MATH 575 ASSIGNMENT 2

1a. Using the Green's identity

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega -\Delta\, u\, v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds,$$

(a special case of the result derived in class), derive the identity

$$\int_\Omega \Delta\, v\, u \, dx = \int_\Omega \Delta\, u\, v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds + \int_{\partial\Omega} \frac{\partial v}{\partial n} u \, ds.$$

Hint: First obtain another expression for $\int_\Omega \nabla u \cdot \nabla v \, dx$ by interchanging roles of u and v.

1b. Use part (a) to derive the identity

$$\int_\Omega \Delta\, u\, \Delta\, v \, dx = \int_\Omega \Delta^2\, u\, v \, dx - \int_{\partial\Omega} \frac{\partial}{\partial n} \Delta\, u\, v \, ds + \int_{\partial\Omega} \Delta\, u \frac{\partial v}{\partial n} \, ds.$$

1c. Consider the variational problem:

Find $u \in V = \{v \in H^2(\Omega) : v = 0 \text{ on } \partial\Omega\}$ such that

$$\int_\Omega \Delta\, u\, \Delta\, v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g \frac{\partial v}{\partial n} \, ds, \ \forall\ v \in V.$$

If u is sufficiently smooth, what boundary value problem does u satisfy?

1d. Consider the boundary value problem

$$\Delta^2\, u = f, \qquad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0, \qquad \text{on } \partial\Omega.$$

Find a variational formulation of this boundary value problem in which u is sought in an appropriate subspace of $H^2(\Omega)$. (The above is a model of a clamped plate.)

2. Consider the boundary value problem (BVP):

$$-u'' = 2, \quad 0 < x < 1, \qquad u(0) = 0, \qquad u'(1) = 3.$$

This problem has a variational formulation of the form:

Find $u \in V \equiv \{v \in H^1(0,1) : v(0) = 0\}$ such that $a(u,v) = F(v)$ for all $v \in V$.

2a. Determine the bilinear form $a(u,v)$ and the linear functional $F(v)$ and show that if u satisfies the BVP, then u is a solution of the variational problem. Hint: Use the integration by parts formula:

$$-\int_0^1 u'' v \, dx = -u'v|_0^1 + \int_0^1 u'v' \, dx.$$

2b. Show that if u satisfies the variational problem and is sufficiently smooth, then u satisfies the BVP.

3. Consider the two-point boundary value problem

$$-u'' = f, \qquad 0 < x < 1, \qquad u(0) = 0, \qquad u(1) = 0.$$

3a. Write a *Matlab* program which approximates the solution of this boundary value problem using a finite element scheme with continuous piecewise linear functions on a uniform mesh of width h. Choose $f = 1$ when $0 < x < c$, and $f = 0$ when $c < x < 1$, where $c = \sqrt{2}/2$. The exact solution is given by $u = -x^2/2 + (c - c^2/2)x$ for $0 < x < c$ and $c^2(1 - x)/2$ for $c < x < 1$. Do this for $N = 4, 8, 16, 32, 64$ subintervals. Use *format long* in all calculations.

3b. Let $u_I(x)$ denote the piecewise linear interpolant of u, i.e., the piecewise linear function satisfying $u_I(x_i) = u(x_i)$, $i = 0, \ldots, N$, and $e(x) = u_I(x) - u_h(x)$. Let $E_h = \max_{0 \le i \le N} |e(x_i)|$, where $h = 1/N$, and $x_i = ih$. Have the computer determine E_h for each value of N and hand in these results along with a copy of your program.

3c. This is a special boundary value problem for which the finite element approximation is much better than we can expect on more general problems. For this problem you are asked to prove that if the computations are done exactly (i.e., there is no roundoff error), then $u_h(x_i) = u(x_i)$, so that $E_h = 0$.

Hint: Use the error equation $a(u - u_h, v_h) = 0$, with $v_h(x) = x/(ih)$ for $0 \le x \le ih$ and $v_h(x) = (1 - x)/(1 - ih)$ for $ih \le x \le 1$, and integration by parts.