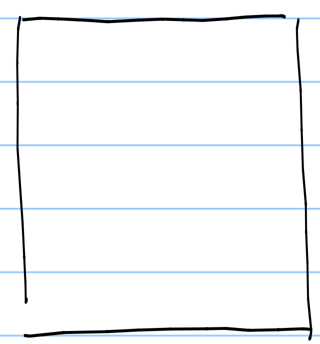
Note Title 3/26/2012

6.3 Multigrid Method

Model Problem : $-\Delta u = f$ in Ω

$u\big|_{\partial\Omega} = 0$



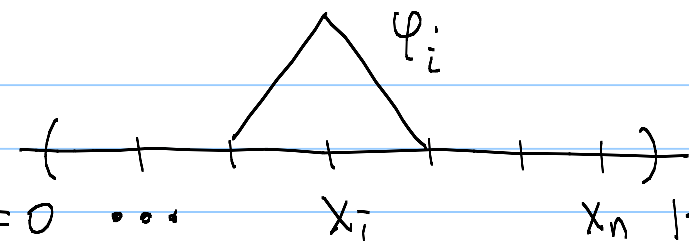
$\Omega = (0,1)^2$ in 2D

$\Omega = (0,1)$ in 1D.

Find $u \in H_0^1(\Omega)$ s.t $a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega)$

, where $a(u,v) = \int_\Omega \nabla u \nabla v \, dx.$

Ex] in 1-D



$h = \max_{1 \le i \le n+1} |x_i - x_{i-1}|$

$V_h = \text{Span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}.$ $\quad \varphi_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

$u_h \in V_h$ s-t $\quad a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.$

$u_h = \sum_{i=1}^{n} u_h(x_i)\varphi_i := \underbrace{\sum_{i=1}^{n} u^i \varphi_i}$, $\quad u^i = u_h(x_i)$

$\Longleftrightarrow \quad a(u_h, \varphi_j) = (f, \varphi_j) \quad \forall j = 1:n \quad$ jth eq

$\Longleftrightarrow \quad \sum_{i=1}^{n} u^i a(\varphi_i, \varphi_j) = (f, \varphi_j) =: \int_\Omega f \varphi_j \, dx. = f^j.$

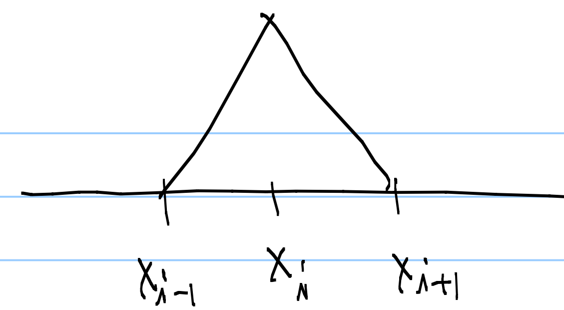
$A_h = \left((A_h)_{ij} \right)_{ij} \qquad (A_h)_{ij} = a(\varphi_j, \varphi_i) \quad \forall i, j = 1:n.$

Ex] (1-D.)

$x_i = ih.$

$$A_h = \begin{pmatrix} 2/h & -1/h & & 0 \\ -1/h & 2/h & -1/h & \\ & -1/h & \ddots & \ddots \\ 0 & & \ddots & 2/h \end{pmatrix} \qquad f^j = \int_0^1 f \varphi_j \, dx.$$

$$(A_h)_{ii} = \int_0^1 \varphi_i' \varphi_i' \, dx$$



$$= \int_{x_{i-1}}^{x_i} \frac{1}{h^2} dx + \int_{x_i}^{x_{i+1}} \frac{1}{h^2} dx = \frac{1}{h^2} \cdot h + \frac{1}{h^2} h = \frac{2}{h}$$

jth local eq :

$$A_h U = F \qquad U = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} \quad F = \begin{pmatrix} f^1 \\ \vdots \\ f^n \end{pmatrix}$$

$$(A_h U)_j = f^j.$$

1 $$\frac{-U^{j+1} + 2U^{j} - U^{j-1}}{h} = f^{j} \qquad j = 1:n.$$

For given $U_k = (U_k^1, U_k^2, \ldots, U_k^n)$, kth iterate

solve for U_{k+1}^{j}

2 $$\frac{-U_k^{j+1} + 2U_{k+1}^{j} - U_k^{j-1}}{h} = f^{j} \qquad \forall\, j = 1:n$$

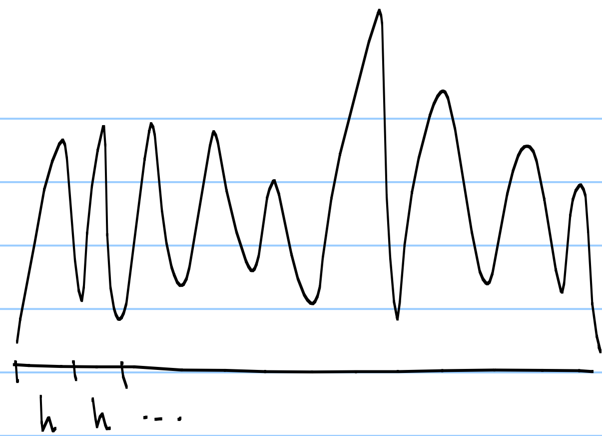
→ Jacobi method

Properties of Jacobi method

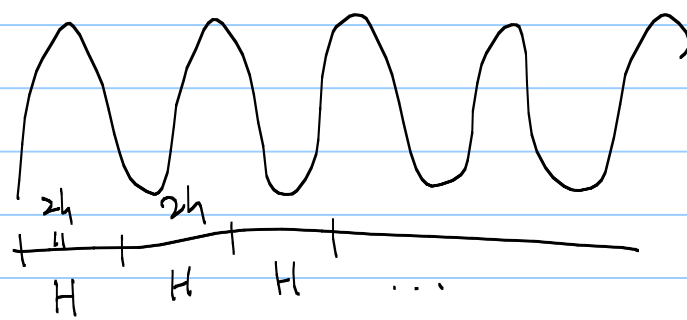
$$e_k^j = U^j - U_k^j \quad \forall \; j = 1:n$$

$$\frac{-e_k^{j+1} + 2\, e_{k+1}^j - e_k^{j-1}}{h} = 0. \qquad e_{k+1}^j = \frac{e_k^{j+1} + e_k^{j-1}}{2}.$$

R.K].



h-scale frequency



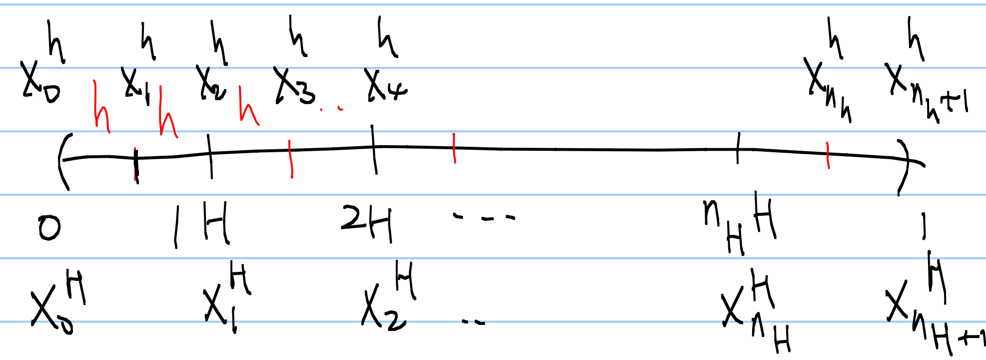
After applying a classical iterative method

$$e = \sum_{i=1}^{n} c_i \mu_i \qquad \mu_n \text{ h-scale frequency}$$

can be handled by Jacobi efficiently. However, smooth component of error is difficult to handle.

Ingredient I]

Ex] 1-D, $\Omega = (0,1)$



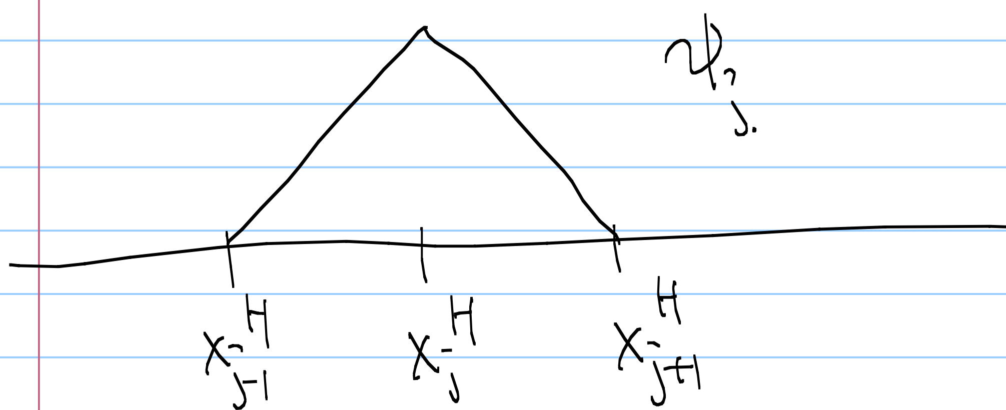
$\varphi_H = \{ \varphi_j^H \}_{j=1}^{n_H+1}$. $\quad \varphi_j^H = (x_j^H, x_{j-1}^H)$. $\quad \bar{\Omega} = \bigcup_{j=1}^{n_H+1} \bar{\varphi}_j^H$.

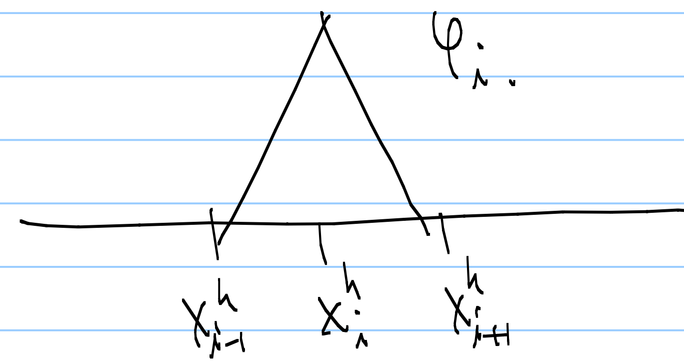
φ_h : obtained by forming 2 subtriangles in each triangle in φ_H

$= \{ \varphi_j^h \}_{j=1}^{n_h+1}$, $\quad \varphi_j^h = (x_j^h, x_{j-1}^h)$. $\quad \bar{\Omega} = \bigcup_{j=1}^{n_h+1} \bar{\varphi}_j^h$.

$V_H = \text{span}\{ \psi_1, \psi_2, \dots, \psi_{n_H} \}$, where

ψ_j is a hat ft st $\psi_j(x_i^H) = \delta_{ij}$.

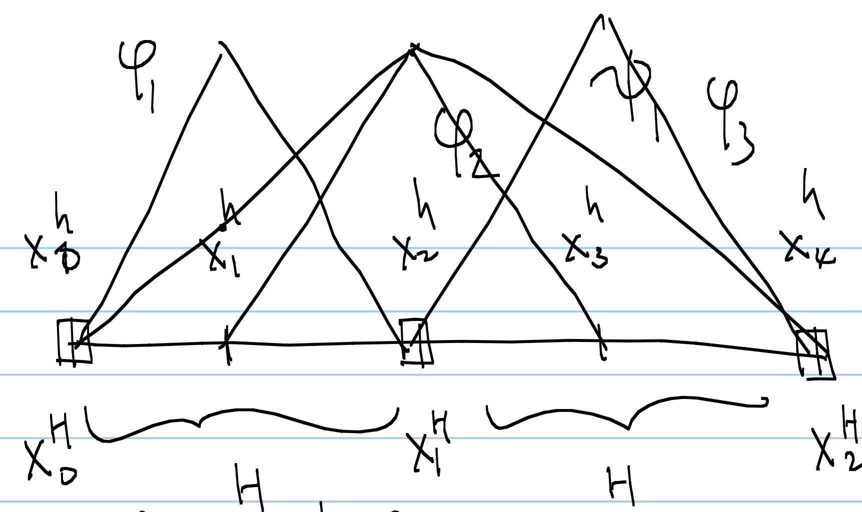




$V_h = \text{Span}\{\varphi_1, \varphi_2, \dots, \varphi_{n_h}\}$, where

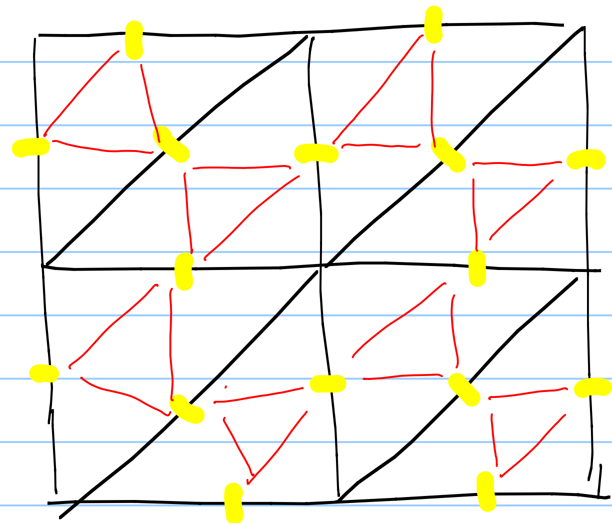
φ_i is a hat ft s.t. $\varphi_i(x_j^h) = \delta_{ij}$.

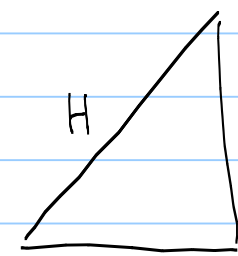
$V_H \subset V_h$ why.



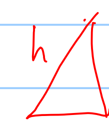
Choose $\psi_1 = \frac{1}{2}\varphi_1 + \varphi_2 + \frac{1}{2}\varphi_3$.

$\Rightarrow \quad \forall\, w_H \in V_H, \quad w_H \in V_h$





$H = 2h$



Notation: $X^h = \{x_i^h\}_{i=1}^{n_h}$ Nodes for $\mathcal{T}_h$

$X^H = \{x_i^H\}_{i=1}^{n_H}$ Nodes for $\mathcal{T}_H$.

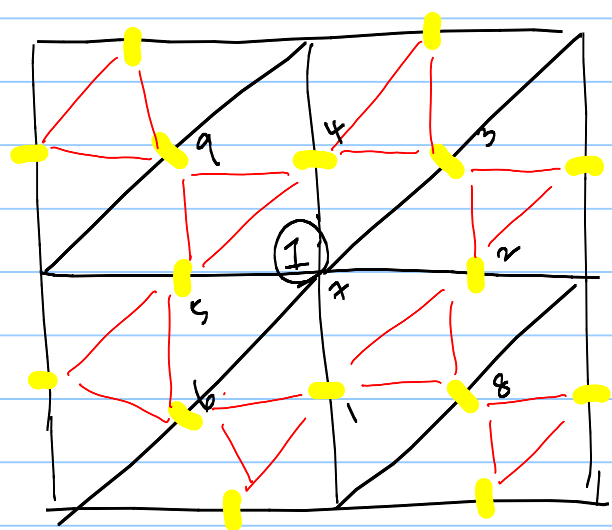
Let $V_h = \text{span}\{\phi_1, \phi_2, \ldots \phi_{n_h}\}$ and $V_H = \text{span}\{\psi_1, \ldots, \psi_{n_H}\}$ where ϕ_i's usual basis (hat ft) defined on $\mathcal{T}_h$. s.t $\phi_i(x_j^h) = \delta_{ij}$ $\forall i, j$. and ψ_j's usual basis (hat ft) defined on $\mathcal{T}_H$. s.t $\psi_j(x_i^H) = \delta_{ij}$ $\forall i, j$ $\Rightarrow$ $V_H \subset V_h$.

Under this setting, $\forall v_h \in V_h$ and $w_H \in V_H$ can be expressed as

$$v_h = \sum_{i=1}^{n_h} v_h(x_i^h)\phi_i \quad \text{and} \quad w_H = \sum_{i=1}^{n_H} w_H(x_i^H)\psi_i.$$

$V_H \subset V_h$

Ex] 2-D case Only 1 - interior node for φ_H.



$$\psi_1 = C_1^1 \varphi_1 + C_1^2 \varphi_2 + \cdots + C_1^9 \varphi_9$$

$C_1^1 = \frac{1}{2}, \quad C_1^2 = \frac{1}{2}, \quad C_1^3 = \frac{1}{2},$

$C_1^4 = \frac{1}{2}, \quad C_1^5 = \frac{1}{2}, \quad C_1^6 = \frac{1}{2},$

$C_1^7 = 1, \quad C_1^8 = C_1^9 = 0.$

$u_h \in V_h, \quad u_h = \sum_{j=1}^{n_h} u^j \phi_j, \quad \tilde{u} = \begin{pmatrix} u^1 \\ \vdots \\ u^{n_h} \end{pmatrix} \in \mathbb{R}^{n_h}.$

Goal: To solve $A_h \tilde{U}_h = \tilde{f}_h$.

Ingredient (II). With $B_h \cong A_h^{-1}$, $(A_h)_{ij} = a(\varphi_j, \varphi_i)$.

$$\tilde{u}_h^{k+1} = \tilde{u}_h^k + B_h(\tilde{f}_h - A_h \tilde{u}_h^k).$$

$(\tilde{f}_h)_j = (f, \varphi_j)$.

<u>Recall</u>: Classical Iterative Method to solve

$$Ax = b, \iff x = A^{-1}b. \qquad A \text{ spd}$$

Step 1] Given x_i ith iterate

Compute the residual $r_i = b - Ax_i$.

Form the residual eq

$$(*) \quad Ae = r_i.$$

Step 2] Solve $(*)$ approximately,

$$\hat{e} = B r_i, \quad \text{where} \quad B \approx A^{-1}.$$

Step 3] Update $\quad x_{i+1} = x_i + \hat{e}.$

If $B = A^{-1}$, then $x_{i+1} = x$.

why ?) $x_{i+1} = \cancel{x_i} + A^{-1}(b - \cancel{A x_i}) = A^{-1}b = x$,

$A = D - L - L^T$,

$B = D^{-1} \rightarrow$ Jacobi $\qquad B = \omega I$

$B = (D-L)^{-1} \rightarrow$ Gauss-Seidel $\qquad$ - Richardson

Two grids method

Let $\tilde{u}_h^k$ be an k-th iterate approximating u_h in V_h.

1. $\tilde{\tilde{u}}_h = \text{Smoothing}(\tilde{u}_h^k, A_h, f_h, \#\text{smoothing})$

For $i = \#$ smoothing (1 or 2)

$$\tilde{u}_h^k = \tilde{u}_h^k + B_h(f_h - A_h \tilde{u}_h^k)$$

End

$\tilde{\bar{u}}_h = \tilde{u}_h^k, \qquad \bar{u}_h = \sum_{j=1}^{m_h} \bar{u}_h^j \phi_j$

2. Coarse-grid correction : Compute the solution $w_H \in V_H$ to the problem

: Find $w_H \in V_H$ s.t

$$a(\bar{u}_h + w_H, v_H) = (f, v_H) \quad \forall v_H \in V_H.$$

$$\Rightarrow \quad u_h^{k+1} = \bar{u}_h + w_H.$$

and obtain $\tilde{u}_h^{k+1}$, a vector representation of u_h^{k+1}.

Ingredient (III) – (For a convenient implementation)

Define intergrid transfer operator.

$I_H^h : V_H \to V_h$, $\quad I_h^H : V_h \to V_H$.

prolongation $\qquad$ restriction.

Note $\forall j, \quad 1 \le j \le n_H.$

$$\psi_j = \sum_{i=1}^{n_h} c_j^i \phi_i. \qquad \mathbb{c}_j = \begin{bmatrix} c_j^1 \\ c_j^2 \\ \vdots \\ c_j^{n_h} \end{bmatrix}.$$

$$P = [\mathbb{c}_1, \mathbb{c}_2, \ldots, \mathbb{c}_{n_H}] \in \mathbb{R}^{n_h \times n_H}.$$

For any given $w_H \in V_H$.

$$w_H = \sum_{j=1}^{n_H} w_H(x_j^H)\,\psi_j = \sum_{j=1}^{n_H} w^j \psi_j.$$

$$= \sum_{j=1}^{n_H} w^j \sum_{i=1}^{m_h} c_j^i \phi_i = \sum_{i=1}^{m_h} \left[\sum_{j=1}^{n_H} c_j^i w^j \right] \phi_i$$

$$=$$

$$I_h^H w_H = \sum_{i=1}^{n_h} \left[P \tilde{w} \right]^i \phi_i . \qquad \tilde{w} = \begin{pmatrix} w^1 \\ \vdots \\ w^{n_H} \end{pmatrix}$$

$$\in V_h .$$

Note that

$$a(\bar{u}_h + w_H, v_H) = (f, v_H) \quad \forall v_H \in V_H .$$

$$a(w_H, \psi_j) = (f, \psi_j) - a(\bar{u}_h, \psi_j), \qquad j = 1 : n_H.$$

set $\quad w_H = \sum_{j=1}^{n_H} w^j \psi_j.$

$$a(\psi_i, \psi_j) = (P^T A_h P)_{ij}, \qquad (A_h)_{kl} = a(\varphi_l, \varphi_k).$$

$$(f, \psi_j) = \left(f, \sum_{k=1}^{n_h} c_j^k \phi_k\right) = \sum_{k=1}^{n_h} c_j^k (f, \phi_k) =$$

$\left[P^T \tilde{f}_h\right]_j$, To compute $a(\bar{u}_h, \psi_j)$,

$$a(\phi_i, \psi_j) = a\left(\phi_i, \sum_{k=1}^{n_h} c_j^k \phi_k\right)$$

$$= \sum_{k=1}^{n_h} c_j^k \, a(\phi_i, \phi_k) = \left(P^T A_h\right)_{ji} \Rightarrow$$

$a(\bar{u}_h, \psi_i) = \left(P^T A_h \tilde{\bar{u}}\right)_j$, where

$\tilde{\bar{u}}$ is the vector representation of $\bar{u}_h$.

$$\left(P^T A_h P\right) \tilde{w} = \left[P^T \tilde{f}_h - P^T A_h \tilde{\bar{u}} \right]$$