MATH 575 ASSIGNMENT 3

1. Consider the two-point boundary value problem

$$-u'' = f, \qquad 0 < x < 1, \qquad u(0) = 0, \qquad u(1) = 0.$$

This is a special boundary value problem for which the finite element approximation is much better than we can expect on more general problems. For this problem you are asked to prove that if the computations are done exactly (i.e., there is no roundoff error), then $u_h(x_i) = u(x_i)$.

Hint: Use the error equation $a(u - u_h, v_h) = 0$, with $v_h(x) = x/(ih)$ for $0 \le x \le ih$ and $v_h(x) = (1-x)/(1-ih)$ for $ih \le x \le 1$, and integration by parts.

2. Consider the two-point boundary value problem

$$-u'' + u = e - e^{-1}, \qquad 0 < x < 1, \qquad u(0) = 0, \qquad u(1) = 0.$$

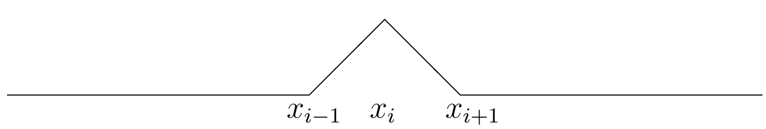
Note that the exact solution of this problem is $u(x) = (e^{-1} - 1)(e^x - 1) + (1 - e)(e^{-x} - 1)$.

2a. If we approximate this problem using a finite element scheme with continuous piecewise linear functions on a uniform mesh of width $h = 1/N$, we obtain a linear system of equations of the form $A\boldsymbol{\alpha} = \boldsymbol{b}$, where A is an $N-1 \times N-1$ matrix and the vector $\boldsymbol{\alpha} = [u_h(h), \ldots, u_h([N-1]h)]^T$.

(i) Show that if $x_i = ih$ and the basis for this finite element space is chosen to be $\phi_1, \ldots, \phi_{N-1}$, where

$$\phi_i(x) = \begin{cases} 0 & 0 \le x \le x_{i-1}, \\ (x - x_{i-1})/(x_i - x_{i-1}) & x_{i-1} \le x \le x_i, \\ (x_{i+1} - x)/(x_{i+1} - x_i) & x_i \le x \le x_{i+1}, \\ 0 & x_{i+1} \le x \le 1, \end{cases}$$

with graph



then

$$\begin{aligned} A_{i,i+1} &= -1/h + h/6, \qquad i = 1, \ldots, N-2, \\ A_{i,i-1} &= -1/h + h/6, \qquad i = 2, \ldots, N-1, \\ A_{i,i} &= 2/h + 2h/3, \qquad i = 1, \ldots, N-1. \end{aligned}$$

(ii) What are the entries of the matrix A, for $j \ne i-1, i, i+1$?

(iii) Calculate the entries of the vector $\boldsymbol{b}$.

2b. Using the results of 4a, write a *Matlab* program which approximates the solution of this boundary value problem using a finite element scheme with continuous piecewise linear functions on a uniform mesh of width h. Do this for $N = 4, 8, 16, 32, 64$ subintervals. Use *format long* in all calculations.

2c. Let $u_I(x)$ denote the piecewise linear interpolant of u, i.e., the piecewise linear function satisfying $u_I(x_i) = u(x_i)$, $i = 0, \ldots, N$, and $e(x) = u_I(x) - u_h(x)$. Let $E_h = \max_{0 \le i \le N} |e(x_i)|$, where $h = 1/N$, and $x_i = ih$. Have the computer determine E_h for each value of N. Then, assuming that the error $E_h \approx Ch^\alpha$ for some constants C and α, determine α for each two successive values of h. Hand in a table showing the values of E_h and the values of α, along with a copy of your program.

3. Let T be the triangle with vertices $\boldsymbol{a}_1 = (1, 0)$, $\boldsymbol{a}_2 = (1, 1)$, and $\boldsymbol{a}_3 = (0, 1)$. Find the barycentric coordinates $\lambda_i(\boldsymbol{x})$, $i = 1, 2, 3$ of a point $\boldsymbol{x} \in T$ with respect to $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$.

4a. Let T be an arbitrary triangle with vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ and let $\lambda_i(\boldsymbol{x})$, $i = 1, 2, 3$ be the barycentric coordinates of a point $\boldsymbol{x} \in T$ with respect to $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$. Let $\boldsymbol{m}_{ij}$ denote the midpoint of the edge joining $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$.

Find a function ϕ_k of the form $c_1\lambda_1(\boldsymbol{x}) + c_2\lambda_2(\boldsymbol{x}) + c_3\lambda_3(\boldsymbol{x})$, where c_1, c_2, c_3 are constants, that satisfies

$$\phi_k(\boldsymbol{m}_{ij}) = 0 \text{ if } k = i \text{ or } k = j, \qquad \phi_k(\boldsymbol{m}_{ij}) = 1, \text{ if } k \neq i \text{ and } k \neq j.$$

4b. Use the result of (2a) to find a linear polynomial P on T that satisfies

$$P(\boldsymbol{m}_{12}) = 4, \qquad P(\boldsymbol{m}_{23}) = 5, \qquad P(\boldsymbol{m}_{31}) = 6.$$

HINT: **Your solution should be expressed in terms of the functions ϕ_1, ϕ_2, ϕ_3.**

5. Show that a quadratic polynomial $P(x)$ on the interval $[a, b]$ is uniquely determined by its values at a and b and its average value over $[a, b]$, i.e., by $P(a)$, $P(b)$ and $(b-a)^{-1}\int_a^b P(x)\,dx$. Hint: show that if these quantities are all zero, then $P(x) \equiv 0$. The computation is simplified if you use the Lagrange interpolating polynomial and the values $P(a)$, $P(b)$, and $P([a+b]/2)$ to represent $P(x)$.

6. What is the dimension of the space of Lagrange finite elements of degree ≤ 3 satisfying homogeneous boundary conditions on the mesh shown? State how you arrived at this dimension.