MATH 575 ASSIGNMENT 7

1. Consider the approximation of $u_t = \sigma u_{xx}$ given for $0 \le \theta \le 1$ by

$$[U_j^{n+1} - U_j^n]/k = \frac{\sigma}{h^2}\{(1-\theta)[U_{j+1}^n - 2U_j^n + U_{j-1}^n] + \theta[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}]\}.$$

Show that for $0 \le \theta \le 1/2$, the scheme is stable when $\sigma k/h^2 \le 1/[2(1-2\theta)]$ and for $1/2 \le \theta \le 1$, the scheme is unconditionally stable. HINT: Use the identity $\cos(x) = 1 - 2\sin^2(x/2)$ to write the amplification factor

$$\lambda = \left[1 - \frac{4\sigma k}{h^2}(1-\theta)\sin^2(ph/2)\right] \Big/ \left[1 + \frac{4\sigma k}{h^2}\theta\sin^2(ph/2)\right]$$

and then show that under the conditions given above, $|\lambda| \le 1$.

2. The simplest three level explicit scheme for the solution of the heat equation $u_t = u_{xx}$ is given by

$$[U_j^{n+1} - U_j^{n-1}]/(2k) = [U_{j+1}^n - 2U_j^n + U_{j-1}^n]/h^2.$$

Show that the scheme is unstable for all values of the constant $r = k/h^2 > 0$.

HINT: Write the equation as a two level system:

$$\begin{aligned} U_j^{n+1} &= 2k[U_{j+1}^n - 2U_j^n + U_{j-1}^n]/h^2 + V_j^n, \\ V_j^{n+1} &= U_j^n, \end{aligned}$$

and show that the eigenvalues of the amplification matrix G are given by

$$\lambda = -4r\sin^2(ph/2) \pm \sqrt{16r^2\sin^4(ph/2) + 1}.$$