MATH 575 ASSIGNMENT 5

1. A subspace of $H(\mathrm{div};\Omega)$ can be constructed from the full set of linear vectors, i.e., functions of the form $\boldsymbol{v} = (a+bx+cy, d+ex+fy)$. On each edge, $\boldsymbol{v}$ will be a linear function of a single variable s. To ensure that $\boldsymbol{v}\cdot\boldsymbol{n}$ will be continuous across triangle edges, we choose as degrees of freedom

$$\int_e \boldsymbol{v}\cdot\boldsymbol{n}\,ds, \qquad \int_e \boldsymbol{v}\cdot\boldsymbol{n}\,s\,ds,$$

i.e., the zeroth and first order moments of $\boldsymbol{v}\cdot\boldsymbol{n}$ on the edge e. The basis functions corresponding to these degrees of freedom will be the vectors $\boldsymbol{\phi}_i$, $i = 1, 2, 3$ with the property:

$$\int_{e_i} \boldsymbol{\phi}_i\cdot\boldsymbol{n}\,ds = 1, \qquad \int_{e_i} \boldsymbol{\phi}_i\cdot\boldsymbol{n}\,s\,ds = 0,$$

and the corresponding values of these two quantities on the other two edges are zero and also the vectors $\boldsymbol{\psi}_i$, $i = 1, 2, 3$ with the property

$$\int_{e_i} \boldsymbol{\psi}_i\cdot\boldsymbol{n}\,ds = 0, \qquad \int_{e_i} \boldsymbol{\psi}_i\cdot\boldsymbol{n}\,s\,ds = 1,$$

and the corresponding values of these two quantities on the other two edges are zero. For the reference triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, find the two basis functions corresponding to the side joining the vertices $(0,0)$ and $(1,0)$.

2. Consider the approximation of the stationary Stokes equations by the mini-element for which $\boldsymbol{V}_h = [V_h]^2 = [S_h + B_h]^2$, S_h consists of continuous piecewise linear functions vanishing on $\partial\Omega$ and B_h is the space of cubic bubble functions, i.e., functions which on each triangle have the form $c_T\lambda_1\lambda_2\lambda_3$. For the space Q_h, we take continuous piecewise linear functions satisfying $\int_\Omega q\,dx = 0$. To prove stability of this element, we need to build an interpolation operator Π_h satisfying

$$\int_\Omega \mathrm{div}(\boldsymbol{u} - \Pi_h\boldsymbol{u})\,q_h\,dx = 0, \quad q_h \in Q_h, \qquad \|\Pi_h\boldsymbol{u}\|_1 \le C\|\boldsymbol{u}\|_1. \tag{1}$$

To build Π_h, we use two operators Π_h^1 and Π_h^2 satisfying for all $\boldsymbol{v} \in [\mathring{H}^1(\Omega)]^2$

$$\|\Pi_h^1\boldsymbol{v}\|_1 \le C_1\|\boldsymbol{v}\|_1, \qquad \|\Pi_h^2(I - \Pi_h^1)\boldsymbol{v}\|_1 \le C_2\|\boldsymbol{v}\|_1, \tag{2}$$

$$\int_\Omega \mathrm{div}(\boldsymbol{v} - \Pi_h^2\boldsymbol{v})\,q_h\,dx = 0, \quad q_h \in Q_h. \tag{3}$$

2a. Show that if $\Pi_h\boldsymbol{u} = \Pi_h^1\boldsymbol{u} + \Pi_h^2(\boldsymbol{u} - \Pi_h^1\boldsymbol{u})$ where Π_h^1 and Π_h^2 satisfy (2) and (3), then Π_h satisfies the conditions in (1).

2b. Show that if $\Pi_h^1\boldsymbol{u} = (w_1, w_2)$, where $w_i \in S_h$ satisfies

$$\int_\Omega (\nabla w_i\cdot\nabla v + w_i v)\,dx = \int_\Omega (\nabla u_i\cdot\nabla v + u_i v)\,dx, \quad v \in S_h,$$

then $\|\Pi_h^1\boldsymbol{u}\|_1 \le C_1\|\boldsymbol{u}\|_1$.

2c. Show that if $\Pi_h^2 \boldsymbol{v} \in [B_h]^2$ is defined by

$$\int_T (\Pi_h^2 \boldsymbol{v} - \boldsymbol{v}) \cdot \operatorname{grad} q_h \, dx = 0, \quad q_h \in P_1(T),$$

then

$$\int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_h^2 \boldsymbol{v})\, q_h \, dx = 0, \quad q_h \in Q_h.$$

Hint: integrate by parts (recall that q_h is continuous).

2d. It is possible to show by an easy scaling argument that

$$\|\Pi_h^2 \boldsymbol{v}\|_1 \leq C h^{-1} \|\boldsymbol{v}\|_{L^2(\Omega)},$$

and by a duality argument that $\|\boldsymbol{u} - \Pi_h^1 \boldsymbol{u}\|_{L^2(\Omega)} \leq Ch\|\boldsymbol{u} - \Pi_h^1 \boldsymbol{u}\|_1$. Using these two facts (you do not need to prove them), show that

$$\|\Pi_h^2 (I - \Pi_h^1)\boldsymbol{u}\|_1 \leq C\|\boldsymbol{u}\|_1.$$

Thus, we have established the inf-sup condition and shown the mini-element is stable.

3. In this problem, we use FreeFem to solve the boundary value problem

$$-\Delta u = 2[x(1-x) + y(1-y)] \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is the unit square $[0,1] \times [0,1]$. The exact solution is given by $u = x(1-x)y(1-y)$.

a) Compute the solution using piecewise linear elements on a 20×20 mesh and then recompute the solution on a refined mesh using a 40×40 mesh. For each of these, compute the errors $\|u - u_h\|_{L^2(\Omega)}$ and $\|\nabla(u - u_h)\|_{L^2(\Omega)}$. Then use these two computations to compute the order of convergence in each of these norms.

b) Repeat the calculations of part (a) using piecewise quadratic elements.

c) How do the numerical results compare with the theory developed in class?

4. In this problem, we solve the boundary value problem in Problem 3 using the mixed finite element method in which the scalar variable u is approximated by piecewise constants and the vector variable $\sigma = \nabla u$ is approximated by lowest order Raviart-Thomas elements.

To define these finite element spaces and compute the solution, we use the following commands, where GMRES is an iterative solver used to solve the linear systems of equations.

```
fespace Sigh(Th,RT0);
fespace Vh(Th,P0);
Sigh [sig1,sig2],[tau1,tau2];
Vh u,v;
solve laplacemixed([sig1,sig2,u],[tau1,tau2,v],solver=GMRES,eps=1.0e-10) =
int2d(Th) (sig1*tau1 + sig2*tau2 + u*(dx(tau1) + dy(tau2))
+ (dx(sig1) + dy(sig2))*v) + int2d(Th) (f*v);
```

Write a FreeFem program to compute the solution using a 20×20 mesh and then recompute the solution on a refined mesh using a 40×40 mesh. For each of these, compute the errors $\|u - u_h\|_{L^2(\Omega)}$, $\|\sigma - \sigma_h\|_{L^2(\Omega)}$, and $\|\operatorname{div}(\sigma - \sigma_h)\|_{L^2(\Omega)}$. Then use these two computations to compute the order of convergence in each of these norms.