MATH 575 ASSIGNMENT 5

1. This is a computational problem to be carried out with Matlab. Consider the boundary value problem

$$-\epsilon u'' + u' = 1, \quad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

where $\epsilon > 0$ is a constant.

a) Verify that the exact solution is

$$u(x) = x - \frac{e^{-(1-x)/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}}.$$

b) Hand in plots of the solution for $\epsilon = 0.1$, 0.01, and 0.001. This may be done using the Matlab commands

```
eps = 0.1;
x = 0:.001:1;
uexact = x - (exp(-(1-x)/eps) - exp(-1/eps))/(1 - exp(-1/eps));
plot(x,uexact)
```

c) A variational formulation of this boundary value problem is:

Find $u \in \mathring{H}^1(0,1) : a(u,v) = F(v)$ for all $v \in \mathring{H}^1(0,1)$, where

$$a(u,v) = \int_0^1 [\epsilon u'v' + u'v]\,dx, \qquad F(v) = \int_0^1 1\,v\,dx.$$

We wish to approximate the solution of this problem using the finite element method based on this variational formulation with continuous piecewise linear elements on two different meshes with general mesh points $0 = x_0 < x_1 < ... < x_{N-1} < x_N = 1$.

Let $\phi_i(x)$ be the piecewise linear basis function corresponding to the mesh point x_i, i.e., $\phi_i(x)$ is given by the formula

$$\phi_i(x) = \begin{cases} 0, & x \le x_{i-1}, \\ (x - x_{i-1})/(x_i - x_{i-1}), & x_{i-1} \le x \le x_i, \\ (x_{i+1} - x)/((x_{i+1} - x_i), & x_i \le x \le x_{i+1}, \\ 0, & x \ge x_{i+1}. \end{cases}$$

Determine formulas for the elements of the matrix $A_{ij} = a(\phi_j, \phi_i)$ and of the right hand side vector $b_i = F(\phi_i)$. Note that for this problem the matrix A is not symmetric. However, it is enough to give formulas for the quantities $a(\phi_i, \phi_i)$, $a(\phi_{i+1}, \phi_i)$, $a(\phi_{i-1}, \phi_i)$, $a(\phi_j, \phi_i)$, $j \le i-2$ and $j \ge i+2$, and $F(\phi_i)$.

d) Compute the approximate solution on a uniform mesh of width $h = 1/N$ for the choices $N = 8, 16, 32, 64, 128$. Hand in a table with the errors $\|u_h - u_I\|_{L^2(0,1)}$ and $\|u_h' - u_I'\|_{L^2(0,1)}$ for each of the following choices: $\epsilon = 0.1, 0.001, 0.00001$, where u_I is the piecewise linear

interpolant of u. Note that these are not exactly the same choices of ϵ as in part (b). Letting $e_i = u_h(x_i) - u_I(x_i)$, these norms may be computed in the following way.

$$\|u_h' - u_I'\|^2_{L^2(0,1)} = \sum_{i=1}^N \frac{[e_i - e_{i-1}]^2}{(x_i - x_{i-1})}, \qquad \|u_h - u_I\|^2_{L^2(0,1)} = \frac{1}{3}\sum_{i=1}^N (x_i - x_{i-1})[e_i^2 + e_{i-1}^2 + e_i e_{i-1}].$$

Also hand in a plot of the approximate solution for each value of ϵ and the choice $N = 128$. The solution of your linear system will produce a vector u with components u_i, $i = 1, \dots, N-1$. This may be plotted in Matlab using the command `plot(u)`. Note that the horizontal axis corresponds to the index i, and not the mesh point x_i. This distorts the graph, but gives a better idea of what is happening near the boundary $x = 1$.

e) Next, let $M = N/2$ and set $h_1 = \bar{x}/M$, $h_2 = (1 - \bar{x})/M$, where $\bar{x} \in (0,1)$ is defined as $\bar{x} = 1 - \min(0.5, 2.5\,\epsilon \ln(N))$. Define mesh points by

$$x_i = ih_1, i = 0, 1, \dots M, \qquad x_{M+i} = \bar{x} + ih_2, i = 1, 2, \dots, M.$$

Thus we use mesh spacing h_1 for the points $0 = x_0 < x_1 < \dots < x_M = \bar{x}$ and mesh spacing h_2 for $\bar{x} = x_M < x_{M+1} < \dots < x_N = 1$. The resulting mesh is called a *Shishkin mesh.* Repeat the computations of part (d) using this mesh, and hand in the corresponding table and plots. Also record the value of $\bar{x}$ for each choice of ϵ when $N = 128$. Be sure to use the correct basis functions at the mesh point $x_M = \bar{x}$, which has unequally spaced neighbors.

f) In one or two sentences, describe what you learned from these computations. In particular, when ϵ is very small, one might expect to have difficulty computing the approximate solution near the boundary $x = 1$ where the true solution is changing rapidly. Were there any problems computing the solution away from the boundary $x = 1$? How did the choice of mesh affect these computations?

2. Let x be the solution of $Ax = b$ and consider approximations to x given by the iteration:

$$x^{k+1} = [I - \alpha A]x^k + \alpha b.$$

Assuming that A is symmetric and positive definite and $\alpha = 2/(\lambda_1 + \lambda_n)$, where λ_1 is the smallest eigenvalue of A and λ_n is the largest eigenvalue of A, show that if

$$k \geq \frac{1}{2}\frac{\lambda_n}{\lambda_1} \ln \frac{1}{\epsilon},$$

then $\|x^k - x\|_2 \leq \epsilon \|x^0 - x\|_2$. Hint: Start from the result proved in class that under the hypotheses of the problem, $\|x - x^k\|_2 \leq \left(\frac{\kappa-1}{\kappa+1}\right)^k \|x - x^0\|$, where $\kappa = \lambda_{\max}(A)/\lambda_{\min}(A)$. Then use the fact that $\ln[(1+x)/(1-x)] \geq 2x$ for $0 < x < 1$. Choose $x = \lambda_1/\lambda_n$.