MATH 575 ASSIGNMENT 3

1. Let T be the triangle with vertices $\boldsymbol{a}_1 = (1, 0)$, $\boldsymbol{a}_2 = (1, 1)$, and $\boldsymbol{a}_3 = (0, 1)$. Find the barycentric coordinates $\lambda_i(\boldsymbol{x})$, $i = 1, 2, 3$ of a point $\boldsymbol{x} \in T$ with respect to $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$.

2a. Let T be an arbitrary triangle with vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ and let $\lambda_i(\boldsymbol{x})$, $i = 1, 2, 3$ be the barycentric coordinates of a point $\boldsymbol{x} \in T$ with respect to $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$. Let $\boldsymbol{m}_{ij}$ denote the midpoint of the edge joining $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$. Find a function ϕ_k of the form $c_1\lambda_1(\boldsymbol{x}) + c_2\lambda_2(\boldsymbol{x}) + c_3\lambda_3(\boldsymbol{x})$, where c_1, c_2, c_3 are constants, that satisfies

$$\phi_k(\boldsymbol{m}_{ij}) = 0 \text{ if } k = i \text{ or } k = j, \qquad \phi_k(\boldsymbol{m}_{ij}) = 1, \text{ if } k \neq i \text{ and } k \neq j.$$

b. Use the result of (2a) to find a linear polynomial P on T that satisfies

$$P(\boldsymbol{m}_{12}) = 4, \qquad P(\boldsymbol{m}_{23}) = 5, \qquad P(\boldsymbol{m}_{31}) = 6.$$

HINT: Your solution should use the functions ϕ_1, ϕ_2, ϕ_3.

3a. Show that a quadratic polynomial P on a triangle T is uniquely determined by the six degrees of freedom consisting of the values of P at the three vertices and the average values of P along the three triangle sides. HINT: Use the representation formula

$$P(x) = \sum_{i=1}^{3} \lambda_i(2\lambda_i - 1)P(a_i) + \sum_{i<j} 4\lambda_i\lambda_j P(m_{ij})$$

to show that if the six degrees of freedom described above are zero, then $P \equiv 0$.
USEFUL FACT: The one-dimensional quadrature formula

$$\int_a^b f(t)\,dt \approx \frac{b-a}{6}[f(a) + 4f([a+b]/2) + f(b)]$$

is exact when $f(t)$ is a cubic polynomial in t.

3b. Show that if $g(t)$ is a quadratic polynomial and

$$\int_a^b g(t)\,dt = 0, \qquad \int_a^b g(t)(t - [a+b]/2)\,dt = 0,$$

then $g(a) = g(b)$ and $g([a+b]/2) = -g(a)/2$. Hint: Use the quadrature formula given above.

3c. Use the result of (3b) and the representation formula of (3a) to show that if P is a quadratic polynomial on a triangle T which satisfies on each edge e of T

$$\int_e P\,ds = 0, \qquad \int_e Ps\,ds = 0,$$

then P is not necessarily zero. Hence, the average value and first moment on each edge do not comprise a unisolvent set of degrees of freedom for the space of quadratic polynomials.