MATH 575 ASSIGNMENT 1

1. Let $L_h u(x,y)$ be an approximation to $(\partial^2 u/\partial x \partial y)(x,y)$ of the form

$$\frac{\alpha[u(x+h,y+h)+u(x-h,y-h)]+\beta[u(x-h,y+h)+u(x+h,y-h)]+\gamma u(x,y)}{h^2}.$$

Assuming u is sufficiently smooth ($\in C^4$), find values of the constants α, β, γ, which give an $O(h^2)$ approximation. Hint: Use the Taylor series expansion:

$$u(x+ah,y+bh) = u + ahu_x + bhu_y + \frac{h^2}{2}[a^2 u_{xx} + 2ab u_{xy} + b^2 u_{yy}] + \frac{h^3}{6}[a^3 u_{xxx} + 3a^2 b u_{xxy} + 3ab^2 u_{xyy} + b^3 u_{yyy}] + O(h^4),$$

for appropriate choices of the constants a and b, where on the right side of the equation, u and all its derivatives are evaluated at (x,y).

2. Consider the approximation to $u''(x)$ given by

$$D_h u(x) = \frac{2}{h^2}\left[\frac{1}{\alpha+1}u(x-h) - \frac{1}{\alpha}u(x) + \frac{1}{\alpha(\alpha+1)}u(x+\alpha h)\right], \qquad 0 < \alpha \le 1.$$

a) Show that if $u \in C^4[x-h, x+\alpha h]$, this formula does not give an $O(h^2)$ approximation to u'' if α is not equal to one.

b) Let $I = [x-h, x+\alpha h]$. Show that if $u \in C^3(I)$ and $0 < \alpha < 1$, then for all $x \in I$,

$$|D_h u(x) - u''(x)| \le M_3 h/3, \qquad M_3 = \max_I |u'''(x)|.$$

c) Show that if $D_h u(x) \ge 0$, $u(x) \ge u(x-h)$, and $u(x) \ge u(x+\alpha h)$, then $u(x) = u(x-h) = u(x+\alpha h)$. Hint: Show that if either $u(x-h)$ or $u(x+\alpha h)$ is strictly less than $u(x)$, then we get a contradiction.

3. Consider the boundary value problem

$$-\epsilon u'' + u' = 1, \quad 0 < x < 1, \qquad u(0) = 0, \quad u(1) = 0, \qquad \text{where} \quad \epsilon > 0.$$

a) Verify that the exact solution is $u(x) = x - [e^{(x-1)/\epsilon} - e^{-1/\epsilon}]/[1 - e^{-1/\epsilon}]$.

The remainder of this problem should be done using Matlab. Some basic instructions and commands can be found at: `http://math.rutgers.edu/~falk/math575/matlab1.html` If you do not have access to a computer that has Matlab, please see me to arrange for an account on a Math Department computer. Rather than entering your commands directly into Matlab, it is usually preferable to write your program in a file, e.g., *assign1.m*, which must end in the extension ".m" You can then execute the program by typing *assign1* at the Matlab prompt (assuming your current directory is the one containing the file *assign1.m*).

b) Plot the solution for $\epsilon = 0.1$, 0.01, and 0.001. (labeling the value of ϵ).

c) Compute the solution using the 3-point finite difference method

$$-\frac{\epsilon}{h^2}(u_{j-1} - 2u_j + u_{j+1}) + \frac{1}{2h}(u_{j+1} - u_{j-1}) = 1, \qquad j = 1, \ldots, n-1,$$

and a uniform mesh of size $h = 1/n$ for the values $n = 64, 128, 256, 512.$. For each of the 3 values of ϵ and 4 values of n, make a table (see below) of errors in each of the 3 norms

$$\|e\|_\infty = \max_{1 \le i \le n} |e(ih)|, \qquad \|e\|_2 = \left(h \sum_{i=1}^{n} |e(ih)|^2 \right)^{1/2}, \qquad \|e\|_1 = h \sum_{i=1}^{n} |e(ih)|,$$

n	$\|e\|_\infty$	$\|e\|_2$	$\|e\|_1$
64			
128			
256			
512			

d) Compute the order of convergence of the method in each of the three norms for each two consecutive values of n for the case $\epsilon = 0.1$ only. To obtain the order of convergence, α, suppose that the error $\|e\|_p = \|e_h\|_p \approx Ch^\alpha$ for some constants C and α. Then

$$\frac{\|e_{h_1}\|_p}{\|e_{h_2}\|_p} \approx \frac{Ch_1^\alpha}{Ch_2^\alpha} \qquad \text{implies} \qquad \alpha \approx \ln \frac{\|e_{h_1}\|_p}{\|e_{h_2}\|_p} \Big/ \ln \frac{h_1}{h_2}.$$

e) Next, let $m = n/2$ (assume n is even) and put $h_1 = \bar{x}/m$, $h_2 = (1-\bar{x})/m$, where $\bar{x} \in (0,1)$ is to be defined. Now we define mesh points by

$$x_i = ih_1, i = 0, 1, \ldots m, \qquad x_{m+i} = x + ih_2, i = 1, 2, \ldots, m.$$

Thus we use mesh spacing h_1 for the points $0 = x_0 < x_1 < \ldots < x_m = \bar{x}$ and mesh spacing h_2 for $\bar{x} = x_m < x_{m+1} < \ldots < x_n = 1$. For the problem above we take $\bar{x} = \max(1 - 2\epsilon \ln n, 1/2)$. The resulting mesh is called a *Shishkin mesh*. Implement the 3-point difference method to solve the problem on this mesh. Note that for the mesh points x_j, $j = 1, \ldots, m-1$, we choose $h = h_1$ and for the mesh points x_j, $j = m+1, \ldots, n-1$, we choose $h = h_2$. At the point x_m, we use the mesh points, $x_m - h_1$, x_m, and $x_m + h_2$, which are not equally spaced. In this case, we can approximate $u''(x_m)$ by the formula of Problem 2 and $u'(x_m)$ by $[u(x_m + \alpha h) - u(x_m - h)]/[(1+\alpha)h]$, where $h = h_1$ and $\alpha = h_2/h_1$. For each of the 3 values of ϵ and 4 values of n, make a table of the errors in each of the 3 norms given above.

f) A statement attributed to R. Hamming is that "the purpose of computing is insight, not numbers." What did you learn from your computations about the approximation of this type of singular perturbation problem? Give a clear explanation, backing up your statements with results from parts (a)-(e).

g) Include a printout of your source code.