MATH 575 ASSIGNMENT 3

1a. Let $\hat{T}$ be the triangle with vertices $\hat{a}_1 = (1,0)$, $\hat{a}_2 = (0,1)$, $\hat{a}_3 = (0,0)$. Find linear polynomials $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_3$ in $\hat{x}_1$, $\hat{x}_2$ such that any linear polynomial $\hat{L}(\hat{x}_1, \hat{x}_2)$ can be written as:

$$\hat{L}(\hat{x}_1, \hat{x}_2) = \hat{L}(1/2, 0)\hat{L}_1(\hat{x}_1, \hat{x}_2) + \hat{L}(0, 1/2)\hat{L}_2(\hat{x}_1, \hat{x}_2) + \hat{L}(1/2, 1/2)\hat{L}_3(\hat{x}_1, \hat{x}_2).$$

b. Let T be an arbitrary triangle with vertices a_1, a_2, a_3. Find linear polynomials L_1, L_2, L_3 in λ_1, λ_2, λ_3 (the barycentric coordinates) such that any linear polynomial $L(x_1, x_2) \equiv L(x)$ can be written as:

$$L(x) = L([a_1 + a_3]/2)L_1 + L([a_2 + a_3]/2)L_2 + L([a_1 + a_2]/2)L_3,$$

i.e., the degrees of freedom are the values of L at the midpoints of the edges of the triangles.

c. Let $\mathcal{T}_h$ be a triangulation of a convex polygon Ω and let $S_h = \{v : v|_T \in P_1$ and v is continuous at the midpoints of triangle edges$\}$, where P_1 denotes the set of polynomials of degree ≤ 1 in x_1, x_2. By considering two adjacent triangles, give an explicit example to show that the functions in S_h need not belong to $C^0(\bar{\Omega})$.

2. Show that the six degrees of freedom consisting of the average values and first moments along the three sides of a triangle T with vertices $a_1 = (1,0)$, $a_2 = (0,1)$, $a_3 = (0,0)$ do NOT uniquely define a polynomial of degree ≤ 2 in x, y. To do this, you need to find a quadratic polynomial $Q(x,y) \neq 0$ such that the quantities

$$(1) \qquad \int_0^1 Q(x,0)\,dx, \qquad \int_0^1 Q(0,y)\,dy, \qquad \int_0^1 Q(x,1-x)\,dx,$$

$$(2) \qquad \int_0^1 xQ(x,0)\,dx, \qquad \int_0^1 yQ(0,y)\,dy, \qquad \int_0^1 xQ(x,1-x)\,dx$$

are all zero. To simplify your search, you may consider only polynomials of the form $Q(x,y) = x^2 + bxy + y^2 + dx + dy + f$, since we expect the polynomial to be symmetric in x and y and may normalize it by setting one coefficient equal to one.

3a. Show that a quadratic polynomial P on a triangle T is uniquely determined by the six degrees of freedom consisting of the values of P at the three vertices and the average values of P along the three triangle sides. HINT: Use the representation formula

$$P(x) = \sum_{i=1}^{3} \lambda_i(2\lambda_i - 1)P(a_i) + \sum_{i<j} 4\lambda_i\lambda_j P(m_{ij})$$

to show that if the six degrees of freedom described above are zero, then $P \equiv 0$.
USEFUL FACT: The one-dimensional quadrature formula

$$\int_a^b f(t)\,dt \approx \frac{b-a}{6}[f(a) + 4f([a+b]/2) + f(b)]$$

is exact when $f(t)$ is a quadratic polynomial in t.

b. Suppose $\mathcal{T}_h$ is a triangulation of a polyon Ω. For any smooth function v defined on Ω, let v^I be a piecewise quadratic (with respect to the triangulation $\mathcal{T}_h$) satisfying $v^I(x_j) = v(x_j)$ for all vertices x_j of $\mathcal{T}_h$ and $\int_e v^I = \int_e v$ for all triangle edges. By part (a) v^I is uniquely defined by these quantities. Now let $\boldsymbol{u}$ be a smooth vector function defined on Ω, i.e., $\boldsymbol{u} = (u_1, u_2)$ and define $\boldsymbol{u}^I = (u_1^I, u_2^I)$. Show that for all piecewise constant functions p,

$$\int_\Omega p \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}^I)\,dx = 0.$$

Hint: Write

$$\int_\Omega p \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}^I)\,dx = \sum_{T \in \mathcal{T}_h} \int_T p_T \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}^I)\,dx$$

where T denotes a triangle in $\mathcal{T}_h$ and p_T is the value of p on the triangle T, and integrate by parts in each triangle. Also note that the unit outward normal vector $\boldsymbol{n}^T$ to each triangle T is constant on each edge of T.

Note: The idea used above will be useful in the construction of stable finite elements for the stationary Stokes equations.