MATH 575 ASSIGNMENT 3

1. Let T be the triangle with vertices $\boldsymbol{a}_1 = (0,0)$, $\boldsymbol{a}_2 = (1,0)$, and $\boldsymbol{a}_3 = (1/2, \sqrt{3}/2)$. Find the barycentric coordinates $\lambda_i(\boldsymbol{x})$, $i = 1,2,3$ of a point $\boldsymbol{x} \in T$ with respect to $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$.

2a. Let T be an arbitrary triangle with vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ and let $\lambda_i(\boldsymbol{x})$, $i = 1,2,3$ be the barycentric coordinates of a point $\boldsymbol{x} \in T$ with respect to $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$. Let $\boldsymbol{m}_{ij}$ denote the midpoint of the edge joining $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$. Find a function ϕ_k depending on the λ_i that satisfies

$$\phi_k(\boldsymbol{m}_{ij}) = 0 \text{ if } k = i \text{ or } k = j, \qquad \phi_k(\boldsymbol{m}_{ij}) = 1, \text{ if } k \neq i \text{ and } k \neq j.$$

2b. Using the result of part(a), find a polynomial P (written in terms of the λ_i) satisfying $P(\boldsymbol{m}_{12}) = 3$, $P(\boldsymbol{m}_{13}) = 2$, and $P(\boldsymbol{m}_{23}) = 1$.

3. Show that the six degrees of freedom consisting of the average values and first moments along the three sides of a triangle T with vertices $a_1 = (1,0)$, $a_2 = (0,1)$, $a_3 = (0,0)$ do NOT uniquely define a polynomial of degree ≤ 2 in x, y. To do this, you need to find a quadratic polynomial $Q(x,y) \neq 0$ such that the quantities

$$\int_0^1 Q(x,0)\,dx, \qquad \int_0^1 Q(0,y)\,dy, \qquad \int_0^1 Q(x,1-x)\,dx, \tag{1}$$

$$\int_0^1 xQ(x,0)\,dx, \qquad \int_0^1 yQ(0,y)\,dy, \qquad \int_0^1 xQ(x,1-x)\,dx \tag{2}$$

are all zero. To simplify your search, you may consider only polynomials of the form $Q(x,y) = x^2 + bxy + y^2 + dx + dy + f$, since we expect the polynomial to be symmetric in x and y and may normalize it by setting one coefficient equal to one.

4a. Show that a quadratic polynomial P on a triangle T is uniquely determined by the six degrees of freedom consisting of the values of P at the three vertices and the average values of P along the three triangle sides. HINT: Use the representation formula

$$P(x) = \sum_{i=1}^{3} \lambda_i(2\lambda_i - 1)P(a_i) + \sum_{i<j} 4\lambda_i\lambda_j P(m_{ij})$$

to show that if the six degrees of freedom described above are zero, then $P \equiv 0$.
USEFUL FACT: The one-dimensional quadrature formula

$$\int_a^b f(t)\,dt \approx \frac{b-a}{6}[f(a) + 4f([a+b]/2) + f(b)]$$

is exact when $f(t)$ is a quadratic polynomial in t.

4b. Suppose $\mathcal{T}_h$ is a triangulation of a polygon Ω. For any smooth function v defined on Ω, let v^I be a piecewise quadratic (with respect to the triangulation $\mathcal{T}_h$) satisfying $v^I(x_j) = v(x_j)$ for all vertices x_j of $\mathcal{T}_h$ and $\int_e v^I = \int_e v$ for all triangle edges. By part (a) v^I is uniquely defined by these quantities. Now let $\boldsymbol{u}$ be a smooth vector function defined on Ω, i.e., $\boldsymbol{u} = (u_1, u_2)$ and define $\boldsymbol{u}^I = (u_1^I, u_2^I)$. Show that for all piecewise constant functions p,

$$\int_\Omega p \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}^I)\, dx = 0.$$

Hint: Write

$$\int_\Omega p \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}^I)\, dx = \sum_{T \in \mathcal{T}_h} \int_T p_T \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}^I)\, dx$$

where T denotes a triangle in $\mathcal{T}_h$ and p_T is the value of p on the triangle T, and integrate by parts in each triangle. Also note that the unit outward normal vector $\boldsymbol{n}^T$ to each triangle T is constant on each edge of T.

Note: The idea used above will be useful in the construction of stable finite elements for the stationary Stokes equations.