MATH 575 ASSIGNMENT 5

1. A subspace of $H(\text{div};\Omega)$ can be constructed from the full set of linear vectors, i.e., functions of the form $\boldsymbol{v} = (a+bx+cy, d+ex+fy)$. On each edge, $\boldsymbol{v}$ will be a linear function of a single variable s. To ensure that $\boldsymbol{v}\cdot\boldsymbol{n}$ will be continuous across triangle edges, we choose as degrees of freedom

$$\int_e \boldsymbol{v}\cdot\boldsymbol{n}\,ds, \qquad \int_e \boldsymbol{v}\cdot\boldsymbol{n}\,s\,ds,$$

i.e., the zeroth and first order moments of $\boldsymbol{v}\cdot\boldsymbol{n}$ on the edge e. The basis functions corresponding to these degrees of freedom will be the vectors $\boldsymbol{\phi}_i$, $i = 1,2,3$ with the property that

$$\int_{e_i} \boldsymbol{\phi}_i\cdot\boldsymbol{n}\,ds = 1, \qquad \int_{e_i} \boldsymbol{\phi}_i\cdot\boldsymbol{n}\,s\,ds = 0,$$

and the corresponding values of these two quantities on the other two edges are zero and also the vectors $\boldsymbol{\psi}_i$, $i = 1,2,3$ with the property that

$$\int_{e_i} \boldsymbol{\psi}_i\cdot\boldsymbol{n}\,ds = 0, \qquad \int_{e_i} \boldsymbol{\psi}_i\cdot\boldsymbol{n}\,s\,ds = 1,$$

and the corresponding values of these two quantities on the other two edges are zero. For the reference triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, find the two basis functions corresponding to the side joining the vertices $(0,0)$ and $(1,0)$.

2. Consider the approximation of the stationary Stokes equations by the mini-element for which $\boldsymbol{V}_h = [V_h]^2 = [S_h + B_h]^2$, S_h consists of continuous piecewise linear functions vanishing on $\partial\Omega$ and B_h is the space of cubic bubble functions, i.e., functions which on each triangle have the form $c_T\lambda_1\lambda_2\lambda_3$. For the space Q_h, we take continuous piecewise linear functions satisfying $\int_\Omega q\,dx = 0$. To prove stability of this element, we need to build an interpolation operator Π_h satisfying

$$(1) \qquad \int_\Omega \text{div}(\boldsymbol{u} - \Pi_h\boldsymbol{u})\,q_h\,dx = 0, \quad q_h \in Q_h, \qquad \|\Pi_h\boldsymbol{u}\|_1 \le C\|\boldsymbol{u}\|_1.$$

To build Π_h, we use two operators Π_h^1 and Π_h^2 satisfying for all $\boldsymbol{v} \in [\mathring{H}^1(\Omega)]^2$

$$(2) \qquad \|\Pi_h^1\boldsymbol{v}\|_1 \le C_1\|\boldsymbol{v}\|_1, \qquad \|\Pi_h^2(I - \Pi_h^1)\boldsymbol{v}\|_1 \le C_2\|\boldsymbol{v}\|_1,$$

$$(3) \qquad \int_\Omega \text{div}(\boldsymbol{v} - \Pi_h^2\boldsymbol{v})\,q_h\,dx = 0, \quad q_h \in Q_h.$$

2a. Show that if $\Pi_h\boldsymbol{u} = \Pi_h^1\boldsymbol{u} + \Pi_h^2(\boldsymbol{u} - \Pi_h^1\boldsymbol{u})$ where Π_h^1 and Π_h^2 satisfy (2) and (3), then Π_h satisfies the conditions in (1).

2b. Show that if $\Pi_h^1\boldsymbol{u} = (w_1, w_2)$, where $w_i \in S_h$ satisfies

$$\int_\Omega (\nabla w_i\cdot\nabla v + w_i v)\,dx = \int_\Omega (\nabla u_i\cdot\nabla v + u_i v)\,dx, \quad v \in S_h,$$

then $\|\Pi_h^1 \boldsymbol{u}\|_1 \le C_1 \|\boldsymbol{u}\|_1$.

2c. Show that if $\Pi_h^2 \boldsymbol{v} \in [B_h]^2$ is defined by

$$\int_T (\Pi_h^2 \boldsymbol{v} - \boldsymbol{v}) \cdot \operatorname{grad} q_h \, dx = 0, \quad q_h \in P_1(T),$$

then

$$\int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_h^2 \boldsymbol{v}) \, q_h \, dx = 0, \quad q_h \in Q_h.$$

Hint: integrate by parts (recall that q_h is continuous).

2d. It is possible to show by an easy scaling argument that

$$\|\Pi_h^2 \boldsymbol{v}\|_1 \le C h^{-1} \|\boldsymbol{v}\|_{L^2(\Omega)},$$

and by a duality argument that $\|\boldsymbol{u} - \Pi_h^1 \boldsymbol{u}\|_{L^2(\Omega)} \le Ch \|\boldsymbol{u} - \Pi_h^1 \boldsymbol{u}\|_1$. Using these two facts (you do not need to prove them), show that

$$\|\Pi_h^2 (I - \Pi_h^1) \boldsymbol{u}\|_1 \le C \|\boldsymbol{u}\|_1.$$

Thus, we have established the inf-sup condition and shown the mini-element is stable.