1. Finite Difference Methods for Elliptic Equations

1.1. The Dirichlet problem for Poisson’s equation. We consider the finite difference approximation of the boundary value problem:

Problem P: \(-\Delta u = f\) in \(\Omega\), \(u = g\) on \(\partial\Omega\).

For simplicity, we first consider the case when \(\Omega\) is the unit square \((0,1) \times (0,1)\). To obtain a finite difference approximation, we place a mesh of width \(h\) with sides parallel to the coordinate axes on \(\bar{\Omega}\) (\(\Omega\) together with its boundary \(\partial\Omega\)) and denote the set of mesh points lying inside \(\Omega\) by \(\Omega_h\) and the set of mesh points lying on the \(\partial\Omega\) by \(\partial\Omega_h\). We then seek numbers \(u_{ij}\) as approximations to the true solution \(u(ih, jh)\), where \(i, j = 0, 1, \ldots, N\) and \(Nh = 1\). To obtain \(u_{ij}\), we derive a system of equations that approximate the equations determining the true solution \(u(ih, jh)\), i.e., the equations

\[-\Delta u(ih, jh) = f(ih, jh), \quad (ih, jh) \in \Omega.\]

To get these approximate equations, we use Taylor series expansions, i.e., we write

\[u(x \pm h, y) = u(x, y) \pm h \frac{\partial u}{\partial x}(x, y) + \frac{h^2}{2} \frac{\partial^2 u}{\partial x^2}(x, y) \pm \frac{h^3}{6} \frac{\partial^3 u}{\partial x^3}(x, y) + \frac{h^4}{24} \frac{\partial^4 u}{\partial x^4}(\xi, y),\]

for some \(x \leq \xi_+ \leq x + h\) and \(x - h \leq \xi_- \leq x\). Adding these equations, we get

\[u(x + h, y) - 2u(x, y) + u(x - h, y) = h^2 \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{h^4}{24} \left( \frac{\partial^4 u}{\partial x^4}(\xi_+, y) + \frac{\partial^4 u}{\partial x^4}(\xi_-, y) \right) = h^2 \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{h^4}{12} \frac{\partial^4 u}{\partial x^4}(\xi, y),\]

where \(x - h \leq \xi \leq x + h\) and we have used the Mean Value Theorem for sums in the last step, i.e., if \(g_i \geq 0\) and \(\sum_{i=1}^{M} g_i = 1\), then there is a number \(c\) satisfying \(\min x_i \leq c \leq \max c_i\) such that \(\sum_{i=1}^{M} g_if(x_i) = f(c)\). Using similar expansions in the \(y\) variable, we get

\[u(x, y + h) - 2u(x, y) + u(x, y - h) = h^2 \frac{\partial^2 u}{\partial y^2}(x, y) + \frac{h^4}{12} \frac{\partial^4 u}{\partial y^4}(x, \eta),\]

where \(y - h \leq \eta \leq y + h\). Adding these equations and dividing by \(h^2\), we get

\[\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = \frac{1}{h^2} \{u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h) - 4u(x, y)\} + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi, y) + \frac{h^2}{12} \frac{\partial^4 u}{\partial y^4}(x, \eta).\]

Defining a finite difference operator

\[\Delta_h u(x, y) = \frac{1}{h^2} \{u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h) - 4u(x, y)\},\]

and supposing that

\[\max_{(x, y) \in \Omega} \left| \frac{\partial^4 u}{\partial x^4}(x, y) \right| \leq M_4, \quad \max_{(x, y) \in \Omega} \left| \frac{\partial^4 u}{\partial y^4}(x, y) \right| \leq M_4,\]
and so on, where we get

\[ |\Delta u(ih, jh) - \Delta_h u(ih, jh)| \leq \frac{M_4}{6} h^2. \]

We then use the discrete Laplace operator \( \Delta_h \) to define a set of discrete equations from which we can determine \( u_{ij} \), i.e., we consider the problem:

**Problem P\(_h\):** Find a mesh function \( U_h = (u_{ij}) \) (i.e., \( U_h \) is only defined at the mesh points), such that

\[ -\Delta_h U_h = f \text{ on } \Omega_h, \quad U_h = g \text{ on } \partial \Omega_h. \]

This is a system of \( (N + 1)^2 \) linear equations for the \( (N + 1)^2 \) unknowns \( u_{ij}, i, j = 0, \ldots, N \). We next consider the form of these equations in the special case \( h = 1/4 \).

Since the boundary values \( u_{00}, u_{10}, u_{20}, u_{30}, u_{40}, u_{01}, u_{02}, u_{03}, u_{04}, u_{11}, u_{21}, u_{31}, u_{32}, u_{33} \) are all given by the corresponding values of \( g \), we need only determine the values \( u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33} \). The equations for these are:

\[ 4u_{11} - u_{21} - u_{12} - u_{01} - u_{10} = h^2 f_{11}, \]
\[ 4u_{21} - u_{31} - u_{11} - u_{20} = h^2 f_{21} \]

and so on, where \( f_{ij} = f(ih, jh) \). Rewriting in matrix form, we get

\[
\begin{pmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix}
u_{11} \\
u_{21} \\
u_{31} \\
u_{12} \\
u_{22} \\
u_{32} \\
u_{13} \\
u_{23} \\
u_{33}
\end{pmatrix}
= \begin{pmatrix}
h^2 f_{11} + u_{10} + u_{01} \\
h^2 f_{21} + u_{20} \\
h^2 f_{31} + u_{30} + u_{41} \\
h^2 f_{12} + u_{02} \\
h^2 f_{22} \\
h^2 f_{32} + u_{42} \\
h^2 f_{33} + u_{33} + u_{43} \\
h^2 f_{23} + u_{24} \\
h^2 f_{33} + u_{34} + u_{43}
\end{pmatrix}
\]

Note several properties of this linear system: (1) The right hand side is known; (2) the symmetry properties of the matrix depend on the ordering of the elements; (3) this is an example of a sparse matrix – many zero elements. If we decreased the mesh size further, we would introduce more zeroes.

**Questions:** (1) Does this type of linear system have a unique solution? (2) How does the error between the true and approximate solutions depend on the mesh size \( h \)? (3) What is an efficient way to solve the linear system when the number of equations becomes large?

To see that this linear system always has a unique solution, we first establish a property of the discrete Laplace operator \( \Delta_h \), known as a discrete maximum principle.

**Theorem 1.** (i) If \( v \) is a function defined on \( \Omega_h \cup \partial \Omega_h \) and satisfies \( \Delta_h v(x, y) \geq 0 \) for all \( (x, y) \in \Omega_h \), then \( \max_{\Omega_h} v \leq \max_{\partial \Omega_h} v \).

(ii) Alternatively, if \( v \) satisfies \( \Delta_h v(x, y) \leq 0 \) for all \( (x, y) \in \Omega_h \), then \( \min_{\Omega_h} v \geq \min_{\partial \Omega_h} v \).
**Proof.** The proof is by contradiction. Let \((x_0, y_0) \in \Omega_h\) at which \(v\) has a maximum, i.e., \(v(x_0, y_0) = M\), where \(M \geq v(x, y)\) for all \((x, y) \in \Omega_h\) and \(M > v(x, y)\) for \((x, y) \in \partial \Omega_h\). By assumption, \(\Delta_h v(x_0, y_0) \geq 0\). Hence,

\[
M = v(x_0, y_0) \leq \frac{1}{4} \{ v(x_0 + h, y_0) + v(x_0 - h, y_0) + v(x_0, y_0 + h) + v(x_0, y_0 - h) \}.
\]

But \(M \geq v(x, y)\) then implies that \(v(x_0 \pm h, y_0) = M\) and \(v(x_0, y_0 \pm h) = M\). Repeating this argument, we eventually conclude that \(v(x, y) = M\) for all \((x, y) \in \Omega_h \cup \partial \Omega_h\). This contradicts our initial assumption, so (i) follows. To establish (ii), we let \(w(x, y) = -v(x, y)\). Then \(\Delta_h w(x, y) = -\Delta_h v(x, y) \geq 0\), so by (i), \(\max_{\Omega_h} [-v(x, y)] \leq \max_{\partial \Omega_h} [-v(x, y)]\). But \(\max [-v(x, y)] = -\min v(x, y)\), so \(-\min_{\Omega_h} v(x, y) \leq -\min_{\partial \Omega_h} [v(x, y)]\). Then (ii) follows by multiplying by \((-1)\), which reverses the sign of the inequality. \(\square\)

We note that we can extend this result to non-square domains.

**Theorem 2.** The linear system of equations corresponding to the difference equations

\[-\Delta_h U_h(x, y) = f(x, y), \quad (x, y) \in \Omega_h, \quad U_h(x, y) = g(x, y), \quad (x, y) \in \partial \Omega_h\]

has a unique solution.

**Proof.** We use the fact that a square linear system \(A z = b\) will have a unique solution if and only if the only solution of the homogeneous system \(A z = 0\) is \(z = 0\). Hence, we need to show that the only solution to Problem \(P_h\) when \(f\) and \(g\) are zero is \(U_h = 0\). But by Theorem 1, since \(\Delta_h U\) is both \(\geq 0\) and \(\leq 0\), both the maximum and minimum of \(U(x, y)\) occur on \(\partial \Omega_h\). Hence, \(0 \leq U(x, y) \leq 0\) for all \((x, y) \in \Omega_h \cup \partial \Omega_h\) and so \(U \equiv 0\). \(\square\)