Lab 1: Exact Solutions of Differential Equations

This Maple lab is closely based on earlier versions prepared by Professors R. Falk and R. Bumby of the Rutgers Mathematics department.

Introduction. In this lab we use Maple to find exact solutions of differential equations and initial value problems, finding both solutions in the form of explicit functions and solutions defined only implicitly.

Please obtain the seed file from the web page and save it in your directory on eden, then prepare the Maple lab according to the instructions and hints in the introduction to Lab 0. Turn in only the printout of your Maple worksheet, after removing any extraneous material and any errors you have made.

In each problem an equation and some initial conditions will be given. As indicated in the worksheet and discussed here in detail for problem 1, *you should divide your work on each problem into three parts*: (a) *Solution* of the equations with prescribed initial conditions; (b) *Plots* of the solutions and a direction field; and (c) *Discussion* of aspects of the solution—especially comparing the Maple solution (as revealed by `odeadvisor`) with methods presented in the textbook—and answers to questions.

In the *plots* section you should plot both some of the closed form solutions that Maple finds and the direction field of the equation. A plot of this sort can be used to check closed-form solutions found by other means: the differential equation asserts that the direction field shows the direction of the tangent line to a solution at each point, and a proposed solution that doesn't follow the direction field is wrong. Plots revealing errors should be taken as signals that something in your worksheet must be changed.

Warning: In this lab description, the equations are written in the notation used in the textbook. This notation must be translated into the Maple language. In addition to the strict requirements on algebraic expressions in the *worksheet mode* used in these projects, Maple requires that a differential equation be described in terms of $y(x)$ consistently, you cannot abbreviate it to y; use the solution of Problem 1 as a guide. This is discussed more fully in the *introduction to Maple features relevant to differential equations*, available from the semester web page `http://www.math.rutgers.edu/courses/244/244-s09/`.

Setup. As usual, the seed file begins with commands which load the required Maple packages: `with(plots):` and `with(DEtools):`.

Equation 1.

We consider the equation

$$\frac{dy}{dx} + 2y = \frac{1}{3+e^x}. \tag{1}$$

The first two of the following commands (already entered into the seed files) show how Maple can find a general solution of the first order equation—Maple expresses this as a solution involving an arbitrary constant called `_C1`. Here, the solution y is found explicitly as a function of x. The next two commands instruct Maple to determine the value of `_C1` which will produce a solution with $y(0) = c$.

```
de1:= diff(y(x),x) + 2*y(x) = 1/(3+ exp(x));
s1:=dsolve(de1);
c1:=solve(eval(s1,{x=0,y(x)=c}),_C1);
s1a:=eval(s1,_C1=c1);
```

(a) Solution. This solution above follows the same procedure that you would use to solve the initial value problem if you were not using Maple. When you *are* using Maple, you can simply execute

the following command to solve the initial value problem more simply and directly (note the use of braces { }).

```
s1b:=dsolve({de1,y(0)=c},y(x));
```

Execute this instruction and compare to the previous result. There should be no difference. In the problems below you should, depending on the problem, *either* leave the solution expressed in terms of `_C1` *or* use the shorter method directly above to express the solution in terms of an initial condition.

In either case, the solution produced by Maple is really a family of solutions, i.e., there is a different solution corresponding to each particular choice of the constant c. Solutions of first order equations contain one parameter. The quantity `_C1` playing that role in our first solution was essentially a *constant of integration* in the *method* used to solve the equation. In the second solution, we forced the parameter to be the *initial value* $y(0)$.

(b) Plots. The plots in this project will use instructions described directly in Maple for parts of the desired plot. These parts will not be shown until they are combined into a single *display.* For this equation, there is no difficulty giving a *complete description* of each component of the plot. If you want to test the individual `plot` instructions, or create them using *interactive tools*, remove the experiments before turning in the assignment, or use the another worksheet. Your submitted worksheet should contain *only* the `plot` and `DEplot` instructions for constructing (but not showing) the individual plots and the `display` instruction for combining them. The same procedure should be used for the other equations in this project. For some of them, experimentation may provide the quickest route to the desired plot.

The seed file contains statements to find the solutions `t1, t2, t3, t4, t5` corresponding to the choices $c = -2, -1, 0, 1, 2$, and construct a plot of those solutions on the same set of axes. The plot is not shown immediately, but saved to be shown as part of a composite display. Note that when naming the output of a plot, changing the ending semicolon to a colon will suppress output that shows details of the plot structure.

The second component of the display is a *direction field* constructed by the `DEplot` instruction (the complete instruction for this example appears in the seed file). The qualitative behavior of the differential equation can often be seen in such direction fields, although some practice is needed to interpret them. Again, the instruction producing the named plot of the direction field ends with a colon to suppress output.

The two plots are combined, and *displayed*, by the `display` command (part of the `plots` library). The instruction in the seed file includes a title. Every plot that appears in your report should have a title.

(c) Discussion. The plot suggests some features of the equation and its solutions that should be discussed. In particular, the picture suggests that, no matter what initial condition we start with, all solutions tend to the same value as $x \to \infty$. This can be verified using the formula `s1` for the general solution using `limit(s1,x=infinity);` (Maple considers `infinity` to be $+\infty$). *Answer in text*: What is the limit?

What happens when $x \to -\infty$? It is not clear from the plot whether or not the solutions blow up before x reaches $-\infty$. *Answer in text:* what does Theorem 2.4.1 of Boyce and DiPrima say about this question? You should also try `limit(s1,x=-infinity);` and *interpret the answer in text* (you may need to use *Maple help* to identify terms used in the answer).

Some of this could be found *without solving the equation.* In particular, *what does the differential equation say* about where the solution $y(x)$ is *increasing*? *Answer in text*: Does this agree with your plot of the *direction field*?

You should also compare the results obtained by Maple with the solution that the textbook encourages you to use. The Maple instruction `odeadvisor(de1);` tells you how maple classifies the equation in order to select a solution method. Does the answer that Maple gives to this instruction agree with the description of that type of equation used in the textbook? Does the solution appear in the form expected from the method of solution proposed by the textbook for this type of equation?

Equation 2.

Introduce the name `de2` for the *equation*

$$\frac{dy}{dx} = \frac{3x - e^{-x}}{2y + e^{y}}, \tag{2}$$

and use `dsolve` to solve the equation. You should name the result `s2`. In this case, you should find that the solution $y(x)$ is only defined *implicitly* as a function of x, i.e., the answer is an equation involving both $y(x)$ and x in which $y(x)$ is not solved explicitly as a function of x. For this reason it is best just to leave the integration constant `_C1` in the problem—don't impose an initial condition.

If the solution were given in the special form $f(x, y) = constant$, the `lhs` function would extract $f(x, y)$ to use as input to the `contourplot` command. Currently, the equation produced by `dsolve` is not given in this special form, so you should examine it to find the constant introduced into the solution—it is likely to be called `_C1`—and solve for it. An expression that accomplishes this, and gives a reasonable name to the result, is `V2:=solve(s2,_C1);` . This identifies an expression that is constant on solutions of the equation.

The second part of the problem is to construct a graph of solutions of the equation and combine it with the direction field produced by `DEplot`. Since we have a function that is constant on solutions of the equation, the graph of solutions may be constructed using `g2c:=contourplot(V2,x=-3..5,y=-3..3,color=black):` . The `display` command should be used to combine `g2` with the slope field produced by the `DEplot` command, and to *provide a title* for this plot.

For the third part, as with the previous equation, compare the result of `odeadvisor` with the classification in the textbook. Also, *answer in text*: many of the solutions of this equation appear to have a minimum or maximum at the same value of x; are these values really the same? Your answer should indicate a *feature of the equation* that supports your conclusion.

Equation 3.

Consider the *logistic equation*

$$\frac{dy}{dt} = \frac{4}{3}y\left(1 - \frac{y}{2}\right). \tag{3}$$

As in previous exercises, use the instructions `dsolve`, `DEplot`, and `odeadvisor` to: *(a)* find the solution—in this case, you should find a solution in which the parameter c is the value of y(0); *(b)* plot a slope field for $-3 \le t \le 3$ and $-1 \le y \le 6$ together with some solutions; and *(c)* interpret Maple's classification of the equation.

As part of your plot in part *(b)*, use appropriate instructions to obtain solutions with initial conditions $y(0) = 1$, 2, and 5. Plot graphs of these on the interval $-3 \le t \le 3$ and use the `display` command to combine this graph with the slope field obtained from the `DEplot` command. Be sure to *provide a title* for the combined plot. Follow the model of Equation 1.

In part *(c)*, describe connections between the report of `odeadvisor`, the form of the solution given by `dsolve`, and the method used in the text for solving this equation. In addition, evaluate the solution of the equation with $y(0) = 5$ at $t = -2$. You should get $y(-2) = -0.2619464043$. *Explain* why this negative value is consistent with the fact that the differential equation predicts that a solution through a point with $y > 5$ is a decreasing function with $y(t)$ *always* greater than 2.

Equation 4.

Consider the equation

$$1 + \big(x/y - \sin y\big)\frac{dy}{dx} = 0, \tag{4}$$

which is essentially the equation of Exercise 27 of Section 2.6. As in previous problems, introduce `de4` as a name for the equation. Then, use the instructions `dsolve`, `DEplot`, and `odeadvisor` to *(a)* find the solution (leave it in terms of the integration constant `_C1`), *(b)* plot a slope field for $-2 \le x \le 2$ and $-1 \le y \le 3$ (just the slope field this time — no solutions), and *(c)* interpret Maple's classification of the equation.

The textbook informs you that this equation has an *integrating factor* $\mu(x, y) = y$. Maple knows this, too! The instruction `iF:=intfactor(de4);` will assign this integrating factor to the name `iF`. Then, use the instruction `de4a:=iF*de4;` to create a new equation that has been multiplied by this integrating factor. In your discussion in part *(c)*, describe any changes in the result of applying `dsolve` and `odeadvisor` to this equation instead of the original `de4`.

End of 244 Lab 1