MATH 574 ASSIGNMENT 1

1a. Let

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 3 & -1 & 1 \\ 1 & 3 & 0 \end{pmatrix}.$$

Find a lower triangular matrix L, a *unit* upper triangular matrix U, and a permutation matrix P, such that $LU = P^T A$. Use the algorithm given in class, incorporating the partial pivoting strategy.

1b. Use Matlab's `lu` command (type `help lu` to learn about it) to find a *unit* lower triangular matrix l, an upper triangular matrix u, and a permutation matrix p such that $pA = lu$. Note that p should be the transpose of the matrix found in part(a). To enter the matrix A into Matlab, type the following (the semicolons separate the rows):

```
A = [1 1 3; 3 -1 1; 1 3 0]
```

1c. Determine a diagonal matrix D such that $LD = l$ and $D^{-1}U = u$.

2a. Show that the LU decomposition of the matrix $\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$ in which L is unit lower triangular and U is upper triangular is not unique by finding the general form of this decomposition.

2b. Does this matrix have an LU decomposition in which L is lower triangular and U is unit upper triangular? If so, find it, and if not, give a reason why not.

3. In this problem we consider the question of whether a small value of the residual $\|Az-b\|_2$ means that z is a good approximation to the solution x of the linear system $Ax = b$. We showed in class that when $\delta A = 0$,

$$\frac{\|x-z\|}{\|x\|} \le \|A\|\|A^{-1}\|\frac{\|Az-b\|}{\|b\|},$$

which implies that if the condition number $\|A\|\|A^{-1}\|$ of A is small, a small relative residual implies a small relative error in the solution. We now show computationally what can happen if the condition number is large. A standard example of a matrix that is ill-conditioned is the Hilbert matrix H, with entries $(H_{ij}) = 1/(i+j-1)$. For $n = 8, 12, 16$ (where H is of dimension $n \times n$), use Matlab to solve the linear system of equations $Hx = b$, where b is the vector Hy and y is the vector with $y_i = 1/\sqrt{n}$, $i = 1 \ldots n$. Clearly, the true solution is given by $x = y$, and we let z denote the approximation obtained by Matlab. Then calculate for each value of n the following quantities: (i) the relative error $\|x-z\|_2/\|x\|_2$, (ii) the relative residual $\|Hz-b\|_2/\|b\|$, (iii) the condition number $\|H\|_2\|H^{-1}\|_2$, and (iv) the product of the quantities in (ii) and (iii). Arrange all these numbers in a table. The Matlab commands `norm` and `cond` can be used to compute the norm and condition numbers, respectively. When vectors are input, Matlab writes them as row vectors. To convert `y` to a column vector, write it as `y'`. To solve the linear system $Hz = b$ in Matlab, type `z = H\b`. An example of a Matlab loop is given below; the semicolon keeps Matlab from writing unwanted output to the screen. To avoid potential problems, type `clear` before running a new value of n.

Example of a Matlab Loop:

```
for i=1:10
y(i) = 1/sqrt(10);
end
```

4. Suppose x is a vector with n components and A is an $n \times n$ matrix.

a) Show that $\|x\|_1 \leq n\|x\|_\infty$.

b) Find a vector x such that equality holds in (a).

c) Show that $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$.

d) Find a vector x such that equality holds in (c).

e) Show that $\|x\|_\infty \leq \|x\|_2$.

f) Use parts (c) and (e) and the facts that

$$\|A\|_2 = \max_{x \neq 0} \|Ax\|_2/\|x\|_2, \qquad \|A\|_\infty = \max_{x \neq 0} \|Ax\|_\infty/\|x\|_\infty$$

to show that

$$\|A\|_2/\sqrt{n} \leq \|A\|_\infty \leq \sqrt{n}\|A\|_2.$$