12. Finite element methods for parabolic problems

We consider the parabolic problem:

$$u_t - \text{div}(p \nabla u) + qu = f, \quad (x,t) \in \Omega \times (0,T],$$
$$u(x,t) = 0, \quad (x,t) \in \partial \Omega \times (0,T], \quad u(x,0) = g(x), \quad x \in \Omega.$$ 

A variational formulation of this problem is to seek $$u(t) \in \dot{H}^1(\Omega)$$ such that

$$\langle \partial u/\partial t, v \rangle + a(u,v) = \langle f, v \rangle, \quad v \in \dot{H}^1(\Omega),$$

where as in the elliptic case, $$(\cdot, \cdot)$$ denotes the $$L^2(\Omega)$$ inner product and

$$a(u,v) = \int_{\Omega} [p \nabla u \cdot \nabla v + quv] \, dx.$$ 

12.1. Continuous time Galerkin scheme. We first consider an approximation in which we discretize by finite elements in the spatial variable, but keep time continuous. Thus, we choose a finite dimensional subspace $$V_h \subset \dot{H}^1(\Omega)$$ and look for an approximation $$u_h(t) \in V_h,$$ satisfying:

$$u_h(0) = g_h, \quad \langle \partial u_h/\partial t, v \rangle + a(u_h,v) = \langle f, v \rangle, \quad v \in V_h.$$ 

To see what is involved in solving this problem, we write

$$u_h(t) = \sum_{j=1}^m \alpha_j(t) \phi_j(x).$$

Inserting this into the variational equations, and choosing $$v$$ to be each of the basis functions $$\phi_i,$$ we get

$$\sum_{j=1}^m \alpha_j'(t) \phi_j(\phi_i) + \sum_{j=1}^m \alpha_j(t) a(\phi_j, \phi_i) = \langle f, \phi_i \rangle, \quad i = 1, \ldots, m.$$ 

Let

$$M_{ij} = \langle \phi_j, \phi_i \rangle, \quad A_{ij} = a(\phi_j, \phi_i), \quad F_i = \langle f, \phi_i \rangle, \quad \alpha = (\alpha_1, \ldots, \alpha_m)^T.$$ 

Our equations then have the form

$$M \alpha' + A \alpha = F,$$

a first order system of ordinary differential equations.

One can obtain a simple error estimate for this approximation scheme by comparing the approximate solution to the elliptic projection $$w_h(t) \in V_h,$$ satisfying

$$a(u(t) - w_h(t), v_h) = 0, \quad v_h \in V_h.$$ 

We showed previously that if $$V_h$$ consists of piecewise polynomials of degree $$\leq r,$$ and $$u$$ is sufficiently smooth, then

$$\|u(t) - w_h(t)\|_1 + h\|u(t) - w_h(t)\|_1 \leq C h^{r+1} |u(t)|_{r+1}.$$ 

**Theorem 12.** If $$V_h$$ consists of piecewise polynomials of degree $$\leq r$$ and $$u$$ is sufficiently smooth, then for $$t \geq 0,$$

$$\|u(t) - u_h(t)\| \leq \|g - g_h\| + C h^{r+1} \left[ \|g\|_{r+1} + \int_0^t \|u\|_{r+1} \, ds \right].$$
Proof. We estimate the error by writing \( u - u_h = (u - w_h) + (w_h - u_h) \). From the above, we have
\[
\|u(t) - w_h(t)\| \leq C h^{r+1} \|u(t)\|_{r+1} \leq C h^{r+1} \|u(0) + \int_0^t u_t(s) \, ds\|_{r+1} \\
\leq C h^{r+1} \left[ \|g\|_{r+1} + \int_0^t \|u_t(s)\|_{r+1} \, ds \right].
\]

It thus remains to estimate \( \|u_h - w_h\| \). Using the continuous and discrete variational formulations and the definition of \( w_h(t) \), we get
\[
(\partial [u_h - w_h]/\partial t, v) + a(u_h - w_h, v) = (\partial [u - u]/\partial t, v) + a(u_h - u, v) \\
+ (\partial [u - w_h]/\partial t, v) + a(u - w_h, v) = (\partial [u - w_h]/\partial t, v), \quad v \in V_h.
\]
Choosing \( v = u_h - w_h \), and observing that
\[
\|u_h - w_h\| \frac{d}{dt} \|u_h - w_h\| = \frac{1}{2} \frac{d}{dt} \|u_h - w_h\|^2 = ([u_h - w_h]_t, u_h - w_h),
\]
we get
\[
\|u_h - w_h\| \frac{d}{dt} \|u_h - w_h\| + \|u_h - w_h\|^2 = ([u_h - w_h]_t, u_h - w_h) \leq \|[u_h]_t\| \|u_h - w_h\|.
\]
Hence,
\[
\frac{d}{dt} \|u_h - w_h\| \leq \|[u_h]_t\|.
\]
Integrating this equation between 0 and \( t \), we get
\[
\|u_h(t) - w_h(t)\| \leq \|u_h(0) - w_h(0)\| + \int_0^t \|[u_h(t)]_t(s)\| \, ds \\
\leq \|u_h(0) - u(0)\| + \|u(0) - w_h(0)\| + \int_0^t \|[u_h]_t(s)\| \, ds.
\]
Using the triangle inequality, and combining estimates, we then obtain
\[
\|u(t) - u_h(t)\| \leq \|u(t) - w_h(t)\| + \|u_h(t) - w_h(t)\| \\
\leq \|g - g_h\| + C h^{r+1} \left[ \|g\|_{r+1} + \int_0^t \|u_t(s)\|_{r+1} \, ds \right].
\]

\[ \square \]

12.2. Fully discrete schemes. One way to get a fully discrete scheme is to combine the use of finite elements to discretize the spatial variable with a finite difference approximation in time. For example, if we approximate \( u_t \) by the backward Euler approximation, we get the scheme: Find \( U^n \in V_h \), satisfying \( U^0 = g_h \) and for \( n \geq 0 \)
\[
([U^{n+1} - U^n]/k, v) + a(U^{n+1}, v) = (f^{n+1}, v) \quad v \in V_h.
\]
Using the matrices defined previously, and defining $U^n(x) = \sum_{j=1}^{m} \alpha^n \phi_j(x)$, the discrete variational formulation above corresponds to the linear system

$$(M + kA)\alpha^{n+1} = M\alpha^n + kF^{n+1}, \quad n = 0, 1, \ldots.$$  

Another choice is the Crank-Nicholson-Galerkin method, which has the form: Find $U^n \in V_h$, satisfying $U^0 = g_h$ and for $n \geq 0$

$$
\left(\frac{[U^{n+1} - U^n]}{k}, v\right) + a(\frac{[U^{n+1} + U^n]}{2}, v) = (\frac{[f^{n+1} + f^n]}{2}), \quad v \in V_h.
$$

In this case, we get the linear system

$$(M + \frac{1}{2}kA)\alpha^{n+1} = (M - \frac{1}{2}kA)\alpha^n + k(F^{n+1} + F^n)/2, \quad n = 0, 1, \ldots.$$  

For the backward Euler method, we have the following error estimate ($t_n = nk$).

**Theorem 13.**

$$
\|u(t_n) - U^n\| \leq \|g - g_h\| + Ch^{r+1} \left[ \|g\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1} \right] + k \int_0^{t_n} \|u_{tt}(s)\| \, ds, \quad n \geq 0.
$$

**Proof.** As before, we write $u(t_n) - U^n = (u(t_n) - W^n) + (W^n - U^n)$, where $W^n = w_h(t_n) \in V_h$ (the elliptic projection) satisfies

$$a(u(t) - w_h(t), v_h) = 0, \quad v_h \in V_h.$$  

From our previous result, we have

$$
\|u(t_n) - W^n\| \leq Ch^{r+1} \left[ \|g\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1} \, ds \right].
$$

To estimate $U^n - W^n$, we again use our continuous and discrete variational formulations, but this time obtaining

$$
\left(\frac{[(U - W)^{n+1} - (U - W)^n]}{k}, v\right) + a((U - W)^{n+1}, v) = \left(\frac{[(U - u)^{n+1} - (U - u)^n]}{k}, v\right)
$$

$$
+ a((U - u)^{n+1}, v) + \left(\frac{[(u - W)^{n+1} - (u - W)^n]}{k}, v\right) + a((u - W)^{n+1}, v)
$$

$$
= (u_t^{n+1} - [u^{n+1} - u^n]/k, v) + \left(\frac{[(u - W)^{n+1} - (u - W)^n]}{k}, v\right) \equiv (\rho^n, v) \quad v \in V_h.
$$

Choosing $v = (U - W)^{n+1}$, we get

$$
\|(U - W)^{n+1}\|^2 + k\|(U - W)^{n+1}\|_E^2 = ((U - W)^n, (U - W)^{n+1}) + k(\rho^n, (U - W)^{n+1})
$$

$$
\leq \|(U - W)^n\| + k\|\rho^n\|\|(U - W)^{n+1}\|.
$$

Hence,

$$
\|(U - W)^{n+1}\| \leq \|(U - W)^n\| + k\|\rho^n\|.
$$

Iterating this equation, we get

$$
\|(U - W)^n\| \leq \|(U - W)^0\| + k \sum_{j=0}^{n-1} \|\rho^j\|.
By Taylor series,
\[ u(x, t) = u(x, t + k) - ku_t(x, t + k) + \int_t^{t+k} (s - t)u_{tt}(s) \, ds. \]

Hence, for \( t_j = jk \),
\[ u^{j+1} - [u^{j+1} - u^j] / k = k^{-1} \int_{t_j}^{t_{j+1}} (s - t_j)u_{tt}(s) \, ds. \]

So
\[ \| \rho^j \| = \left\| k^{-1} \int_{t_j}^{t_{j+1}} [(s - t_j)u_{tt}(s) + (u - w_h)_t(s)] \, ds \right\| \]
\[ \leq k^{-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + \|u - w_h\|_t(s)] \, ds \]
\[ \leq k^{-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + Ch^{r+1}\|u_t(s)\|_{r+1}] \, ds. \]

Hence,
\[ k \sum_{j=0}^{n-1} \| \rho^j \| \leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + Ch^{r+1}\|u_t(s)\|_{r+1}] \, ds \]
\[ \leq k \int_{t_0}^{t_n} \|u_{tt}(s)\| \, ds + Ch^{r+1} \int_{t_0}^{t_n} \|u_t(s)\|_{r+1} \, ds. \]

The theorem follows by combining all these results. \( \square \)