MATH 575 ASSIGNMENT 6

1. Consider the approximation of the one dimensional heat equation $u_t = \sigma u_{xx}$ by the generalized Crank-Nicholson method:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{\sigma}{h^2}\{(1-\theta)[u_{j+1}^n - 2u_j^n + u_{j-1}^n] + \theta[u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}]\}, \qquad 0 \le \theta \le 1.$$

a) The local trunctation error for this method can be defined either as

$$\begin{aligned}\tau(x,t) = u_t(x,t) - \sigma u_{xx}(x,t) - &\frac{u(x,t+k) - u(x,t)}{k} \\ &+ \frac{\sigma}{h^2}\{(1-\theta)[u(x+h,t) - 2u(x,t) + u(x-h,t)] \\ &\qquad + \theta[u(x+h,t+k) - 2u(x,t+k) + u(x-h,t+k)]\},\end{aligned}$$

if we think of the difference operator as an approximation to the PDE at (x,t) or as

$$\tau(x,t) = (1-\theta)[u_t(x,t) - \sigma u_{xx}(x,t)] + \theta[u_t(x,t+k) - \sigma u_{xx}(x,t+k)] - \frac{u(x,t+k) - u(x,t)}{k}$$
$$+ \frac{\sigma}{h^2}\{(1-\theta)[u(x+h,t) - 2u(x,t) + u(x-h,t)] + \theta[u(x+h,t+k) - 2u(x,t+k) + u(x-h,t+k)]\},$$

if we think of the difference operator as an approximation to a weighted combination of the PDE at two time levels. Choose one of these definitions and show that the local truncation error of this method is $O(k + h^2)$.

b) If u satisfies the heat equation and is sufficiently smooth and $\theta = 1/2$, show that the error in part (a) is $O(k^2 + h^2)$.

2. Write a Matlab program to approximate the initial boundary value problem:

$$u_t = u_{xx}, \qquad 0 < x < 1, \qquad t > 0,$$
$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = x(1-x)$$

by the explicit method

$$[u_j^{n+1} - u_j^n]/k = [u_{j+1}^n - 2u_j^n + u_{j-1}^n]/h^2$$

for the three mesh sizes (i) $k = 1/256$, $h = 1/16$, (ii) $k = 1/512$, $h = 1/16$, and (iii) $k = 1/1024$, $h = 1/16$. Plot the approximate solution and true solution on the same plot at $t = 1$. Explain your results. Note that the true solution

$$u(x,t) = \frac{8}{\pi^3} \sum_{n=1,3,5,\ldots} \frac{1}{n^3} e^{-n^2\pi^2 t} \sin(n\pi x)$$

and the first two terms of the series give a good approximation. See the file *heatsoln.m* for some help in defining and plotting the exact solution.

3. Consider the approximation of the transport equation $u_t + au_x = 0$ by the difference scheme:

$$L_{h,k}U_j^n \equiv \left[U_j^{n+1} - \frac{1}{2}(U_{j+1}^n + U_{j-1}^n)\right]/k + a[U_{j+1}^n - U_{j-1}^n]/(2h) = 0.$$

a) Show that if $h = O(k)$, the local truncation error of the method is $O(k)$.

b) Show that if $\lambda \equiv ak/h$ satisfies $|\lambda| \leq 1$, then

$$\max_{j,\ 0\leq n\leq N} |W_j^n| \leq \max_j |W_j^0| + T \max_{j,\ 0\leq n\leq N-1} |L_{h,k}W_j^n|$$

for all mesh functions W_j^n, where $T = Nk$.

c) Show that for the pure initial value problem

$$u_t + au_x = 0, \qquad u(x,0) = \phi(x),$$

the error estimate, $\max_j |u - U| \leq O(k)$ is valid for $0 \leq n \leq N$ under the hypothesis of part (b).