MATH 575 ASSIGNMENT 2

1. Consider the differential operator

$$Lu = (pu')', \quad 0 < x < 1$$

where $p(x) \in C^1[0,1]$ and $p(x) \geq \gamma > 0$. Suppose we discretize this operator by rewriting it in the form $Lu = pu'' + p'u'$ and using the difference approximation:

$$L_h u = p(x_j)[u_{j+1} - 2u_j + u_{j-1}]/h^2 + p'(x_j)[u_{j+1} - u_{j-1}]/(2h).$$

1a) Let I_h denote the mesh points $x_i = ih$, $i = 1, \ldots, N-1$, with $h = 1/N$. Show that if

$$p(x_j) > 0 \qquad \text{and} \qquad 0 \leq \frac{h}{4}\frac{p'(x_j)}{p(x_j)} \leq \frac{1}{2},$$

then L_h satisfies a discrete maximum principle, i.e., show that if v is a function defined on the mesh points and $L_h v(x) \geq 0$ for all $x \in I_h$, then

$$\max_{I_h} v \leq \max[v(0), v(1)].$$

1b) Show that for $h = 1/2$ and $p(x) = e^{8x}$, the discrete maximum principle is not satisfied, i.e., give specific values of $v(0)$, $v(1/2)$, and $v(1)$, for which the discrete maximum principle does not hold.

Now consider the difference approximation:

$$L_h v = \frac{1}{h}\left[[p(x_{j+1/2})\frac{(v_{j+1} - v_j)}{h} - p(x_{j-1/2})\frac{(v_j - v_{j-1})}{h} \right]$$

1c) Show that this scheme satisfies the discrete maximum principle for all h.

2. Consider the boundary value problem (BVP):

$$-u'' = 3, \quad 0 < x < 1, \qquad u'(0) = 2, \qquad u(1) = 0.$$

This problem has a variational formulation of the form:

Find $u \in V \equiv \{v \in H^1(0,1) : v(1) = 0\}$ such that $a(u,v) = F(v)$ for all $v \in V$.

2a. Determine the bilinear form $a(u,v)$ and the linear functional $F(v)$ and show that if u satisfies the BVP, then u is a solution of the variational problem. Hint: Use the integration by parts formula:

$$-\int_0^1 u''v\,dx = -u'v|_0^1 + \int_0^1 u'v'\,dx.$$

2b. Show that if u satisfies the variational problem and is sufficiently smooth, then u satisfies the BVP.

3a. Using the Green's identity

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega - \Delta \, u \, v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds,$$

(a special case of the result derived in class), derive the identity

$$\int_\Omega \Delta \, v \, u \, dx = \int_\Omega \Delta \, u \, v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds + \int_{\partial\Omega} \frac{\partial v}{\partial n} u \, ds.$$

Hint: First obtain another expression for $\int_\Omega \nabla u \cdot \nabla v \, dx$ by interchanging roles of u and v.

3b. Use part (a) to derive the identity

$$\int_\Omega \Delta \, u \, \Delta \, v \, dx = \int_\Omega \Delta^2 \, u \, v \, dx - \int_{\partial\Omega} \frac{\partial}{\partial n} \Delta \, u \, v \, ds + \int_{\partial\Omega} \Delta \, u \frac{\partial v}{\partial n} \, ds.$$

3c. Consider the variational problem:

Find $u \in V = \{v \in H^2(\Omega) : v = 0 \text{ on } \partial\Omega\}$ such that

$$\int_\Omega \Delta \, u \, \Delta \, v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g \frac{\partial v}{\partial n} \, ds, \ \forall \ v \in V.$$

If u is sufficiently smooth, what boundary value problem does u satisfy?

3d. Consider the boundary value problem

$$\Delta^2 \, u = f, \qquad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0, \qquad \text{on } \partial\Omega.$$

Find a variational formulation of this boundary value problem in which u is sought in an appropriate subspace of $H^2(\Omega)$. (The above is a model of a clamped plate.)