MATH 575 ASSIGNMENT 5

1. Let x be the solution of $Ax = b$ and consider approximations to x given by the iteration:

$$x^{k+1} = [I - \alpha A]x^k + \alpha b,$$

where A is the $n \times n$ matrix given by

$$A = \begin{pmatrix} 1+c & c & \cdots & c \\ c & 1+c & \cdots & c \\ \vdots & & \ddots & \vdots \\ c & c & \cdots & 1+c \end{pmatrix}.$$

1a. Show that if $c \geq 0$, then for all $x \neq 0$, $x^T Ax > 0$, i.e., A is positive definite.

1b. Show that $x = (1, 1, \ldots, 1)^T$ is an eigenvector of A and find the corresponding eigenvalue.

1c. For $i = 1, \ldots, n-1$, let x^i denote the vector defined by $x^i_i = 1$, $x^i_n = -1$, and $x^i_j = 0$ if $j \neq i$ and $j \neq n$. Show that x^i is an eigenvector of A and find the corresponding eigenvalue.

1d. Use these results and the result proved in class about the convergence of this iterative method to find the values of α for which this iteration converges when $c = 1/n$.

2. Let x be the solution of $Ax = b$ and consider approximations to x given by the iteration:

$$x^{k+1} = [I - \alpha A]x^k + \alpha b.$$

Assuming that A is symmetric and positive definite and $\alpha = 2/(\lambda_1 + \lambda_n)$, where λ_1 is the smallest eigenvalue of A and λ_n is the largest eigenvalue of A, show that if

$$k \geq \frac{1}{2}\frac{\lambda_n}{\lambda_1} \ln \frac{1}{\epsilon},$$

then $\|x^k - x\|_2 \leq \epsilon \|x^0 - x\|_2$. Hint: Start from the result proved in class that under the hypotheses of the problem, $\|x - x^k\|_2 \leq \left(\frac{\kappa-1}{\kappa+1}\right)^k \|x - x^0\|$, where $\kappa = \lambda_{\max}(A)/\lambda_{\min}(A)$. Then use the fact that $\ln\left[(1+x)/(1-x)\right] \geq 2x$ for $0 < x < 1$. Choose $x = \lambda_1/\lambda_n$.

3. Use FeNiCS to approximate the solution of the stationary Stokes equations

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \qquad \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \qquad \boldsymbol{u} = 0 \text{ on } \partial\Omega,$$

where $\Omega = (0,1) \times (0,1)$, and

$$f_1 = 12(1-2y)x^2(1-x)^2 - 4y(1-y)(1-2y)(6x^2 - 6x + 1) + 5(x - 1/2)^4,$$
$$f_2 = -12(1-2x)y^2(1-y)^2 + 4x(1-x)(1-2x)(6y^2 - 6y + 1) + 5(y - 1/2)^4.$$

The exact solution of this problem is given by

$$\boldsymbol{u} = \operatorname{curl}[x^2(1-x)^2y^2(1-y)^2] = \begin{pmatrix} 2x^2(1-x)^2y(1-y)(1-2y) \\ -2y^2(1-y)^2x(1-x)(1-2x) \end{pmatrix},$$

$$p = (x-1/2)^5 + (y-1/2)^5.$$

Do this computation for two choices of finite element spaces: (i) V_h = continuous piecewise quadratic functions and Q_h = piecewise constants, and (ii) V_h = continuous piecewise quadratic functions and Q_h = continuous piecewise linear functions. Use the mesh command `mesh = UnitSquare(n,n)`, where n takes the values 4,8,16,64. Hand in a table of L^2 and H^1 errors for the approximation of $\boldsymbol{u}$ and L^2 errors for the approximation of p. Also include in the table a calculation of the rates of convergence.

It is a little tricky to compute the errors and rates of convergence for p because p is not uniquely defined (i.e., if p is a solution, so is $p+c$ for any constant c). The formula given for the exact p has average value zero on the domain. Hence, the computed p_h must be modified so that it also has mean value zero, before the errors and convergence rates are computed.

Some useful FeNiCS commands to help you write the programs are:

`pow(x[0]-.5,5.)` implements $(x-1/2)^5$.

```
# Define function spaces
     deg = 2
     V = VectorFunctionSpace(mesh, "CG", deg) # CG -- continuous elements
     Q = FunctionSpace(mesh, "DG", 0) # DG -- discontinuous elements
     W = V*Q
     U = Function(W)  # finite element solution (includes both u and p)

# Define variational problem
     (v,q) = TestFunctions(W)
     (u,p) = TrialFunctions(W)
      b = inner(grad(u), grad(v))*dx - div(v)*p*dx + q*div(u)*dx
```

Once the variational problem is defined and U computed, the command
`u, p = U.split()` will split U into the two functions u and p.