LIE ALGEBRA EXTENSION ENDOMORPHISM AND DERIVATION COORDINATES

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Abstract. If $L \simeq V_1 \oplus V_2$ is a direct sum of vector spaces, we may associate to any linear endomorphism f of L a 2×2 matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$, where $f_{ij} : V_i \to V_j$ are linear maps. We generalize this to the case where L is an arbitrary group extension, determine the functional equations satisfied by the f_{ij} and how such matrices multiply. We also carry this out for f a derivation.

I. Endomorphisms

Suppose we have a short exact sequence

$$0 \to N \xrightarrow{i} L \xrightarrow{j} E \to 0$$

of lie algebras, together with a cross-section $s : E \to G$ for j, i.e., $j \circ s = 1_E$, and a projection $t : G \to N$ along s, i.e., $t \circ i = 1_N$, $t \circ s = 0$, $i \circ t + s \circ j = 1_L$. We write this in diagrammatic form as follows:

$$0 \to N \underset{t}{\overset{i}{\rightleftarrows}} L \underset{s}{\overset{j}{\rightleftarrows}} E \to 0 \ .$$

Neither s nor t are necessarily homomorphisms of lie algebras, and their failure to be such is measured by a 2−cocycle σ defined by means of s as

$$\sigma(y_1, y_2) = t(s([y_1, y_2]) - [s(y_1), s(y_2)]) \text{ or } s([y_1, y_2]) = [s(y_1), s(y_2)] + i(\sigma(y_1, y_2)) \ .$$

We also have a function $D : E \times N \to N$ defined as

$$D(y, x) = D_y(x) = x^y = t([s(y), i(x)]) \ .$$

If we look at the map $y \to D_y$ we do not quite get an lie algebra homomorphism from E to the lie algebra $Der(N)$, the lie algebra of derivations of N but miss by the 2−cocycle σ.

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Lemma 1.1. *For $y_1, y_2 \in E, x \in N$, we have*

$$D([y_1,y_2],x) = [D_{y_1}, D_{y_2}](x) + [\sigma(y_1,y_2),x] \ .$$

Proof.

$$\begin{aligned}
D([y_1,y_2],x) &= t([s([y_1,y_2]),i(x)]) \\
&= t([[s(y_1),s(y_2)] + i(\sigma(y_1,y_2)),i(x)]) \\
&= t([[s(y_1),s(y_2)],i(x)]) + [\sigma(y_1,y_2),x] \\
&= t([s(y_1),[s(y_2),i(x)]]) - t([s(y_2),[s(y_1),i(x)]]) + [\sigma(y_1,y_2),x] \\
&= t([s(y_1),(i\circ t)[s(y_2),i(x)]]) - t([s(y_2),(i\circ t)[s(y_1),i(x)]]) + [\sigma(y_1,y_2),x] \\
&= t([s(y_1),i(t([s(y_2),i(x)]))]) - t([s(y_2),i(t([s(y_1),i(x)]))]) + [\sigma(y_1,y_2),x] \\
&= D_{y_1}(D_{y_2}(x)) - D_{y_2}(D_{y_1}(x)) + [\sigma(y_1,y_2),x] \\
&= [D_{y_1},D_{y_2}](x) + [\sigma(y_1,y_2),x]
\end{aligned}$$

since t is a homomorphism when restricted to N.

Lemma 1.2. *For $y_1, y_2 \in E$,*

$$\sigma(y_1,[y_2,y_3]) + \sigma(y_2,[y_3,y_1]) + \sigma(y_3,[y_1,y_2]) + \sigma(y_2,y_3)^{y_1} + \sigma(y_3,y_1)^{y_2} + \sigma(y_1,y_2)^{y_3} = 0 \ .$$

Proof.

This is well known. See, e.g., [H2] or [Ja1]

Lemma 1.3. *For $g_1, g_2 \in L$,*

$$t([g_1,g_2]) = [t(g_1),t(g_2)] + t(g_2)^{j(g_1)} - t(g_1)^{j(g_2)} - \sigma(j(g_1),j(g_2)) \ .$$

Proof.

$$\begin{aligned}
t([g_1,g_2]) &= t([(i\circ t + s\circ j)(g_1),(i\circ t + s\circ j)(g_2)]) \\
&= t([(i\circ t)(g_1),(i\circ t)(g_2)]) + t([(i\circ t)(g_1),(s\circ j)(g_2)]) \\
&\quad + t([(s\circ j)(g_1),(i\circ t)(g_2)]) + t([(s\circ j)(g_1),(s\circ j)(g_2)]) \\
&= (t\circ i)([t(g_1),t(g_2)]) - t([(s\circ j)(g_2),(i\circ t)(g_1)]) + t([(s\circ j)(g_1),(i\circ t)(g_2)]) \\
&\quad + t(s([j(g_1),j(g_2)]) - i(\sigma(j(g_1),s(g_2)))) \\
&= [t(g_1),t(g_2)] + t(g_2)^{j(g_1)} - t(g_1)^{j(g_2)} - \sigma(j(g_1),j(g_2))
\end{aligned}$$

Now for $f\colon L \to L$ an endomorphism, define the *coordinates* f_{ij} for $1 \le i,j \le 2$ of f as follows:

$$\begin{aligned}
f_{11} &= t\circ f\circ i\colon N \to N \\
f_{21} &= j\circ f\circ i\colon N \to E \\
f_{12} &= t\circ f\circ s\colon E \to N \\
f_{22} &= j\circ f\circ s\colon E \to E \ .
\end{aligned}$$

Then

Theorem 1.1. *The f_{ij} satisfy the following functional equations:*

(1.1)
$$f_{11}([x_1,x_2]) = [f_{11}(x_1), f_{11}(x_2)] + f_{11}(x_2)^{f_{21}(x_1)} - f_{11}(x_1)^{f_{21}(x_2)} - \sigma(f_{21}(x_1), f_{21}(x_2))$$
(1.2)
$$f_{21}([x_1,x_2]) = [f_{21}(x_1), f_{21}(x_2)]$$
(1.3)
$$f_{12}([y_1,y_2]) = [f_{12}(y_1), f_{12}(y_2)] + f_{12}(y_2)^{f_{22}(y_1)} - f_{12}(y_1)^{f_{22}(y_2)} - \sigma(f_{22}(y_1), f_{22}(y_2)) + f_{11}(\sigma(y_1,y_2))$$
(1.4)
$$f_{22}([y_1,y_2]) = [f_{22}(y_1), f_{22}(y_2)] + f_{21}(\sigma(y_1,y_2))$$

Proof.

$$\begin{aligned}
f_{11}([x_1,x_2]) &= t((f\circ i)([x_1,x_2])) = t([(f\circ i)(x_1),(f\circ i)(x_2)]) \\
&= [(t\circ f\circ i)(x_1),(t\circ f\circ i)(x_2)] + (t\circ f\circ i)(x_2)^{(j\circ f\circ i)(x_1)} \\
&\quad - (t\circ f\circ i)(x_1)^{(j\circ f\circ i)(x_2)} - \sigma((j\circ f\circ i)(x_1),(j\circ f\circ i)(x_2)) \\
&= [f_{11}(x_1), f_{11}(x_2)] + f_{11}(x_2)^{f_{21}(x_1)} - f_{11}(x_1)^{f_{21}(x_2)} - \sigma(f_{21}(x_1), f_{21}(x_2)) \\
f_{21}([x_1,x_2]) &= (j\circ f\circ i)([x_1,x_2]) \\
&= [(j\circ f\circ i)(x_1),(j\circ f\circ i)(x_2)] = [f_{21}(x_1), f_{21}(x_2)] \\
f_{12}([y_1,y_2]) &= (t\circ f\circ s)([y_1,y_2]) = (t\circ f)\,([s(y_1),s(y_2)] + (i\circ\sigma)(y_1,y_2)) \\
&= t\,([(f\circ s)(y_1),(f\circ s)(y_2)] + (f\circ i\circ\sigma)(y_1,y_2)) \\
&= [(t\circ f\circ s)(y_1),(t\circ f\circ s)(y_2)] + (t\circ f\circ s)(y_2)^{(j\circ f\circ s)(y_1)} - (t\circ f\circ s)(y_1)^{(j\circ f\circ s)(y_2)} \\
&\quad s((j\circ f\circ s)(y_1),(j\circ f\circ s)(y_2)) + (t\circ f\circ i\circ\sigma)(y_1,y_2) \\
&= [f_{12}(y_1), f_{12}(y_2)] + f_{12}(y_2)^{f_{22}(y_1)} - f_{12}(y_1)^{f_{22}(y_2)} - \sigma(f_{22}(y_1), f_{22}(y_2)) + f_{11}(\sigma(y_1,y_2)) \\
f_{22}([y_1,y_2]) &= (j\circ f\circ s)([y_1,y_2]) = (j\circ f)\,([s(y_1),s(y_2)] = i(\sigma(y_1,y_2))) \\
&= [(j\circ f\circ s)(y_1),(j\circ f\circ s)(y_2)] + (j\circ f\circ i)(\sigma(y_1,y_2)) \\
&= [f_{22}(y_1), f_{22}(y_2)] + f_{21}(\sigma(y_1,y_2))
\end{aligned}$$

Theorem 1.2.
$$f = i\circ f_{11}\circ t + s\circ f_{21}\circ t + i\circ f_{12}\circ j + s\circ f_{22}\circ j$$

Proof.

$$\begin{aligned}
f &= (i\circ t + s\circ j)\circ f\circ(i\circ t + s\circ j) \\
&= i\circ t\circ f\circ i\circ t + s\circ j\circ f\circ i\circ t i\circ t\circ f\circ s\circ j + s\circ j\circ f\circ s\circ j \\
&= f = i\circ f_{11}\circ t + s\circ f_{21}\circ t + i\circ f_{12}\circ j + s\circ f_{22}\circ j
\end{aligned}$$

Theorem 1.3. *If we define f as in theorem 1.2, where the f_{ij} satisfy the functional equations in theorem 1.1, then f is an endomorphism of L.*

Before we prove this, we would do well to separate our task into two stages. First suppose that we have an endomorphism $f\colon L\to L$. Then define the two maps $f_1 = f\circ i\colon N\to L$ and $f_2 = f\circ s\colon E\to L$. f_1 is a homomorphism and f_2 satisfies

Lemma 1.4.

$$f_2([y_1, y_2]) = [f_2(y_1), f_2(y_2)] + f_1(\sigma(y_1, y_2)) \tag{1.5}$$

Proof.

$$\begin{aligned} f_2([y_1, y_2]) &= (f \circ s)([y_1, y_2]) = f([s(y_1), s(y_2)] + i(\sigma(y_1, y_2))) \\ &= [(f \circ s)(y_1), (f \circ s)(y_2)] + (f \circ i)(\sigma(y_1, y_2)) \\ &= [f_2(y_1), f_2(y_2)] + f_1(\sigma(y_1, y_2)) \end{aligned}$$

Lemma 1.5. $f = f_1 \circ t + f_2 \circ j$.

Proof.

$$f = f \circ (i \circ t + s \circ j) = f \circ i \circ t + f \circ s \circ j = f_1 \circ t + f_2 \circ j$$

Conversely,

Lemma 1.6. *If we define $f = f_1 \circ t + f_2 \circ j$, where f_1 is a homomorphism and f_2 satisfies* (1.5)*, then f is an endomorphism of L.*

Proof.

$$\begin{aligned} f([g_1, g_2]) &= (f_1 \circ t)([g_1, g_2]) + (f_2 \circ j)([g_1, g_2]) \\ &= f_1 \left([t(g_1), t(g_2)] + t(g_2)^{j(g_1)} - t(g_1)^{j(g_2)} - \sigma(j(g_1), j(g_2))\right) \\ &\quad + [(f_2 \circ j)(g_1), (f_2 \circ j)(g_2)] + f_1(\sigma(j(g_1), j(g_2))) \\ &= [(f_1 \circ t)(g_1), (f_1 \circ t)(g_2)] + f([(s \circ j)(g_1), (i \circ t)(g_2)]) - f\left([(s \circ j)(g_2), (i \circ t)(g_1)]\right) \\ &\quad - f_1(\sigma(j(g_1), j(g_2))) + [(f_2 \circ j)(g_1), (f_2 \circ j)(g_2)] + f_1(\sigma(j(g_1), j(g_2))) \\ &= [(f_1 \circ t)(g_1), (f_1 \circ t)(g_2)] + [(f_2 \circ j)(g_1), (f_1 \circ t)(g_2)] \\ &\quad [(f_1 \circ t)(g_1), (f_2 \circ j)(g_2)] + [(f_2 \circ j)(g_1), (f_2 \circ j)(g_2)] \\ &= [f(g_1), f(g_2)] \end{aligned}$$

Next, suppose we have a homomorphism $f_1 \colon N \to L$. Then $f_{11} = t \circ f_1 \colon N \to N$ and $f_{21} = j \circ f_1 \colon N \to E$ must satisfy equations (1.1), (1.2) respectively and

Lemma 1.7. $f_1 = i \circ f_{11} + s \circ f_{21}$

Proof.

$$f_1 = (i \circ t + s \circ j) \circ f_1 = i \circ t \circ f_1 + s \circ j \circ f_1 = i \circ f_{11} + s \circ f_{21} \ .$$

Conversely,

Lemma 1.8. *If we define* $f_1 = (i\circ f_{11})(s\circ f_{21})$*, where* f_{11}, f_{21} *satisfy* (1.1), (1.2)*, then* f_1 *is a homomorphism.*

Proof.

$$\begin{aligned}
f_1([x_1,x_2]) &= (i\circ f_{11})([x_1,x_2]) + (s\circ f_{21})([x_1,x_2]) \\
&= i\left([f_{11}(x_1), f_{11}(x_2)] + f_{11}(x_2)^{f_{21}(x_1)} - f_{11}(x_1)^{f_{21}(x_2)} - \sigma(f_{21}(x_1), f_{21}(x_2))\right) \\
&\quad + s([f_{21}(x_1), f_{21}(x_2)]) \\
&= [(i\circ f_{11})(x_1), (i\circ f_{11})(x_2)] + [s\circ f_{21}(x_1), (i\circ f_{11})(x_2)] \\
&\quad - [(s\circ f_{21})(x_2), (i\circ f_{11})(x_1)] - (i\circ\sigma)(f_{21}(x_1), f_{21}(x_2)) \\
&\quad + [(s\circ f_{21})(x_1), (s\circ f_{21})(x_2)] + (i\circ\sigma)(f_{21}(x_1), f_{21}(x_2)) \\
&= [(i\circ f_{11} + s\circ f_{21})(x_1), (i\circ f_{11} + s\circ f_{21})(x_2)] \\
&= [f_1(x_1), f_1(x_2)]
\end{aligned}$$

Now suppose we have a map $f_2\colon E \to L$ satisfying (1.5). Then $f_{12} = t\circ f_2\colon E \to N$ and $f_{22} = j\circ f_2\colon E \to E$ satisfy equations (1.3), (1.4) and

Lemma 1.9. $f_2 = i\circ f_{12} + s\circ f_{22}$

Proof.

Similar to the proof of lemma 1.7.

Lemma 1.10. *If we define* $f_2 = i\circ f_{12} + s\circ f_{22}$*, where* f_{12}, f_{22} *satisfy equations* (1.3), (1.4)*, then* f_2 *satisfies* (1.5)

Proof.

$$\begin{aligned}
f_2([y_1,y_2]) &= (i\circ f_{12} + s\circ f_{22})([y_1,y_2]) \\
&= i\,(f_{12}([y_1,y_2])) + s\,(f_{22}([y_1,y_2])) \\
&= i\left([f_{12}(y_1), f_{12}(y_2)] + f_{12}(y_2)^{f_{22}(y_1)} - f_{12}(y_1)^{f_{22}(y_2)}\right) \\
&\quad - \sigma(f_{22}(y_1), f_{22}(y_2) + f_{11}(\sigma(y_1,y_2))) \\
&= [(i\circ f_{12})(y_1), (i\circ f_{12})(y_2)] + [(s\circ f_{22})(y_1), (i\circ f_{12})(y_2)] - [(s\circ f_{22})(y_2), (i\circ f_{12})(y_1)] \\
&\quad - (i\circ\sigma)(f_{22}(y_1), f_{22}(y_2)) + (i\circ f_{11})(\sigma(y_1,y_2)) + [(s\circ f_{22})(y_1), (s\circ f_{22})(y_2)] \\
&\quad + (j\circ\sigma)(f_{22}(y_1), f_{22}(y_2)) + (s\circ f_{21})(\sigma(y_1,y_2)) \\
&= [(i\circ f_{12})(y_1), (i\circ f_{12})(y_2)] + [(i\circ f_{12})(y_1), (s\circ f_{22})(y_2)] \\
&\quad + [(s\circ f_{22})(y_1), (i\circ f_{12})(y_2)] + [(s\circ f_{22})(y_1), (s\circ f_{22})(y_2)] \\
&= [f_2(y_1), f_2(y_2)] + f_1(\sigma(y_1,y_2))\ .
\end{aligned}$$

We may combine lemmas 1.4 – 1.10 to prove theorem 1.3, since theorem 1.2 says that $f = f_1\circ t + f_2\circ j$.

Now let us associate the 2×2 matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{pmatrix}$ of functions f_{ij} to an endomorphism f of L. We may let the above matrix operate on the set $N\times E$ as

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} f_{11}(x)f_{12}(y) \\ f_{21}(x)f_{22}(y)\end{pmatrix}\ .$$

If we have two endomorphisms f, g of L, how do we compute the matrix of $f\circ g$?

Theorem 1.4. *If* $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ *and* $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ *are the matrices of endomorphisms* f, g *of* L *respectively, then the matrix of* $f \circ g$ *is*

$$\begin{pmatrix} f_{11} \circ g_{11} + f_{12} \circ g_{21} & f_{11} \circ g_{12} + f_{12} \circ g_{22} \\ f_{21} \circ g_{12} + f_{22} \circ g_{22} & f_{21} \circ g_{12} + f_{22} \circ g_{22} \end{pmatrix} .$$

Proof.

All the maps f_{ij} are linear maps of vector spaces.

2. Derivations

If we assume now that f is a derivation of L and define the functions f_{ij} as before, we get

Theorem 2.1. *The* f_{ij} *satisfy the following functional equations:*

(2.1)
$$f_{11}([x_1, x_2]) = [f_{11}(x_1), x_2] + [x_1, f_{11}(x_2)] + x_1^{f_{21}(x_1)} - x_1^{f_{21}(x_2)}$$
(2.2)
$$f_{21}([x_1, x_2]) = 0$$
(2.3)
$$f_{12}([y_1, y_2]) = f_{12}(y_2)^{y_1} - f_{12}(y_1)^{y_2} + f_{11}(\sigma(y_1, y_2)) - (\sigma(f_{22}(y_1), y_2) + \sigma(y_1, f_{22}(y_2)))$$
(2.4)
$$f_{22}([y_1, y_2]) = [f_{22}(y_1), y_2] + [y_1, f_{22}(y_2)] + f_{21}(\sigma(y_1, y_2))$$
(2.5)
$$f_{11}(x^y) = [f_{12}(y), x] + x^{f_{22}(y)} + f_{11}(x)^y$$
(2.6)
$$f_{21}(x^y) = [y, f_{21}(x)]$$

Theorem 2.2.
$$f = i \circ f_{11} \circ t + s \circ f_{21} \circ t + i \circ f_{12} \circ j + s \circ f_{22} \circ j$$

Proof.

The same as theorem 1.2.

Theorem 2.3. *If we define* f *as in the previous theorem, where the* f_{ij} *satisfy* (2.1) – (2.6), *then* f *is a derivation.*

As in the proof of theorem 1.3, we split our task in two and start with

Lemma 2.1. *If* $f : L \to L$ *is a derivation, define*

$$f_1 = f \circ i : N \to L$$
$$f_2 = f \circ s : E \to L$$

then f_1, f_2 *satisfy the following functional equations:*

(2.7) $$f_1([x_1, x_2]) = [f_1(x_1), i(x_2)] + [i(x_1), f_1(x_2)]$$
(2.8) $$f_2([y_1, y_2]) = [f_2(y_1), s(y_2)] + [s(y_1), f_2(y_2)] + f_1(\sigma(y_1, y_2))$$
(2.9) $$f_1(x^y) = [f_2(y), i(x)] + [s(y), f_1(x)]$$

Lemma 2.2. $f = f_1 \circ t + f_2 \circ j$

Proof.

The same as Lemma 1.5.

Lemma 2.3. *If we define* $f = f_1 \circ t + f_2 \circ j$, *where* f_1, f_2 *satisfy* (2.7) – (2.9), *then* f *is a derivation.*

Proof.

$$\begin{aligned}
f([g_1, g_2]) &= (f_1 \circ t)([g_1, g_2]) + (f_2 \circ j)([g_1, g_2]) \\
&= f_1 \left([t(g_1), t(g_2)] + t(g_2)^{j(g_1)} - t(g_1)^{j(g_2)} - \sigma(j(g_1), j(g_2)) \right) \\
&\quad + f_2([j(g_1), j(g_2)]) \\
&= [(f_1 \circ t)(g_1), (i \circ t)(g_2)] + [(i \circ t)(g_1), (f_1 \circ t)(g_2)] \\
&\quad [(f_2 \circ j)(g_1), (i \circ t)(g_2)] + [(s \circ j)(g_1), (f_1 \circ j)(g_2)] \\
&\quad - [(f_2 \circ j)(g_2), (i \circ t)(g_1)] - [(s \circ j)(g_2), (f_1 \circ j)(g_1)] \\
&\quad - (f_1 \circ \sigma)(j(g_1), j(g_2)) \\
&\quad + [(f_2 \circ j)(g_1), (s \circ j)(g_2)] + [(s \circ j)(g_1), (f_2 \circ j)(g_2)] \\
&\quad + (f_1 \circ \sigma)(j(y_1), j(y_2)) \\
&= [f(g_1), g_2] + [g_1, f(g_2)]
\end{aligned}$$

We now state a sequence of lemmas necessary to prove theorem 2.3 and omit the fairly straightfoward proofs, which, as is to be expected, involve similar algebraic manipulations as have been presented above.

Lemma 2.4. *Suppose we have two maps* $f_1 : N \to L$ *and* $f_2 : E \to L$ *satisfying* (2.7)–(2.9). *Then if we define*

$$\begin{aligned}
f_{11} &= j \circ f_1 : N \to E \\
f_{21} &= j \circ f_1 : N \to E \\
f_{12} &= t \circ f_2 : E \to N \\
f_{22} &= j \circ f_2 : E \to E
\end{aligned}$$

the f_{ij} *satisfy equations* (2.1) – (2.6).

Lemma 2.5. *If we define*

$$\begin{aligned}
f_1 &= i \circ f_{11} + s \circ f_{21} \\
f_2 &= i \circ f_{12} + s \circ f_{22} \ ,
\end{aligned}$$

where the f_{ij} *satisfy* (2.1) – (2.6), *then* f_1, f_2 *satisfy* (2.7) – (2.9).

It is now a simple matter to combine lemmas 2.1 – 2.5 to prove theorem 2.3.

3. Applications

Let h_n be the real Heisenberg lie algebra of dimension $2n+1$ over $\mathbb{R}$, given by a nondegenerate $\mathbb{R}$-valued skew 2-form on $\mathbb{R}^{2n}$. Such a form is automatically a 2-cocycle and gives a central extension

$$0 \to \mathbb{R} \xrightarrow{i} h_n \xrightarrow{j} \mathbb{R}^{2n} \to 0 .$$

Suppose that f is an endomorphism of h_n. Since $\mathbb{R}$ is the center of h_n, $f_{21}=0$, f_{11}, f_{22} are endomorphisms of $\mathbb{R}, \mathbb{R}^{2n}$ respectively, and f_{12} satisfies

$$f_{12}([y_1,y_2]) - [f_{12}(y_1), f_{12}(y_2)] = f_{11}(\sigma(y_1,y_2)) - \sigma(f_{22}(y_1), f_{22}(y_2)) \ . \tag{3.1}$$

The left side of this is zero, since $\mathbb{R}, \mathbb{R}^{2n}$ are each abelian, and thus f_{12} is an arbitrary linear map $\mathbb{R}^{2n} \to \mathbb{R}$. Since the right side is zero, this says that if $f_{11}=0$ then $f_{22}=0$, because of the nondegeneracy of σ. If $f_{11}>0$ then f_{22} must be a scalar multiple $\sqrt{f_{11}}$ of a symplectic map f_{22}' with respect to σ. If $f_{11}<0$, then f_{22} must be $\tau\sqrt{-\lambda} f_{22}'$, where f_{22}' is as above, and $\sigma(\tau(x),\tau(y)) = \sigma(x,y)$. If we consider how the matrices multiply, we can gather all these results into

Theorem 3.1. *The monoid of endomorphisms of h_n has a closed ideal consisting of elements whose matrices are of the form* $\begin{pmatrix} 0 & f_{12} \\ 0 & 0 \end{pmatrix}$ *and is thus homeomorphic to $\mathbb{R}^{2n}$ with the zero multiplication. The group of units of this monoid is the group of automorphisms of h_n and is a semidirect product of the closed normal subgroup of elements whose matrices are of the form* $\begin{pmatrix} 1 & f_{12} \\ 0 & 1 \end{pmatrix}$ *by the closed subgroup of elements whose matrices are of the form* $\begin{pmatrix} \lambda & 0 \\ 0 & \tau^{\frac{\lambda-|\lambda|}{2\lambda}} \sqrt{|\lambda|} f_{22}' \end{pmatrix}$, *where $\varphi_{12} \in \mathbb{R}^{2n}, \lambda \neq 0$, f_{22}' is a symplectic map with respect to σ and $\sigma(\tau(x),\tau(y)) = -\sigma(x,y)$.*

The automorphism group is homeomorphic to $\mathbf{Z}_2 \times \mathbb{R}^{2n+1} \times Sp(n,\mathbb{R})$ and thus has two connected components. The endomorphism monoid thus is connected and is homeomorphic to $\mathbb{R}^{2n+1} \times Sp(n,\mathbb{R})$

Theorem 3.2. *The lie algebra of derivations of h_n is a semidirect product of an ideal whose elements have matrices of the form* $\begin{pmatrix} 0 & f_{12} \\ 0 & 0 \end{pmatrix}$, *where $f_{12} \in \mathbb{R}^{2n}$ by an subalgebra which is a semidirect product of an ideal whose elements have matrices of the form* $\begin{pmatrix} 0 & 0 \\ 0 & f_{22} \end{pmatrix}$, *where f_{22} satisfies*

$$\sigma(f_{22}(y_1), y_2) + \sigma(y_1, f_{22}(y_2)) = 0 \tag{3.2}$$

and a subalgebra whose elements have matrices of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda f_{22}' \end{pmatrix}$, *where $\lambda \neq 0$ and f_{22}' satisfies*

$$\sigma(f_{22}'(y_1), y_2) + \sigma(y_1, f_{22}'(y_2)) = \sigma(y_1, y_2) \ . \tag{3.3}$$

Proof.

Of the equations (2.1) – (2.6) only (2.3), (2.4) impose any conditions on the f_{ij}. (2.3) becomes

$$\sigma(f_{22}(y_1), y_2) + \sigma(y_1, f_{22}(y_2)) = f_{11}\sigma(y_1, y_2) \tag{3.4}$$

and (2.4) forces $f_{21} = 0$ because of the nondegeneracy of σ.Thus the matrices are all upper triangular, and the rules of matrix multiplication together with (3.4), yield the assertion of the theorem.

The difficulty, in general, in explicitly solving the obvious equations distinguishing the automorphisms from the endomorphisms, lies in the fact that, even in the case of a direct product of abelian groups, their automorphism groups are not abelian. Consider the problem, for example, in trying obtain a closed form for the inverse of an $n \times n$ matrix of real numbers which has been divided into four blocks, reflecting the decomposition of $\mathbb{R}^n$ into the direct product of two subspaces, in terms of the four matrices constituting the blocks.

We are now working on extending these calculations to three-step nilpotent lie algebras as a first step toward calculating in a straightfoward way the automorphisms and derivations of lie algebras by means of the technique introduced here of introducing coordinates by assigning them 2×2 matrices of linear maps satisfying equations which ensure that the combined map is an automorphism, endomorphism or derivation.

We are also working out the situation in case there is a tower of ideals, and we want to associate larger sized matrices with endomorphisms of lie algebras with such towers.

After this paper had been written, I became aware of the paper of Hsu [H1], in which similar techniques are introduced for the case of a semidirect product of groups. In that paper, the automorphism group of a holomorph of a perfect group is computed.

References

[**DL**] J. Dixmier and G. Lister, *Derivations of Nilpotent Lie Algebras*, Proc. Am. Math. Soc. **8** (1957), 155—158.

[**H2**] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.

[**H1**] N. Hsu, *The Group of Automorphisms of the Holomorph of a Group*, Pacific J. Math. **11** (1961), 999—1012.

[**J1**] N. Jacobson, *Lie Algebras*, Interscience, New York, 1962.

[**J2**] N. Jacobson, *A Note on Automorphisms of Lie Algebras*, Pacific J. Math. **12** (1962), 303—317.

Newark, N.J.