REAL HEISENBERG GROUP EXTENSION ISOMORPHISM CLASSES

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ABSTRACT. The problem of determining when two central extension groups E_1,E_2 of H_n in $0 \longrightarrow \mathbb{R} \longrightarrow E_1 \rightarrow H_n \rightarrow 1$ and $0 \rightarrow \mathbb{R} \rightarrow E_2 \rightarrow H_n \longrightarrow 1$ are isomorphic as Lie groups is reduced to the classical problem of the equivalence of pencils of antisymmetric matrices. A technique of cohomologically trivializing trilinear functions from $\mathbb{R}^2$ to $\mathbb{R}$ is used to explicitly calculate 2-cocycles on H_1 which are not cohomologous to bilinear ones, thus providing a counterexample to a recent paper of Moskowitz, although for the higher groups H_n, $n \geq 2$, all $\mathbb{R}$-valued 2-cocycles *are* cohomologous to bilinear ones. Finally, a generalization, from semidirect products to arbitrary extensions, of a set of cocycle equations of Mackey and Tahara is derived and solved in the particular case of the Heisenberg group H_n. Consequences for the projective representation theory of H_n are given.

I. INTRODUCTION

In this paper the following question is completely answered for $n = 1$ in theorem 8.3: When are two one-dimensional central extensions

$$1 \rightarrow \mathbb{R} \rightarrow E \rightarrow H_n \rightarrow 1$$

of the (2n+1)-dimensional Heisenberg group H_n by $\mathbb{R}$ isomorphic as groups, and not necessarily as extensions? It should be remarked here that this terminology slightly conflicts with another, which calls E an extension of $\mathbb{R}$ by H_n. For $n > 1$, this is reduced to a classical problem in pencils of antisymmetric matrices in theorem 9.12. Here H_n is defined to be the unique central extension

$$0 \rightarrow \mathbb{R} \rightarrow H_n \rightarrow \mathbb{R}^{2n} \rightarrow 0$$

which does not contain any line $\mathbb{R}$ as a direct factor. This condition of not containing $\mathbb{R}$ as a factor is equivalent to the 2-cocycle of the extension, which can always be taken to be alternating bilinear ([Kl], theorem 7.1), being nondegenerate. The uniqueness follows from the fact that every pair of nondegenerate such forms are congruent in Gl(n,$\mathbb{R}$), the outer automorphism group of $\mathbb{R}^{2n}$.

This type of question is the mathematical core of a cohomological treatment of two- and three dimensional crystallography. In [Sch], one is first interested in classifying, up to congruence as defined in [McL], p.64, the extensions

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$$1 \to \mathbf{Z}^m \to \mathrm{E} \to \mathrm{O} \to 1$$

of a finite orthogonal group O, also known by crystallographers as a point group, by a free abelian group $\mathbf{Z}^m$ of rank two or three respectively, also known as a translation group, where O acts on $\mathbf{Z}^m$. The equivalence classes of such extensions under congruence are classified by $\mathrm{H}^2(\mathrm{O},\mathbf{Z}^m)$, the second Eilenberg-MacLane cohomology group as defined in [Mac], p.115. Secondly, it is also of interest to answer the question of when two extensions E,E′ are i somorphic by an isomorphism $\theta\colon \mathrm{E} \to \mathrm{E}'$ which is not required to respect the structures of E,E′ as extensions. The answer to this more interesting and significantly harder question is that there are 17 such groups in the plane, 230 in three dimensions, and 4783 in four (see [Sch], preface).

A careful study of the extensions of the Heisenberg group have interesting applications in the theory of optical wave guides, which includes the technologies of lasers and fiber optics. This is explained in [S].

The Heisenberg group is the simplest nonabelian Lie group and as such, its ordinary representation theory has been studied extensively. For further bibliography see the survey articles [H] and [R]. In a forthcoming paper, I will show how the results of the present paper can be applied to the projective representation theory of the Heisenberg group, which is what [R] and [S] are really studying.

In order to address this question for the Heisenberg groups, it is convenient to have explicit cocycle representatives for the cohomology classes in $\mathrm{H}^2_b(\mathrm{H},\mathbb{R})$, the second Moore cohomology group ([Mo1], p.42), based on Borel measurable cochains, which classify all locally compact central extensions of H_n by $\mathbb{R}$. We find these representatives in theorems 3.10,9.4. It is shown in [Du1], p.257 and [Wg], theorem 3, that we may replace the Moore groups by $\mathrm{H}^2_c(\mathrm{H}_n,\mathbb{R})$, cohomology based on continuous cochains. In [Mst], theorem 3.4.1, it is shown that each of these groups is isomorphic to $\mathrm{H}^2_\infty(\mathrm{H}_n,\mathbb{R})$, cohomology based on C^∞ cochains.

We show in theorems 3.10,9.4 that we may reduce still further the class of functions which may serve as representatives of a cohomology class, to either alternating bilinear functions $\sigma\colon \mathrm{H}_n \times \mathrm{H}_n \to \mathbb{R}$ in case $n \geq 2$, or in the case of H_1, polynomials in six variables, the parameters of H_1, at least one term of which is of degree two in one variable and linear in the others, the remaining terms being linear in all variables. This theorem, which may be thought of as a cohomological multilinearizability stability result, answers a question raised by a mistake in [Mos], theorem 4.3, where it is claimed to have been proved for a class of groups containing H_n, that $\mathrm{H}^2_c(\mathrm{H}_n,\mathbb{R})$ may always be calculated with bilinear functions. We give a counterexample to this assertion by exhibiting in theorem 3.1 a 2-cocycle σ on H_1 which is not cohomologous to a bilinear one. Also, this same cocycle has the property that its transpose $\sigma^t(g,g') = \sigma(g',g)$ is *not* a 2-cocycle, contrary to lemma 1.2 of [Mos].

The method of calculating explicit cocycle representatives is an elaboration to general group extensions of one developed by Mackey in [Ma1], theorem 9.4, for semidirect products of locally compact separable groups, hereafter known as polonais locally compact groups, polonais meaning complete separable metrizable. This involves showing how a 2-cocycle on a group extension can be built up from three types of functions, each of which arises as the restriction of σ to one of three appropriately defined subsets (theorem 4.1). These three types of functions jointly satisfy certain functional equations ((4.4) − (4.7)), which are then solved in the case of H_n in theorems 3.10,9.4 . It is also further of interest to determine when such a cocycle is cohomologous to zero, and this is done in theorems 4.5,4.8.

The determination of $\mathrm{H}^2(\mathrm{H}_n,\mathbb{R})$ in terms of an explicit description of the type of polynomial cocycles which are necessary and sufficient to compute this group is new, and part of a continuing program ,begun by Kleppner in [Kl], who showed in theorem 4.1, for a large class of locally compact abelian polonais groups A, which includes discrete, compact and connected such, that every Borel measurable 2-cocycle σ on A with values in the circle group $\mathbf{T}$, A acting trivially on $\mathbf{T}$, is cohomologous to a continuous bilinear function, and that $\mathrm{H}^2_b(\mathrm{A},\mathbf{T})$ is isomorphic to the group of these bilinear functions divided by the group of symmetric such.

Continuing this program, we showed in [Du1], p.255, for any polonais locally compact abelian group A acting trivially on $\mathbb{R}$, that $\mathrm{H}^n_b(\mathrm{A},\mathbb{R})\cong \wedge^n(\mathrm{A},\mathbb{R})$, the group of continuous alternating multilinear functions on A. $\mathrm{H}^m_b(\mathrm{H}_n,\mathbb{R})\cong\mathrm{H}^m_b(\mathrm{H}_n,\mathbf{T})$ since H_n is simply connected. This latter group arises in physics in the study of the projective representations $\rho\colon \mathrm{H}_n \to \mathrm{PU}(\infty)$, the projective unitary unitary group of a separable Hilbert space. This is described in [Ma1], theorem 2.1. The techniques introduced in this paper are further elaborated in [Du3].

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II. $\mathrm{H}^2(\mathrm{H}_n,\mathbb{R})$

Definition 2.1. *If α is a nondegenerate alternating $\mathbb{R}$-bilinear form $\mathbb{R}^{2n}\times\mathbb{R}^{2n}\xrightarrow{\alpha}\mathbb{R}$, then the $(2n+1)$-dimensional Heisenberg group H^α_n fits into an exact sequence*

$$0\to\mathbb{R}\to\mathrm{H}^\alpha_n\to\mathbb{R}^{2n}\to 0\ , \tag{2.1}$$

and is the set of pairs $(t,\mathbf{v})\in\mathbb{R}\times\mathbb{R}^{2n}$, with the group operation being given as

$$(t_1,\mathbf{v}_1)(t_2,\mathbf{v}_2)=(t_1+t_2+\alpha(\mathbf{v}_1,\mathbf{v}_2),\mathbf{v}_1+\mathbf{v}_2)\ .$$

H^α_n is a connected, simply-connected, 2-step nilpotent Lie group with center and commutator subgroup $\{(t,0)|t\in\mathbb{R}\}\cong\mathbb{R}$ and quotient group $\mathrm{H}_n/\mathbb{R}\cong\mathbb{R}^{2n}$.

Lemma 2.2. *If α,α' are two nondegenerate alternating bilinear forms on $\mathbb{R}^{2n}$ then there is an $\mathrm{L}\in\mathrm{Gl}(\mathbb{R}^{2n},\mathbb{R})$ such that $\alpha\circ(\mathrm{L}\times\mathrm{L})=\alpha'$ and $\mathrm{H}^\alpha_n\cong\mathrm{H}^{\alpha'}_n$.*

Proof.

The existence of such an L can be found in many books on algebra, e.g., [Ja], theorem 6.3. For the isomorphism $\theta\colon\mathrm{H}^\alpha_n\to\mathrm{H}^{\alpha'}_n$, define $\theta(t,\mathbf{v})=(t,L\mathbf{v})$.

We may now drop the α from H^α_n and assume that the defining matrix for α is either $\begin{bmatrix}0 & -\mathrm{I}\\ \mathrm{I} & 0\end{bmatrix}$ or $\begin{bmatrix}0 & -\mathrm{J}\\ \mathrm{J} & 0\end{bmatrix}$, where I is the $n\times n$ identity matrix and J is the $n\times n$ matrix with 1's in the secondary diagonal and zeros elsewhere.

From [Ma2], thèorème 2, we see that $\mathrm{H}^2_b(\mathrm{H}_n,\mathbb{R})$, the second Moore cohomology group with Borel measurable cochains and trivial action of H_n on the coefficient group $\mathbb{R}$, classifies the one-dimensional central extensions

$$0\to\mathbb{R}\xrightarrow{i}\mathrm{E}\xrightarrow{j}\mathrm{H}_n\to 1 \tag{2.2}$$

up to equivalence of extensions, the group operation on extensions being the Baer sum as defined in [Mac], theorem 2.1. H_n, being continuously solvable of finite length as defined in [Du1], p.257, or equivalently, since H_n is simply connected, it follows from [Du1], p.257 or [Wg], theorem 3, that $H^2_b(H_n,\mathbb{R}) \cong H^2_c(H_n,\mathbb{R})$, cohomology with continuous cochains.

Since this means that there is a continuous cross-section s for j, we have

Theorem 2.3(Sah). *There is a seven-term exact sequence*

$$0 \to E_2^{10} \to H^1(H,\mathbb{R}) \to E_2^{01} \to E_2^{20} \to H^2(H_n,\mathbb{R})_1 \to E_2^{11} \to E_2^{30}\ , \tag{2.3}$$

where $H^2(H_n,\mathbb{R})_1$ *is defined by the exact sequence*

$$0 \to H^2(H_n,\mathbb{R})_1 \to H^2(H_n,\mathbb{R}) \xrightarrow{res} H^2(\mathbb{R},\mathbb{R})\ , \tag{2.4}$$

res being the restriction map.

Proof.

The arguments referred to in [Sa] p.257 are also valid in case the cocycles are continuous, the existence of the spectral sequence E_r^{pq} following from [Mst], theorem 4.1, [Du1], p.246, or [Mo2], theorem 9. We now drop the subscript c, with the understanding that it is understood to be there throughout the remainder of this paper.

Remark 2.4.

From [Du1], p.250, [Mo2], theorem 9, we may identify the second term E_2^{pq} of the spectral sequence with $H^p(\mathbb{R}^{2n},H^q(\mathbb{R},\mathbb{R}))$. Remembering that $H^1(\mathbb{R},\mathbb{R}) \cong \mathbb{R} \cong H^0(\mathbb{R},\mathbb{R})$, the sequence (2.3) first becomes

$$(2.3^*)\quad 0 \to H^1(\mathbb{R}^{2n},H^0(\mathbb{R},\mathbb{R})) \xrightarrow{inf} H^1(H_n,\mathbb{R}) \to H^0(\mathbb{R}^{2n},H^1(\mathbb{R},\mathbb{R}))$$
$$\xrightarrow{d_2^{01}} H^2(\mathbb{R}^{2n},H^0(\mathbb{R},\mathbb{R})) \xrightarrow{inf} H^2(H_n,\mathbb{R})_1 \xrightarrow{tg} H^1(\mathbb{R}^{2n},H^1(\mathbb{R},\mathbb{R})) \xrightarrow{d_2^{11}} H^3(\mathbb{R}^2 n,\mathbb{R})\ ,$$

where *inf* and *tg* are inflation and transgression, respectively, and then becomes

$$(2.3^{**})\quad 0 \to \mathbb{R} \xrightarrow{d_2^{01}} H^2(\mathbb{R}^{2n},\mathbb{R}) \xrightarrow{inf} H^2(H_n,\mathbb{R}) \xrightarrow{tg} H^1(\mathbb{R}^{2n},\mathbb{R}) \xrightarrow{d_2^{11}} H^3(\mathbb{R}^{2n},\mathbb{R})$$

since the restriction map is zero (any homomorphism of H_n into $\mathbb{R}$ must vanish on the commutator subgroup).

III. $H^2(H_1,\mathbb{R})$

Now let us examine H_1 in more detail.

Theorem 3.1. H_1 *has a faithful matrix representation* θ: $H_1 \to Sl(3,\mathbb{R})$.

Proof.

As we said earlier, we will take for α the form with the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which means that $\alpha((y_1,y_2),(y_1',y_2')) = y_1y_2' - y_1'y_2$, so if we define

$$\theta(t,(y_1,y_2)) = \begin{bmatrix} 1 & \sqrt{2}y_1 & t+y_1y_2 \\ 0 & 1 & \sqrt{2}y_2 \\ 0 & 0 & 1 \end{bmatrix}\ ,$$

θ gives our desired representation. The range of this representation is the set of all strictly upper triangular matrices $\begin{pmatrix} 1 & y_1 & t \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix}$.

Remark 3.2.

Notice here that the form

$$\beta((y_1, y_2), (y_1', y_2')) = y_1 y_2' ,$$

which is the one defining the matrix multiplication, is neither antisymmetric nor nondegenerate. This is explained by

Theorem 3.3. *The form β above is cohomologous to the form*

$$\alpha((y_1, y_2), (y_1', y_2')) = \frac{y_1 y_2' - y_1' y_2}{2} .$$

Proof.

Let $c(y_1, y_2) = \dfrac{-y_1 y_2}{4}$. Then $\beta = \alpha + \delta c$.

Remark 3.4.

This shows that it is not necessary to define H_1^{α} by using a nondegenerate alternating form α, but it suffices to use any form β which is cohomologous to α. In the case above, β was a degenerate form, viz., had rank one. But we may find another more appropriate faithful matrix representation of H_1.

Theorem 3.5. *There is a faithful representation $\theta \colon \mathrm{H}_1 \to \mathrm{Sl}(4, \mathbb{R})$ such that the 2-cocycle defining the image group as an extension and arising via matrix multiplication, is identical to the cocycle α defining H_1.*

Proof.

$$\text{Let } \theta(t, (y_1, y_2)) = \begin{bmatrix} 1 & y_1 & y_2 & t \\ 0 & 1 & 0 & y_2 \\ 0 & 0 & 1 & -y_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Until further notice, we will assume that H_1 is given as the set of matrices $\begin{pmatrix} 1 & y_1 & t \\ ) & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix}$.

Now suppose, as follows from [Mos], theorem 4.3, that $\mathrm{H}^2(\mathrm{H}_1, \mathbb{R})$ may be calculated by using continuous bilinear functions. Then such a function must vanish on sets of the form $\mathbb{R} \times \mathbb{R}^2$ and $\mathbb{R}^2 \times \mathbb{R}$, since $\mathbb{R}$, the center of H_1, is also its commutator subgroup. Since we already know from [Du1], p.256, that $\mathrm{H}^2(\mathbb{R}^2, \mathbb{R}) \cong \wedge^2(\mathbb{R}^2, \mathbb{R})$, and is thus one-dimensional, we may suppose that this vector space is generated cohomologically by the nondegenerate alternating form α above, which becomes a cocycle on H_1 by pullback. But then we have

Theorem 3.6. *α is a trivial 2-cocycle on H_1.*

Proof.

Let $c(t, (y_1, y_2)) = t$. Then $\delta c = \alpha$.

Now that we know where *not* to look for nontrivial 2-cocycles, where *do* we look? Suppose we look at the space P of polynomials in $t, y_1, y_2, t', y_1', y_2'$ which are sums of monomials of degree two or less in each of the variables, of total degree two or three, and which involve variables both primed and unprimed. This gives us forty five monomials as a basis of this space. The coboundary map δ acts on P, and we have

Theorem 3.7. Dim Ker $\delta\colon \mathrm{P} \to \mathrm{P} = 6$, *and a basis for this kernel is*

$$
\begin{aligned}
&\alpha_1 : y_1^2 y_2' + 2y_1 t \\
&\alpha_2 : y_1 (y_2')^2 + 2ty_2' \\
&\alpha_3 : y_2^2 y_1 + 2y_2 y_1' y_2' - 2y_2 t' \\
&\alpha_4 : y_2 (y_1')^2 + 2y_1 y_2 y_1' - 2ty_1' \\
&\alpha_5 : y_1 y_2 y_2' - ty_2' + y_2 t' \\
&\alpha_6 : y_1 y_1' y_2' - y_1 t' + ty_1'
\end{aligned}
\tag{3.1}
$$

Proof.

Use linear algebra.

Theorem 3.8. *The six cocycles in (2.1) are cohomologous respectively to*

$$\alpha_1, \quad \alpha_2, \quad -\alpha_2, \quad -\alpha_1, \quad -\alpha_1, \quad -\alpha_2 \,.$$

Proof.

Introduce

$$
\begin{aligned}
&\alpha_1' : \; y_1^2 y_2' + 2y_1 y_1' y_2' + 2ty_1' \\
&\alpha_2' : \; y_1 (y_2')^2 + 2y_1 y_2 y_2' + 2y_2 t' \,,
\end{aligned}
$$

which are cohomologous to $-\alpha_1, -\alpha_2$ via

$$c_1(t,(y_1,y_2)) \;=\; 2y_1 t \;\text{ and }\; c_2(t,(y_1,y_2)) \;=\; 2ty_2 \,,$$

respectively. Thus $\alpha_6 = \dfrac{\alpha_1' - \alpha_1}{2}$, $\alpha_5 = \dfrac{\alpha_2' - \alpha_2}{2}$, showing α_6, α_5 cohomologous to $-\alpha_1, \alpha_2$. Notice too, that $\alpha_3 + \alpha_2', \alpha_4 - \alpha_1'$ are defined on $\mathrm{H}_1/\mathbb{R}$ and are symmetric there, and hence trivial, which proves the theorem.

Theorem 3.9. *If $\beta = \lambda\alpha_1 + \mu\alpha_2$ is cohomologous to zero, then $\lambda = \mu = 0$.*

Proof.

Suppose that we can find a triple commutator $[[g_1, g_2], g_3]$ which is not the identity in the extension group furnished by a cocycle β, which group we can assume is given by 4-tuples (s, t, y_1, y_2) with the composition

$$(s,t,y_1,y_2)(s',t',y_1',y_2') = (s+s'+\beta((t,y_1,y_2),(t',y_1',y_2')), t+t'+y_1y_2'-y_1'y_2, y_1+y_1', y_2+y_2')$$

In order to compute the commutator, we take the inverse

$$(s,t,y_1,y_2)^{-1} \;=\; (-s + 2y_1 t - y_1^2 y_2, -t + y_1 y_2, -y_1, -y_2) \,.$$

The two triple commutators

$$[[(0,0,1,0),(0,0,0,1)],(0,1,1,1)] \quad [[(0,0,1,0),(0,0,0,1)],(0,0,1,0)] \,,$$

when computed with respect to $\lambda\alpha_1 + \mu\alpha_2$, since we are assuming that the extension splits, must both be equal to the identity, which forces $\lambda = \mu = 0$, showing that α_1 and α_2 are cohomologically linearly independent, and in particular, each one is nontrivial.

Theorem 3.10. *Every 2-cocycle in* $\mathrm{Z}^2(\mathrm{H}_1,\mathbb{R})$ *is cohomologous to a linear combination of* α_1, α_2.

Proof.

It follows from theorem 3.6 that the inflation map inf in (1.3^{**}) is zero. Since this makes tg one-to-one, $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$ injects into $\mathrm{H}^2(\mathbb{R}^{2n},\mathbb{R})$, which is two-dimensional. But we have found a basis α_1, α_2 of this space.

Corollary 3.11. $\alpha_1^t((t,y_1,y_2),(t',y_1',y_2')) = \alpha_1((t',y_1',y_1'),(t,y_1,y_2))$ *is not a cocycle.*

Proof.

Calculational.

IV. Functional Equations

We now want to show how a 2-cocycle on a group extension is determined by means of its restrictions to certain very special subsets, and then to derive equations (4.4) – (4.7) subsisting between these restrictions which are necessary and sufficient to ensure that we can reconstruct the original cocycle on the extension group. We also answer, in theorem 4.5, when a cocycle, torn apart in this manner, is cohomologous to zero, this answer being given as conditions holding for the pieces.

The equations we are about to derive for an arbitrary group extension of locally compact polonais groups were obtained in the special case of semidirect products by Mackey in [Ma1], theorem 9.4, and Tahara in [Ta], prop.4. Tahara worked in the category of discrete groups, Mackey in the same category as we. It is possible to utilize the work of Brown in [Br] to extend the category to polonais groups, and this work is in progress. Therefore, for the present, we work in the category of locally compact polonais groups and continuous homomorphisms, and the definition of exactness is as in [Mo1], definition 1.3.

Theorem 4.1. *Suppose* $1 \longrightarrow \mathrm{N} \xrightarrow{i} \mathrm{G} \underset{s}{\overset{j}{\rightleftarrows}} \mathrm{H} \longrightarrow 1$ *is exact,* s *is a Borel section of* j*, and* α *is a factor set defined as*

$$\alpha(y_1,y_2) = s(y_1)s(y_2)s(y_1y_2)^{-1} \quad \textit{and} \quad \beta(y)(x) = s(y)xs(y)^{-1} \ .$$

Also suppose $\gamma\colon \mathrm{A} \to \mathrm{Aut}(\mathrm{A})$*, where* $\gamma(a_1)(a_2) = a_1a_2a_1^{-1}$*. Then if we have data*

$$\varphi\colon \mathrm{G} \to \mathrm{Aut}(\mathrm{A}), \quad \sigma\colon \mathrm{G}\times\mathrm{G} \to \mathrm{A}\ ,$$

satisfying

$$\text{(4.1a)} \qquad \varphi(g_1)\varphi(g_2) = \gamma\big(\sigma(g_1,g_2)\big)\varphi(g_1g_2)$$

$$\text{(4.1b)} \qquad \varphi(g_1)\big(\sigma(g_2,g_3)\big)\sigma(g_1,g_2g_3) = \sigma(g_1,g_2)\sigma(g_1g_2,g_3) \quad \textit{for } g_1,g_2,g_3 \in \mathrm{G}\ .$$

Then (φ,σ) *may be described in terms of data on* N *and* H *as follows: Let* $\varphi_1 = \varphi_{|\mathrm{N}} = \varphi\circ i$ *,* $\varphi_2 = \varphi_{|s(\mathrm{H})} = \varphi\circ s$*, and also let*

$$\sigma_{11} = \sigma\circ(i\times i),\ \ \sigma_{12} = \sigma\circ(i\times s),\ \ \sigma_{21} = \sigma\circ(s\times i),\ \ \sigma_{22} = \sigma\circ(s\times s)\ .$$

Then we have the following equations:

$$\varphi_1(x_1)\varphi_1(x_2)\big(\varphi_1(x_1x_2)\big)^{-1} = \gamma\big(\sigma_{11}(x_1,x_2)\big) \tag{4.2}$$

$$\varphi_2(y_1)\varphi_2(y_2)\big(\varphi_2(y_1y_2)\big)^{-1} = \gamma\big(\sigma_{22}(y_1,y_2)\big)\varphi_1\big(\alpha(y_1,y_2)\big) \tag{4.3}$$

$$\varphi_1(x_1)\big(\sigma_{11}(x_2,x_3)\big)\sigma_{11}(x_1,x_2x_3) = \sigma_{11}(x_1,x_2)\sigma_{11}(x_1x_2,x_3) \tag{4.4}$$

$$\begin{aligned}(4.5)\quad & \sigma_{21}(y,x_1x_2)\\ &= \big(\varphi_2(y)\big(\sigma_{11}(x_1,x_2)\big)\big)^{-1}\sigma_{21}(y,x_1)\varphi_1\big(\beta(y)(x_1)\big)\big(\sigma_{21}(y,x_2)\big)\sigma_{11}\big(\beta(y)(x_1),\beta(y)(x_2)\big)\end{aligned}$$

$$\begin{aligned}(4.6)\quad & \varphi_1\big(\alpha(y_1,y_2)\big)\big(\sigma_{21}(y_1y_2,x)\big)\sigma_{11}\big(\alpha(y_1,y_2),\beta(y_1y_2)(x)\big)\\ &= \big(\sigma_{22}(y_1,y_2)\big)^{-1}\varphi_2(y_1)\big(\sigma_{21}(y_2,x)\big)\sigma_{21}\big(y_1,\beta(y_2)(x)\big)\\ &\qquad \varphi_1\big(\beta(y_1)\beta(y_2)(x)\big)\big(\sigma_{22}(y_1,y_2)\big)\sigma_{11}\big(\beta(y_1)\beta(y_2)(x),\alpha(y_1,y_2)\big)\end{aligned}$$

$$\begin{aligned}(4.7)\quad & \big(\varphi_2(y_1)\big(\sigma_{22}(y_2,y_3)\big)\big)^{-1}\sigma_{22}(y_1,y_2)\varphi_1\big(\alpha(y_1,y_2)\big)\big(\sigma_{22}(y_1y_2,y_3)\big)\\ &= \sigma_{21}\big(y_1,\alpha(y_2,y_3)\big)\varphi_1\,\big(\beta(y_1)\big(\alpha(y_2,y_3)\big)\big)\,\big(\sigma_{22}(y_1,y_2y_3)\big)\\ &\qquad \sigma_{11}\,\big(\beta(y_1)\big(\alpha(y_2,y_3)\big),\alpha(y_1,y_2y_3)\big)\,\big(\sigma_{11}\big(\alpha(y_1,y_2),\alpha(y_1y_2,y_3)\big)\big)^{-1}\end{aligned}$$

Conversely, if we have data satisfying these equations, and we define

$$\begin{aligned}(4.8)\quad & \sigma\big(x_1s(y_1),x_2s(y_2)\big) =\\ &\qquad \varphi_1(x_1)\big(\sigma_{21}(y_1,x_2)\big)\sigma_{11}\big(x_1,\beta(y_1)(x_2)\big)\cdot\\ &\qquad\qquad \varphi_1\big(x_1\beta(y_1)(x_2)\big)\big(\sigma_{22}(y_1,y_2)\big)\sigma_{11}\big(x_1\beta(y_1)(x_2),\alpha(y_1,y_2)\big)\end{aligned}$$

$$\varphi\big(xs(y)\big) = \varphi_1(x)\varphi_2(y)\ ,$$

then (σ,φ) *is a 2-cocycle on* G.

Proof.

For the proof, we first need a lemma. Notice that σ_{12} did not appear in these equations. This is the result of

Lemma 4.2. *If* σ *is as in the theorem, then* σ *is cohomologous to a* σ' *such that* $\sigma'_{12} = e$.

Proof.

Assume σ normalized, i.e., $\sigma(g,e) = \sigma(e,g) = e$ for all $g \in$ G. Then we let $c(xs(y)) = \sigma(x,s(y))$, and define

$$\sigma'(g_1,g_2) = c(g_1)\varphi(g_1)\,(c(g_2))\,\sigma(g_1,g_2)\big(c(g_1g_2)\big)^{-1}.$$

Continuing the proof, (4.2) follows from (4.1a) by restricting to N. For (4.3), substituting $s(y_i)$ for g_i in (4.1a), we get

$$\varphi(s(y_1))\varphi(s(y_2)) \;=\; \gamma(\sigma(s(y_1),s(y_2)))\varphi(s(y_1)s(y_2)) \;,$$

which may be rewritten as

$$\varphi_2(y_1)\varphi_2(y_2) \;=\; \gamma(\sigma_{22}(y_1,y_2))\varphi(\alpha(y_1,y_2)s(y_1y_2)) \;.$$

Using (4.1a) and lemma 4.2,

$$\begin{aligned}\varphi(\alpha(y_1,y_2)s(y_1y_2)) = \\ &[\gamma(\sigma(\alpha(y_1,y_2),s(y_1y_2)))]^{-1}\,\varphi_1(\alpha(y_1,y_2))\varphi_2(y_1y_2) \\ &= \varphi_1(\alpha(y_1,y_2))\varphi_2(y_1y_2)\;,\end{aligned}$$

and putting this back in our equation, we arrive at (4.3). (4.4) follows by restricting g_1, g_2, g_3 to lie in N. For (4.5) replace g_1, g_2, g_3 by $s(y), x_1, x_2$ respectively in (4.1b), and reduce the right side of the resulting equation. (4.1b) becomes

$$\varphi_2(y)(\sigma_{11}(x_1,x_2))\sigma_{21}(y,x_1x_2) \;=\; \sigma_{21}(y,x_1)\sigma(s(y)x_1,x_2) \;.$$

We rewrite $\sigma(s(y)x_1,x_2)$ as $\sigma(\beta(y)(x_1)|s(y),x_2)$. We always use a vertical bar between $\beta(y)(x_1)$ and $s(y)$ to indicate how we will use (4.1b), namely, we assume that the term $\sigma(\beta(y)(x_1)|s(y),x_2)$ is the term $\sigma(g_1|g_2,g_3)$, with g_1, g_2, g_3 in this term now being identified with $\beta(y)(x_1), s(y), x_2$ respectively. Using this procedure, we get

$$\sigma(\beta(y)(x_1)|s(y),x_2) = [\sigma_{12}(\beta(y)(x_1),s(y))]^{-1}\,\varphi_1(\beta(y)(x_1))\sigma_{21}(y,x_2)\sigma_{21}(y,x_1x_2)\;,$$

in which the σ_{12} term is e, according to lemma 4.2. This reduction yields (4.5). We will not even write down any σ_{12} terms should they occur from now on, as we know they are all $= e$. Also, we will indicate how to apply (4.1b) by putting a vertical bar in the appropriate place, as we did in this derivation.

For (4.6), we substitute $s(y_1), s(y_2), x$ for g_1, g_2, g_3 in (4.1b), obtaining

$$\varphi_2(y_1)\sigma_{21}(y_2,x)\sigma(s(y_1),s(y_2)x) \;-\; \sigma_{22}(y_1,y_2)\sigma(s(y_1)s(y_2),x) \;.$$

Rewriting the left side first, we have

$$\sigma(s(y_1),s(y_2)x) = \sigma(s(y_1),\beta(y_2)(x)|s(y_2))\;,$$

and using (4.1b), this is

$$\varphi_2(y_1)\sigma_{21}(y_1,\beta(y_2)(x))\sigma(s(y_1)|\beta(y_2)(x),s(y_2))\;.$$

The rightmost term must be reduced by using (4.1b).

$$\sigma(s(y_1)|\beta(y_2)(x),s(y_2)) \;=\; \sigma(\beta(y_1)\beta(y_2)(x)|s(y_1),s(y_2))\;,$$

to which we apply (4.1b) again, obtaining

$$\varphi_1(\beta(y_1)\beta(y_2)(x))\sigma_{22}(y_1,y_2)\sigma(\beta(y_1)\beta(y_2)(x),s(y_1)s(y_2))$$

in which we rewrite the right most term as $\sigma(\beta(y_1)\beta(y_2)(x),\alpha(y_1,y_2)|s(y_1y_2))$ which reduces to

$$\sigma_{11}(\beta(y_1)\beta(y_2)(x),\alpha(y_1,y_2))\ .$$

Collecting all the reduced terms, the left side becomes

$$\varphi_2(y_1)\,(\sigma_{21}(y_2,x))\,\sigma_{21}(y_1,\beta(y_2)(x))\cdot \\ \varphi_1(\beta(y_1)\beta(y_2)(x))\,(\sigma_{22}(y_1,y_2))\,\sigma_{11}(\beta(y_1)\beta(y_2)(x).\alpha(y_1,y_2))\ .$$

Rewriting the rightmost term on the right side, and reducing, we get

$$\sigma(\alpha(y_1,y_2)|s(y_1y_2),x)=\varphi_1(\alpha(y_1,y_2))\sigma_{21}(y_1y_2,x)\sigma(\alpha(y_1,y_2),s(y_1y_2)x)\ .$$

Rewriting the rightmost term and reducing,

$$\sigma(\alpha(y_1,y_2),\beta(y_1y_2)(x)|s(y_1y_2))=\sigma_{11}(\alpha(y_1,y_2),\beta(y_1y_2)(x))\ ,$$

and collecting all the reduced terms on the right side, we get

$$\sigma_{22}(y_1,y_2)\varphi_1(\alpha(y_1,y_2))\,(\sigma_{21}(y_1y_2,x))\,\sigma_{11}(\alpha(y_1,y_2),\beta(y_1y_2)(x))\ ,$$

showing that if we multiply the left and right sides by $(\sigma_{22}(y_1,y_2))^{-1}$, we get equation (4.6).

In order to derive equation (4.7), we first replace g_1,g_2,g_3 by $s(y_1),s(y_2),s(y_3)$ respectively, and obtain

$$\varphi_2(y_1)\,(\sigma_{22}(y_2,y_3))\,\sigma(s(y_1),s(y_2)s(y_3))=\sigma_{22}(y_1,y_2)\sigma(s(y_1)s(y_2),s(y_3))\ .$$

Rewriting the rightmost term on the left side and reducing, we get

$$\sigma(s(y_1),s(y_2)s(y_3))=\sigma(s)y_1),\alpha(y_1,y_2)|s(y_1y_2)) \\ =\sigma_{21}(y_1,\alpha(y_2,y_3))\sigma(s(y_1)\alpha(y_2,y_3),s(y_2y_3))\ .$$

Again rewriting the rightmost term and reducing, this is

$$\sigma(s(y_1)\alpha(y_2,y_3),s(y_2y_3))=\sigma(\beta(y_1)(\alpha(y_2,y_3))|s(y_1),s(y_2y_3)) \\ =\varphi_1(\beta(y_1)(\alpha(y_2,y_3)))\,(\sigma_{22}(y_1,y_2y_3))\,\sigma(\beta(y_1)(\alpha(y_2,y_3)),s(y_1)s(y_2y_3))\ .$$

rewriting and reducing the rightmost term,

$$\sigma(\beta(y_1)(\alpha(y_2,y_3)),s(y_1)s(y_2y_3))=\sigma(\beta(y_1)(\alpha(y_2,y_3)),\alpha(y_1,y_2y_3)|s(y_1y_2y_3)) \\ =\sigma_{11}(\beta(y_1)(\alpha(y_2,y_3)),\alpha(y_1,y_2y_3))\ .$$

Collecting all the reduced terms on the left side, it becomes

$$\varphi_2(y_1)\,(\sigma_{22}(y_2,y_3))\,\sigma_{21}(y_1,\alpha(y_2,y_3))\cdot \\ \varphi_1(\beta(y_1)(\alpha(y_2,y_3)))\,(\sigma_{22}(y_1,y_2y_3))\,\sigma_{11}(\beta(y_1)(\alpha(y_2,y_3)),\alpha(y_1,y_2y_3))\ .$$

We now rewrite and reduce the rightmost term on the right side of our transformed (4.7).

$$\begin{aligned}\sigma(s(y_1)s(y_2), s(y_3)) &= \sigma(\alpha(y_1, y_2)|s(y_1y_2)) \\ &= \varphi_1(\alpha(y_1, y_2))\,(\sigma_{22}(y_1y_2, y_3))\,\sigma(\alpha(y_1, y_2), s(y_1y_2)s(y_3))\ ,\end{aligned}$$

and applying this process again, this is

$$\begin{aligned}\sigma(\alpha(y_1, y_2), s(y_1y_2)s(y_3)) &= \sigma(\alpha(y_1, y_2), \alpha(y_1y_2, y_3)|s(y_1y_2y_3)) \\ &= \sigma_{11}(\alpha(y_1, y_2), \alpha(y_1y_2, y_3))\ ,\end{aligned}$$

and the right side becomes

$$\sigma_{22}(y_1, y_2)\varphi_1(\alpha(y_1, y_2))\,(\sigma_{22}(y_1y_2, y_3))\,\sigma_{11}(\alpha(y_1, y_2), \alpha(y_1y_2, y_3))\ ,$$

which yields (4.7). We may establish in the same manner the fact that if (σ, φ) are defined as in (4.8), then we obtain a cocycle.

Remark 4.3.

A set of equations similar to that given in the previous theorem can be derived for 3-cocycles taking values in an abelian group A, but these will not be needed here.

Theorem 4.4. *Suppose σ as in theorem 4.1 takes values in an abelian group* A *on which* G *operates trivially and* N *is central in* G*. Then the above equations become*

$$\sigma_{11}(x_2, x_3)\sigma_{11}(x_1, x_2x_3) = \sigma_{11}(x_1, x_2)\sigma_{11}(x_1x_2, x_3) \tag{4.9}$$

$$\sigma_{21}(y, x_1x_2) = \sigma_{21}(y, x_1)\sigma_{21}(y, x_2) \tag{4.10}$$

$$\sigma_{21}(y_1y_2, x) = \sigma_{21}(y_1, x)\sigma_{21}(y_2, x)\sigma_{11}\big(x, \alpha(y_1, y_2)\big)\,\big(\sigma_{11}\big(\alpha(y_1, y_2), x\big)\big)^{-1} \tag{4.11}$$

$$\begin{aligned}&\big(\sigma_{22}(y_2, y_3)\big)^{-1}\sigma_{22}(y_1y_2, y_3)\big(\sigma_{22}(y_1, y_2y_3)\big)^{-1}\sigma_{22}(y_1, y_2) \\ &\quad = \sigma_{21}\big(y_1, \alpha(y_2, y_3)\big)\sigma_{11}\big(\alpha(y_2, y_3), \alpha(y_1, y_2y_3)\big)\big(\sigma_{11}(\alpha(y_1, y_2), \alpha(y_1y_2, y_3))\big)^{-1}\end{aligned} \tag{4.12}$$

and if we define

$$\sigma\big(x_1s(y_1), x_2s(y_2)\big) = \sigma_{11}(x_1, x_2)\sigma_{21}(y_1, x_2)\sigma_{22}(y_1, y_2)\sigma_{11}\big(x_1x_2, \alpha(y_1, y_2)\big), \tag{4.13}$$

we recover σ.

Proof.

Notice that γ and φ become maps having the identity map as range in the case stipulated.

These equations are only half the story. They will determine the group of 2-cocycles Z^2(G,A), but we must still derive conditions for such cocycles to be coboundaries. For this we have

Theorem 4.5. *If $\sigma \in Z^2(G,A)$ as in theorem 4.1, then σ is a coboundary iff*

(4.14) $\sigma_{11}(x_1.x_2) = \varphi_1(x_1)(c_1(x_2))\,[c_1(x_1x_2)]^{-1}\,c_1(x_1)$

(4.15) $\sigma_{21}(y,x) = \varphi_2(y)(c_1(x))\,[\varphi_1(\beta(y)(x))(c_2(y))]^{-1}\,[c_1(\beta(y)(x))]^{-1}\,c_2(y)$

(4.16) $\sigma_{22}(y_1,y_2) = c_2(y_2)\,[c_2(y_1y_2)]^{-1}\,[c_1(\alpha(y_1,y_2))]^{-1}\,c_2(y_1)$

for some $c_1 \in C^1(N,A)$, $c_2 \in C^1(H,A)$.

Proof.

First we need a lemma.

Lemma 4.6. *If $\sigma \in B^2(G,A)$ and $\sigma = \delta c$, $\sigma_{12} = e$, for $c \in C^1(G,A)$, then c may be taken to have the property $c(xs(y)) = c(x)\beta(x)(c(s(y)))$.*

Proof.

$$(\delta(c))_{12}(x,s(y)) = \beta(x)\,(c(s(y)))\,[c(xs(y))]^{-1}\,c(x) = e\ .$$

To finish the proof, let $c_1(x) = c(x)$, $c_2(y) = c(s(y))$, and in the equation

(4.17)
$$\delta c(x_1s(y_1),x_2s(y_2)) = \varphi(x_1s(y_1))\,(c(x_2s(y_2)))\,[c(x_1x_2\alpha(y_1,y_2)s(y_1y_2))]^{-1}\,c(x_1s(y_1))\ ,$$

which is just

(4.18) $$\delta c(g_1,g_2) = \varphi(g_1)\,(c(g_2))\,[c(g_1g_2)]^{-1}\,c(g_1)\ ,$$

let g_1, g_2 be, successively, x_1, x_2; $s(y), x$; $s(y_1), s(y_2)$, which will then be seen to yield the equations (4.14),(4.15),(4.16), respectively.

Theorem 4.7. *If* A *is abelian with* G *operating trivially, and* N *is central in* G, *then equations (4.14)-(4.16) become*

(4.14*) $\sigma_{11}(x_1,x_2) = c_1(x_1)c_1(x_2)\,[c_1(x_1x_2)]^{-1}$

(4.15*) $\sigma_{21}(y,x) = e$

(4.16*) $\sigma_{22}(y_1,y_2) = c_2(y_1)c_2(y_2)\,[c_2(y_1y_2)]^{-1}\,[c_1(\alpha(y_1,y_2))]^{-1}$

Proof.

Clear.

Putting this together with equation (4.8) in theorem 4.1, we get

Theorem 4.8. *$\sigma \in Z^2(G,A)$ is a coboundary iff*

(4.19) $$\sigma(x_1s(y_1),x_2s(y_2) =$$
$$\varphi_1(x_1)\left[\varphi_2(y_1)(c_1(x_2))\,[\varphi_1(\beta(y_1)(x_2))(c_2(y_1))]^{-1}\,[c_1(\beta(y_1)(x_2))]^{-1}\,c_2(y_2)\right]\cdot$$
$$\varphi_1(x_1)\,(c_1(\beta(y_1)(x_2)))\,[c_1(x_1\beta(y_1)(x_2))]^{-1}\,c_1(x_1)\cdot$$
$$\varphi_1(x_1\beta(y_1)(x_2))\left[c_2(y_2)\,[c_2(y_1y_2)]^{-1}\,[c_1(\alpha(y_1,y_2))]^{-1}\,c_2(y_1)\right]\cdot$$
$$\varphi_1(x_1\beta(y_1)(x_2))\,(c_1(\alpha(y_1,y_2)))\,[c_1(x_1\beta(y_1)(x_2)\alpha(y_1,y_2))]^{-1}\,c_1(x_1\beta(y_1)(x_2))$$

for some $c_1 \in C^1(N,A)$, $c_2 \in C^1(H,A)$

Proof.

Substitute expressions (4.14),(4.15),(4.16) in (4.8).

Theorem 4.9. *If* A *is abelian,* G *operates trivially on* A *and* N *is central in* G *then* (4.19) *becomes*
(4.19*)

$$\sigma(x_1 s(y_1), x_2 s(y_2)) = c_1(x_1)c_1(x_2)c_1(x_1x_2)^{-1}c_2(y_1)c_2(y_2)c_2(y_1y_2)^{-1}c_1(x_1x_2\alpha(y_1,y_2))^{-1}.$$

Proof.

Clear.

Remark 4.10.

Now we have the full picture in terms of functional equations. In general, these appear quite impossible to solve at present, but by restricting our attention first to simple cases, and then progressively building up solutions, we shall see that such an approach is eminently feasible, with very explicit results. See theorem 6.7 for example.

V. Trivializing Trilinear Functions.

Theorem 5.1. *If* $\sigma\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is a continuous bilinear function, or, which amounts to the same thing, an* $\mathbb{R}$*-bilinear function, then* σ *is symmetric iff* σ *is a coboundary, considering* $\mathbb{R}^n$ *operating trivially on the coefficient group* $\mathbb{R}$.

Proof.

If σ is symmetric and bilinear, it is certainly a 2-cocycle. If we define $c(\mathbf{v}) = -\dfrac{\sigma(\mathbf{v},\mathbf{v})}{2}$ for $\mathbf{v} \in \mathbb{R}^n$, then

$$\sigma(\mathbf{v},\mathbf{w}) = -\frac{\sigma(\mathbf{w},\mathbf{w})}{2} + \frac{\sigma(\mathbf{v}+\mathbf{w},\mathbf{v}+\mathbf{w})}{2} - \frac{\sigma(\mathbf{v},\mathbf{v})}{2},$$

showing that $\delta c = \sigma$. For the converse, it is clear that a coboundary must always be symmetric.

Remark 5.2.

The reader may recognize this relationship between σ and c as that holding between an inner product and its associated norm$^2/2$, or between the polarization of a quadratic form and itself/2 over a field of characteristic $\neq 2$.

From [Du1], p.253, we have

Theorem 5.3. *If* $\sigma \in (\mathbb{R}^n)^{\otimes 3}$*, where*

$$(\mathbb{R}^n)^{\otimes 3} = \{\sigma\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\ \sigma \text{ is } \mathbb{R}\text{-trilinear}\}$$

is a continuous trilinear function, then it is a coboundary iff it is annihilated by the alternating map

$$\mathrm{A}(\sigma)(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) = \sum_{s\in \mathrm{S}_3} (-1)^{sgn(s)} \frac{\sigma(\mathbf{v}_{s(1)},\mathbf{v}_{s(2)},\mathbf{v}_{s(3)})}{6},$$

S_3 *being the symmetric group on three letters.*

Theorem 5.4. *Every $\sigma \in (\mathbb{R}^n)^{\otimes 3}$ can be written as a sum*

$$\sigma(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \sigma_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \; + \; \sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \; + \; \sigma_a(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \quad ,$$

where σ_s is a symmetric function called the symmetrization or symmetric part of σ, σ_a is an antisymmetric function called the antisymmetrization or alternation or antisymmetric part of σ and $\sigma_{[12]}$ satisfies the identity

$$\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) + \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) + \sigma_{[12]}(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2) \; = \; 0 \quad ,$$

and is called the Jacobization or simply the Jacobi part of σ.

Proof.

Let

$$\begin{aligned}
\sigma_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \sum_{s \in S_3} \frac{\sigma(\mathbf{v}_{s(1)}, \mathbf{v}_{s(2)}, \mathbf{v}_{s(3)})}{6} \\
\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{2\sigma(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) - \sigma(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) - \sigma(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2)}{3} \\
\sigma_a(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \sum_{s \in S_3} (-1)^{sgn(s)} \frac{\sigma(\mathbf{v}_{s(1)}, \mathbf{v}_{s(2)}, \mathbf{v}_{s(3)})}{6} \; .
\end{aligned}$$

Remark 5.5.

This splitting of σ is a special case of a splitting of $(\mathbb{R}^n)^{\otimes m}$ into irreducible subspaces under the action of the symmetric group S_n acting by permuting the m coordinates. For this, see [Wey], theorem 4.4D.

Remark 5.6.

Now the cocycles σ_s and $\sigma_{[21]}$ are each annihilated by the alternating map, thus they are trivial, but theorem 5.4 does not provide us a way of constructing a trivialization, i.e., of finding a 2-cochain c so that $\sigma = \delta c$.

Theorem 5.7. *$\sigma_s, \sigma_{[12]}$ are the coboundaries, respectively, of*

(5.1)

$$\begin{aligned}
c_s(\mathbf{v}_1, \mathbf{v}_2) &= -\frac{\sigma_s(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2)}{2} \\
c_{[12]}(\mathbf{v}_1, \mathbf{v}_2) &= \frac{\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1) - \sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) - \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_1) - 2\sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_2)}{3}
\end{aligned}$$

Proof.

Remembering what we did in the case of 2-cocycles in theorem 5.1 above, we set pairs of variables equal in the various functions which arise by permuting the variables. Hence we look for a cochain of the form

$$\begin{aligned}
c_s(\mathbf{v}_1, \mathbf{v}_2) &= \lambda_1 \sigma_s(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) + \lambda_2 \sigma_s(\mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_1) + \lambda_3 \sigma_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1) \\
&\quad + \lambda_4 \sigma_s(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_2) + \lambda_5 \sigma_s(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_1) + \lambda_6 \sigma_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2) \\
&= (\lambda_1 + \lambda_3 + \lambda_5)\sigma_s(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) + (\lambda_2 + \lambda_4 + \lambda_6)\sigma_s(\mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_1) \\
&= \mu_1 c_1(\mathbf{v}_1, \mathbf{v}_2) + \mu_2 c_2(\mathbf{v}_1, \mathbf{v}_2) \; .
\end{aligned}$$

Noticing that $\delta c(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = -2\sigma_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, we may choose $\mu_1 = -\frac{1}{2}$, $\mu_2 = 0$.

For the second part, we use the Jacobi identity for $\sigma_{[12]}$, and look for a $c_{[12]}(\mathbf{v}_1, \mathbf{v}_2)$ of the form

$$(\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 + \lambda_3 c_4)(\mathbf{v}_1, \mathbf{v}_2)$$
$$= \lambda_1 \sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) + \lambda_2 \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_1) + \lambda_3 \sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1) + \lambda_4 \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_2)$$

Then

$$\begin{aligned}
\delta c_1(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= -\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) - \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3) \\
\delta c_2(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) + \sigma_{[12]}(\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1) \\
\delta c_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= -\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) - \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) \\
\delta c_4)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3) + \sigma_{[12]}(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2).
\end{aligned}$$

Now we again apply the Jacobi identity to eliminate $\sigma_{[12]}(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2)$ and $\sigma_{[12]}(\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1)$, obtaining

$$\delta c(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (\lambda_1 \delta c_1 + \lambda_2 \delta c_2 + \lambda_3 \delta c_3 + \lambda_4 \delta c_4(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

$$\begin{aligned}
&= \lambda_1 \left(\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) - \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3)\right) \\
&\quad + \lambda_2 \left(\sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) - \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3) - \sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2)\right) \\
&\quad + \lambda_3 \left(-\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) - \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1)\right) \\
&\quad + \lambda_4 \left(\sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3) - \sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) - \sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3)\right) \\
&= -(\lambda_1 + \lambda_4)\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) - (\lambda_2 + \lambda_3)\sigma_{[12]}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) \\
&\quad + (\lambda_2 - \lambda_3 - \lambda_4)\sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) \\
&\quad + (-\lambda_1 - \lambda_2 + \lambda_4)\sigma_{[12]}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3),
\end{aligned}$$

so if we set

$$\lambda_1 + \lambda_4 = -1, \quad \lambda_2 + \lambda_3 = 0, \quad \lambda_2 - \lambda_3 - \lambda_4 = 0, \quad \lambda_1 + \lambda_2 - \lambda_4 = 0 ,$$

this has the unique solution

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1/3, -1/3, 1/3, -2/3) ,$$

which proves the theorem.

If we notice that all this works for trilinear functions with values in $\mathbb{R}^n$ as well as $\mathbb{R}$, then we have a

Corollary 5.8. *If* L *is a Lie algebra of dimension n over any field of characteristic* $\neq 3$, *and if*

$$\sigma(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = [\mathbf{v}_1, [\mathbf{v}_2, \mathbf{v}_3]] ,$$

then σ *is a trivial* 3*-cocycle on* $\mathbb{R}^n$ *with values in* $\mathbb{R}^n$.

Proof.

Using the antisymmetry of [,], the Jacobi identity, and arguments similar to the proof of theorem 5.7, we see that we may take

$$c(\mathbf{v}_1, \mathbf{v}_2) = \frac{2[\mathbf{v}_1, [\mathbf{v}_2, \mathbf{v}_1]] - [\mathbf{v}_2, [\mathbf{v}_1, \mathbf{v}_2]]}{3}$$

and get

$$\delta c(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = [\mathbf{v}_1, [\mathbf{v}_2, \mathbf{v}_3]] .$$

Remark 5.9.

This shows that the Lie triple bracket evaluated at three distinct elements in L is rationally linearly determined by the triple bracket evaluated at a three-tuple of vectors of which two are equal. This is well-known in the theory of free Lie algebras ([Ba], p.49).

Corollary 5.10. *If* M *is an* n*-dimensional Riemannian manifold,* g *a Riemannian structure, and* R *the corresponding curvature tensor, then*

$$\sigma(X,Y,Z,T) = g(R(X,Y)Z,T)$$

is a quadrilinear function which is trivial as a 4-cocycle on $\mathbb{R}^n$*.*

Proof.

For definitions of these notions, see [He], p.68ff. The Bianchi identities show that σ is annihilated by the alternating map, and by techniques similar to those used in proving theorem 5.4, we get that $\delta c = \sigma$ if

$$c(X,Y,Z) = \frac{2\sigma(X,Y,Y,Z) - \sigma(Z,Y,Y,X)}{3} .$$

Remark 5.11.

The σ in corollary 5.10 is a member of the similarity class $[2^2]$ in $(\mathbb{R}^n)^{\otimes 4}$. Incidentally, this result is somewhat interesting, inasmuch as it is already known that σ is determined by its restriction to sets of four vectors which are equal in pairs, and this corollary gives a determination by restriction to sets of four vectors of which only one pair is equal. Thus this gives a kind of intermediate reduction.

Remark 5.12.

As we mentioned, we have been using the irreducible representations of the symmetric group acting on the tensor space $(\mathbb{R}^n)^{\otimes m}$, $m = 3,4$, in order to trivialize cocycles within each irreducible symmetry class of tensors, i.e., multilinear functions. It is possible in the case of three and four variables to handle the problem without splitting it up into symmetry classes. For three variables, this is a matter of solving a linear system of six equations in six unknowns; for four variables, a system of 24 equations in 36 unknowns. The general problem for n variables lead to $n!$ equations in $\binom{n-1}{2} n!$ unknowns. Although the matrices are quite sparse and somewhat regular, I have not suceeded in solving them in general. I certainly feel that it must always be possible to trivialize a multilinear cocycle which is annihilated by the alternating map by setting pairs of its variables equal in various permutations of its variables.

VI. $H^2(H_1,\mathbb{R})$ via Extension Equations.

We are ready to apply the results of sections IV and V to the calculation of 2-cocycles on H_1. The exact sequence we are working with is

$$1 \to \mathbb{R} \xrightarrow{i} H_1 \underset{s}{\overset{j}{\rightleftarrows}} \mathbb{R}^2 \to 1,$$

where we take H_1 in its realization as 3×3 upper triangular matrices $\begin{bmatrix} 1 & y_1 & t \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix}$, or, which is equivalent, the triples $(t, y_1, y_2) \in \mathbb{R}^3$ with multiplication

$$(t, y_1, y_2)(t', y_1', y_2') = (t + t' + y_1 y_2',\ y_1 + y_1',\ y_2 + y_2') \ .$$

and

$$i(y) = (t, 0, 0), \quad j(t, y_1, y_2) = (y_1, y_2), \quad s(y_1, y_2) = (0, y_1, y_2) \ .$$

Lemma 6.1. *Let $\sigma \in Z^2(H_1, \mathbb{R})$. Then σ is cohomologous to σ' with $\sigma'_{11} = \sigma'_{12} = 0$.*

Proof.

First, since $H^2(\mathbb{R},\mathbb{R}) = 0$, σ_{11} is cohomologous to zero. Let $\sigma_{11} = \delta c$, $c \in C^1(\mathbb{R},\mathbb{R})$. If we define

$$c'(t, y_1, y_2) = c(y) \text{ then } (\sigma')_{11} = (\sigma - \delta c')_{11} = 0 \ ,$$

and if we take

$$c''(t, y_1, y_2) = \sigma'((t, 0, 0), (0, y_1, y_2)) \text{ then } (\sigma'' + \delta c'')_{11} = (\sigma'' + \sigma c'')_{12} = 0 \ .$$

Remark 6.2.

If we assume that $\sigma \in Z^2(H_1, \mathbb{R})$ has the property of σ' in the previous lemma, then equations (4.10), (4.11) say that σ_{21} is *bilinear* and (4.12) says that

$$\delta(\sigma_{22})((y_1, y_2), (y_1', y_2'), (y_1'', y_2'')) = \sigma_{21}((y_1, y_2), \alpha((y_1', y_2'), (y_1'', y_2''))) \ .$$

This last equation just says that its right side is a trivial 3-cocycle and σ_{22} trivializes it.

Lemma 6.3. *We may find constants λ_1, λ_2 such that*

$$\sigma_{21}((y_1, y_2), t) \ = \ (\lambda_1 y_1 + \lambda_2 y_2) t \ .$$

Proof.

σ_{21} is bilinear.

Lemma 6.4.

$$\sigma_{21}((y_1, y_2), \alpha((y_1', y_2'), (y_1'', y_2''))) = \lambda_1 y_1 y_1' y_2'' + \lambda_2 y_2 y_1' y_2'' \ .$$

Proof.

$\alpha((y_1', y_2'), (y_1'', y_2'')) = y_1' y_2''$.

Lemma 6.5. *If*

$$c((t,y_1,y_2),(t',y_1',y_2')) = -\frac{y_1^2 y_2'}{2}$$
$$c_s((t,y_1,y_2),(t',y_1',y_2')) = \frac{y_1(y_2')^2 + 2y_2y_1'y_2'}{6}$$
$$c_{[12]}((t,y_1,y_2),(t',y_1',y_2')) = \frac{y_2^2y_1' - y_1(y_2')^2 - 2y_1y_2y_2'}{3}$$

then

$$\delta(c) = y_1y_1'y_2'' \quad \text{and} \quad \delta(c_s + c_{[12]}) = y_2y_1'y_2'' \ .$$

Proof.

$\delta(c) = y_1y_1'y_2'$ follows by calculation. For the second part, use theorem 5.3 to write $y_2y_1'y_2''$ as the sum of a symmetric and a Jacobi trilinear function

$$y_2y_1'y_2'' = \frac{y_1y_1'y_2'' + y_1y_2'y_2'' + y_2y_2'y_1''}{3} + \frac{2y_2y_1'y_2'' - y_2y_2'y_1'' - y_1y_2'y_2''}{3}$$

and trivialize the symmetric part by using theorem 5.7 to take

$$c_s((t,y_1,y_2),(t',y_1',y_2')) = \frac{y_1(y_2')^2 + 2y_2y_1'y_2'}{6} \ ,$$

the Jacobi part by taking

$$c_{[12]}((t,y_1,y_2),(t',y_1',y_2')) = \frac{y_2^2y_1' - y_1(y_2')^2 - 2y_1y_2y_2'}{3} \ ,$$

and adding the two to get

$$(c_s + c_{[12]})((t,y_1,y_2),(t',y_1',y_2')) = \frac{2y_2^2 - y_1(y_2')^2 - 2y_1y_2y_2'}{6} \ .$$

This proves the lemma. But we have also proved

Lemma 6.6.

$$\sigma_{22}((y_1,y_2),(y_1',y_2')) = \lambda_1y_1^2y_2 + \lambda_2(2y_2^2y_1' - y_1(y_2')^2 + 4y_2y_1'y_2' - 2y_1y_2y_2') + \lambda_3y_1y_2' \ .$$

Theorem 6.7. *Every 2-cocycle on* H_1 *can be written in the form*

$$\lambda_1(y_1^2y_2' + 2y_1t') + \lambda_2(y_2^2y_1' - y_1(y_2')^2 + 4y_2y_1'y_2' - 2y_1y_2y_2' - 6y_2t') \ .$$

Proof.

From (4.13) we have

$$\sigma((t,y_1,y_2),(t',y_1',y_2')) = \sigma_{21}(t',(y_1,y_2)) + \sigma_{22}((y_1,y_2),(y_1',y_2')) \ ,$$

and combining the results of lemmas 6.4 and 6.6, we get that the right side of this equation can be written in the form

$$\lambda_1 y_1 t' + \lambda_2 y_2 t' + \lambda_3 y_1^2 y_2' + \lambda_4 (2y_2^2 y_1' - y_1 (y_2')^2 + 4y_2 y_1' y_2' - 2y_1 y_2 y_2') + \lambda_5 y_1 y_2' \ .$$

A little linear algebra shows that in order for this to be a cocycle, we must then have $\lambda_1 = 2\lambda_3$, $\lambda_2 = -6\lambda_4$ and λ_5 arbitrary, yielding

$$\lambda_1 (y_1^2 y_2' + 2y_1 t') + \lambda_2 (y_2^2 y_1' - y_1 (y_2')^2 + 4y_2 y_1' y_2' - 2y_1 y_2 y_2' - 6y_2 t') + \lambda_3 (y_1 y_2') \ .$$

But $y_1 y_2'$ is a trivial cocycle on H_1, since it is the pullback to H_1 of the cocycle defining the extension, and this is always trivial, as follows from the exactness of (2.3**) at $\mathrm{H}^2(\mathbb{R}^{2n},\mathbb{R})$. This is because the image of d_2^{01} is the line determined by α. We also notice that λ_2 is the coefficient of a cocycle which is just $2\alpha_3 - \alpha_2'$, from (3.1),(3.2), and thus it is cohomologous to $-\alpha_1$.

When is a cocycle of the form $\lambda_1\alpha_1 + \lambda_2\alpha_2$ cohomologous to zero?

Theorem 6.8. *If* $\mathrm{G}=\mathrm{H}_1$, $\mathrm{N}=\mathbb{R}$, $\sigma \in \mathrm{Z}^2(\mathrm{H}_1,\mathbb{R})$ *with* $\sigma_{11} = \sigma_{12} = 0$, *then equations (4.14*)-(4.16*) become*

(4.14**) $$\sigma_{11}(x_1, x_2) = 0$$

(4.15**) $$\sigma_{21}(y, x) = 0$$

(4.16**) $$\sigma_{22}(y_1, y_2) = c_2(y_1) + c_2(y_2) - c_2(y_1, y_2) - \alpha(y_1, y_2)$$

Proof.

Since, as we have remarked immediately above, $\alpha(y_1, y_2)$ is always a trivial 2-cocycle when pulled back to H_1, (4.16**) says that σ_{22} must be a trivial 2-cocycle on $\mathbb{R}^2$, i.e., it must be symmetric. But neither α_1 nor α_2 is symmetric, and the only linear combination $\lambda_1\alpha_1 + \lambda_2\alpha_2$ which is, is the zero cocycle, which proves that α_1, α_2 form a basis of $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$

Remark 6.9.

In order to exhibit the versatility of the method we have just used by looking at H_1 as a one-dimensional central extension of $\mathbb{R}^2$, we now compute the same cohomology group $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$, this time conceiving H_1 as a semidirect product, and using as functional equations, those of Mackey in [Ma], theorem 9.4, which may be derived here as a special case of equations (4.4)-(4.7).

Theorem 6.10. *If* G *is a semidirect product of* N *and* H, *and* G *operates trivially on the abelian group* A, *we have*

(6.1) $$\delta(\sigma_{11})(x_1, x_2, x_3) = 0$$

(6.2) $$\sigma_{21}(y, x_1x_2) - \sigma_{21}(y, x_1) - \sigma_{21}(y, x_2) = \sigma_{11}(\beta(y)(x_1), \beta(y)(x_2)) - \sigma_{11}(x_1, x_2)$$

(6.3) $$\sigma_{21}(y_1y_2, x) = \sigma_{21}(y_1, \beta(y_2)(x)) + \sigma_{21}(y_2, x)$$

(6.4) $$\delta(\sigma_{22})(y_1, y_2, y_3) = 0$$

Proof.

By [Ma1], theorem 9.4, [Ta], prop.1, or by assuming that $\alpha = 0$, and that φ,γ are maps into the identity map, and taking account of this restriction in equations (4.4)-(4.7).

Remark 6.11.

These are the equations obtained by Mackey and Tahara. Mackey solved them in the case of a semidirect product of $\mathbb{R}$ and $\mathbb{R}$ and coefficient group $\mathbb{R}$, Tahara for two finite cyclic groups and arbitrary coefficient group A.

Lemma 6.12. H_1 *is the semidirect product of its subgroups*

$$\mathrm{N} = \{(t,0,y_2)|\ t,y_2 \in \mathbb{R}\} \cong \mathbb{R}^2 \quad \textit{and} \quad \mathrm{H} = \{(0,y_1,0)|\ y_1 \in \mathbb{R}\} \cong \mathbb{R}\ ,$$

and H_1 *fits into the exact sequence*

$$0 \to \mathbb{R}^2 \xrightarrow{i} \mathrm{H}_1 \xrightarrow{j} \mathbb{R} \to 0\ ,$$

where $i(t,y_2) = (t,0,y_2)$, $j(t,y_1,y_2) = y_1$ *and a section for* j *is* $s(y_1) = (0,y_1,0)$.

Proof.

Take H_1 in its 3×3 matrix form.

Theorem 6.13. α_1, α_2 *of theorem 3.8 form a basis of* $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$.

Proof.

This is another proof of theorem 3.8. Since $\mathrm{H}^2(\mathbb{R},\mathbb{R}) = 0$, we may take $\sigma_{22} = 0$ by an argument similar to that in the proof of lemma 6.1, which argument also shows that we still have $\sigma_{12} = 0$. Choose an alternating 2-form σ_{11} on $\mathbb{R}^2$, and we may as well assume that it is ty_2', which is cohomologous to α.

In order to determine β, i.e., how $\mathbb{R}$ acts on $\mathbb{R}^2$, we compute

$$(0,y_1,0)(t,0,y_2)(0,-y_1,0) = (t + y_1y_2,0,y_2)\ ,$$

so $\beta(y_1)(t,0,y_2) = (t + y_1y_2,0,y_2)$. The matrix for this action is $\begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix}$ for the basis $(1,0,0),(0,0,1)$.

Next, equation (6.2) says that we should "cross-homomorphically" trivialize

$$\sigma_{11}((t + y_1y_2,y_2),(t' + y_1y_2',y_2')) - \sigma_{11}(t,y_2),(t',y_2')) = (t + y_1y_2)y_2' - ty_2' = y_1y_2y_2'\ ,$$

i.e., we want to determine σ_{21} so that

$$\sigma_{21}(y_1,t + t',y_2 + y_2') - \sigma_{21}(y_1,y,y_2) - \sigma_{21}(y_1,t',y_2') = y_1y_2y_2'\ ,$$

but in such a way that σ_{21} is, for each (t,y_2), a crossed homomorphism in the variable y_1. This is the content of (6.3), which says that

$$(6.5) \quad \sigma_{21}(y_1 + y_1',t,y_2)$$
$$= \sigma_{21}(y_1,\beta(y_1')(t,y_2)) + \sigma_{21}(y_1',t,y_2)$$
$$= \sigma_{21}(y_1,t + y_1'y_2,y_2) + \sigma_{21}(y_1',t,y_2)\ ,$$

where y_1 acts on functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ as

$$(y_1 f)(t, y_2) = f(\beta(y_1)(t, y_2)) = f(t + y_1 y_2, y_2)\ .$$

Now for each fixed y_1 we have

$$\sigma_{21}(y_1, t, y_2) = \lambda_1(y_1) y_2^2 + \lambda_2(y_1) + \lambda_3(y_1) y_2\ ,$$

but (6.5) yields

$$\begin{aligned} \lambda_1(y_1 + y_1') y_2^2 \ &+\ \lambda_2(y_1 + y_1')t \ +\ \lambda_3(y_1 + y_1') y_2 \\ &= \lambda_1(y_1) y_2^2 + \lambda_2(y_1)(t + y_1' y_2) + \lambda_3(y_1) y_2 + \lambda_1(y_1') y_2^2 + \lambda_2(y_1')t + \lambda_3(y_1') y_2\ , \end{aligned}$$

giving

$$\begin{array}{ll} \lambda_1(y_1 + y_1') = \lambda_1(y_1) + \lambda_1(y_1') \text{ or } \lambda_1(y_1) = \lambda_1 y_1 & \\ \lambda_2(y_1 + y_1') = \lambda_2(y_1) + \lambda_2(y_1') \text{ or } \lambda_2(y_1) = \lambda_2 y_1 & \text{for constants } \lambda_1, \lambda_2\ , \end{array}$$

and finally,

$$\lambda_3(y_1 + y_1') = y_1' \lambda_2(y_1) + \lambda_3(y_1) + \lambda_3(y_1') = \lambda_2 y_1' y_1 + \lambda_3(y_1) + \lambda_3(y_1')\ ,$$

which gives $\lambda_3(y_1) = \dfrac{\lambda_2 y_1^2}{2} + \lambda_3 y_1$, for some constant λ_3. Then we may write

$$\sigma_{21}(y_1, t, y_2) \ =\ \lambda_1 y_1 y_2^2 \ +\ \lambda_2 y_1 t \ +\ \lambda_2 \frac{y_1^2 y_2}{2} \ +\lambda_3 y_1 y_2\ .$$

Now every 2-cocycle on H_1 can be written as

$$\sigma(y_1, t, y_2), (y_1', t', y_2')) = \sigma_{21}(y_1, (t', y_2')) + \sigma_{11}((t, y_2), (t' + y_1 y_2', y_2'))$$

the specialization of (3.8) to semidirect products; but this is just

$$\lambda_1 y_1 (y_2')^2 \ +\ \lambda_2 y_1 t' \ +\ \lambda_2 \frac{y_1^2 y_2'}{2} \ +\ \lambda_3 y_1 y_2' \ +\ \lambda_4 t y_2'\ .$$

If we want this to be a 2-cocycle, linear algebra shows us that we must have $\lambda_4 = 2\lambda_1$ and λ_3 arbitrary. But, we saw earlier, $y_1 y_2'$ is a trivial 2-cocycle, and so we are left with

$$\lambda_1 \alpha_1 + \lambda_2 \alpha_2.$$

Theorem 6.14. *The groups furnished by the cocycles* $\frac{\alpha_1}{2}$ *and* $\frac{\alpha_2}{2}$ *of theorem 3.8 have faithful representations in* $\mathrm{Sl}(4,\mathbb{R})$.

Proof.

For $\frac{\alpha_1}{2}$, $\frac{\alpha_2}{2}$, respectively, these are

$$(s, t, y_1, y_2) \to \begin{bmatrix} 1 & y_1 & \frac{y_1^2}{2} & s \\ 0 & 1 & y_1 & t \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & y_1 & t & s \\ 0 & 1 & y_2 & \frac{y_2^2}{2} \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Theorem 6.15. *The two groups in theorem 6.14 are isomorphic.*

Proof.

The first of the two groups above is also a semidirect product of the normal subgroup of elements of the form $(s,t,0,y_2)$ and the nonnormal subgroup of elements of the form $(0,0,y_1,0)$. The second is a semidirect product of the normal subgroup of elements of the form $(s,t,y_1,0)$ and the nonnormal subgroup of elements of the form $(0,0,0,y_2)$. If we are going to show these extensions isomorphic, it is easier to first ask whether they are isomorphic as semidirect product extensions, i.e., we want to find an equivariant map $\mathbb{R}^3 \to \mathbb{R}^3$ for the two actions. These actions are, respectively,

$$T_{y_1}(s,t,y_2) = (s - y_1 t + (y_1^2/2)y_2, t - y_1 y_2, y_2) \text{ with matrix } \begin{bmatrix} 1 & -y_1 & \frac{y_1^2}{2} \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{and } T_{y_2}(s,t,y_1) = (s + y_2 t + (y_2^2/2)y_1, t + y_2 y_1, y_1) \text{ with matrix } \begin{bmatrix} 1 & y_2 & \frac{y_2^2}{2} \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix},$$

and thus these actions differ by the automorphism $y_1 \to -y_1$. Since the map $(s,t,y_1,y_2) \to (s,t,y_2,y_1)$ is an anti-automorphism between the two extension groups, we can, by composing with the inverse map $(s,t,y_1,y_2) \to (s,t,y_1,y_2)^{-1}$, obtain our desired isomorphism.

Remark 6.16.

Now consider the following automorphism of H_1 :

$$\theta(t,y_1,y_2) = (t,y_2,y_1)^{-1} = (-t + y_1 y_2, -y_2, -y_1) .$$

If we define

$$\begin{aligned} \alpha_1^\theta((t,y_1,y_2),(t',y_1',y_2')) &= \alpha_1(\theta^{-1}(t,y_1,y_2),\theta^{-1}(t',y_1',y_2')) \\ &= \alpha_1((-t + y_1 y_2, -y_2, -y_1),(-t' + y_1' y_2', -y_2', -y_1')) \\ &= -\frac{y_2^2 y_1'}{2} + y_2 t' - y_2 y_1' y_2' , \end{aligned}$$

then this is cohomologous to $\frac{\alpha_2}{2}$, which of course gives an extension isomorphic to that given by α_2. We shall see in theorem 8.3 that every two nontrivial 2-cocycles in $H^2(H_1,\mathbb{R})$ yield isomorphic extension groups. Anticipating the discussion in the next section, this fact can be stated as "every nontrivial one-dimensional central extension of H_1 is *rigid*", i.e., extensions near it are isomorphic to it. Now let us examine this idea in more detail.

VII. Restricted Deformations

In its usual setting, as in [Ge], the deformation theory of algebras is concerned with the set of algebra structures on a fixed vector space. Nijenhuis in [Ni] has considered deformations of ideals, which is a more restricted type of variation. In this paper we fix a normal subgroup not only setwise, but fix the subgroup and quotient group structure as well. Another way of looking at the matter is this. Classical deformation theory takes a local point of view and here we take a global one. Also, classical deformation theory would consider all the possible group laws on a fixed locally compact space and study when two such were isomorphic, whereas here we demand that the locally compact space of the extension group be the product as Borel spaces of two locally compact spaces, one representing the space of the normal subgroup and the other representing that of the quotient group. Further, we also assume group structures on these two locally compact spaces as fixed. This is why it is more fitting to call what we do in this paper a studying of isomorphism classes of group extensions, or, if we wish to focus on how the isomorphism class changes as a function of the natural topology in H_2, whenever there is one, of *restricted* deformations.

The question is this: when do two elements of $H^2(G,A)$ represent isomorphic extension groups? It is to be emphasized that we speak here of the isomorphism of two groups, *not just* as extensions, but purely as topological groups. A large component of this looser type of equivalence is the outer automorphism group of G and its action on $H^2(G,A)$, an example of which action was given in remark 6.16 in the previous section. This is explained fully in theorem 9.10.

Lemma 7.1. *The extensions of* $\mathbb{R}^2$ *given by the two cohomologous 2-cocycles* y_1y_2' *and* $\dfrac{y_1y_2' - y_1'y_2}{2}$ *are isomorphic via*

$$T(t, y_1, y_2) = (t + \frac{y_1y_2}{2}, y_1, y_2) .$$

Proof.

Calculational.

Lemma 7.2. *The eight 2-cocycles on* H_1, α_i, $1 \le i \le 6, \alpha_1', \alpha_2'$, *when composed with* $T\times T$ *of the previous lemma, become, respectively*

$$\begin{aligned} \tilde{\alpha}_1 &= y_1^2y_2' + 2y_1t' + y_1y_1'y_2' \\ \tilde{\alpha}_2 &= y_1(y_2')^2 + 2ty_2' + y_1y_2y_2' \\ \tilde{\alpha}_3 &= y_2^2y_1' + y_2y_1'y_2' - 2y_2t' \\ \tilde{\alpha}_4 &= y_2(y_1')^2 + y_1y_2y_1' - 2ty_1' \\ \tilde{\alpha}_5 &= (y_1 + y_1')(y_2y_2'/2) + y_2t' - ty_2' \\ \tilde{\alpha}_6 &= (y_1y_1'/2)(y_2 + y_2') + ty_1' - y_1t' \\ \tilde{\alpha}_1' &= y_1^2y_2' + 2y_1y_1'y_2' + 2ty_1' + y_1y_2y_1' \\ \tilde{\alpha}_2' &= y_1(y_2')^2 + y_1y_2y_2' + y_2t' + y_2y_1'y_2' . \end{aligned} \tag{7.3}$$

Proof.

Calculational.

Remark 7.3.

We could also explicitly write down the outer automorphism sending $\tilde{\alpha}_1$ into $\tilde{\alpha}_2$. We want now to look more carefully at what it means for two elements of $H^2(H_1,\mathbb{R})$ to yield isomorphic extension groups. The first thing we ask ourselves is : how large can the center of the extension group be?

Theorem 7.4. *The center of the extension group of* H_1 *given by the cocycle* $\lambda_1\alpha_1 + \lambda_2\alpha_2$ *is one-dimensional.*

Proof.

Let the extension group be given by the four-tuples (s, t, y_1, y_2) with the multiplication

$$(s,t,y_1,y_2)(s',t',y_1',y_2')$$
$$(s + s' + \lambda_1(y_1^2 y_2' + 2y_1 t') + \lambda_2(y_1(y_2')^2 + 2ty_2'), t + t' + y_1 y_2', y_1 + y_1', y_2 + y_2') ,$$

and suppose $(s_0, t_0, (y_1)_0, (y_2)_0)$ is in the center of the group. Then we see that we must have $y_1(y_2)_0 = (y_1)_0 y_2$ for all $y_1, y_2 \in \mathbb{R}$ and

$$\begin{aligned}(7.4)\quad \alpha_1(y_1^2(y_2)_0 + 2y_1 t_0) + \lambda_2(y_1(y_2)_0^2 + 2t(y_2)_0 \\ = \lambda_1(y_1)_0^2 y_2 + 2(y_1)_0 t) + \lambda_2((y_1)_0 y_2^2 + 2t_0 y_2) ,\end{aligned}$$

which implies that $(y_1)_0 = (y_2)_0 = 0$, and (7.4) becomes

$$\lambda_1(2y_1 t_0) = \lambda_2(2t_0 y_2) , \tag{7.5}$$

showing that $t_0 = 0$ if $(\lambda_1, \lambda_2) \neq (0,0)$ and proving the theorem.

Before we look at isomorphisms of extensions of H_1, we must look at automorphisms of H_1.

Theorem 7.5. *We may represent an automorphism* $a\colon \mathrm{H}_1 \to \mathrm{H}_1$ *by a matrix* $\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$, *where* a_{11}, a_{22} *are, respectively, an automorphism of* $\mathbb{R}$, *the center of* H_1, *and an automorphism of* $\mathbb{R}^2$, *the quotient* $\mathrm{H}_1/\mathbb{R}$, a_{12} *a homomorphism* $\mathbb{R}^2 \to \mathbb{R}$, *and composition of automorphisms corresponds to multiplication of matrices*

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \end{bmatrix} = \begin{bmatrix} a_{11}a_{11}' & a_{11}a_{12}' + a_{12}a_{22}' \\ 0 & a_{22}a_{22}' \end{bmatrix} .$$

Proof.

Since the center of H_1 must go onto itself under any automorphism $a\colon \mathrm{H}_1 \to \mathrm{H}_1$, a must be of the form

$$(t, y_1, y_2) \xrightarrow{a} (a_{11}(t) + a_{12}(y_1,y_2), a_{22}(y_1,y_2)) . \tag{7.6}$$

By letting $y_1 = y_2 = 0$, we see that a_{11} is an automorphism of $\mathbb{R}$, and hence of the form $a_{11}(t) = \lambda t$, for some constant $\lambda \neq 0$. If we let $t = 0$, the automorphism property of a gives us

$$a_{22}(y_1 + y_1', y_2 + y_2') = a_{22}(y_1, y_2) + a_{22}(y_1', y_2') ,$$

showing that a_{22} is an automorphism of $\mathbb{R}^2$, and

$$\begin{aligned}(7.7)\quad a_{12}(y_1', y_2') - a_{12}(y_1 + y_1', y_2 + y_2') + a_{12}(y_1, y_2) \\ = \lambda\alpha((y_1, y_2), (y_1', y_2')) - \alpha(a_{22}(y_1, y_2), a_{22}(y_1', y_2')) ,\end{aligned}$$

which says that the two 2-cocycles $\lambda\alpha$ and $\alpha \circ (a_{22} \times a_{22})$ are cohomologous. But they are each antisymmetric and thus identical, so a_{22} has the property that it takes α into a multiple of itself. If $l > 0$ then a_{22} is some scalar multiple $\sqrt{l}a'_{22}$ of a symplectic automorphism a'_{22} of $\mathbb{R}^2$ with respect to σ. If $l < 0$, then by following a_{22} by any automorphism θ of $\mathbb{R}^2$ which replaces α by $-\alpha$, say the one with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\theta \circ a_{22}$ replaces σ by a positive multiple of itself, and we may write $\theta \circ a_{22} = \sqrt{l}a'_{22}$ as before. Since $\theta^2 = 1_{\mathbb{R}^2}$, we have that the group of such a_{22} is isomorphic to $Z_2 \times \mathbb{R}^* \times Sp(2)$, or $\mathbb{R} \times Sp(2)$.

As for any group, the inner automorphisms of H_1 correspond to $\mathrm{H}_1/\mathbb{R} \cong \mathbb{R}^2$. By representing automorphisms as two by two matrices, it becomes a nice way to see the known fact that

Corollary 7.6. *The inner automorphisms are a normal subgroup of the group of all automorphisms, which is a semidirect product of this normal subgroup and the group of outer automorphisms.*

Corollary 7.7. *The outer automorphism group of* H_1 *is isomorphic to* $\mathbb{R} \oplus \mathrm{Sl}(2,\mathbb{R})$.

Proof.

$\mathrm{Sl}(2,\mathbb{R})$ is known to be the symplectic group for the determinant form $y_1y'_2 - y'_1y_2$.

Theorem 7.8. *Any isomorphism* $i\colon \mathrm{E} \to \mathrm{E}'$ *between two one-dimensional central extensions of* H_1 *by* $\mathbb{R}$ *can be represented by a* 2×2 *matrix* $\begin{bmatrix} i_{11} & i_{12} \\ 0 & i_{22} \end{bmatrix}$, *where* i_{11} *is an isomorphism between centers,* i_{22} *an automorphism of* H_1, *and* i_{12} *is a map* $\mathrm{H}_1 \to \mathbb{R}_1$ *whose coboundary is*

$$\lambda\sigma_1((t, y_1, y_2), (t', y'_1, y'_2)) - \sigma_2(i_{22}(t, y_1, y_2), i_{22}(t', y'_1, y'_2)) \ ,$$

where σ_1, σ_2 *are two 2-cocycles in* $\mathrm{Z}^2(\mathrm{H}_1,\mathbb{R})$ *and* $i_{11}(s) = \lambda s$.

Proof.

We have shown that the centers of E,E′ are $\cong \mathbb{R}$, and from there on the proof is similar to that of theorem 7.5.

VIII. The Action of Aut(H_1) on $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$.

We now find it more convenient to conceive of H_1 as being defined by the form

$$\alpha((y_1, y_2), (y'_1, y'_2)) = \frac{y_1y'_2 - y'_1y_2}{2} \ .$$

This makes it easier to identify the group of automorphisms of $\mathbb{R}^2$ which fix this form as $\mathrm{Sl}(2,\mathbb{R})$ instead of some conjugate group. The group of outer automorphisms is thus $\mathbb{R}^* \oplus \mathrm{Sl}(2,\mathbb{R})$.

Theorem 8.1. *If we take* $\tilde{\alpha}_2, \tilde{\alpha}_1$ *as a basis of* $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$, *then the matrix of the action of* $a_{11} \in \mathrm{Sl}(2,\mathbb{R})$ *as part of the outer automorphism group of* H_1 *on* $\mathrm{H}^1(\mathrm{H}_1,\mathbb{R})$, *expressed in this basis is identical to the matrix of* a_{11} *expressed in the usual basis* $(1,0), (0,1)$.

Proof.

Sl(2,$\mathbb{R}$) is generated by matrices of the form

$$\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}, \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} .$$

The symplectic mapping corresponding to the first matrix is $(y_1, y_2) \to (\lambda y_1, (\frac{1}{\lambda})y_2)$, yielding the automorphism

$$\theta_\lambda(t, y_1, y_2) = (t, \lambda y_1, (\tfrac{1}{\lambda})y_2)$$

of H_1. But remember, in order to have a representation and not an antirepresentation of Sl(2,$\mathbb{R}$) on $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$, we define

$$\alpha^\theta(g_1, g_2) = \alpha(\theta^{-1}g_1, \theta^{-1}g_2) ,$$

for $g_1, g_2 \in \mathrm{H}_1$ and $\alpha \in \mathrm{Z}^2(\mathrm{H}_1,\mathbb{R})$. Choosing the cohomology classes of $\tilde{\alpha}_2, \tilde{\alpha}_1$ as a basis of $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$, we calculate the effect of θ_λ above on $\tilde{\alpha}_2, \tilde{\alpha}_1$.

$$(\tilde{\alpha}_2)^{\theta_\lambda}((t,y_1,y_2),(t',y_1',y_2')) = \tilde{\alpha}_2((t,(\tfrac{1}{\lambda})y_1,\lambda y_2),(t',(\tfrac{1}{\lambda})y_1',\lambda y_2'))$$
$$= \lambda(y_1(y_2')^2 + 2ty_2' + y_1y_2y_2') = \lambda\tilde{\alpha}_2((t,y_1,y_2),(t',y_1',y_2'))$$

$$(\tilde{\alpha}_1)^{\theta_\lambda}(t,y_1,y_2),(t',y_1',y_2')) = (\tfrac{1}{\lambda})\tilde{\alpha}_1((t,y_1,y_2),(t',y_1',y_2')) .$$

For $\theta_\beta(t,y_1,y_2) = (t, y_1, \beta y_1 + y_2)$, we get

$$(\tilde{\alpha}_2)^{\theta_\beta}((t,y_1,y_2),(t',y_1',y_2')) = \tilde{\alpha}_2((t,y_1,-\beta y_1+y_2),(t',y_1',-\beta y_1'+y_2'))$$
$$= y_1(-\beta y_1' + y_2')^2 + 2t(-\beta y_1' + y_2') + y_1(-\beta y_1 + y_2)(-\beta y_1' + y_2')$$
$$= y_1(y_2')^2 + 2ty_2' + y_1y_2y_2' + \beta(-2y_1y_1'y_2' - 2ty_1' - y_1^2y_2' - y_1y_2y_1') + \beta^2(y_1(y_1')^2 + y_1^2y_1') .$$

Since $y_1(y_1')^2 + y_1^2y_1'$ is trivial, this is cohomologous to $\tilde{\alpha}_2 + \beta(-\tilde{\alpha}_1')$, and remembering that $-\tilde{\alpha}_1'$ is cohomologous to $\tilde{\alpha}_1$, we get $\begin{bmatrix} 1 \\ \beta \end{bmatrix}$ as the first column of our matrix. Similarly, $(\tilde{\alpha}_1)^{\theta_\beta} = \tilde{\alpha}_1$, and so too for the remaining generators of Sl(2,$\mathbb{R}$).

Corollary 8.2. *There are just two orbits of the cohomology classes in* $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$ *under the action of* Sl(2,$\mathbb{R}$) *- the origin, a one-point orbit, and the complement of the origin, the other orbit, which is open.*

Proof.

The actions of Sl(2,$\mathbb{R}$) on $\mathrm{H}^2(\mathrm{H}_1,\mathbb{R})$ and on $\mathbb{R}^2$ are identical

In terms of our definition of restricted deformations and rigidity, we may state this result as follows:

Theorem 8.3. *Any restricted deformation of a nontrivial one-dimensional central extension of* H_1 *by* $\mathbb{R}$ *within the space of such extensions is of the same isomorphism class.*

Remark 8.4.

In this particular case, we did not have to look at all outer automorphisms of H_1, just the subgroup Sl(2,$\mathbb{R}$) of codimension one. Furthermore, the group of outer automorphisms was only a part of what has to be looked at in general to obtain the complete set of isomorphisms between two extensions. This is indicated by theorem 7.8. When we look at groups other than H_1, e.g., H_n, we will not be able to get open orbits even by taking these other isomorphisms into account, and this is indicated in theorem 9.10.

IX. Isomorphism Classes in $H^2(H_n, \mathbb{R})$.

Rewriting $H^1(\mathbb{R}^{2n}, H^1(\mathbb{R},\mathbb{R}))$, we get

$$0 \to \mathbb{R} \xrightarrow{d_2^{01}} H^2(\mathbb{R}^{2n},\mathbb{R}) \xrightarrow{j^*} H^2(H_n,\mathbb{R})_1 \xrightarrow{tg} H^1(\mathbb{R}^{2n}, H^1(\mathbb{R},\mathbb{R})) \xrightarrow{d_2^{11}} H^3(\mathbb{R}^{2n},\mathbb{R}). \tag{9.1}$$

Theorem 9.1. *If* $G \cong H_n$, $A \cong \mathbb{R}$, *with* H_n *operating trivially on* $\mathbb{R}$, *with* $N \cong \mathbb{R}$, *the center and commutator subgroup of* H_n, *and* $H \cong H_n/\mathbb{R} \cong \mathbb{R}^{2n}$, *and* $\sigma \in Z^2(H_n,\mathbb{R})$, *with* $\sigma_{12} = 0$, *then equations (4.10)-(4.12) become*

$$\sigma_{21}(y, x_1 + x_2) = \sigma_{21}(y, x_1) + \sigma_{21}(y, x_2) \tag{9.2}$$

$$\sigma_{21}(y_1 + y_2, x) = \sigma_{21}(y_1, x) + \sigma_{21}(y_2, x) \tag{9.3}$$

$$\delta(\sigma_{22})(y_1, y_2, y_3) = \sigma_{21}(y_1, (\alpha(y_2, y_3)). \tag{9.4}$$

Proof.

Clear.

Remark 9.2.

Using these equations, it is possible to say more about the maps in (8.1). The map d_2^{11} is given at the cocycle level as

$$d_2^{11}(\sigma_{21})(y_1, y_2, y_3) \ = \ \sigma_{21}(y_1, \alpha(y_2, y_3)) \ ,$$

where we have identified the σ_{21} which appears on the left side of this equation as the argument of d_2^{11} with its counterpart in the set of functions from H to the set of functions from N to A.

Theorem 9.3. *The map* d_2^{11}, *as given in (9.1), is injective for* $n \geq 2$.

Proof.

Let us look at the kernel of d_2^{11} in $H^1(\mathbb{R}^{2n}, H^1(\mathbb{R},\mathbb{R}))$, which is the subspace of forms σ_{21} which become trivial when we consider $d_2^{11}(\sigma_{21})(y_1, y_2, y_3) = \sigma_{21}(y_1, \alpha(y_2, y_3))$ as a trilinear function on $\mathbb{R}^{2n}$. We may take as a basis for the subspace of trilinear forms of the form $\sigma_{21}(y_1, \alpha(y_2, y_3))$, for some σ_{21}, the $2n$ forms

$$y_i \alpha(\mathbf{y}', \mathbf{y}''), \quad i = 1, 2, \ldots, 2n \ ,$$

where $\mathbf{y} = (y_1, \ldots, y_n)$, and similarly for $\mathbf{y}', \mathbf{y}''$. Now it is precisely here that we see the difference between H_1 and H_n for $n \geq 2$. If $n = 1$, the two trilinear functions $y_1 \alpha(\mathbf{y}', \mathbf{y}'')$, $y_2 \alpha(\mathbf{y}', \mathbf{y}'')$ are each annihilated by the alternating map. But for $n \geq 2$ this is no longer the case.

Let us consider the effect of the alternating map on each of the $2n$ trilinear functions $y_i \alpha(\mathbf{y}', \mathbf{y}'')$ $i = 1, \ldots, 2n$. The alternating form α is the sum of n alternating forms

$$y'_k y''_{2n+1-k} - y''_k y'_{2n+1-k} \ .$$

Now exactly one of these forms contains both y'_i and y''_i. The alternating map A annihilates the two terms $y_i y'_i y''_{2n+1-i}$ and $y_i y'_{2n+1-i} y''_i$. On the other hand, A does not take to zero any other product $y_i(y'_k y''_{2n+1-k} - y'_{2n+1-k} y''_k)$ for $i \neq k, 2n+1-k$. Not only this, but it is the case that for $i = 1, \ldots, 2n$, the set of forms $A(y_i \alpha(\mathbf{y}', \mathbf{y}''))$ is linearly independent, since the sets of basis vectors $y_j y'_k y''_l$, by means of which these forms can be expressed as linear combinations, are disjoint.

Thus we observe what might be aptly termed a cohomological multilinearizability stability theorem.

Theorem 9.4. $\mathrm{H}^2(\mathrm{H}_n,\mathbb{R})$ *may be computed using alternating biilinear functions, as soon as* $n \geq 2$. *For* H_1, *every alternating bilinear function on* H_1 *is trivial, and every measurable 2-cocycle is cohomologous to a cocycle which is a polynomial in six variables at least one of whose terms is of degree two in one variable and linear in the others, and whose terms not of this form are bilinear.*

Proof.

Exactness of (9.1) at $\mathrm{H}^1(\mathbb{R}^{2n},\mathrm{H}^1(\mathbb{R},\mathbb{R}))$ and injectivity of d_2^{11} means that the transgression map has image zero, which is another way of saying that we may choose cocycles σ so that $\sigma_{21} = \sigma_{12} = \sigma_{11} = 0$, and this is necessary and sufficient for σ to be defined on $\mathrm{H}_n/\mathbb{R} \cong \mathbb{R}^{2n}$, yielding the result stated in the theorem for $n \geq 2$. We have already proved the stated results in theorem 3.9 for H_1.

Corollary 9.5. $\mathrm{H}^2(\mathrm{H}_n,\mathbb{R})$ *fits into an exact sequence*

$$0 \to \mathbb{R} \xrightarrow{i} \mathrm{H}^2(\mathbb{R}^{2n},\mathbb{R}) \xrightarrow{j^*} \mathrm{H}^2(\mathrm{H}_n,\mathbb{R}) \to 0 ,$$

where $i(\mathbb{R})$ *is the one-dimensional subspace of* $\mathrm{H}^2(\mathbb{R}^{2n},\mathbb{R})$ *generated by* α, *the form defining* H_n.

Proof.

This follows from the exactness of (9.1) and injectivity of d_2^{11}.

Lemma 9.6. *If* $n \geq 2$ *and* $0 \to \mathbb{R} \to \mathrm{E} \to \mathrm{H}_n \to 1$ *is a central extension, then the center of* E *is at least dimension two.*

Proof.

Let β be the alternating form defining the extension. Since $n \geq 2$, β is defined modulo the center of H_n, i.e., it is defined on $\mathrm{H}_n/\mathbb{R} \cong \mathbb{R}^{2n}$. Since we may write the group multiplication in E as

$$(s,t,\mathbf{y})(s',t',\mathbf{y}') = (s+s'+\beta(\mathbf{y},\mathbf{y}'), t+t'+\alpha(\mathbf{y},\mathbf{y}'), \mathbf{y}+\mathbf{y}') ,$$

we see that

$$(s,t,0)(s',t',\mathbf{y}') = (s+s', t+t', \mathbf{y}') = (s',t',\mathbf{y}')(s,t,0) .$$

Theorem 9.7. *With the hypotheses as in the previous lemma,* E *is a two-step nilpotent group with center and commutator subgroup* $= \{(s,t,0,0) | s,t \in \mathbb{R}\}$.

Proof.

Suppose that

$$(s_0,t_0,\mathbf{y}_0)(s,t,\mathbf{y}) = (s_0+s+\beta(\mathbf{y}_0,\mathbf{y}), t_0+t+\alpha(\mathbf{y}_0,\mathbf{y}), \mathbf{y}_0+\mathbf{y})$$

and

$$(s,t,y)(s_0,t_0,\mathbf{y}_0) = (s+s_0+\beta(\mathbf{y},\mathbf{y}_0), t+t_0+\alpha(\mathbf{y},\mathbf{y}_0), \mathbf{y}+\mathbf{y}_0)$$

are equal. Then $\alpha(\mathbf{y}_0,\mathbf{y}) = \alpha(\mathbf{y},\mathbf{y}_0)$ for all $\mathbf{y} \in \mathbb{R}^{2n}$. But α is nondegenerate and so $\mathbf{y}_0 = 0$. In order to compute a commutator, first compute the inverse of $(s,t,\mathbf{y})$ to be $(-s,-t-\mathbf{y})$. The commutator is then

$$(s,t,\mathbf{y})(s',t',\mathbf{y}')(-s,-t,-\mathbf{y})(-s',-t',-\mathbf{y}') = (2\beta(\mathbf{y},\mathbf{y}'), 2\alpha(\mathbf{y},\mathbf{y}'), 0) ,$$

and because we may take β not to be a multiple of α, we see that this is the entire center.

Theorem 9.8. *If $n \geq 2$ and*

$$0 \to \mathbb{R} \to \mathrm{E}_1 \to \mathrm{H}_n \to 1, \quad 0 \to \mathbb{R} \to \mathrm{E}_2 \to \mathrm{H}_n \to 1,$$

are two central extensions of H_n with cocycles β_1, β_2 respectively, then $\mathrm{E}_1 \cong \mathrm{E}_2$ if there exists a symplectic map $S\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$\beta_2(\mathrm{Sy}, \mathrm{Sy}') = \beta_1(\mathbf{y}, \mathbf{y}'), \quad \mathbf{y}, \mathbf{y}' \in \mathbb{R}^{2n} .$$

Proof.

If such an S exists, then we define the isomorphism

$$\theta\colon (s, t, \mathbf{y}) \to (s, t, \mathrm{Sy})$$

from E_1 to E_2, where E_1 is the set of triples $(s, t, \mathbf{y})$ with the multiplication

$$(s, t, \mathbf{y})(s', t', \mathbf{y}') = (s + s' + \beta_1(\mathbf{y}, \mathbf{y}'), t + t' + \alpha(\mathbf{y}, \mathbf{y}'), \mathbf{y} + \mathbf{y}')$$

and E_2 is the same set of triples with the multiplication using β_2 in place of β. The remainder of the proof is a straightfoward calculation.

Lemma 9.9. *If E_1,E_2 are as in the previous theorem and $\theta\colon \mathrm{E}_1 \to \mathrm{E}_2$ is an isomorphism, then we can associate to θ a 3×3 matrix*

$$\begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{12} & \theta_{13} \\ 0 & 0 & \theta_{33} \end{bmatrix} , \quad \textit{where} \quad \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \in \mathrm{Gl}(2,\mathbb{R}) ,$$

$$\begin{bmatrix} \theta_{13} \\ \theta_{23} \end{bmatrix} \in \mathrm{Hom}(\mathbb{R}^{2n},\mathbb{R}^2) , \ \theta_{33} \in \mathrm{Gl}(2\mathrm{n},\mathbb{R}) ,$$

and multiplication of matrices corresponds to composition of automorphisms.

Proof.

Since θ must take the center of E_1 onto that of E_2, we must have

$$\theta(s, t, \mathbf{y}) = (\theta_{11}(s) + \theta_{12}(t) + \theta_{13}(\mathbf{y}), \theta_{21}(s) + \theta_{22}(t) + \theta_{23}(\mathbf{y}), \theta_{33}(\mathbf{y})) .$$

Letting $\mathbf{y} = 0$, we see that $\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \in \mathrm{Gl}(2,\mathbb{R})$ and $\theta_{33} \in \mathrm{Gl}(2\mathrm{n},\mathbb{R})$, using the homomorphism property of θ. On the other hand, let $s = t = 0$. The homomorphism property for θ in this case is that the following two expressions are equal.

$$\begin{aligned} \theta(0,0,\mathbf{y})\theta(0,0,\mathbf{y}') &= (\theta_{13}(\mathbf{y}), \theta_{23}(\mathbf{y}), \theta_{33}(\mathbf{y}))(\theta_{13}(\mathbf{y}'), \theta_{23}(\mathbf{y}'), \theta_{33}(\mathbf{y}')) \\ &= (\theta_{13}(\mathbf{y}) + \theta_{13}(\mathbf{y}') + \beta_2(\theta_{33}(\mathbf{y}), \theta_{33}(\mathbf{y}')), \theta_{23}(\mathbf{y}) + \theta_{23}(\mathbf{y}') + \alpha(\theta_{33}(\mathbf{y}), \theta_{33}(\mathbf{y}')), \theta_{33}(\mathbf{y}) + \theta_{33}(\mathbf{y}')) \end{aligned}$$

$$\begin{aligned} &\theta((0,0,\mathbf{y})(0,0,\mathbf{y}')) = \theta(\beta_1(\mathbf{y},\mathbf{y}'), \alpha(\mathbf{y},\mathbf{y}'), \mathbf{y}+\mathbf{y}') \\ &= (\theta_{11}(\beta_1(\mathbf{y},\mathbf{y}')) + \theta_{12}(\alpha(\mathbf{y},\mathbf{y}')) + \theta_{13}(\mathbf{y}+\mathbf{y}'), \theta_{21}(\beta_1(\mathbf{y},\mathbf{y}')) + \theta_{22}(\alpha(\mathbf{y},\mathbf{y}')) + \theta_{23}(\mathbf{y}+\mathbf{y}'), \theta_{33}(\mathbf{y}+\mathbf{y}')) \end{aligned}$$

Equating the right side of these equations gives

(9.5)

$$\theta_{13}(\mathbf{y}') - \theta_{13}(\mathbf{y} + \mathbf{y}') + \theta_{13}(\mathbf{y}) = \theta_{11}(\beta_1(\mathbf{y}, \mathbf{y}')) + \theta_{12}(\alpha(\mathbf{y}, \mathbf{y}')) - \beta_2(\theta_{33}(\mathbf{y}), \theta_{33}(\mathbf{y}'))$$

(9.6)

$$\theta_{23}(\mathbf{y}') - \theta_{13}(\mathbf{y} + \mathbf{y}') + \theta_{13}(\mathbf{y}) = \theta_{21}(\beta_1(\mathbf{y}, \mathbf{y}')) + \theta_{22}(\alpha(\mathbf{y}, \mathbf{y}')) - \alpha(\theta_{33}(\mathbf{y}), \theta_{33}(\mathbf{y}'))$$

Since the right side of these equations are alternating forms which the left side says are cohomologous to zero, both sides must be identically zero. This forces θ_{13}, θ_{23} to be homomorphisms from $\mathbb{R}^{2n}$ to $\mathbb{R}$ and $\begin{bmatrix} \theta_{13} \\ \theta_{23} \end{bmatrix} \in \mathrm{Hom}(\mathbb{R}^{2n},\mathbb{R}^2)$, as stated.

Theorem 9.10. *Two one-dimensional central extensions* E_1,E_2 *of* H_n *by* $\mathbb{R}$ *are isomorphic iff there are maps* θ_1,θ_2 *in* $Gl(2n,\mathbb{R}),Gl(2,\mathbb{R})$ *respectively, for which*

$$\theta_1 \circ \begin{bmatrix} \beta_1 \\ \alpha \end{bmatrix} = \begin{bmatrix} \beta_2 \\ \alpha \end{bmatrix} \circ \theta_2 \tag{9.7}$$

as alternating bilinear functions $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^2$.

Proof.

The right side of equations (9.5),(9.6), being identically zero, may be written in the form stated in the theorem, where $\theta_1 = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$, $\theta_2 = \theta_{33}$. It is clear that these equations are necessary, and certainly sufficient, since we have already proved sufficiency in theorem 9.8 for a special case.

Definition 9.11.

Let the group $Sl(2,\mathbb{R}) \oplus Sl(2n,\mathbb{R})$ act on the space of bilinear maps

$$\{b : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^2\}$$

as

$$(\theta_1, \theta_2)(b) = \theta_1 b \theta_2^{-1} .$$

Theorem 9.12. *The extensions* E_1,E_2 *of the previous theorem are isomorphic iff the two bilinear maps* $b_1 = \begin{bmatrix} \beta_1 \\ \alpha \end{bmatrix}$, $b = \begin{bmatrix} \beta_2 \\ \alpha \end{bmatrix}$ *are in the same orbit with respect to the action of* $Sl(2,\mathbb{R}) \oplus Sl(2n,\mathbb{R})$ *as defined above.*

Proof.

Equation 9.7 can be rewritten to read

$$\theta_1 \circ \begin{bmatrix} \beta_1 \\ \alpha \end{bmatrix} \circ \theta_2^{-1} = \begin{bmatrix} \beta_2 \\ \alpha \end{bmatrix} .$$

If we express this result in terms of the matrices A,B_1,B_2,T_1,T_2 which we may associate to $\alpha,\beta_1,\beta_2,\theta_1,\theta_2$, respectively, we obtain

Corollary 9.13. *The extensions* E_1,E_2 *above are isomorphic iff the two pencils* $\{\lambda A + \mu B_1 | \lambda,\mu \in \mathbb{R}\}$ *and* $\{\lambda A + \mu B_2 | \lambda,\mu \in \mathbb{R}\}$ *are congruent in the extended sense, i.e., if we have*

$$\lambda' A + \mu' B_1 \text{ , where } T_1 \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \lambda' \\ \mu' \end{bmatrix} ,$$

$$\text{and } T_2(\lambda A + \mu B_1) = \lambda T_2^t A T_2 + \mu T_2^t B_1 T_2 ,$$

then the two pencils are congruent in this extended sense.

Theorem 9.14. *In theorem 9.12, we may assume that the* θ_2 *belong to* $\mathrm{Sp}(2\mathrm{n},\mathbb{R})\oplus\mathbb{R}$.

Proof.

First of all, we are not interested in the case that β_2 is in the subspace $\lambda\beta_1 + \mu\alpha$, because this would make β_1 cohomologous to $\lambda\beta_1$, since $\mu\alpha$ is cohomologous to zero. Thus we may assume that β_1, β_2 lie in different two dimensional subspaces $\{\lambda\beta + \mu\alpha\}$.Each of these two dimensional subspaces generated by α and β passes through the line generated by α and so we may assume that θ_2 takes this line into itself. But this means that θ_2 is a multiple of a symplectic map for the form α.

Corollary 9.15. *Since the planes generated by* α *and* β *pass through* α*, we may assume that* θ_1 *is of the form* $\begin{bmatrix} \theta_{11} & \theta_{12} \\ 0 & \theta_{22} \end{bmatrix}$.

Proof.

Clear.

Theorem 9.16. *We may write an automorphism a of* H_n *in the form*

$$a(t, \mathbf{y}) = (rx + \alpha(\boldsymbol{\gamma}, \mathbf{y}), \mathrm{S}\mathbf{y}) ,$$

where $r \in \mathbb{R}^*$*, and we have represented a linear form on* $\mathbb{R}^{2n}$ *using the nondegenerate alternating form* α*. We may write the matrix for a in the form* $\begin{bmatrix} r & \alpha(\boldsymbol{\gamma},\ ) \\ 0 & \mathrm{S} \end{bmatrix}$*, and the product of two such matrices is*

$$\begin{bmatrix} r_1 & \alpha(\boldsymbol{\gamma}_1,\ ) \\ 0 & \mathrm{S}_1 \end{bmatrix} \begin{bmatrix} r_2 & \alpha(\boldsymbol{\gamma}_2,\ ) \\ 0 & \mathrm{S}_2 \end{bmatrix} \begin{bmatrix} r_1 r_2 & \alpha(r_1\boldsymbol{\gamma}_2 + \mathrm{S}_2^{-1}\boldsymbol{\gamma}_1,\ ) \\ 0 & \mathrm{S}_1\mathrm{S}_2 \end{bmatrix} .$$

Proof.

Note that $\alpha(\mathbf{y}, \mathrm{S}_2\mathbf{y}') = \alpha(\mathrm{S}_2^{-1}\mathbf{y}, \mathbf{y}')$ and use methods similar to those used in proving theorem 7.5.

Theorem 9.17. *If* $a \in \mathrm{Aut}(\mathrm{H}_n)$ *and* $\beta \in \wedge^2(\mathrm{H}_n, \mathbb{R})$*, then*

$$\beta^{a^{-1}}((t, \mathbf{y}), (t', \mathbf{y}')) = r\beta((t, \mathrm{S}\mathbf{y}), (t', \mathrm{S}\mathbf{y}')) ,$$

if $a = \begin{bmatrix} r & \alpha(\boldsymbol{\gamma},\ ) \\ 0 & \mathrm{S} \end{bmatrix}$.

Proof.

The outer automorphism group of H_n is $\begin{bmatrix} r & 0 \\ 0 & \mathrm{S} \end{bmatrix}$, and the inner automorphism group $\alpha(\boldsymbol{\gamma},\ )$ has no effect on cohomology, as is well known, e.g., [Mac], prop.5.6.

X. Projective σ-representations

In this section, we interpret our result in theorem 8.3 on the isomorphism classes of central extensions of H_1 by $\mathbb{R}$ in terms of projective representations. We recall some definitions and theorems from [Ma1], p.267ff.

Definition 10.1.

If G is a locally compact polonais group, then a projective representation ρ of G is a continuous map $\rho\colon \mathrm{G} \to \mathrm{PU}(\infty)$, the projective unitary group, with the quotient topology induced from $\mathrm{U}(\infty)$, the group of unitary operators, with the strong operator topology, on a separable Hilbert space.

Remark 10.2.

There is an exact sequence of polonais groups

$$1 \to \mathbf{T} \xrightarrow{i} \mathrm{U}(\infty) \xrightarrow{j} \mathrm{PU}(\infty) \to 1 .$$

It can be shown that j has a Borel measurable cross-section s, which we may compose with a projective representation ρ in order to get a measurable map $\tilde{\rho} = s \circ \rho$ from G into $\mathrm{U}(\infty)$, which satisfies the relation

$$\tilde{\rho}(g_1 g_2) = \sigma(g_1, g_2)\tilde{\rho}(g_1)\tilde{\rho}(g_2) ,$$

where $\sigma(g_1, g_2)$ is a Borel measurable 2-cocycle with values in $\mathbf{T}$, the kernel of j in the exact sequence above. This gives an alternate way of defining projective representation, and such a measurable function $\tilde{\rho}$ is called a projective σ-representation. Most of the theory of ordinary Hilbert Space representations of locally compact groups goes through for projective σ-representations([Ma1], p267ff).

Theorem 10.3(Mackey). *If*

$$1 \to \mathbf{T} \xrightarrow{i} \mathrm{E} \xrightarrow{j} \mathrm{G} \to 1$$

is a central extension with the usual conditions on i and j, then by Schur's lemma, if we restrict an irreducible representation $r\colon \mathrm{G} \to \mathrm{U}(\infty)$ to $s(\mathrm{G})$, we obtain a projective representation of G *by composing r with s. The σ associated to this projective representation, when looked at as a function on* G×G, *is a Borel-measurable* 2*-cocycle with values in* $\mathbf{T}$ *which is cohomologous to the* 2*-cocycle of the extension above.*

Theorem 10.4(Mackey). *Given a Borel measurable* 2*-cocycle σ in $\mathrm{Z}^2_b(\mathrm{G},\mathbf{T})$, we construct the extension G^σ associated to σ. Then there is a $1-1$ correspondence between ordinary irreducible representations of G^σ and irreducible prejective σ-representations of* G.

Definition 10.5.

If $1 \to \mathrm{N} \to \mathrm{R(G)} \to \mathrm{G} \to 1$ is a central extension, then R(G) is called a σ-representation group for G if every irreducible σ-projective representation $\tilde{\rho}_\sigma$ of G occurs as the restriction to s(G) of an ordinary irreducible representation ρ of R(G).

Definition 10.6.

If $1 \to \mathrm{N} \to \mathrm{R}_g(\mathrm{G}) \to \mathrm{G} \to 1$ is a central extension, $\mathrm{R}_g(\mathrm{G})$ is called a *generalized* representation group for G if every irreducible σ-projective representation $\tilde{\rho}_\sigma$ of G occurs as the composition with an automorphism θ of G of the restriction to s(G) of an ordinary irreducible representation ρ of $\mathrm{R}_g(\mathrm{G})$.

As we have remarked earlier, the simple connectedness of H_1 results in the isomorphism $\mathrm{H}_b^2(\mathrm{H}_1,\mathbb{R}) \cong \mathrm{H}_b^2(\mathrm{H}_1,\mathbf{T})$. Using the isomorphism exhibited between H_1^σ and $\mathrm{H}_1^{\sigma'}$, for σ, σ' in $\mathrm{Z}_b^2(\mathrm{H}_1,\mathbb{R})$, which carries over to σ, σ' in $\mathrm{Z}_b^2(\mathrm{H}_1,\mathbf{T})$, we get the

Theorem 10.7. *Let*

$$1 \to \mathbf{T} \xrightarrow{i} \mathrm{H}_1^\sigma \xrightarrow{j} \mathrm{H}_1 \to 1$$

be the extension of $\mathbf{T}$ *by* H_1 *defined by* $\sigma \in \mathrm{Z}^2(\mathrm{H}_1,\mathbf{T})$, *and suppose that* $\theta \in \mathrm{Aut}(\mathrm{H}_1)$ *so that* $\theta(\sigma) = \sigma'$. *Then if* $\rho_{\sigma'}$ *is any irreducible* σ'*-projective representation of* H_1, *we may find an ordinary representation* ρ *of* H_1^σ *so that* $\rho \circ s \circ \theta = \rho_{\sigma'}$.

Proof.

Since $\theta(\sigma) = \sigma'$, $\delta(s \circ \theta) = \sigma'$, since $\delta s = \sigma$.

Remark 10.8.

Of course, theorem 10.7 just says, according to definition 10.6, that H_1^σ is a generalized representation group $\mathrm{R}_g(\mathrm{H}_1)$ for H_1 for any σ. The ordinary representation group $\mathrm{R}(\mathrm{H}_1)$ of H_1 is five-dimensional, as opposed to $\mathrm{R}_g(\mathrm{H}_1)$, which is only four-dimensional, but twisting by outer automorphisms is not allowed in $\mathrm{R}(\mathrm{H}_1)$.

Remark 10.9.

The next question which naturally presents itself is this: is there any way of combining the various G^σ for σ running through a set of representatives of isomorphism classes, so as to construct a generalized representation group $\mathrm{R}_g(\mathrm{G})$ of G?

XI. Conclusion.

Describing the congruence of antisymmetric matrices with respect to symplectic ones is equivalent to describing the geometry of the orbit structure of the second fundamental representation of the symplectic group $\mathrm{Sp}(2\mathrm{n},\mathbb{R})$ on $\wedge^2(\mathbb{R}^{2n},\mathbb{R})$. This representation decomposes into two irreducible pieces: one, a one-dimensional piece generated by scalar multiples of a, the form defining Sp(2n), and a $\left(\binom{2n}{2} - 1\right) = (2n^2 - n + 1)$-dimensional piece. To begin to understand the geometry of these orbits, it is necessary to first study the classical literature [Kr],[Fr],[We],[Wi]. These authors give various cross-sections from the set of orbits into the representation space.

The methods introduced in this paper can also be applied to extensions of H_n by a vector group $\mathbb{R}^n$, in which we are looking at the problem of symplectic congruence of webs of antisymmetric forms, which are just higher dimensional analogues of pencils.

Because H_n is simply connected, all the results in this paper not involving unique divisibility hold for 2-cocycles on H_n with values in the circle group $\mathbf{T}$, which arise in physics in the study of projective representations. These representations are of interest in quantum mechanics because many of the studied phenomena are independent of phase angle. In fact, we have shown for H_1 that, by taking any one nontrivial 2-cocycle σ with values in $\mathbf{T}$, and constructing the extension G^σ as in [Ma1], theorem 2.1, we obtain *all* the projective representations of H_n for *each* multiplier σ', by looking at the ordinary representations of G^σ, and twisting by an outer automorphism.

Notice too, that this inductive procedure of building up 2-cocycles can lead only to ones which are at most polynomial in the parameters defining the nilpotent group, and the extensions they define are all linear algebraic groups. It is an interesting question in general to determine what conditions on a cocycle will ensure that the extension group is linear and algebraic if the range and domain groups are. For affine algebraic groups, it appears that the correct condition may be that the graph of the 2-cocycle is a variety. This then would reduce the question to that of asking when a variety with a group map having a varietal graph and an inverse having a varietal graph is an affine algebraic group. As far as I know, there is no general theory of cohomology which addresses itself to this question.

There is also a generalization of the methods and results of this paper to H^3 of the Heisenberg groups, based on a set of equations which generalize (4.9) – (4.13), and this is done in [Du2].

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