MATH 575 ASSIGNMENT 2

1a. Using the Greens's identity

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega - \Delta\, u\, v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds,$$

(a special case of the result derived in class), derive the identity

$$\int_\Omega \Delta\, u\, \Delta\, v \, dx = \int_\Omega \Delta^2\, u\, v \, dx - \int_{\partial\Omega} \frac{\partial}{\partial n}\, \Delta\, u\, v \, ds + \int_{\partial\Omega} \Delta\, u \frac{\partial v}{\partial n} \, ds.$$

1b. Consider the variational problem:

Find $u \in V = \{v \in H^2(\Omega) : v = 0 \text{ on } \partial\Omega\}$ such that

$$\int_\Omega \Delta\, u\, \Delta\, v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g \frac{\partial v}{\partial n} \, ds, \ \forall\, v \in V.$$

If u is sufficiently smooth, what boundary value problem does u satisfy?

1c. Consider the boundary value problem

$$\Delta^2\, u = f, \qquad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0, \qquad \text{on } \partial\Omega.$$

Find a variational formulation of this boundary value problem in which u is sought in an appropriate subspace of $H^2(\Omega)$. (The above is a model of a clamped plate.)

2a. Consider the variational problem:

Find $\boldsymbol{\sigma} \in \mathrm{H}(\mathrm{div}, \Omega) = \{\boldsymbol{\sigma} \in L^2(\Omega) : \mathrm{div}\, \boldsymbol{\sigma} \in L^2(\Omega)\}$ and $u \in L^2(\Omega)$ such that

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx - \int_\Omega u \, \mathrm{div}\, \boldsymbol{\tau} \, dx, = 0, \quad \text{for all } \boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}, \Omega),$$

$$\int_\Omega \mathrm{div}\, \boldsymbol{\sigma}\, v \, dx = \int_\Omega f v \, dx, \quad \text{for all } v \in L^2(\Omega).$$

If $\boldsymbol{\sigma}$ and u are sufficiently smooth, what partial differential equations and boundary conditions (if any) do $\boldsymbol{\sigma}$ and u satisfy?

2b. If instead of the boundary condition $u = 0$, we want to consider the boundary condition $\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$, what would be the variational formulation in this case?

2c. By eliminating $\boldsymbol{\sigma}$ from these two systems, find the boundary value problems satisfied by u alone.