14]. One yields a discontinuous approximation, the other a continuous one.

In the discontinuous Galerkin method, the approximate solution u_h, of degree $n \geq 0$, is taken to be an interpolant (perhaps discontinuous) of the given initial data g on $\Gamma_{\text{in}}(\Omega)$. The triangles are then processed in an explicit order, with the following

*Received by the editors October 15, 1990; accepted for publication (in revised form) June 25, 1991. This research was supported by National Science Foundation grant DMS-8902120 (RSF).
†Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903.
‡Department of Computer Science, Rutgers University, New Brunswick, New Jersey 08903.

inner product relations enforced in individual triangles T:

$$(1.1)\qquad (\boldsymbol{\alpha}\cdot\nabla u_h, v_h) - \int_{\Gamma_{\text{in}}(T)} (u_h^+ - u_h^-) v_h \boldsymbol{\alpha}\cdot n d\tau = (f, v_h) \quad \text{for all } v_h \in \mathbf{P}_n(T).$$

Here $\mathbf{P}_k(T)$ denotes the space of polynomials of degree $\leq k$ over T, and u_h^- (u_h^+) denotes the upstream (downstream) limit of u_h on interelement boundaries, the latter parameterized by arclength τ. The continuous method is similar, except that $n \geq 2$, the interpolant of g on $\Gamma_{\text{in}}(\Omega)$ must be continuous, and the inner product relations (1.1) are changed to

$$(1.2)\qquad (\boldsymbol{\alpha}\cdot\nabla u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in \mathbf{P}_{n-l}(T),$$

where l denotes the number of inflow sides that T has. The test spaces are thus different for type I and type II triangles, reflecting the different number of degrees of freedom remaining for u_h in the two types of triangles. An advantage of this method is the reduced number of unknowns to be solved for in the approximate solution. In the vicinity of a discontinuity, however, the additional degrees of freedom of the discontinuous Galerkin method could be potentially useful. We remark that the continuous method is also applicable over rectangular meshes (see [21]).

The discontinuous Galerkin method was first analyzed by Lesaint and Raviart [11], and subsequently by Johnson and Pitkäranta [9], who established the error estimates

$$(1.3)\qquad \|u_h - u\|_\Omega + |u_h - u|_{\Gamma_{\text{out}}(\Omega)} \leq Ch^{n+1/2}\|u\|_{n+1,\Omega},$$

$$(1.4)\qquad \|\boldsymbol{\alpha}\cdot\nabla(u_h - u)\|_\Omega \leq Ch^n\|u\|_{n+1,\Omega}.$$

Here and throughout the paper, C denotes a generic constant, independent of u and h, and $\|\cdot\|_\Omega$, $|\cdot|_{\Gamma_{\text{out}}(\Omega)}$, and $\|\cdot\|_{k,\Omega}$ denote the norms on $L^2(\Omega), L^2(\Gamma_{\text{out}}(\Omega))$, and $H^k(\Omega)$, respectively. In [8], a methodology is given which can be used to extend (1.3) and (1.4) to a corresponding set of local error estimates, showing that crosswind propagation of the numerical solution is limited to a distance $O(\sqrt{h}\log\frac{1}{h})$. The continuous finite element method (1.2) was analyzed in [3], where the following global bounds were derived:

$$(1.5)\qquad \|u_h - u\|_\Omega + |u_h - u|_{\Gamma_{\text{out}}(\Omega)} \leq Ch^{n+1/4}\|u\|_{n+1,\Omega},$$

$$(1.6)\qquad \|\nabla(u_h - u)\|_\Omega + |(u_h - u)_\tau|_{\Gamma_{\text{out}}(\Omega)} \leq Ch^{n-1/2}\|u\|_{n+1,\Omega},$$

$$(1.7)\qquad \|\boldsymbol{\alpha}\cdot\nabla(u_h - u)\|_\Omega \leq Ch^n\|u\|_{n+1,\Omega}.$$

In deriving these estimates, it was assumed that $\boldsymbol{\alpha}\cdot\boldsymbol{n}$ is uniformly bounded away from zero and that u_h can be computed layer by layer (a precise characterization is given in §3) in $O(h^{-1})$ steps. These assumptions are retained here. (We remark that although one can construct a mesh which violates the latter assumption, we view such a situation as anomalous.) Local versions of (1.5), (1.6), and (1.7), indicating the same $O(\sqrt{h}\log\frac{1}{h})$ crosswind spread as for the discontinuous method, will be obtained in this paper. The existence of such local error estimates is an important attribute of a numerical method for hyperbolic equations, indicating a correspondence in domain of dependence properties of the discrete and continuous problems.

For a convection-diffusion equation dominated by convection, the standard Galerkin method typically exhibits instabilities (see, for example [4]), and an appropriate remedy is to instead use a finite element discretization geared to the hyperbolic limit. Both the continuous and discontinuous Galerkin methods can be extended to convection-diffusion equations in ways that preserve their explicitness and yield similar global error estimates, provided the diffusion term is of strength no greater than $O(h)$, where h is the mesh size. These extensions are given for the continuous method in [15], [16] and for the discontinuous method in [17]. For the continuous method, the simplest option is to include the additional diffusion term in the inner product relations (1.2), otherwise implementing the method exactly as in the diffusionless limit. Alternatively, for either method, we may treat diffusion in a way analogous to the discontinuous Galerkin discretization of the convection term in (1.1), resulting in an integral over $\Gamma_{\text{in}}(T)$ involving the diffusion term. These schemes will not use any boundary data given for u on $\Gamma_{\text{out}}(\Omega)$ since they are explicit. The resulting outflow boundary layer, whose width is of the order of the diffusion coefficient [20], will therefore not be present in u_h.

Another finite element method that can be applied to convection dominated convection-diffusion problems is the streamline diffusion method, developed by Hughes and Brooks [5]. This method is like the standard Galerkin method except that the test functions are augmented by a multiple of their streamline derivatives. The resulting discrete system is implicit, and has the same connectivity as the standard Galerkin discretization. Johnson et al. [6], [7], [8] have shown that for nth degree polynomials, the resulting finite element approximation u_h has $O(h^{n+1/2})$ accuracy for $u \in H^{n+1}(\Omega)$. They have also shown [8] that such an estimate holds locally in regions of smoothness, with crosswind numerical spread limited to a layer of width $O(\sqrt{h}\log\frac{1}{h})$. For a diffusion coefficient of size $< O(h)$, improved estimates of crosswind spread were obtained in [10] for the case of linear approximation, in part by adding an artificial crosswind diffusion term. Pointwise error estimates were also given in [10], with further refinements in [12]. The survey paper [4] contains additional references on the streamline diffusion method.

Our purpose in this paper is to establish local error estimates for the continuous method of Reed and Hill via an approach that is also applicable to the discontinuous Galerkin method, and which facilitates the elucidation of basic interrelationships between the two methods. In §2, we state our basic assumptions and notation, and derive some preliminary results pertaining to the weighting function to be used later in obtaining local error estimates. In §3 we give unified, parallel analyses of the discontinuous and continuous methods (1.1) and (1.2), extending work begun in [18]. We use as independent variables the characteristic and crosswind variables s and t, respectively, and apply rather simple test functions depending only on the crosswind variable t. This will lead directly to stability results, expressed in terms of this variable, on triangle boundaries. It will be seen that an important feature of both methods is the role of L^2 projections across the boundaries of type II triangles. For the continuous method, this analysis is a considerable simplification over that given in [3].

In §§4 and 5, we derive local error estimates for the continuous method (1.2). Since the method is explicit, numerical effects cannot propagate upwind; thus only the crosswind direction needs to be dealt with. To obtain local error estimates, we follow the basic approach in [8]. However, our task is facilitated by the crosswind variable analysis of §3, which lends itself naturally to the introduction of a t-dependent weighting function. For a strip $D \subset \Omega$ contained between two characteristics, and a larger subset D^+ of Ω consisting of points lying a distance no greater than $O(\sqrt{h}\log(1/h))$ from D, we derive the following "local" analogue of (1.5):

$$(1.8)\quad \|u_h - u\|_D + |u_h - u|_{\Gamma_{\text{out}}(D)} \le Ch^{n+1/4}\left(\|u\|_{n+1,D_h^+} + \|u\|_\Omega + |u|_{\Gamma_{\text{in}}(\Omega)} + \|f\|_\Omega\right),$$

and similar analogues of (1.6) and (1.7). Here $D_h^+ \equiv \{T \in \Omega | T \cap D^+ \neq \emptyset\}$. These localization results generalize (1.5)–(1.7).

In §6, we show how the analysis can be extended to convection-diffusion equations of the form

$$\boldsymbol{\alpha} \cdot \nabla u - (au_{xx} + bu_{xy} + cu_{yy}) = f \quad \text{in } \Omega,$$

with appropriate boundary conditions specified for u on the boundary of Ω. We assume the diffusion coefficients a, b, c are of magnitude no greater than $O(h)$, and that $\boldsymbol{\alpha}$, as before, is a unit vector. In addition, we require that the diffusion term, when expressed in terms of s and t, have a dominant nonnegative u_{tt} coefficient. Our framework includes as special cases the elliptic and parabolic equations

$$\boldsymbol{\alpha} \cdot \nabla u - \epsilon \Delta u = f, \qquad \alpha_2 u_t + \alpha_1 u_x - \epsilon u_{xx} = f,$$

where $\epsilon = O(h)$. We shall obtain local error estimates for the method

$$(\boldsymbol{\alpha} \cdot \nabla u_h - (a(u_h)_{xx} + b(u_h)_{xy} + c(u_h)_{yy}), v_h) = (f, v_h) \quad \text{for all } v_h \in \mathbf{P}_{n-l}(T).$$

The result will be a pair of local error estimates, of the type (1.8), analogous to (1.6) and (1.7). Of perhaps greater significance, this section serves as an illustration of how the crosswind variable analysis of §3 can be applied to problems with diffusion. We believe it is also applicable to the other extensions of the continuous and discontinuous methods alluded to previously.

2. Notation and preliminary results. For a domain Ω with boundary Γ, we have defined $\Gamma_{\text{in}}(\Omega)$, the inflow portion of Γ, as $\{(x, y) \in \Gamma : \boldsymbol{\alpha} \cdot \boldsymbol{n} < 0\}$. We further denote by $\Gamma_{\text{out}}(\Omega)$, the outflow portion of Γ corresponding to points on Γ at which $\boldsymbol{\alpha} \cdot \boldsymbol{n} > 0$. Most of the analysis of the paper will be done using the variables $s = \alpha_1 x + \alpha_2 y$, along the characteristic direction $\boldsymbol{\alpha}$, and $t = \alpha_2 x - \alpha_1 y$, in the direction $\boldsymbol{\beta} = (\alpha_2, -\alpha_1)$ perpendicular to $\boldsymbol{\alpha}$. Note that

$$u_s = \boldsymbol{\alpha} \cdot \nabla u \quad \text{and} \quad u_t = \boldsymbol{\beta} \cdot \nabla u.$$

In this notation, a generic triangle T may be described by

$$T = \{(s, t) : s \in [s_{\text{in}}(t), s_{\text{out}}(t)], t \in [t_0, t_1]\}$$

(see Fig. 1.) and $\Gamma_{\text{in}}(T)$ and $\Gamma_{\text{out}}(T)$ can be parameterized by $t \in [t_0, t_1]$.

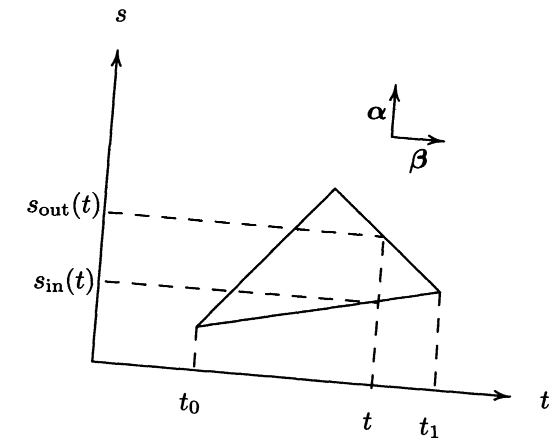

FIG. 1

To simplify notation, we denote the interval $[t_0, t_1]$ by L, set $h = t_1 - t_0$, and define for an arbitrary function v, $v_{\text{in}}(t) = v|_{\Gamma_{\text{in}}}$ and $v_{\text{out}}(t) = v|_{\Gamma_{\text{out}}}$. For a function $v(t)$ defined on $[t_0, t_1]$, we define an extension Ev to T by $Ev(s,t) = v(t)$, i.e., Ev extends v as a constant in the characteristic direction $\boldsymbol{\alpha}$. We shall frequently make use of the facts

$$(Ev)_{\text{out}}(t) = (Ev)_{\text{in}}(t) = v(t), \tag{2.1}$$

$$(Ev)_t = Ev', \tag{2.2}$$

and for any function $z(t)$,

$$E[z(t)v(t)] = z(t)E[v(t)]. \tag{2.3}$$

We assume that T satisfies a minimum angle condition independent of h. The notation $(\cdot,\cdot)$ is used to denote the L^2 inner product over T and $\|\cdot\|_k$ is used to denote the norm in the Sobolev space $H^k(T)$, with k omitted when it has value zero. The L^2-norm over $L = [t_0, t_1]$ is denoted by $|\cdot|$.

In our analysis, we shall make use of several interpolants and projections. Let $\mathbf{P}_n(D)$ denote the space of polynomials of degree $\leq n$ over the set D. We denote by P_n the L^2 projection over T into $\mathbf{P}_n(T)$ and Q_n the L^2 projection on L into $\mathbf{P}_n(L)$. For $u \in C^0(\Omega)$, we define an interpolant $u_I \in S_h^n \equiv \{v_h \in C^0(\Omega) : v_h|_T \in \mathbf{P}_n(T)\}$ as follows.

(i) $u_I(a_i) = u(a_i)$ for all triangle vertices a_i;

(ii) $\int_{\Gamma_i}(u_I - u)\tau^l\, d\tau = 0,\ l = 0, 1, \cdots, n-2$ for all triangle sides Γ_i;

(iii) $\int_T (u_I - u)q\, dx\, dy = 0$ for all $q \in \mathbf{P}_{n-3}(T)$ and all triangles T.

It is straightforward to show (for example, using the techniques in [1, Chap. 3]), that u_I has the following approximation properties.

$$\|u - u_I\|_j \leq Ch^{n+1-j}\|u\|_{n+1}, \qquad j = 0, 1, \cdots, n, \tag{2.4}$$

and

$$|u - u_I|_{j,\Gamma(T)} \leq Ch^{n+1/2-j}\|u\|_{n+1}, \qquad j = 0, 1, \cdots, n. \tag{2.5}$$

It will also be convenient to have a one-dimensional version of this interpolant defined on the interval $L = [t_0, t_1]$. Let $Q_n^*\phi \in \mathbf{P}_n(L)$ be defined by

$$(Q_n^*\phi - \phi)(t_0) = (Q_n^*\phi - \phi)(t_1) = 0,$$
$$\int_{t_0}^{t_1} (Q_n^*\phi - \phi)(t)q(t)\,dt = 0 \quad \text{for all } q \in \mathbf{P}_{n-2}(L).$$

Choosing $q = -r''(t)$ for $r \in \mathbf{P}_n(L)$ and integrating by parts, it follows that

$$\int_{t_0}^{t_1} (Q_n^*\phi - \phi)'(t)r'(t)\,dt = 0,$$

and so $[Q_n^*\phi]' = Q_{n-1}\phi'$.

In deriving local error estimates for our approximation schemes, it will also be useful to have the following results. Consider a quasi-uniform partition of $I = [a, b]$, comprised of subintervals I_h of width h. Letting t_0 and t_1 denote the endpoints of the subinterval I_h, define for a positive function ψ, the weighted L^2 inner product on a typical subinterval I_h by

$$\langle f, g\rangle_\psi = \int_{t_0}^{t_1} \psi f(t)g(t)\,dt$$

and the ψ weighted norm on I_h by

$$|f|_\psi = \left[\int_{t_0}^{t_1} \psi f(t)^2\,dt\right]^{1/2}.$$

We shall assume that ψ satisfies the following hypotheses:

$$\frac{\max_{I_h}\psi}{\min_{I_h}\psi} \le C, \tag{2.6}$$

$$\max_{I_h}|\psi'| \le Ch^{-1/2}\max_{I_h}\psi, \tag{2.7}$$

where C denotes a generic constant independent of h. (We note that (2.6) follows from (2.7) for h sufficiently small.) The two choices of ψ that we shall use, for which the above are easily verified, are $\psi = 1$ and for fixed $t^* \in I$,

$$\psi(t, t^*) \equiv \frac{1}{2\sqrt{h}}e^{-|t-t^*|/\sqrt{h}}.$$

Using (2.6) and (2.7), we find that ψ must satisfy the following additional properties.

LEMMA 2.1. *Suppose ψ satisfies* (2.6) *and* (2.7). *If v and w are L^2 orthogonal on I_h, then*

$$|\langle v, w\rangle_\psi| \le Ch^{1/2}|v|_\psi|w|_\psi. \tag{2.8}$$

If $|f_1| + \|f_2\| \le C(|g_1| + \|g_2\|)$, *then*

$$|f_1|_\psi + \|f_2\|_\psi \le C(|g_1|_\psi + \|g_2\|_\psi). \tag{2.9}$$

Proof. To prove (2.8), we let $\bar{t} \in I_h$. Then since $\psi_0 \equiv \psi(\bar{t}) \in \mathbf{P}_0(I_h)$, we get, using (2.6) and (2.7), that

$$\begin{aligned}|\langle v, w\rangle_\psi| = \left|\int_{t_0}^{t_1} \psi v w\, dt\right| &\le \left|\int_{t_0}^{t_1} \psi_0 v w\, dt\right| + \left|\int_{t_0}^{t_1} (\psi - \psi_0) v w\, dt\right| \\ &\le Ch \max_{I_h} |\psi'| \int_{t_0}^{t_1} |v||w|\, dt \\ &\le Ch^{1/2} \frac{\max_{I_h} \psi}{\min_{I_h} \psi} \int_{t_0}^{t_1} \psi |v||w|\, dt \\ &\le Ch^{1/2} |v|_\psi |w|_\psi.\end{aligned}$$

Inequality (2.9) follows from (2.6) by observing that if $|f_1| + \|f_2\| \le C(|g_1| + \|g_2\|)$, then

$$\begin{aligned}|f_1|_\psi^2 + \|f_2\|_\psi^2 &\le \max_{[t_0,t_1]} \psi(|f_1|^2 + \|f_2\|^2) \le C \max_{[t_0,t_1]} \psi(|g_1|^2 + \|g_2\|^2) \\ &\le C \frac{\max_{[t_0,t_1]} \psi}{\min_{[t_0,t_1]} \psi} (|g_1|_\psi^2 + \|g_2\|_\psi^2) \le C(|g_1|_\psi^2 + \|g_2\|_\psi^2).\end{aligned}$$ □

3. An analysis of the basic methods. Our aim in this section is to explore the relationships between the continuous and discontinuous Galerkin methods and to show how a simple and parallel analysis of these methods can be given by using the characteristic and crosswind variables. These results are obtained for the model problem $\boldsymbol{\alpha} \cdot \nabla u = f$ and are also intended to provide some basic motivation for the more general, but more technical results to follow in the remainder of the paper.

We first consider the continuous method for $\boldsymbol{\alpha} \cdot \nabla u = f$:

$$((u_h)_s, v_h) = (f, v_h), \qquad v_h \in \mathbf{P}_{n-l}(T). \tag{3.1}$$

For a type I triangle, it follows from the fact that $(u_h)_s \in \mathbf{P}_{n-1}(T)$ that

$$(u_h)_s = P_{n-1} f.$$

Thus

$$u_h(s,t) = u_{h,\mathrm{in}}(t) + \int_{s_{\mathrm{in}}(t)}^{s} P_{n-1} f\, ds \tag{3.2}$$

for a type I triangle.

To characterize u_h locally on a type II triangle, it is convenient to define a quantity $U(s,t)$, $(s,t) \in T$ as the solution of the equation $U_s = f$ with initial data $U_{\mathrm{in}} = u_{h,\mathrm{in}}$ on $\Gamma_{\mathrm{in}}(T)$. Observe that for $v_h \in \mathbf{P}_{n-2}(T)$,

$$0 = ((u_h - U)_s, v_h) = -(u_h - U, (v_h)_s) + \int_{t_0}^{t_1} (u_{h,\mathrm{out}} - U_{\mathrm{out}}) v_h dt.$$

We first take $v_h = -Ew'(t)$, where $w \in \mathbf{P}_{n-1}(L)$, and we integrate by parts to obtain

$$0 = -\int_{t_0}^{t_1} (u_{h,\mathrm{out}} - U_{\mathrm{out}}) w'(t) dt = \int_{t_0}^{t_1} (u'_{h,\mathrm{out}} - U'_{\mathrm{out}}) w(t) dt,$$

where the fact that $u_{h,\text{out}} = U_{\text{out}}$ at $t = t_0$ and $t = t_1$ has been used. Since $u'_{h,\text{out}} \in \mathbf{P}_{n-1}(L)$, we infer that

$$u'_{h,\text{out}} = Q_{n-1}U'_{\text{out}} \tag{3.3}$$

for a type II triangle. We next take $v_h \in \mathbf{P}_{n-2}(T)$ to vanish on $\Gamma_{\text{out}}(T)$; $(v_h)_s$ can then be an arbitrary member of $\mathbf{P}_{n-3}(T)$. We conclude that

$$P_{n-3}u_h = P_{n-3}U. \tag{3.4}$$

Equation (3.3) and the given data $u_h = u_{h,\text{in}}$ on $\Gamma_{\text{in}}(T)$ determine u_h on the boundary of a type II triangle T. The moment conditions (3.4) then complete the specification of a unique u_h in the interior.

We now obtain a similar characterization for the discontinuous method. For our model problem, u_h satisfies

$$((u_h)_s, v_h) + \int_{t_0}^{t_1} \left(u^+_{h,\text{in}} - u^-_{h,\text{in}}\right) v_h dt = (f, v_h), \qquad v_h \in \mathbf{P}_n(T).$$

We first consider the case of a type I triangle, where the situation is somewhat more complicated than for the continuous method. Choosing $v_h = [s(t) - s_{\text{in}}(t)]q$ with $q \in \mathbf{P}_{n-1}(T)$, we get, from the fact that $s(t) - s_{\text{in}}(t) = 0$ on $\Gamma_{\text{in}}(T)$ for a type I triangle,

$$([s(t) - s_{\text{in}}(t)](u_h)_s, q) = ([s(t) - s_{\text{in}}(t)]f, q).$$

Since $(u_h)_s \in \mathbf{P}_{n-1}(T)$ and $s(t) - s_{\text{in}}(t) \geq 0$ in T, we get that

$$(u_h)_s = R_{n-1}f,$$

where $R_{n-1}f$ denotes the projection of f into $\mathbf{P}_{n-1}(T)$ with respect to the weighted L^2 inner product $[f, q] = ([s(t) - s_{\text{in}}(t)]f, q)$. Using this result and choosing $v_h = w(t)$, we get

$$\begin{aligned}\int_{t_0}^{t_1} (u^+_{h,\text{in}} - u^-_{h,\text{in}})w(t)\, dt &= (f - R_{n-1}f, w) \\ &= \int_{t_0}^{t_1} \left[\int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} (f - R_{n-1}f)\, ds\right] w(t)\, dt.\end{aligned}$$

Since $u^+_{h,\text{in}} - u^-_{h,\text{in}} \in \mathbf{P}_n(L)$ on a type I triangle, we conclude that

$$u^+_{h,\text{in}} - u^-_{h,\text{in}} = Q_n \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} (f - R_{n-1}f)\, ds.$$

Now $u_h(s,t) = u^-_{h,\text{in}} + [u^+_{h,\text{in}} - u^-_{h,\text{in}}] + \int_{s_{\text{in}}(t)}^{s} (u_h)_s\, ds$. Thus

$$u_h(s,t) = u^-_{h,\text{in}} + Q_n \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} (f - R_{n-1}f)\, ds + \int_{s_{\text{in}}(t)}^{s} R_{n-1}f\, ds \tag{3.5}$$

for a type I triangle.

For a type II triangle, we define U as before with $u_{h,\text{in}}^-$ as the given inflow data. For $v_h \in \mathbf{P}_n(T)$, we have

$$0 = ((u_h - U)_s, v_h) + \int_{t_0}^{t_1} (u_{h,\text{in}}^+ - U_{\text{in}}) v_h dt$$
$$= -(u_h - U, (v_h)_s) + \int_{t_0}^{t_1} (u_{h,\text{out}}^- - U_{\text{out}}) v_h dt$$

after integrating by parts. Taking $v_h = Ew$, where $w \in \mathbf{P}_n(L)$, we conclude that

$$u_{h,\text{out}}^- = Q_n U_{\text{out}}. \tag{3.6}$$

Moreover, for arbitrary $w_h \in \mathbf{P}_{n-1}(T)$, we may take $v_h \in \mathbf{P}_n(T)$ to satisfy $(v_h)_s = w_h$, $v_h = 0$ on $\Gamma_{\text{out}}(T)$. Thus

$$P_{n-1} u_h = P_{n-1} U. \tag{3.7}$$

It is easy to check that u_h is completely defined in a type II triangle by (3.6)–(3.7). Note the close correspondence of (3.2)–(3.4) with (3.5)–(3.7). For the continuous method, (3.2) and (3.3) can be used to derive a global stability result for u_h' on interelement boundaries, while for the discontinuous method, (3.5) and (3.6) lead to global stability of u_h^- on interelement boundaries.

We now give an error analysis for the two methods using test functions that depend only on the crosswind variable t. This will illustrate the basic idea of the analysis to follow in this paper. We first consider the continuous method. The error $e \equiv u_h - u$ satisfies $(e_s, v_h) = 0$, $v_h \in \mathbf{P}_{n-2}(T)$, for a triangle of either type. For $v_h = -Ew'(t)$, where $w \in \mathbf{P}_{n-1}(L)$, we get

$$0 = -\int_{t_0}^{t_1} (e_{\text{out}} - e_{\text{in}}) w'(t) dt = \int_{t_0}^{t_1} (e_{\text{out}}' - e_{\text{in}}') w(t) dt$$

after integrating by parts. We choose $w = Q_{n-1}(e_{\text{out}}' + e_{\text{in}}')$ to obtain

$$|Q_{n-1} e_{\text{out}}'|^2 = |Q_{n-1} e_{\text{in}}'|^2.$$

This is equivalent to

$$|e_{\text{out}}'|^2 + |(I - Q_{n-1}) e_{\text{in}}'|^2 = |e_{\text{in}}'|^2 + |(I - Q_{n-1}) e_{\text{out}}'|^2.$$

An estimate of the error will follow upon estimating $|(I - Q_{n-1}) e_{\text{out}}'|^2$ and summing over all triangles. For a type II triangle T, $(I - Q_{n-1}) u_{h,\text{out}}' = 0$ since $u_{h,\text{out}}' \in \mathbf{P}_{n-1}(L)$, so

$$|(I - Q_{n-1}) e_{\text{out}}'| = |(I - Q_{n-1}) u_{\text{out}}'| \le C h^{n-1/2} \|u\|_{n+1,T}.$$

For a type I triangle, from (3.2) and the corresponding relation for u,

$$u(s,t) = u_{\text{in}}(t) + \int_{s_{\text{in}}(t)}^{s} f ds, \tag{3.8}$$

we obtain

$$u_{h,\text{out}}' - u_{\text{out}}' = u_{h,\text{in}}' - u_{\text{in}}' - \frac{d}{dt} \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} (I - P_{n-1}) f ds.$$

Hence, since $u'_{h,\text{in}} \in \mathbf{P}_{n-1}(L)$,

$$|(I-Q_{n-1})e'_{\text{out}}| \leq |(I-Q_{n-1})u'_{\text{in}}| + \left|\frac{d}{dt}\int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} (I-P_{n-1})f\,ds\right| \leq Ch^{n-1/2}\|u\|_{n+1,T}$$

via standard estimates. Thus for a triangle of either type,

$$|e'_{\text{out}}|^2 \leq |e'_{\text{in}}|^2 + Ch^{2n-1}\|u\|^2_{n+1,T}. \tag{3.9}$$

In formulating a global error estimate, it is convenient to think of u_h as evolving in layers S_i, defined by:

$$S_0 = \emptyset,$$
$$S_i = \{T \in \Omega : \Gamma_{\text{in}}(T) \subset \Gamma_{\text{in}}(\Omega - \cup_{j<i}S_j)\}, \qquad i = 1, 2, \cdots.$$

Note that u_h can be developed in parallel in the triangles within a layer. Thus we have, in analogy with (3.9), that

$$|e'_{\text{out}}|^2_{F_i} \leq |e'_{\text{out}}|^2_{F_{i-1}} + Ch^{2n-1}\|u\|^2_{S_i,n+1},$$

where $F_0 = \Gamma_{\text{in}}(\Omega)$ and F_i is the "front line" to which u_h has advanced after it has been computed in $\Omega_i \equiv \cup_{j\leq i}S_j$. Iterating the above inequality, we obtain the global error estimate

$$|e'_{\text{out}}|^2_{F_i} \leq |e'_{\text{in}}|^2_{F_0} + Ch^{2n-1}\|u\|^2_{\Omega_i,n+1} \leq Ch^{2n-1}\|u\|^2_{\Omega_i,n+1}. \tag{3.10}$$

The discontinuous method can be treated in an analogous fashion. The error satisfies

$$0 = (e_s, v_h) - \int_{t_0}^{t_1} (e^+_{\text{in}} - e^-_{\text{in}})v_h\,dt = -(e, (v_h)_s) + \int_{t_0}^{t_1} (e^-_{\text{out}} - e^-_{\text{in}})v_h\,dt.$$

For $v_h = Ew$, $w \in \mathbf{P}_n(L)$, this becomes

$$\int_{t_0}^{t_1} (e^-_{\text{out}} - e^-_{\text{in}})w\,dt = 0.$$

The choice $w = Q_n(e^-_{\text{out}} + e^-_{\text{in}})$ thus yields

$$|Q_n e^-_{\text{out}}|^2 = |Q_n e^-_{\text{in}}|^2.$$

Equivalently,

$$|e^-_{\text{out}}|^2 + |(I - Q_n)e^-_{\text{in}}|^2 = |e^-_{\text{in}}|^2 + |(I - Q_n)e^-_{\text{out}}|^2.$$

For a type II triangle, $u^-_{h,\text{out}} \in \mathbf{P}_n(L)$, so

$$|(I - Q_n)e^-_{\text{out}}| = |(I - Q_n)u^-_{\text{out}}| \leq Ch^{n+1/2}\|u\|_{n+1,T}.$$

This bound also applies to $|(I - Q_n)e^-_{\text{out}}|$ on a type I triangle, as can be seen by using (3.5) and (3.8). We therefore conclude that

$$|e'_{\text{out}}|^2 \leq |e^-_{\text{in}}|^2 + Ch^{2n+1}\|u\|^2_{n+1,T}$$

for a triangle of either type. The corresponding global error estimate is

$$|e^-_{\text{out}}|^2_{F_i} \leq |e^-_{\text{in}}|^2_{F_0} + Ch^{2n+1}\|u\|^2_{\Omega_i,n+1} \leq Ch^{2n+1}\|u\|^2_{\Omega_i,n+1}. \tag{3.11}$$

The basic global error estimates (3.10) and (3.11) for the two methods may be used to derive interior estimates over Ω_i. In the next section, we shall obtain such estimates for the continuous method in a more general weighted norm setting.

4. Stability results for a single triangle. We begin our local analysis of the continuous finite element method by deriving weighted norm stability results over a single triangle for the simple hyperbolic problem $\boldsymbol{\alpha}\cdot\nabla u = f$. These results will provide the basic tools for the global stability and error analysis for this problem and also for the convection-diffusion problem in §6. As a consequence of the fact that u_h is well defined, we have the following.

LEMMA 4.1. *If* u_h *satisfies* (3.1), *then*

$$\|u_h\|_\psi \le C(\sqrt{h}|u_{h,\text{in}}|_\psi + h\|P_{n-\iota}f\|_\psi), \tag{4.1}$$

$$\|\nabla u_h\|_\psi \le C(\sqrt{h}|u'_{h,\text{in}}|_\psi + \|P_{n-\iota}f\|_\psi). \tag{4.2}$$

Proof. We will prove these bounds for $\psi = 1$. They will then follow for general ψ satisfying (2.6) and (2.7) by (2.9).

To prove (4.1), we will show that $|u_{h,\text{in}}| = \|P_{n-\iota}f\| = 0$ implies $u_h \equiv 0$. The desired result then follows by scaling, in view of the fact that $P_{n-\iota}f = P_{n-\iota}(u_h)_s$. For a type I triangle, the representation (3.2) leads directly to this conclusion. For a type II triangle, it is easy to see that the characterization (3.3), (3.4) of u_h remains valid for U defined by $U_s = P_{n-2}f$, $U_{\text{in}} = u_{h,\text{in}}$. Moreover, if $|u_{h,\text{in}}| = \|P_{n-2}f\| = 0$, then $U \equiv 0$. Thus by (3.3) and (3.4), $|u'_{h,\text{out}}| = \|P_{n-3}u_h\| = 0$. From $|u_{h,\text{in}}| = |u'_{h,\text{out}}| = 0$, we deduce that u_h vanishes on $\Gamma(T)$ and write

$$u_h = \lambda_1\lambda_2\lambda_3 w_h, \qquad w_h \in \mathbf{P}_{n-3}(T),$$

where $\lambda_i(s,t) \in \mathbf{P}_1(T)$ is the distance from (s,t) to side Γ_i of T, $i = 1,2,3$. We take the inner product of u_h with w_h and use $\|P_{n-3}u_h\| = 0$ and the positivity of $\lambda_1\lambda_2\lambda_3$ inside T to conclude that $w_h \equiv 0$, implying $u_h \equiv 0$. This establishes (4.1) for a type II triangle.

To verify (4.2), we apply (4.1) to $u_h - EQ_0u_{h,\text{in}}$ to get

$$\|u_h - EQ_0u_{h,\text{in}}\| \le C\left(\sqrt{h}\,|u_h - EQ_0u_{h,\text{in}}| + h\|P_{n-\iota}f\|\right),$$

and use this in combination with the bounds

$$\|\nabla u_h\| = \|\nabla(u_h - EQ_0u_{h,\text{in}})\| \le Ch^{-1}\|u_h - EQ_0u_{h,\text{in}}\|,$$
$$|u_h - EQ_0u_{h,\text{in}}| \le Ch\,|u'_{h,\text{in}}|. \qquad \square$$

Using Lemma 4.1, we are able to prove the following.

LEMMA 4.2. *If* u_h *satisfies* (3.1), *then*

$$\begin{aligned}&\|(u_h)_s\|_\psi + \|(u_h)_t - EQ_{n-1}u'_{h,\text{in}}\|_\psi + \sqrt{h}|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi \\ &\quad\le C(\sqrt{h}|(I - Q_{n-1})u'_{h,\text{in}}|_\psi + \|P_{n-\iota}f\|_\psi).\end{aligned} \tag{4.3}$$

Proof. By (2.9), it is enough to prove the result for $\psi = 1$. First note that

$$|u'_{h,\text{out}}| \le |(u_h)_s|_{\Gamma_{\text{out}}(T)} + |(u_h)_t|_{\Gamma_{\text{out}}(T)} \le Ch^{-1/2}\|\nabla u_h\|.$$

Combining this result with (4.2) (with $\psi = 1$), we get

$$\|\nabla u_h\| + \sqrt{h}\,|u'_{h,\text{out}}| \le C(\sqrt{h}\,|u'_{h,\text{in}}| + \|P_{n-\iota}f\|). \tag{4.4}$$

Next observe that $w_h \equiv u_h - EQ_n^* u_{h,\text{in}} \in \mathbf{P}_n(T)$ and satisfies $(w_h)_s = (u_h)_s$. Applying (4.4), with u_h replaced by w_h, we obtain

$$\|(u_h)_s\| + \|[u_h - EQ_n^* u_{h,\text{in}}]_t\| + \sqrt{h}\,|u'_{h,\text{out}} - (Q_n^* u_{h,\text{in}})'| \\ \leq C(\sqrt{h}\,|u'_{h,\text{in}} - (Q_n^* u_{h,\text{in}})'| + \|P_{n-l} f\|).$$

Since

$$(Q_n^* u_{h,\text{in}})' = Q_{n-1} u'_{h,\text{in}}$$

and

$$[EQ_n^* u_{h,\text{in}}]_t = E(Q_n^* u_{h,\text{in}})' = EQ_{n-1} u'_{h,\text{in}},$$

we have that

$$\|(u_h)_s\| + \|(u_h)_t - EQ_{n-1} u'_{h,\text{in}}\| + \sqrt{h}\,|u'_{h,\text{out}} - Q_{n-1} u'_{h,\text{in}}| \\ \leq C(\sqrt{h}\,|(I - Q_{n-1}) u'_{h,\text{in}}| + \|P_{n-l} f\|).$$

The result follows by writing

$$u'_{h,\text{out}} - Q_{n-1} u'_{h,\text{in}} = u'_{h,\text{out}} - u'_{h,\text{in}} + (I - Q_{n-1}) u'_{h,\text{in}}$$

and applying the triangle inequality. $\square$

Using these basic estimates, we now derive stability results for type I and type II triangles, which have the property that they can be iterated to prove global stability for the method. We shall make extensive use of the test function $-[EMu_h]_t \in P_{n-2}(T)$, where

$$Mu_h = Q_{n-1} \psi Q_{n-1}(u'_{h,\text{out}} + u'_{h,\text{in}}).$$

The following lemma indicates the effect of this test function on the two terms of (3.1).

LEMMA 4.3. *For a triangle of type* I,

$$-((u_h)_s, [EMu_h]_t) \geq (1 - Ch)|u'_{h,\text{out}}|_\psi^2 - |u'_{h,\text{in}}|_\psi^2 - C|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi^2. \tag{4.5}$$

For a triangle of type II,

$$-((u_h)_s, [EMu_h]_t) \geq |u'_{h,\text{out}}|_\psi^2 - (1 + Ch)|u'_{h,\text{in}}|_\psi^2 + \tfrac{1}{2}|(I - Q_{n-1}) u'_{h,\text{in}}|_\psi^2. \tag{4.6}$$

For a triangle of either type,

$$|(f, [EMu_h]_t)| \leq C(h^{-2}\|P_{n-2} f\|_\psi^2 + \|\nabla u_h\|_\psi^2). \tag{4.7}$$

Proof. Integrating by parts and using the fact that $u^h_{\text{out}} = u^h_{\text{in}}$ at $t = t_0$ and $t = t_1$, we obtain

$$-((u_h)_s, [EMu_h]_t) = -\int_{\Gamma(T)} u_h [EMu_h]_t \boldsymbol{\alpha} \cdot \boldsymbol{n}\, d\tau = \int_{t_0}^{t_1} (u'_{h,\text{out}} - u'_{h,\text{in}}) Mu_h dt \\ = \int_{t_0}^{t_1} \psi[(Q_{n-1} u'_{h,\text{out}})^2 - (Q_{n-1} u'_{h,\text{in}})^2]\, dt \\ = |Q_{n-1} u'_{h,\text{out}}|_\psi^2 - |Q_{n-1} u'_{h,\text{in}}|_\psi^2.$$

On a type II triangle, we have $Q_{n-1}u'_{h,\text{out}} = u'_{h,\text{out}}$, and hence using (2.8) and the arithmetic-geometric mean inequality, we get

$$\begin{aligned}
|Q_{n-1}u'_{h,\text{in}}|^2_\psi &= |u'_{h,\text{in}}|^2_\psi - |(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi - 2\langle Q_{n-1}u'_{h,\text{in}}, (I-Q_{n-1})u'_{h,\text{in}}\rangle_\psi \\
&\le |u'_{h,\text{in}}|^2_\psi - |(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi + Ch^{1/2}|Q_{n-1}u'_{h,\text{in}}|_\psi |(I-Q_{n-1})u'_{h,\text{in}}|_\psi \\
&\le |u'_{h,\text{in}}|^2_\psi - \tfrac{1}{2}|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi + Ch|Q_{n-1}u'_{h,\text{in}}|^2_\psi \\
&\le (1+Ch)|u'_{h,\text{in}}|^2_\psi - \tfrac{1}{2}|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi.
\end{aligned}$$

Inequality (4.6) follows directly. On a type I triangle, $Q_{n-1}u'_{h,\text{in}} = u'_{h,\text{in}}$ and hence using (2.8) and the arithmetic-geometric mean inequality, we get

$$\begin{aligned}
|Q_{n-1}u'_{h,\text{out}}|^2_\psi &= |u'_{h,\text{out}}|^2_\psi - |(I-Q_{n-1})u'_{h,\text{out}}|^2_\psi - 2\langle Q_{n-1}u'_{h,\text{out}}, (I-Q_{n-1})u'_{h,\text{out}}\rangle_\psi \\
&\ge |u'_{h,\text{out}}|^2_\psi - |(I-Q_{n-1})u'_{h,\text{out}}|^2_\psi \\
&\qquad - Ch^{1/2}|Q_{n-1}u'_{h,\text{out}}|_\psi |(I-Q_{n-1})u'_{h,\text{out}}|_\psi \\
&\ge |u'_{h,\text{out}}|^2_\psi - 2|(I-Q_{n-1})u'_{h,\text{out}}|^2_\psi - Ch|Q_{n-1}u'_{h,\text{out}}|^2_\psi. \\
&\ge (1-Ch)|u'_{h,\text{out}}|^2_\psi - 2|(I-Q_{n-1})u'_{h,\text{out}}|^2_\psi.
\end{aligned}$$

Moreover, by (2.9),

$$|(I-Q_{n-1})u'_{h,\text{out}}|_\psi = |(I-Q_{n-1})(u'_{h,\text{out}} - u'_{h,\text{in}})|_\psi \le C|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi.$$

Inequality (4.5) follows by combining these results.

To establish (4.7), we use the following sequence of inequalities.

$$\begin{aligned}
|(f, [EMu_h]_t)| &= |(P_{n-2}f, [EMu_h]_t) \le \|P_{n-2}f\| \|[EMu_h]_t\| \\
&\le \|P_{n-2}f\| Ch^{-1/2} \,|\!|\!| Q_{n-1}\psi Q_{n-1}(u'_{h,\text{out}} + u'_{h,\text{in}}) |\!|\!| \\
&\le \|P_{n-2}f\| Ch^{-1/2} \,|\!|\!| \psi Q_{n-1}(u'_{h,\text{out}} + u'_{h,\text{in}}) |\!|\!| \\
&\le \|P_{n-2}f\| Ch^{-1/2}[\max\psi]^{1/2}|Q_{n-1}(u'_{h,\text{out}} + u'_{h,\text{in}})|_\psi \\
&\le [\max\psi^{-1}]^{1/2}\|P_{n-2}f\|_\psi Ch^{-1/2}[\max\psi]^{1/2}|u'_{h,\text{out}} + u'_{h,\text{in}}|_\psi \\
&\le Ch^{-1}\|P_{n-2}f\|_\psi \|\nabla u_h\|_\psi \\
&\le C\left(h^{-2}\|P_{n-2}f\|^2_\psi + \|\nabla u_h\|^2_\psi\right). \qquad \square
\end{aligned}$$

The inequalities (4.5), (4.6), and (4.7) can now be used in conjunction with (4.2) and (4.3) to obtain a global stability result for u'_h. Before doing so, we derive a bound which will enable us to control u_h as well.

LEMMA 4.4. *For a triangle of type* I,

(4.8)
$$|u_{h,\text{out}}|^2_\psi - |u_{h,\text{in}}|^2_\psi \le C\left\{h\left[|u_{h,\text{in}}|^2_\psi + h^{3/2}|u'_{h,\text{in}}|^2_\psi\right] + h^{1/2}\|P_{n-1}f\|^2_\psi + \|P_{n-2}f\|^2_\psi\right\}.$$

For a triangle of type II *and for any positive bounded* ϵ,

(4.9)
$$\begin{aligned}
|u_{h,\text{out}}|^2_\psi - |u_{h,\text{in}}|^2_\psi \le & C\left\{\epsilon h^{3/2}|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi + \epsilon^{-1}h\left[|u_{h,\text{in}}|^2_\psi + h^{3/2}|u'_{h,\text{in}}|^2_\psi\right]\right. \\
&\left. + \epsilon^{-1}\|P_{n-2}f\|^2_\psi\right\}.
\end{aligned}$$

Proof. First note that

$$\begin{aligned}|u_{h,\text{out}}|_\psi^2 - |u_{h,\text{in}}|_\psi^2 &= \int_{\Gamma(T)} u_h^2 \psi \boldsymbol{\alpha}\cdot \boldsymbol{n}\, d\tau = \int_T (u_h^2)_s \psi\, dx = \int_T (u_h)_s 2\psi u_h\, dx \\ &= \int_T (u_h)_s (I - P_{n-2})(2\psi u_h)\, dx + \int_T f P_{n-2}(2\psi u_h)\, dx \\ &\le \|(u_h)_s\| \|(I-P_{n-2})(2\psi u_h)\| + 2\|P_{n-2} f\|_\psi \|u_h\|_\psi \\ &\le Ch\|(u_h)_s\| \|\nabla(\psi u_h)\| + 2\|P_{n-2}f\|_\psi \|u_h\|_\psi .\end{aligned}$$

Now using (2.6) and (2.7), it follows that

$$\begin{aligned}\|(u_h)_s\|\|\nabla(\psi u_h)\| &\le \|(u_h)_s\|(\|\psi\nabla u_h\| + \|u_h \nabla \psi\|) \\ &\le C\|(u_h)_s\|(\|\psi \nabla u_h\| + h^{-1/2}\|\psi u_h\|) \\ &\le C\|(u_h)_s\|_\psi (\|\nabla u_h\|_\psi + h^{-1/2}\|u_h\|_\psi).\end{aligned}$$

Combining these results, we obtain for positive ϵ that

$$\begin{aligned}|u_{h,\text{out}}|_\psi^2 - |u_{h,\text{in}}|_\psi^2 &\le Ch\|(u_h)_s\|_\psi(\|\nabla u_h\|_\psi + h^{-1/2}\|u_h\|_\psi) + 2\|P_{n-2}f\|_\psi\|u_h\|_\psi \\ &\le C[\epsilon h^{1/2}\|(u_h)_s\|_\psi^2 + \epsilon^{-1}h^{3/2}\|\nabla u_h\|_\psi^2 + \|P_{n-2}f\|_\psi^2 \\ &\quad + (1 + h^{1/2}\epsilon^{-1})\|u_h\|_\psi^2].\end{aligned}$$

Applying (4.1), (4.2), and (4.3), we get

$$\begin{aligned}|u_{h,\text{out}}|_\psi^2 - |u_{h,\text{in}}|_\psi^2 &\le C(\epsilon h^{3/2}|(I - Q_{n-1})u'_{h,\text{in}}|_\psi^2 + \epsilon^{-1}h^{5/2}|u'_{h,\text{in}}|_\psi^2 \\ &\quad + h(1 + h^{1/2}\epsilon^{-1})|u_{h,\text{in}}|_\psi^2 \\ &\quad + \|P_{n-2}f\|_\psi^2 + (\epsilon h^{1/2} + \epsilon^{-1}h^{3/2} + h^2)\|P_{n-l}f\|_\psi^2).\end{aligned}$$

On a type II triangle, (4.9) is now easily established. On a type I triangle, we choose $\epsilon = 1$ and use the fact that $(I - Q_{n-1})u'_{h,\text{in}} = 0$ to obtain (4.8). □

We now combine these results to get a single stability result for a triangle of either type, which we shall be able to iterate to obtain global stability for the method.

LEMMA 4.5. *For a triangle T of either type, there exists a positive constant λ such that*

(4.10)
$$\begin{aligned}(1 - Ch)\left[|u_{h,\text{out}}|_\psi^2 + h^{3/2}|u'_{h,\text{out}}|_\psi^2\right] + \lambda h^{1/2}\|(u_h)_s\|_\psi^2 + \|u_h\|_\psi^2 + h^{3/2}\|\nabla u_h\|_\psi^2 \\ \le (1 + Ch)\left[|u_{h,\text{in}}|_\psi^2 + h^{3/2}|u'_{h,\text{in}}|_\psi^2\right] + C\left[h^{1/2}\|P_{n-1}f\|_\psi^2 + h^{-1/2}\|P_{n-2}f\|_\psi^2\right].\end{aligned}$$

Proof. Setting $v_h = -[EMu_h]_t$ in (3.1) and using (4.5) and (4.7), we obtain for a type I triangle that

$$\begin{aligned}(1 - Ch)|u'_{h,\text{out}}|_\psi^2 - |u'_{h,\text{in}}|_\psi^2 - C|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi^2 &\le -((u_h)_s, [EMu_h]_t) = -(f, [EMu_h]_t) \\ &\le C(h^{-2}\|P_{n-2}f\|_\psi^2 + \|\nabla u_h\|_\psi^2).\end{aligned}$$

Applying (4.3) to estimate $|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi$ and using (4.3) and (4.2) to also gain control over $\|(u_h)_s\|_\psi$ and $\|\nabla u_h\|_\psi$, we get after rearranging terms that

$$\begin{aligned}(1-Ch)|u'_{h,\text{out}}|^2_\psi + h^{-1}\|(u_h)_s\|^2_\psi + \|\nabla u_h\|^2_\psi \\ \le (1+Ch)|u'_{h,\text{in}}|^2_\psi + C(h^{-2}\|P_{n-2}f\|^2_\psi + h^{-1}\|P_{n-1}f\|^2_\psi).\end{aligned}$$

Multiplying this result by $h^{3/2}$, adding it to (4.8), and using (4.1) to also gain control over $\|u_h\|_\psi$, we obtain (4.10) with $\lambda = 1$.

For a type II triangle, we proceed in a similar fashion, using (4.6) and (4.7) to first obtain

$$|u'_{h,\text{out}}|^2_\psi + \tfrac{1}{2}|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi \le (1+Ch)|u'_{h,\text{in}}|^2_\psi + C(h^{-2}\|P_{n-2}f\|^2_\psi + \|\nabla u_h\|^2_\psi).$$

Again using (4.3) and (4.2) to gain control over $\|(u_h)_s\|_\psi$ and $\|\nabla u_h\|_\psi$, we get after rearranging terms that

$$\begin{aligned}|u'_{h,\text{out}}|^2_\psi + \tfrac{1}{2}|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi + \epsilon h^{-1}\|(u_h)_s\|^2_\psi + \|\nabla u_h\|^2_\psi \\ \le (1+Ch)|u'_{h,\text{in}}|^2_\psi + Ch^{-2}\|P_{n-2}f\|^2_\psi + C\epsilon|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi.\end{aligned}$$

Multiplying this result by $h^{3/2}$, adding it to (4.9), and using (4.1) to also gain control over $\|u_h\|_\psi$, we then obtain the following for ϵ sufficiently small:

$$\begin{aligned}\left[|u_{h,\text{out}}|^2_\psi + h^{3/2}|u'_{h,\text{out}}|^2_\psi\right] + \epsilon h^{1/2}\|(u_h)_s\|^2_\psi + \tfrac{1}{4}h^{3/2}|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi \\ +\|u_h\|^2_\psi + h^{3/2}\|\nabla u_h\|^2_\psi \le (1+Ch)\left[|u_{h,\text{in}}|^2_\psi + h^{3/2}|u'_{h,\text{in}}|^2_\psi\right] + Ch^{-1/2}\|P_{n-2}f\|^2_\psi.\end{aligned}$$

The result follows immediately. □

5. Global stability and error estimates for the model problem. Using the stability results of the previous section, it is now fairly easy to derive a global stability result and error estimates for the model problem $\boldsymbol{\alpha} \cdot \nabla u = f$.

As was done in §3, we consider the solution u_h as developing in layers F_i. Thus we have, in analogy with (4.10),

$$\begin{aligned}(1-Ch)\left[|u_{h,\text{out}}|^2_{\psi,F_i} + h^{3/2}|u'_{h,\text{out}}|^2_{\psi,F_i}\right] + \lambda h^{1/2}\|(u_h)_s\|^2_{\psi,S_i} + \|u_h\|^2_{\psi,S_i} \\ + h^{3/2}\|\nabla u_h\|^2_{\psi,S_i} \\ \le (1+Ch)\left[|u_{h,\text{in}}|^2_{\psi,F_{i-1}} + h^{3/2}|u'_{h,\text{in}}|^2_{\psi,F_{i-1}}\right] \\ + C\left[h^{1/2}\|P_{n-1}f\|^2_{\psi,S_i} + h^{-1/2}\|P_{n-2}f\|^2_{\psi,S_i}\right],\end{aligned}$$

where again F_i is the "front line" to which u_h has advanced after it has been computed in $\Omega_i \equiv \cup_{j\le i} S_j$.

To convert this bound into a global stability result for the method, we use the following lemma.

LEMMA 5.1. *If*

$$(1 - Ch)x_i + a_i \le (1 + Ch)x_{i-1} + b_i, \qquad i = 1, 2, \cdots,$$

where $0 < h < \frac{1}{C}$, $x_0 > 0$, *and* $a_i > 0$, $b_i > 0$ *for all* i, *then*

$$x_i + \frac{1}{1 - Ch}\sum_{j=1}^{i} a_j \le M^{ih}\left(x_0 + \frac{1}{1 - Ch}\sum_{j=1}^{i} b_j\right), \qquad i = 1, 2, \cdots,$$

where $M \to e^{2C}$ *as* $h \to 0$.

Proof. The solution of the above inequality is

$$x_i \le \left(\frac{1 + Ch}{1 - Ch}\right)^i x_0 + \frac{1}{1 - Ch}\sum_{j=1}^{i}\left(\frac{1 + Ch}{1 - Ch}\right)^{i-j}(b_j - a_j).$$

Thus,

$$x_i + \frac{1}{1 - Ch}\sum_{j=1}^{i} a_j \le \left(\frac{1 + Ch}{1 - Ch}\right)^i \left[x_0 + \frac{1}{1 - Ch}\sum_{j=1}^{i} b_j\right].$$

The desired result follows from the fact that

$$\lim_{h \to 0}\left(\frac{1 + Ch}{1 - Ch}\right)^{1/h} = e^{2C}.$$

□

Assuming there are at most $O(h^{-1})$ layers per triangulation, we apply Lemma 5.1 to obtain the following global stability result.

THEOREM 5.2. *For* h *sufficiently small,*

$$\begin{aligned} &|u_{h,\text{out}}|^2_{\psi,F_i} + h^{3/2}|u'_{h,\text{out}}|^2_{\psi,F_i} + h^{1/2}\|(u_h)_s\|^2_{\psi,\Omega_i} \\ &\qquad + \|u_h\|^2_{\psi,\Omega_i} + h^{3/2}\|\nabla u_h\|^2_{\psi,\Omega_i} \\ &\le C\Big[|u_{h,\text{in}}|^2_{\psi,\Gamma_{\text{in}}(\Omega)} + h^{3/2}|u'_{h,\text{in}}|^2_{\psi,\Gamma_{\text{in}}(\Omega)} + h^{1/2}\|P_{n-1}f\|^2_{\psi,\Omega_i} \\ &\qquad + h^{-1/2}\|P_{n-2}f\|^2_{\psi,\Omega_i}\Big]. \end{aligned} \tag{5.1}$$

We now show how Theorem 5.2 may be used to derive various error estimates. Let us first assume that $u \in H^{n+1}(\Omega)$ and let $u_I \in S_h^n$ be an interpolant of u. Defining $e_h \equiv u_h - u_I$, we have, for all $v_h \in \mathbf{P}_{n-l}(T)$, that

$$((e_h)_s, v_h) = (f - (u_I)_s, v_h) = ([u - u_I]_s, v_h).$$

Hence, we may apply (5.1) with u_h and f replaced by e_h and $(u - u_I)_s$, respectively. Assuming, for convenience, that $u_h = u_I$ on $\Gamma_{\text{in}}(\Omega)$, we get

$$\begin{aligned} &|e_{h,\text{out}}|^2_{\psi,F_i} + h^{3/2}|e'_{h,\text{out}}|^2_{\psi,F_i} + h^{1/2}\|(e_h)_s\|^2_{\psi,\Omega_i} + \|e_h\|^2_{\psi,\Omega_i} \\ &\qquad + h^{3/2}\|\nabla e_h\|^2_{\psi,\Omega_i} \\ &\le C(h^{-1/2}\|P_{n-2}(u - u_I)_s\|^2_{\psi,\Omega_i} + h^{1/2}\|(u - u_I)_s\|^2_{\psi,\Omega_i}). \end{aligned} \tag{5.2}$$

We shall first take u_I to be the interpolant of §2. From the defining properties of this interpolant, it follows immediately from integration by parts that

$$([u - u_I]_s, v_h) = 0 \quad \text{for all } v_h \in \mathbf{P}_{n-2}(T). \tag{5.3}$$

Thus, using the approximation properties (2.4), (2.5) of u_I, we obtain the following error estimates for the method in the case where the solution u is smooth.

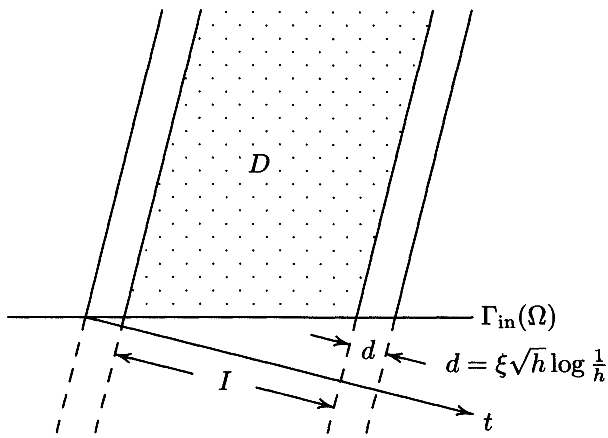

FIG. 2

THEOREM 5.3.

$$|u_{h,\text{out}} - u_{\text{out}}|_{\psi,F_i} + \|u_h - u\|_{\psi,\Omega_i} \le Ch^{n+1/4}\|u\|_{n+1,\psi,\Omega_i},$$
$$|u'_{h,\text{out}} - u'_{\text{out}}|_{\psi,F_i} + \|\nabla(u_h - u)\|_{\psi,\Omega_i} \le Ch^{n-1/2}\|u\|_{n+1,\psi,\Omega_i},$$
$$\|(u_h)_s - u_s\|_{\psi,\Omega_i} \le Ch^{n}\|u\|_{n+1,\psi,\Omega_i}.$$

Up until this point, all our estimates apply to a general ψ satisfying (2.6) and (2.7). We now indicate how (5.2) can be applied locally in problems where u is not globally smooth, for example, in a problem with a discontinuous initial condition. To accomplish this, we now make, for fixed t^*, the specific choice

$$\psi = \psi(t, t^*) \equiv \frac{1}{2\sqrt{h}}e^{-|t-t^*|/\sqrt{h}}.$$

For an interval $I \subset \Gamma_{\text{in}}(\Omega)$, let $D = \{(s,t) \in \Omega : t \in I\}$ (see Fig. 2).

For fixed $\xi > 0$, we define

$$D^+ = \left\{(s,t) \in \Omega : \min_{t' \in I}|t - t'| \le \xi\sqrt{h}\log\frac{1}{h}\right\}$$

and

$$D_h^+ = \{T \in \Omega : T \cap D^+ \ne \emptyset\}.$$

Note that for $t^* \in I$ and $(s,t) \in \Omega - D^+$,

$$\psi(t,t^*) = \frac{1}{2\sqrt{h}}e^{-|t-t^*|/\sqrt{h}} \le \frac{1}{2\sqrt{h}}e^{-\xi\log(1/h)} \le \frac{1}{2}h^{\xi-1/2}.$$

We shall obtain local $L^2(D)$ error estimates for u_h, assuming $u \in H^{n+1}(D_h^+)$ and minimal regularity in $\Omega - D_h^+$. To get these local results, we make use of the following lemma.

LEMMA 5.4. *There is a positive constant λ such that*

$$\lambda\|v\|_D^2 \le \int_I \|v\|_{\psi,\Omega}^2\, dt^* \le \|v\|_{D_h^+}^2 + \frac{|I|}{2} h^{\xi-1/2}\|v\|_{\Omega-D_h^+}^2, \tag{5.4}$$

$$\lambda|v|_{\Gamma_{\text{out}}(D)}^2 \le \int_I \|v\|_{\psi,\Gamma_{\text{out}}(\Omega)}^2\, dt^*, \tag{5.5}$$

where $|I|$ denotes the length of I.

Proof.

$$\begin{aligned}\int_I \|v\|_{\psi,\Omega}^2\, dt^* &\le \int_{D_h^+} \psi(t,t^*)v^2(s,t)\,ds\,dt + \int_{\Omega-D_h^+} \psi(t,t^*)v^2(s,t)\,ds\,dt\\ &\le \int_{D_h^+}\left[\frac{1}{2\sqrt{h}}\int_{-\infty}^{\infty} e^{-|t-t^*|/\sqrt{h}}\,dt^*\right]v^2(s,t)\,ds\,dt\\ &\quad + \int_{\Omega-D_h^+} \frac{1}{2}h^{\xi-1/2}|I|v^2(s,t)\,ds\,dt\\ &\le \|v\|_{D_h^+}^2 + \frac{|I|}{2}h^{\xi-1/2}\|v\|_{\Omega-D_h^+}^2.\end{aligned}$$

We also have that

$$\int_I \|v\|_{\psi,\Omega}^2\, dt^* \ge \int_D J_h(t)v^2(s,t)\,ds\,dt,$$

where

$$J_h(t) = \frac{1}{2\sqrt{h}}\int_I e^{-|t-t^*|/\sqrt{h}}\,dt^*.$$

A simple computation shows that if $t \in I = [t', t'']$, then

$$J_h(t) = \tfrac{1}{2}\left(1 - e^{-(t''-t)/\sqrt{h}}\right) + \tfrac{1}{2}\left(1 - e^{-(t-t')/\sqrt{h}}\right) \ge \left(1 - e^{-|I|/(2\sqrt{h})}\right).$$

Hence, $J_h(t)$ is uniformly positive (independent of h) for $t \in I$, which completes the proof of (5.4). The proof of (5.5) is obtained in an analogous manner. □

Because of the lack of global smoothness of the solution u, we cannot use the same interpolant as in the smooth case. We redefine u_I for this situation as follows. In D_h^+, we take u_I to be the interpolant of §2, as before. At degrees of freedom which belong neither to D_h^+ nor $\Gamma_{\text{in}}(\Omega)$, we use the method of Clement [2], i.e., by letting Λ_i denote the support of the basis function associated with the point v_i, we define $u_I(v_i) = q(v_i)$, where q is the L^2 projection of u over Λ_i into $\mathbf{P}_n(\Lambda_i)$. Finally, at degrees of freedom lying on $\Gamma_{\text{in}}(\Omega)$, we use the one-dimensional version of the Clement method, involving projections of u only along $\Gamma_{\text{in}}(\Omega)$. Note the similarity to the interpolant developed in [19]. The approximation properties (2.4), (2.5) remain valid for this new interpolant. Using standard techniques, it is not difficult to show that u_I also satisfies the following bound, whose proof we omit.

LEMMA 5.5.

$$\|u_I\|_\Omega \le C(\|u\|_\Omega + |u|_{\Gamma_{\text{in}}(\Omega)} + \|u\|_{2,D_h^+}).$$

Integrating (5.2) with $\Omega_i = \Omega$ over $t^* \in I$ and using the preceding results, we obtain

$$|e_{h,\text{out}}|_I^2 + h^{3/2}|e'_{h,\text{out}}|_I^2 + h^{1/2}\|(e_h)_s\|_D^2 + \|e_h\|_D^2 + h^{3/2}\|\nabla e_h\|_D^2$$
$$\le C\left(h^{1/2}\|(u-u_I)_s\|^2_{D_h^+} + h^{\xi-1}\|(u-u_I)_s\|^2_{\Omega-D_h^+}\right)$$
$$\le C\left(h^{1/2}\|(u-u_I)_s\|^2_{D_h^+} + h^{\xi-1}(\|u_s\|_\Omega^2 + h^{-2}\|u_I\|_\Omega^2)\right)$$
$$\le C\Big(h^{2n+1/2}\|u\|^2_{n+1,D_h^+} + h^{\xi-1}(\|f\|_\Omega^2 + h^{-2}\|u\|_\Omega^2 + h^{-2}|u|^2_{\Gamma_{\text{in}}(\Omega)} + h^{-2}\|u\|^2_{2,D_h^+})\Big).$$

Again, using the approximation properties of u_I, we conclude the following.

THEOREM 5.6. *For* $\xi > 2n + 7/2$,

$$|u_{h,\text{out}} - u_{\text{out}}|_I^2 + \|u_h - u\|_D^2 + h^{3/2}|u'_{h,\text{out}} - u'_{\text{out}}|_I^2 + h^{3/2}\|\nabla(u_h - u)\|_D^2 + h^{1/2}\|(u_h-u)_s\|_D^2$$
$$\le Ch^{2n+1/2}\left[\|u\|^2_{n+1,D_h^+} + \epsilon(\|u\|_\Omega^2 + |u|^2_{\Gamma_{\text{in}}(\Omega)} + \|f\|_\Omega^2)\right],$$

where $\epsilon \to 0$ *as* $h \to 0$.

These localization results generalize (1.4)–(1.6). Computational results in [18] for $n = 2$ show a somewhat more favorable crosswind spread, $\approx O(h^{.7})$. This corresponds closely to the discontinuous Galerkin method with $n = 1$, via the parallelism of §3. For the discontinuous Galerkin method, computational estimates of crosswind spread can be found in [13]. These results indicate a decrease in crosswind spread with increasing n.

6. The general convection-diffusion problem. We next consider a class of convection-diffusion equations of the form

$$\boldsymbol{\alpha} \cdot \nabla u - (a(u_h)_{xx} + b(u_h)_{xy} + c(u_h)_{yy}) = f,$$

and we consider the corresponding finite element method

$$(6.1)\quad (\boldsymbol{\alpha}\cdot\nabla u_h, v_h) - (a(u_h)_{xx} + b(u_h)_{xy} + c(u_h)_{yy}, v_h) = (f, v_h), \qquad v_h \in \mathbf{P}_{n-l}(T),$$

as described in the introduction.

Now since $s = \alpha_1 x + \alpha_2 y$ and $t = \alpha_2 x - \alpha_1 y$, we get for any function v that

$$v^h_{xx} = \alpha_2^2 v^h_{tt} + 2\alpha_1\alpha_2 v^h_{ts} + \alpha_1^2 v^h_{ss},$$
$$v^h_{xy} = -\alpha_1\alpha_2 v^h_{tt} + (\alpha_2^2 - \alpha_1^2)v^h_{ts} + \alpha_1\alpha_2 v^h_{ss},$$
$$v^h_{yy} = \alpha_1^2 v^h_{tt} - 2\alpha_1\alpha_2 v^h_{ts} + \alpha_2^2 v^h_{ss}.$$

Hence, we may express

$$av^h_{xx} + bv^h_{xy} + cv^h_{yy} = [a\alpha_2^2 - b\alpha_1\alpha_2 + c\alpha_1^2]v^h_{tt} + [2a\alpha_1\alpha_2 + b(\alpha_2^2 - \alpha_1^2) - 2c\alpha_1\alpha_2]v^h_{ts} + [a\alpha_1^2 + b\alpha_1\alpha_2 + c\alpha_2^2]v^h_{ss}$$
$$\equiv \sigma v^h_{tt} + \delta v^h_{ts} + \gamma v^h_{ss}.$$

In the analysis of this problem which follows, we shall assume that the coefficients σ, δ, and γ satisfy

$$0 \leq \sigma \leq qh, \quad |\delta| \leq C\sigma, \quad |\gamma| \leq C\sigma \tag{6.2}$$

for some constant q independent of h. As mentioned in the introduction, this framework includes as special cases the parabolic and elliptic equations

$$\begin{aligned} \boldsymbol{\alpha} \cdot \nabla u - \sigma u_{xx} &= f, \\ \boldsymbol{\alpha} \cdot \nabla u - \sigma \Delta u &= f, \end{aligned}$$

provided $\sigma \leq O(h)$. To analyze the effect of the additional terms now present, it will be convenient to first prove some preliminary results. We begin with a stability result for (6.1), valid over a single triangle.

LEMMA 6.1. *Assuming q is sufficiently small,* (4.1) *and* (4.2) *remain valid, and in place of* (4.3), *we have*

$$\begin{aligned} \|(u_h)_s\|_\psi + \|(u_h)_t - EQ_{n-1}u'_{h,\text{in}}\|_\psi + \sqrt{h}|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi \\ \leq C\left[\sqrt{h}|(I - Q_{n-1})u'_{h,\text{in}}|_\psi + \sigma\|(u_h)_{tt}\|_\psi + \|P_{n-l}f\|_\psi\right]. \end{aligned}$$

Proof. The convection-diffusion problem (6.1) is of the form (3.1) with f replaced by

$$\bar{f} = f + \sigma(u_h)_{tt} + \delta(u_h)_{ts} + \gamma(u_h)_{ss}.$$

Via repeated use of inverse inequalities, we have

$$\begin{aligned} \|P_{n-l}\bar{f}\|_\psi &\leq \|P_{n-l}f\|_\psi + \sigma\|(u_h)_{tt}\|_\psi + Cq\|(u_h)_s\|_\psi \\ &\leq \|P_{n-l}f\|_\psi + Cq\|\nabla u_h\|_\psi \\ &\leq \|P_{n-l}f\|_\psi + Cqh^{-1}\|u_h\|_\psi. \end{aligned}$$

Using the above three bounds together with (4.3), (4.2), and (4.1), respectively, we obtain the desired results. □

The next three lemmas contain some technical results which will facilitate the analysis.

LEMMA 6.2. *Let v_h and $w_h \in \mathbf{P}_n(T)$. Then*

$$\|(v_h)_t + \psi(w_h)_t\|_\psi \leq C\left(h^{-1}\|v_h + \psi w_h\|_\psi + h^{-1/2}\max\psi\|w_h\|_\psi\right).$$

Proof. Let ψ_0 denote the average value of ψ on T. Then

$$\begin{aligned} \|(v_h)_t + \psi(w_h)_t\|_\psi &\leq \|(v_h + \psi_0 w_h)_t\|_\psi + \|(\psi - \psi_0)(w_h)_t\|_\psi \\ &\leq Ch^{-1}\|v_h + \psi_0 w_h\|_\psi + \|(\psi - \psi_0)(w_h)_t\|_\psi \\ &\leq Ch^{-1}\|v_h + \psi w_h\|_\psi + Ch^{-1}\|(\psi - \psi_0)w_h\|_\psi \\ &\quad + \|(\psi - \psi_0)(w_h)_t\|_\psi. \end{aligned}$$

Now using (2.7) and standard inverse estimates,

$$\begin{aligned} \|(\psi - \psi_0)w_h\|_\psi &\leq Ch\max|\psi'|\|w_h\|_\psi \leq Ch^{1/2}\max\psi\|w_h\|_\psi, \\ \|(\psi - \psi_0)(w_h)_t\|_\psi &\leq Ch\max|\psi'|\|(w_h)_t\|_\psi \leq Ch^{-1/2}\max\psi\|w_h\|_\psi. \end{aligned}$$

Combining these results establishes the lemma. □

Recalling that $Mu_h = Q_{n-1}\psi Q_{n-1}(u'_{h,\text{out}} + u'_{h,\text{in}})$, we next prove the following.

LEMMA 6.3.

$$|Mu_h - 2\psi Q_{n-1}u'_{h,\text{in}}|_\psi \le C \max\psi \left[|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi + h^{1/2}|u'_{h,\text{in}}|_\psi\right].$$

Proof. Letting ψ_0 again denote the average value of ψ on $[t_0, t_1]$, we have that

$$\begin{aligned}
&|Mu_h - 2\psi Q_{n-1}u'_{h,\text{in}}|_\psi \\
&\le C\left[|Mu_h - 2Q_{n-1}\psi Q_{n-1}u'_{h,\text{in}}|_\psi + |(I - Q_{n-1})\psi Q_{n-1}u'_{h,\text{in}}|_\psi\right] \\
&\le C\Big[\max\psi|Q_{n-1}(u'_{h,\text{out}} - u'_{h,\text{in}})|_\psi \\
&\qquad + |(I - Q_{n-1})(\psi - \psi_0)Q_{n-1}u'_{h,\text{in}}|_\psi\Big].
\end{aligned}$$

Now

$$\begin{aligned}
|(I - Q_{n-1})(\psi - \psi_0)Q_{n-1}u'_{h,\text{in}}|_\psi &\le C|(\psi - \psi_0)Q_{n-1}u'_{h,\text{in}}|_\psi \\
&\le Ch \max|\psi'||Q_{n-1}u'_{h,\text{in}}|_\psi \\
&\le Ch^{1/2}\max\psi|Q_{n-1}u'_{h,\text{in}}|_\psi.
\end{aligned}$$

Combining these results, we obtain

$$\begin{aligned}
|Mu_h - 2\psi Q_{n-1}u'_{h,\text{in}}|_\psi &\le C\Big[\max\psi|Q_{n-1}(u'_{h,\text{out}} - u'_{h,\text{in}})|_\psi \\
&\qquad + h^{1/2}\max\psi|Q_{n-1}u'_{h,\text{in}}|_\psi\Big] \\
&\le C\max\psi\left[|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi + h^{1/2}|u'_{h,\text{in}}|_\psi\right].
\end{aligned}$$ □

LEMMA 6.4.

$$\|\psi^{-1}[EMu_h]_t - 2(u_h)_{tt}\|_\psi \le C(q\|(u_h)_{tt}\|_\psi + h^{-1/2}R),$$

where

$$R = |(I - Q_{n-1})u'_{h,\text{in}}|_\psi + h^{1/2}|u'_{h,\text{in}}|_\psi + h^{-1/2}\|P_{n-l}f\|_\psi.$$

Proof. Using Lemma 6.2 and the triangle inequality, we have

$$\begin{aligned}
&\|\psi^{-1}[EMu_h]_t - 2(u_h)_{tt}\|_\psi \\
&\le \max\psi^{-1}\|[EMu_h]_t - 2\psi(u_h)_{tt}\|_\psi \\
&\le C\max\psi^{-1}\left[h^{-1}\|EMu_h - 2\psi(u_h)_t\|_\psi + h^{-1/2}\max\psi\|(u_h)_t\|_\psi\right] \\
&\le C\max\psi^{-1}\Big[h^{-1}\|EMu_h - 2E\psi Q_{n-1}u'_{h,\text{in}}\|_\psi \\
&\qquad + 2h^{-1}\|\psi(u_h)_t - E\psi Q_{n-1}u'_{h,\text{in}}\|_\psi + h^{-1/2}\max\psi\|(u_h)_t\|_\psi\Big].
\end{aligned}$$

Now using Lemma 6.3 and standard inverse estimates, we get

$$\begin{aligned}
\|EMu_h - 2E\psi Q_{n-1}u'_{h,\text{in}}\|_\psi &\le Ch^{1/2}|Mu_h - 2\psi Q_{n-1}u'_{h,\text{in}}|_\psi \\
&\le C\max\psi\left[h^{1/2}|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi + h|u'_{h,\text{in}}|_\psi\right].
\end{aligned}$$

Using (2.2) and (2.3), we get

$$\|\psi(u_h)_t - E\psi Q_{n-1}u'_{h,\text{in}}\|_\psi = \|\psi(u_h)_t - \psi E Q_{n-1}u'_{h,\text{in}}\|_\psi \\ \le \max\psi\|(u_h)_t - EQ_{n-1}u'_{h,\text{in}}\|_\psi.$$

Combining these results and using Lemma 6.1, we obtain

$$\begin{aligned}
&\|\psi^{-1}[EMu_h]_t - 2(u_h)_{tt}\|_\psi \\
&\le C\Big[h^{-1/2}|u'_{h,\text{out}} - u'_{h,\text{in}}|_\psi + |u'_{h,\text{in}}|_\psi + h^{-1}\|(u_h)_t - EQ_{n-1}u'_{h,\text{in}}\|_\psi \\
&\qquad + h^{-1/2}\|(u_h)_t\|_\psi\Big] \\
&\le C\left[h^{-1/2}|(I-Q_{n-1})u'_{h,\text{in}}|_\psi + h^{-1}\sigma\|(u_h)_{tt}\|_\psi + h^{-1}\|P_{n-l}f\|_\psi + |u'_{h,\text{in}}|_\psi\right].
\end{aligned}$$

The lemma follows immediately. □

Using these results, we now proceed as in §4, deriving, for a single triangle, a stability result that can be iterated to obtain a global stability result for the method. To avoid technical problems involving terms which are not central to the analysis of the new diffusion terms, we limit our stability result to the derivative of u_h, rather than also include u_h itself. The key result is then to obtain an analogue of Lemma 4.5 and the essential new feature of the analysis is the handling of the terms

$$(a(u_h)_{xx} + b(u_h)_{xy} + c(u_h)_{yy}, [EMu_h]_t) = (\sigma(u_h)_{tt} + \delta(u_h)_{ts} + \gamma(u_h)_{ss}, [EMu_h]_t)$$

and $(f, [EMu_h]_t)$. The next lemma contains the neccessary estimates for these terms.

LEMMA 6.5. *For arbitrary $\phi \in L^2(T)$ and q sufficiently small, we have*

$$(\sigma(u_h)_{tt} + \delta(u_h)_{ts} + \gamma(u_h)_{ss} + \sigma\phi, [EMu_h]_t) \ge \sigma\|(u_h)_{tt}\|_\psi^2 - C(qR^2 + \sigma\|\phi\|_\psi^2).$$

Proof. Using Lemma 6.4, the triangle inequality, and the arithmetic-geometric mean inequality, we get, for arbitrary $\epsilon > 0$,

$$\begin{aligned}
(\sigma(u_h)_{tt}, [EMu_h]_t) &= 2\sigma\|(u_h)_{tt}\|_\psi^2 + (\sigma(u_h)_{tt}, \psi^{-1}[EMu_h]_t - 2(u_h)_{tt})_\psi \\
&\ge 2\sigma\|(u_h)_{tt}\|_\psi^2 - C\sigma\|(u_h)_{tt}\|_\psi(h^{-1/2}R + q\|(u_h)_{tt}\|_\psi) \\
&\ge \sigma\|(u_h)_{tt}\|_\psi^2(2 - Cq) - C(\sqrt{\sigma}\|(u_h)_{tt}\|_\psi)(\sqrt{q}R) \\
&\ge \sigma\|(u_h)_{tt}\|_\psi^2(2 - Cq - \epsilon) - C\epsilon^{-1}qR^2.
\end{aligned}$$

Now using Lemma 6.4, the triangle inequality, and the fact that $q = O(1)$, we infer

$$\|\psi^{-1}[EMu_h]_t\|_\psi \le C(\|(u_h)_{tt}\|_\psi + h^{-1/2}R).$$

Using this result, together with Lemma 6.1, the arithmetic-geometric mean inequality, and the fact that $\sigma h^{-1} \le q$, we get

$$\begin{aligned}
&\big|(\delta(u_h)_{ts} + \gamma(u_h)_{ss} + \sigma\phi, [EMu_h]_t)\big| \\
&\quad\le (\|\delta(u_h)_{ts} + \gamma(u_h)_{ss}\|_\psi + \sigma\|\phi\|_\psi)\|\psi^{-1}[EMu_h]_t\|_\psi \\
&\quad\le C(\sigma h^{-1}\|(u_h)_s\|_\psi + \sigma\|\phi\|_\psi)(\|(u_h)_{tt}\|_\psi + h^{-1/2}R) \\
&\quad\le C(\sigma h^{-1/2}R + q\sigma\|(u_h)_{tt}\|_\psi + \sigma\|\phi\|_\psi)(\|(u_h)_{tt}\|_\psi + h^{-1/2}R) \\
&\quad\le C(\sqrt{q}R + q\sqrt{\sigma}\|(u_h)_{tt}\|_\psi + \sqrt{\sigma}\|\phi\|_\psi)(\sqrt{\sigma}\|(u_h)_{tt}\|_\psi + \sqrt{q}R) \\
&\quad\le \epsilon(\sigma\|(u_h)_{tt}\|_\psi^2 + qR^2) + C\epsilon^{-1}(qR^2 + q^2\sigma\|(u_h)_{tt}\|_\psi^2 + \sigma\|\phi\|_\psi^2) \\
&\quad\le (\epsilon + Cq^2\epsilon^{-1})\sigma\|(u_h)_{tt}\|_\psi^2 + (\epsilon + C\epsilon^{-1})qR^2 + C\epsilon^{-1}\sigma\|\phi\|_\psi^2.
\end{aligned}$$

The lemma follows by combining these results and choosing first ϵ and then q sufficiently small. □

Using Lemma 6.5, we get the following analogue of Lemma 4.5 when $f \equiv f_1 + \sigma f_2$.

LEMMA 6.6. *For a triangle of either type, there exists a positive constant λ such that*

$$(1-Ch)|u'_{h,\text{out}}|^2_\psi + \lambda h^{-1}\|(u_h)_s\|^2_\psi + \|\nabla u_h\|^2_\psi + \tfrac{1}{2}\sigma\|(u_h)_{tt}\|^2_\psi$$
$$\leq (1+Ch)|u'_{h,\text{in}}|^2_\psi + C\left(h^{-1}\|P_{n-1}f\|^2_\psi + h^{-2}\|P_{n-2}f_1\|^2_\psi + \sigma\|f_2\|^2_\psi\right).$$

Proof. From Lemma 6.5 and the definition of R, we have

(6.3)
$$(\sigma(u_h)_{tt} + \delta(u_h)_{ts} + \gamma(u_h)_{ss} + \sigma f_2, [EMu_h]_t)$$
$$\geq \sigma\|(u_h)_{tt}\|^2_\psi - Cq\left(|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi + h|u'_{h,\text{in}}|^2_\psi + h^{-1}\|P_{n-l}f\|^2_\psi\right) - C\sigma\|f_2\|^2_\psi.$$

For a type II triangle, we get from (4.6), (4.7), and Lemma 6.1 that

$$\begin{aligned}&-((u_h)_s - f_1, [EMu_h]_t)\\ &\quad\geq (1-Ch)|u'_{h,\text{out}}|^2_\psi - (1+Ch)|u'_{h,\text{in}}|^2_\psi\\ &\qquad + \tfrac{1}{2}|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi - C\left(h^{-2}\|P_{n-2}f_1\|^2_\psi + \|P_{n-2}f\|^2_\psi\right).\end{aligned} \tag{6.4}$$

Combining these results via (6.1) and then using Lemma 6.1 to also control $\|\nabla u_h\|^2_\psi$ and adding a suitably small positive multiple of $h^{-1}\|(u_h)_s\|^2_\psi$, we obtain

$$(1-Ch)|u'_{h,\text{out}}|^2_\psi + (\tfrac{1}{2} - Cq - C\epsilon))|(I-Q_{n-1})u'_{h,\text{in}}|^2_\psi + (1-C\epsilon)\sigma\|(u_h)_{tt}\|^2_\psi + \epsilon h^{-1}\|(u_h)_s\|^2_\psi$$
$$+ \|\nabla u_h\|^2_\psi \leq (1+Ch)|u'_{h,\text{in}}|^2_\psi + C\left(h^{-2}\|P_{n-2}f_1\|^2_\psi + h^{-1}\|P_{n-2}f\|^2_\psi + \sigma\|f_2\|^2_\psi\right).$$

The desired inequality now follows on a type II triangle by choosing q and ϵ sufficiently small.

For a type I triangle, we use (4.5), (4.7), and Lemma 6.1 to get, in place of (6.4),

$$\begin{aligned}&-((u_h)_s - f_1, [EMu_h]_t)\\ &\quad\geq (1-Ch)|u'_{h,\text{out}}|^2_\psi - (1+Ch)|u'_{h,\text{in}}|^2_\psi\\ &\qquad - C\left(q\sigma\|(u_h)_{tt}\|^2_\psi + h^{-2}\|P_{n-2}f_1\|^2_\psi + h^{-1}\|P_{n-1}f\|^2_\psi\right).\end{aligned}$$

Combining this result with (6.3) via (6.1), and again using Lemma 6.1 to also control $\|\nabla u_h\|^2_\psi$ and a suitably small positive multiple of $h^{-1}\|(u_h)_s\|^2_\psi$, we obtain

$$(1-Ch)|u'_{h,\text{out}}|^2_\psi + \|\nabla u_h\|^2_\psi + \epsilon h^{-1}\|(u_h)_s\|^2_\psi + (1-Cq-C\epsilon)\sigma\|(u_h)_{tt}\|^2_\psi$$
$$\leq (1+Ch)|u'_{h,\text{in}}|^2_\psi + C\left(h^{-2}\|P_{n-2}f_1\|^2_\psi + h^{-1}\|P_{n-1}f\|^2_\psi + \sigma\|f_2\|^2_\psi\right).$$

The desired inequality now follows on a type I triangle by again choosing q and ϵ sufficiently small. □

As in §5, this result may now be converted to an analogous result along fronts and then iterated using Lemma 5.1. When summed over layers, we get the following global stability result, analogous to Theorem 5.2.

THEOREM 6.7. *For h sufficiently small,*

(6.5)
$$|u'_{h,\text{out}}|^2_{\psi,F_i} + h^{-1}\|(u_h)_s\|^2_{\psi,\Omega_i} + \|\nabla u_h\|^2_{\psi,\Omega_i} + \sigma\|(u_h)_{tt}\|^2_{h,\psi,\Omega_i}$$
$$\leq C[|u'_{h,\text{in}}|^2_{\psi,\Gamma_{\text{in}}(\Omega)} + h^{-1}\|P_{n-1}f\|^2_{h,\psi,\Omega_i} + h^{-2}\|P_{n-2}f_1\|^2_{h,\psi,\Omega_i} + \sigma\|f_2\|^2_{h,\psi,\Omega_i}],$$

where we use the notation

$$\|v\|^2_{h,\psi,\Omega_i} = \sum_{T\in\Omega_i} \|v\|^2_{\psi,T}.$$

The reason for introducing this notation is that some of the terms in the estimate do not belong to $L^2(\Omega_i)$.

To obtain error estimates, we again set $e_h \equiv u_h - u_I$. It easily follows that for $v_h \in \mathbf{P}_{n-l}(T)$,

$$\begin{aligned}((e_h)_s, v_h) &- (\sigma(e_h)_{tt} + \delta(e_h)_{ts} + \gamma(e_h)_{ss}, v_h)\\ &= ([u - u_I]_s, v_h) - (\sigma[u - u_I]_{tt} + \delta[u - u_I]_{ts} + \gamma[u - u_I]_{ss}, v_h).\end{aligned}$$

Hence, we may apply (6.5) with

$$u_h = e_h, \quad f_1 = (u - u_I)_s, \quad f_2 = [u - u_I]_{tt} + \frac{\delta}{\sigma}[u - u_I]_{ts} + \frac{\gamma}{\sigma}[u - u_I]_{ss}.$$

From (5.3), we have that $P_{n-2}f_1 = 0$. Again assuming $u_h = u_I$ on $\Gamma_{\text{in}}(\Omega)$, and using (6.2), (6.5) gives

(6.6)
$$\begin{aligned}&|e'_{h,\text{out}}|^2_{\psi,F_i} + h^{-1}\|(e_h)_s\|^2_{\psi,\Omega_i} + \|\nabla e_h\|^2_{\psi,\Omega_i} + \sigma\|(e_h)_{tt}\|^2_{h,\psi,\Omega_i}\\ &\quad\leq C(h^{-1}\|u - u_I\|^2_{1,\psi,\Omega_i} + \sigma\|u - u_I\|^2_{2,h,\psi,\Omega_i}).\end{aligned}$$

Thus, using the approximation properties (2.4), (2.5) of u_I, we obtain the following error estimates for the method in the case of a smooth solution u.

THEOREM 6.8.

$$|u'_{h,\text{out}} - u'_{\text{out}}|_{\psi,F_i} + \|\nabla(u_h - u)\|_{\psi,\Omega_i} \leq Ch^{n-1/2}\|u\|_{n+1,\psi,\Omega_i},$$
$$\|(u_h)_s - u_s\|_{\psi,\Omega_i} \leq Ch^n\|u\|_{n+1,\psi,\Omega_i}.$$

Local estimates can also be obtained from (6.6), following the method used in §5. Since the main ideas are essentially the same, we only give a statement of the main result.

THEOREM 6.9. *For $\xi > 2n + 7/2$,*

$$\begin{aligned}&|u'_{h,\text{out}} - u'_{\text{out}}|^2_I + h^{-1}\|(u_h - u)_s\|^2_D + \|\nabla(u_h - u)\|^2_D\\ &\quad\leq Ch^{2n-1}\left[\|u\|^2_{n+1,D_h^+} + \epsilon(\|u\|^2_\Omega + |u|^2_{\Gamma_{\text{in}}(\Omega)} + \|f\|^2_\Omega)\right],\end{aligned}$$

where $\epsilon \to 0$ as $h \to 0$.

We have thus established a crosswind spread of $O(\sqrt{h}\log\frac{1}{h})$ for the continuous method (6.1) for convection-diffusion equations with an $O(h)$ diffusion term. The same crosswind spread was shown for the streamline diffusion method [8] and later improved, for the case of linear approximation, to $O(h^{3/4} \log \frac{1}{h})$ for $\sigma = h^{3/2}$ [10].

REFERENCES

[1] P. G. CIARLET, *The Finite Element Method for Elliptic Equations*, North-Holland, Amsterdam, 1978.

[2] P. CLEMENT, *Approximation by finite element functions using local regularization*, RAIRO Modél. Math. Anal. Numér., R-2 (1975), pp. 77–83.

[3] R. S. FALK AND G. R. RICHTER, *Analysis of a continuous finite element method for hyperbolic equations*, SIAM J. Numer. Anal., 24 (1987), pp. 257–278.

[4] T. J. R. HUGHES, *Recent progress in the development and understanding of supg methods with special reference to the compressible Euler and Navier–Stokes equations*, in Finite Elements in Fluids, R. H. Gallagher et al., eds., John Wiley, New York, 1987, pp. 273–287.

[5] T. J. R. HUGHES AND A. BROOKS, *A multidimensional upwind scheme with no crosswind diffusion*, in Finite Element Methods for Convection Dominated Flows, Vol. 34, T. J. R. Hughes, ed., American Society of Mechanical Engineers, 1979, pp. 19–35.

[6] C. JOHNSON, *Finite element methods for convection-diffusion problems*, in Computing Methods in Engineering and Applied Sciences, V, R. Glowinski and J. L. Lions, eds., North-Holland, Amsterdam, 1981.

[7] C. JOHNSON AND U. NÄVERT, *An analysis of some finite element methods for advection-diffusion problems*, in Analytical and Numerical Approaches to Asymptotic Problems in Analysis, O. Axelsson, L. S. Frank, and A. van der Sluis, eds., North-Holland, Amsterdam, 1981, pp. 99–116.

[8] C. JOHNSON, U. NÄVERT, AND J. PITKÄRANTA, *Finite element methods for linear hyperbolic problems*, Comput. Methods Appl. Mech. Engrg., 45 (1984), pp. 285–312.

[9] C. JOHNSON AND J. PITKÄRANTA, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp., 46 (1986), pp. 1–26.

[10] C. JOHNSON, A. H. SCHATZ, AND L. B. WAHLBIN, *Crosswind smear and pointwise errors in streamline diffusion finite element methods*, Math. Comp., 49 (1987), pp. 25–38.

[11] P. LESAINT AND P. A. RAVIART, *On a finite element method for solving the neutron transport equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. deBoor, ed., Academic Press, New York, 1974, pp. 89–123.

[12] K. NIIJIMA, *Pointwise error estimates for a streamline diffusion finite element scheme*, Numer. Math., 56 (1990), pp. 707–719.

[13] T. E. PETERSON, *Towards understanding the discontinuous Galerkin method*, Centre for Mathematics and its Applications Tech. Report, CMA-MR02-91, Canberra, Australia.

[14] W. H. REED AND T. R. HILL, *Triangular mesh methods for the neutron transport equation*, Los Alamos Scientific Laboratory Report, LA-UR-73-479, 1973, Los Alamos, NM.

[15] G. R. RICHTER, *A finite element method for time dependent convection-difbreak fusion equations*, Math. Comp., 54 (1990), pp. 81–106.

[16] ———, *An explicit finite element method for convection-dominated steady state convection-diffusion equations*, SIAM J. Numer. Anal., 28 (1991), pp. 744–759.

[17] ———, *The discontinuous Galerkin method with diffusion*, Math Comp., to appear.

[18] ———, *Explicit finite element methods for scalar hyperbolic equations*, Appl. Numer. Math., 7 (1991), pp. 509–521.

[19] L. R. SCOTT AND S. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.

[20] M. I. VISHIK AND L. A. LYUSTERNIK, *Regular degeneration and boundary layer for linear differential equations with a small parameter*, Uspekhi Mat. Nauk., 12 (1957), pp. 3–122; Amer. Math. Soc. Transl., Ser. 2, 20 (1962), pp. 239–364.

[21] R. WINTHER, *A stable finite element method for first-order hyperbolic systems*, Math. Comp., 36 (1981), pp. 65–86.