

MATH 575 ASSIGNMENT 5

1. Let x be the solution of $Ax = b$ and consider approximations to x given by the iteration:

$$x^{k+1} = [I - \alpha A]x^k + \alpha b,$$

where A is the $n \times n$ matrix given by

$$A = \begin{pmatrix} 1+c & c & \cdots & c \\ c & 1+c & \cdots & c \\ \vdots & & \ddots & \vdots \\ c & c & \cdots & 1+c \end{pmatrix}.$$

- 1a. Show that if $c \geq 0$, then for all $x \neq 0$, $x^T Ax > 0$, i.e., A is positive definite.
- 1b. Show that $x = (1, 1, \dots, 1)^T$ is an eigenvector of A and find the corresponding eigenvalue.
- 1c. For $i = 1, \dots, n-1$, let x^i denote the vector defined by $x^i_i = 1$, $x^i_n = -1$, and $x^i_j = 0$ if $j \neq i$ and $j \neq n$. Show that x^i is an eigenvector of A and find the corresponding eigenvalue.
- 1d. Use these results and the result proved in class about the convergence of this iterative method to find the values of α for which this iteration converges when $c = 1/n$.

2. Consider the solution of the linear system $Ax = b$ by the iterative method

$$x^{k+1} = (1 - \alpha)x^k + \alpha P^{-1}(b - Qx^k),$$

where $A = P + Q$ and α is a positive constant.

- 2a. Find the iteration matrix M which satisfies

$$x - x^{k+1} = M(x - x^k),$$

where x is the exact solution of the linear system.

- 2b. Show that M can also be written in the form $I - \alpha P^{-1}A$, where I is the identity matrix.
- 2c. If $P^{-1}A$ is symmetric and positive definite, for what values of α will this iteration converge?

3. We wish to approximate the solution of the problem

$$-\Delta u = 1, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega.$$

where $\Omega = (0, 1) \times (0, 1)$. Write a *Matlab* program to approximate this problem using a finite difference method where Δu is approximated by the 5-point difference approximation and a mesh of size $h = 1/N$. Order the unknowns from left to right in each row, starting from the smallest value of y , and then proceed to the next row.

3a. To make sure the matrix A is being generated properly, print out and hand in the matrix A for the case $N = 4$ that is generated by your computer code.

3b. Next, solve the linear system of equations $Ax = b$ for the choice $N = 100$ using *Matlab's* built in `pcg` function, which solves by the preconditioned conjugate gradient method.

First try the command `[x0,f10,rr0,it0,rv0] = pcg(A,b,1e-8,100);`

Read the document page to see what the various arguments stand for. Check the flag `f10` to see if the method converges before the maximum number of iterations is reached.

Next try the preconditioned version using the incomplete Cholesky factorization with zero fill-in as the preconditioner. Since this requires that A be a sparse matrix, we first need to convert A to a sparse matrix, if you have not stored it in this form already. This is done by typing:

```
As = sparse(A);
```

```
L = ichol(As);
```

```
[x1,f11,rr1,it1,rv1] = pcg(As,b,1e-8,100,L,L');
```

Hand in the values of `rr1` and `it1`, along with a copy of your computer program.