The Local-Global Principle for Integral Crystallographic Sphere Packings

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Soddy Sphere Packings: The Construction

Given four mutually tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

Figure: Four tangent spheres.

Figure: Four tangent spheres with two additional tangent spheres.
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Figure: A Soddy sphere packing.
Figure: A Soddy sphere packing made by Nicolas Hannachi.

Label on sphere:
\[
\text{bend} = \frac{1}{\text{radius}}
\]

What do you notice about the bends that you can see on this Soddy sphere packing?
Figure: A Soddy sphere packing made by Nicolas Hannachi.

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They are all integers.
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They are all integers.

Which integers appear as bends?
Lemma (Kontorovich, 2019)

For a primitive integral Soddy sphere packing $\mathcal{P}$, there is an $\varepsilon = \varepsilon(\mathcal{P}) \in \{\pm 1\}$ such that each bend of the packing is

$$\equiv 0 \text{ or } \varepsilon \pmod{3}.$$ 

Example

Each bend $\equiv 0$ or $1 \pmod{3}$. 
Admissible Integers

Definition

An integer $m$ is *admissible* if for every $q \geq 1$

$$m \equiv \text{bend of some sphere in the packing (mod } q).$$
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Theorem (Kontorovich, 2019)

$m$ is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

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$m$ is admissible $\iff$

$$m \equiv 0 \text{ or } 1 \pmod{3}.$$
The bends of a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a local-to-global principle. That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and $m$ is admissible, then $m$ is the bend of a sphere in the packing.

Example

If $m \equiv 0$ or 1 (mod 3) and $m$ is sufficiently large, then $m$ is the bend of a sphere in the packing.
Goal: Prove a local-global principle for bends of more general integral sphere packings (called crystallographic sphere packings).

Figure: An integral crystallographic (more specifically, an orthoplicial) packing made by Kei Nakamura.

Figure: A fundamental domain of an integral crystallographic packing made by Arseniy (Senia) Sheydvasser.
Möbius Transformations

Automorphism group $\Gamma$ of Möbius transformations that map a packing $\mathcal{P}$ to itself

$$\Gamma : \hat{\mathbb{R}}^n \rightarrow \hat{\mathbb{R}}^n$$

$$z \mapsto g(z) = (a \cdot z + b)(c \cdot z + d)^{-1},$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$a, b, c, d$ in a Clifford algebra (which is the set of quaternions when $n = 3$)
\begin{itemize}
  \item $n \geq 3$
  \item $\mathcal{P}$ is a primitive integral $(n - 1)$-sphere packing in $\mathbb{R}^n$ with an automorphism group $\Gamma$ of Möbius transformations
\end{itemize}
Theorem in Progress

- \( n \geq 3 \)
- \( \mathcal{P} \) is a primitive integral \((n - 1)\)-sphere packing in \( \mathbb{R}^n \) with an automorphism group \( \Gamma \) of Möbius transformations
- \( S_0 \) an \((n - 1)\)-sphere in \( \mathcal{P} \)
- There exists an \((n - 1)\)-sphere \( S_1 \) in \( \mathcal{P} \) tangent to \( S_0 \).
Theorem in Progress

- $n \geq 3$
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- There exists an $(n-1)$-sphere $S_1$ in $\mathcal{P}$ tangent to $S_0$.
- Stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(O_K)$, $K$ imaginary quadratic field.

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- Stabilizer of $S_0$ in $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$, $K$ imaginary quadratic field.

Then every sufficiently large admissible integer is a bend of a $(n - 1)$-sphere in $\mathcal{P}$. 

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Definition

A principal congruence subgroup of \( \text{PSL}_2(\mathcal{O}_K) \) is a subgroup of \( \text{PSL}_2(\mathcal{O}_K) \) of the form

\[
\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathcal{O}_K) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varrho} \right\}
\]

for a fixed \( \varrho \in \mathcal{O}_K \).

Example (Soddy sphere packing, Kontorovich, 2019)

There exists a sphere \( S_0 \in \mathcal{P} \) such that the stabilizer of \( S_0 \) in \( \Gamma \) contains (up to conjugacy) the congruence subgroup

\[
\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathcal{O}) : b, c \equiv 0 \pmod{\varrho} \right\},
\]

where \( \mathcal{O} = \mathbb{Z}[e^{\pi i/3}] \) and \( \varrho = 1 + e^{\pi i/3} \).
The assumption that $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$ shows that the set of bends of $\mathcal{P}$ contains the “primitive” values of a shifted quaternary quadratic form.
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This shifted quaternary quadratic form gives you enough to work with so that you can apply the circle method to show that every sufficiently large admissible number is represented as a bend.
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- Major arcs: use spectral theory
1. The assumption that $\Gamma$ contains (up to conjugacy) a congruence subgroup of $\text{PSL}_2(\mathcal{O}_K)$ shows that the set of bends of $\mathcal{D}$ contains the “primitive” values of a shifted quaternary quadratic form.

2. This shifted quaternary quadratic form gives you enough to work with so that you can apply the circle method to show that every sufficiently large admissible number is represented as a bend.

   - Major arcs: use spectral theory
   - Minor arcs: use Kloosterman circle method
Example (Quadratic form for Soddy sphere packings)

Shifted quaternary quadratic form in $a_0, a_1, c_0, c_1 \in \mathbb{Z}$:

$$\hat{\beta} | C(a_0 + a_1 \omega) + D \varrho (c_0 + c_1 \omega)|^2 - Dj \tilde{C} + Cj \tilde{D},$$

$$\omega = e^{\pi i/3}$$

$$\varrho = 1 + \omega$$

$$\gcd(a_0 + a_1 \omega, \varrho (c_0 + c_1 \omega)) = 1$$

$$\hat{\beta} \in \mathbb{R}$$ and $C, D$ in the Clifford algebra depend on packing.

(Scale appropriately to obtain a primitive integral quadratic form.)

Many of (but not all of) the pictures used in this presentation are from this paper.
Thank you for listening!
Definition

A \((n - 1)\)-sphere packing is *crystallographic* if its limit set is that of a geometrically finite reflection group \(\Gamma < \text{Isom}(\mathbb{H}^{n+1})\).

**Figure:** Apollonian circle packing as the limit set of \(\Gamma\). Figure created by Alex Kontorovich.