GROUP EXTENSION ENDOMORPHISM COORDINATES

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Abstract. If $G \simeq V_1 \oplus V_2$ is a direct sum of vector spaces, we may associate to any linear endomorphism f of G a 2×2 matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$, where $f_{ij}: V_j \to V_i$ are linear maps. We generalize this to the case where G is an arbitrary group extension, determine the functional equations satisfied by the f_{ij} and how such matrices multiply. We extend these results to the case where G carries a locally compact polish topology, and apply them to calculate the continuous endomorphisms and automorphisms of H_n, the $2n+1-$dimensional real Heisenberg group.

I. Discrete Groups

Suppose we have a short exact sequence

$$1 \to N \xrightarrow{i} G \xrightarrow{j} H \to 1$$

of groups, together with a cross-section $s\colon H \to G$ for j, i.e., $j \circ s = 1_H$, and a projection $t\colon G \to N$ along s, i.e., $t \circ i = 1_N$, $t \circ s = e$, $(i \circ t)(s \circ j) = 1_G$, where for $f_1, f_2\colon X \to G$, $f_1 f_2$ stands for the pointwise product $(f_1 f_2)(x) = (f_1(x))(f_2(x))$.We write this in diagrammatic form as follows:

$$1 \to N \underset{t}{\overset{i}{\rightleftarrows}} G \underset{s}{\overset{j}{\rightleftarrows}} H \to 1 \ .$$

s defines a $2-$cocycle σ

$$\sigma(y_1, y_2) = t(s(y_1)s(y_2)(s(y_1 y_2))^{-1}) \text{ or } s(y_1)s(y_2) = i(\sigma(y_1, y_2))s(y_1 y_2) \ .$$

We also have a function $\varphi\colon H \times N \to N$ defined as $\varphi(y, x) = t(s(y)i(x)(s(y))^{-1})$, which we write x^y for short. φ does not quite give an operation of H on N, but misses by the $2-$cocycle σ.

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Lemma 1.1. *For $y_i \in H, x \in N$, we have*

$$\varphi(y_1, \varphi(y_2, x)) = (\varphi(y_1 y_2, x))^{\sigma(y_1, y_2)} ,$$

where, for $x_1, x_2 \in N$, $x_1^{x_2} = x_2 x_1 x_2^{-1}$.

Proof.

$$\begin{aligned} t(s(y_1)(i \circ t)(s(y_2)i(x)s(y_2)^{-1})s(y_1)^{-1}) &= t(s(y_1)s(y_2)i(x)(s(y_2))^{-1}(s(y_1))^{-1}) \\ &= t(i(\sigma(y_1, y_2))s(y_1 y_2)i(x)(s(y_1 y_2))^{-1}(i(\sigma(y_1, y_2)))^{-1}) \\ &= \sigma(y_1, y_2)t(s(y_1 y_2)i(x)(s(y_1 y_2))^{-1})(\sigma(y_1, y_2))^{-1} , \end{aligned}$$

since t is a homomorphism when restricted to $i(N)$.

Lemma 1.2. *For $y_i \in H$,*

$$\sigma(y_1, y_2)\sigma(y_1 y_2, y_3) = (\sigma(y_2, y_3))^{y_1}\sigma(y_1, y_2 y_3) .$$

Proof.

This is well known. See, e.g., [B].

Lemma 1.3. *For $g_i \in G$,*

$$t(g_1 g_2) = t(g_1)[t(g_2)]^{j(g_1)}\sigma(j(g_1), j(g_2)) .$$

Proof.

$$\begin{aligned} t(g_1 g_2) &= t((i \circ t)(s \circ j)(g_1)(i \circ t)(s \circ j)(g_2)) \\ &= t(i(t(g_1))s(j(g_1))i(t(g_2))s(j(g_2))) \\ &= t(i(t(g_1))s(j(g_1))i(t(g_2))(s(j(g_1)))^{-1}s(j(g_1))s(j(g_2)) \\ &= t(i(t(g_1))s(j(g_1))i(t(g_2))(s(j(g_1)))^{-1}i(\sigma(j(g_1), j(g_2)))s(j(g_1))s(j(g_2)) \\ &= t(g_1)t(s(j(g_1))i(t(g_2))(s(j(g_1)))^{-1})\sigma(j(g_1), j(g_2)) , \\ &= t(g_1)[t(g_2)]^{j(g_1)}\sigma(j(g_1), j(g_2)) \end{aligned}$$

Now for $f\colon G \to G$ an endomorphism, define the *coordinates* f_{ij} for $1 \leq i, j \leq 2$ of f as follows:

$$\begin{aligned} f_{11} &= t \circ f \circ i\colon N \to N \\ f_{21} &= j \circ f \circ i\colon N \to H \\ f_{12} &= t \circ f \circ s\colon H \to N \\ f_{22} &= j \circ f \circ s\colon H \to H . \end{aligned}$$

Then

Theorem 1.1. *The f_{ij} satisfy the following functional equations:*

(1.1)
$$f_{11}(x_1x_2) = f_{11}(x_1)[f_{11}(x_2)]^{f_{21}(x_1)}\sigma(f_{21}(x_1), f_{21}(x_2))$$

(1.2)
$$f_{21}(x_1x_2) = f_{21}(x_1)f_{21}(x_2)$$

(1.3)
$$f_{12}(y_1y_2) = f_{11}(\sigma(y_1,y_2)^{-1})\left[f_{12}(y_1)[f_{12}(y_2)]^{f_{22}(y_1)}\sigma(f_{22}(y_1), f_{22}(y_2))\right]^{f_{21}(\sigma(y_1,y_2)^{-1})}\cdot$$
$$\sigma(f_{21}(\sigma(y_1,y_2)^{-1}), f_{22}(y_1)f_{22}(y_2))$$

(1.4)
$$f_{22}(y_1y_2) = f_{21}(\sigma(y_1,y_2)^{-1})f_{22}(y_1)f_{22}(y_2)$$

Proof.

$$\begin{aligned}
f_{11}(x_1x_2) &= (t\circ f\circ i)(x_1x_2) = t((f\circ i)(x_1)(f\circ i)(x_2)) \\
&= (t\circ f\circ i)(x_1)[(t\circ f\circ i)(x_2)]^{(j\circ f\circ i)(x_1)}\sigma((j\circ f\circ i)(x_1),(j\circ f\circ i)(x_2)) \\
&= f_{11}(x_1)[f_{11}(x_2)]^{f_{21}(x_1)}\sigma(f_{21}(x_1), f_{21}(x_2)) \\
f_{21}(x_1x_2) &= (j\circ f\circ i)(x_1x_2) = (j\circ f\circ i)(x_1)(j\circ f\circ i)(x_2) = f_{21}(x_1)f_{21}(x_2) \\
f_{12}(y_1y_2) &= (t\circ f\circ s)(y_1y_2) = (t\circ f)(i(\sigma(y_1,y_2)^{-1})s(y_1)s(y_2)) \\
&= t((f\circ i)(\sigma(y_1,y_2)^{-1})(f\circ s)(y_1)(f\circ s)(y_2)) \\
&= (t\circ f\circ i)(\sigma(y_1,y_2)^{-1})[t((f\circ s)(y_1)(f\circ s)(y_2))]^{(j\circ f\circ i)(\sigma(y_1,y_2)^{-1})}\cdot \\
&\quad \sigma((j\circ f\circ i)(\sigma(y_1,y_2)^{-1}), j((f\circ s)(y_1)(f\circ s)(y_2))) \\
&= f_{11}(\sigma(y_1,y_2)^{-1})\cdot \\
&\quad \left[(t\circ f\circ s)(y_1)[(t\circ f\circ s)(y_2)]^{(j\circ f\circ s)(y_1)}\sigma((j\circ f\circ s)(y_1),(j\circ f\circ s)(y_2))\right]^{f_{21}(\sigma(y_1,y_2)^{-1})}\cdot \\
&\quad \sigma(f_{21}(\sigma(y_1,y_2)^{-1}), f_{22}(y_1)f_{22}(y_2)) \\
&= f_{11}(\sigma(y_1,y_2)^{-1})\left[f_{12}(y_1)[f_{12}(y_2)]^{f_{22}(y_1)}\sigma(f_{22}(y_1), f_{22}(y_2))\right]^{f_{21}(\sigma(y_1,y_2)^{-1})}\cdot \\
&\quad \sigma(f_{21}(\sigma(y_1,y_2)^{-1}), f_{22}(y_1)f_{22}(y_2)) \\
f_{22}(y_1y_2) &= (j\circ f\circ s)(y_1y_2) = (j\circ f)(i(\sigma(y_1,y_2)^{-1})s(y_1)s(y_2)) \\
&= (j\circ f\circ i)(\sigma(y_1,y_2)^{-1})(j\circ f\circ s)(y_1)(j\circ f\circ s)(y_2) \\
&= f_{21}((\sigma(y_1,y_2))^{-1})f_{22}(y_1)f_{22}(y_2)
\end{aligned}$$

Theorem 1.2.

$$f = (i\circ f_{11}\circ t)(s\circ f_{21}\circ t)(i\circ f_{12}\circ j)(s\circ f_{22}\circ j)$$

Proof.

For $f_1 : G\to N$, $f_2 : G\to H$, define

$$f_1^{f_2} = t\circ(s\circ f_2)(i\circ f_1)(s\circ f_2)^{-1}\ .$$

Then

$$\begin{aligned} f &= (i\circ t)(s\circ j)\circ f\circ(i\circ t)(s\circ j)\\ &= (i\circ t)(s\circ j)\circ(f\circ i\circ t)(f\circ s\circ j)\\ &= [(i\circ t)\circ(f\circ i\circ t)(f\circ s\circ j)][(s\circ j)\circ(f\circ i\circ t)(f\circ s\circ j)]\\ &= [i\circ(t\circ f\circ i\circ t)(t\circ f\circ s\circ j)^{j\circ f\circ i\circ t}\sigma(j\circ f\circ i\circ t, j\circ f\circ s\circ j)][s\circ(j\circ f\circ i\circ t)(j\circ f\circ s\circ j)]\\ &= [i\circ(t\circ f\circ i\circ t)(t\circ(s\circ j\circ f\circ i\circ t)(i\circ t\circ f\circ s\circ j)(s\circ j\circ f\circ i\circ t)^{-1})\sigma(j\circ f\circ i\circ t, j\circ f\circ s\circ j)]\cdot\\ &\quad [(i\circ\sigma(j\circ f\circ i\circ t, j\circ f\circ s\circ j)^{-1})(s\circ j\circ f\circ i\circ t)(s\circ j\circ f\circ s\circ j)]\\ &= (i\circ t\circ f\circ i\circ t)(s\circ j\circ f\circ i\circ t)(i\circ t\circ f\circ s\circ j)(s\circ j\circ f\circ i\circ t)^{-1}\cdot\\ &\quad (i\circ\sigma)(j\circ f\circ i\circ t, j\circ f\circ s\circ j)(i\circ\sigma)(j\circ f\circ i\circ t, j\circ f\circ s\circ j)^{-1}\cdot\\ &\quad (s\circ j\circ f\circ i\circ t)(s\circ j\circ f\circ s\circ j)\\ &= (i\circ f_{11}\circ t)(s\circ f_{21}\circ t)(i\circ f_{12}\circ j)(s\circ f_{22}\circ j) \end{aligned}$$

Theorem 1.3. *If we define f as in theorem* 1.2, *where the f_{ij} satisfy the functional equations in theorem* 1.1, *then f is an endomorphism of G.*

Before we prove this, it behooves us to separate our task into two parts. First suppose that we have an endomorphism $f\colon G\to G$. Then define the two maps $f_1 = f\circ i\colon N\to G$ and $f_2 = f\circ s\colon H\to G$. f_1 is a homomorphism and f_2 satisfies

Lemma 1.4.

$$f_2(y_1y_2) = f_1(\sigma(y_1,y_2)^{-1})f_2(y_1)f_2(y_2)\ . \tag{1.5}$$

Proof.

$$\begin{aligned} f_2(y_1y_2) = (f\circ s)(y_1y_2) &= f(i(\sigma(y_1,y_2)^{-1})s(y_1)s(y_2))\\ &= (f\circ i)(\sigma(y_1,y_2)^{-1})(f\circ s)(y_1)(f\circ s)(y_2)\\ &= f_1(\sigma(y_1,y_2)^{-1})f_2(y_1)f_2(y_2) \end{aligned}$$

Lemma 1.5. $f = (f_1\circ t)(f_2\circ j)$.

Proof.

$$f = f\circ(i\circ t)(s\circ j) = (f\circ i\circ t)(f\circ s\circ j) = (f_1\circ t)(f_2\circ j)$$

Conversely,

Lemma 1.6. *If we define $f = (f_1\circ t)(f_2\circ j)$, where f_1 is a homomorphism and f_2 satisfies* (1.5), *then f is an endomorphism of G.*

Proof.

$$
\begin{aligned}
f(g_1 g_2) &= (f_1 \circ t)(g_1 g_2)(f_2 \circ j)(g_1 g_2) \\
&= f_1 \left(t(g_1)[t(g_2)]^{j(g_1)} \sigma(j(g_1), j(g_2)) \right) f_2(j(g_1) j(g(2))) \\
&= (f_1 \circ t)(g_1) f_1 \left(t(s(j(g_1)) i(t(g_2)) s(j(g_1))^{-1}) \right) (f_1 \circ \sigma)(j(g_1), j(g_2)) \cdot \\
&\quad f_1(\sigma(j(g_1), j(g_2))^{-1}) f_2(j(g_1)) f_2(j(g_2)) \\
&= (f_1 \circ t)(g_1) f(s(j(g_1)) i(t(g_2)) s(j(g_1))^{-1}) f_1(\sigma(j(g_1), j(g_2))) f_1(\sigma(j(g_1), j(g_2))^{-1}) f_2(j(g_1)) f_2(j(g_2)) \\
&= (f_1 \circ t)(g_1)(f_2 \circ j)(g_1)(f_1 \circ t)(g_2)(f_2 \circ j)(g_2) \\
&= f(g_1) f(g_2)
\end{aligned}
$$

Next, suppose we have a homomorphism $f_1 : N \to G$ and we define $f_{11} = t \circ f_1 : N \to N$ and $f_{21} = j \circ f_1 : N \to H$. Then f_{11}, f_{21} satisfy equations (1.1), (1.2) respectively and

Lemma 1.7. $f_1 = (i \circ f_{11})(s \circ f_{21})$

Proof.

$$f_1 = (i \circ t)(s \circ j) \circ f_1 = (i \circ t \circ f_1)(s \circ j \circ f_1) = (i \circ f_{11})(s \circ f_{21})$$

Conversely,

Lemma 1.8. *If we define* $f_1 = (i \circ f_{11})(s \circ f_{21})$, *where* f_{11}, f_{21} *satisfy* (1.1), (1.2), *then* f_1 *is a homomorphism.*

Proof.

$$
\begin{aligned}
f_1(x_1 x_2) &= (i \circ f_{11})(x_1 x_2)(s \circ f_{21})(x_1 x_2) \\
&= i \left(f_{11}(x_1)[f_{11}(x_2)]^{f_{21}(x_1)} \sigma(f_{21}(x_1), f_{21}(x_2)) \right) s(f_{21}(x_1) f_{21}(x_2)) \\
&= i(f_{11}(x_1)) s(f_{21}(x_1)) i(f_{11}(x_2)) s(f_{21}(x_1))^{-1} i(\sigma(f_{21}(x_1), f_{21}(x_2))) \cdot \\
&\quad i(\sigma(f_{11}(x_1), f_{11}(x_2))^{-1}) s(f_{21}(x_1)) s(f_{21}(x_2)) \\
&= (i \circ f_{11})(x_1)(s \circ f_{21})(x_1)(i \circ f_{11})(x_2)(s \circ f_{21})(x_2) \\
&= f_1(x_1) f_1(x_2)
\end{aligned}
$$

Now suppose we have a map $f_2 : H \to G$ satisfying (1.5), and we define

$$
\begin{aligned}
f_{12} &= t \circ f_2 : H \to N \\
f_{22} &= j \circ f_2 : H \to H \ .
\end{aligned}
$$

Then f_{12}, f_{22} satisfy equations (1.3), (1.4) and

Lemma 1.9. $f_2 = (i \circ f_{12})(s \circ f_{22})$

Proof.

Similar to the proof of lemma 1.7.

Lemma 1.10. *If we define* $f_2 = (i \circ f_{12})(s \circ f_{22})$, *where* f_{12}, f_{22} *satisfy equations* (1.3), (1.4), *then* f_2 *satisfies* (1.5)

Proof.

$$\begin{aligned} f_2(y_1y_2) &= (i\circ f_{12})(s\circ f_{22})(y_1y_2) = (i\circ f_{12})(y_1y_2)(s\circ f_{22})(y_1y_2) \\ &= i\Big(f_{11}(\sigma(y_1,y_2)^{-1}) \left[f_{12}(y_1)[f_{12}(y_2)]^{f_{22}(y_1)}\sigma(f_{22}(y_1),f_{22}(y_2))\right]^{f_{21}(\sigma(y_1,y_2)^{-1})} \cdot \\ &\quad \sigma(f_{21}(\sigma(y_1,y_2)^{-1}), f_{22}(y_1)f_{22}(y_2)) \Big) s(f_{21}(\sigma(y_1,y_2)^{-1})f_{22}(y_1)f_{22}(y_2)) \\ &= (i\circ f_{11})(\sigma(y_1,y_2)^{-1})(s\circ f_{21})(\sigma(y_1,y_2)^{-1}) i\left(f_{12}(y_1)[f_{12}(y_2)]^{f_{22}(y_1)}\sigma(f_{22}(y_1),f_{22}(y_2))\right)\cdot \\ &\quad ((s\circ f_{21})(\sigma(y_1,y_2)^{-1}))^{-1} i(\sigma(f_{21}(\sigma(y_1,y_2)^{-1}), f_{22}(y_1)f_{22}(y_2)))\cdot \\ &\quad i(\sigma(f_{21}(\sigma(y_1,y_2)^{-1}), f_{22}(y_1)f_{22}(y_2)))^{-1} s(f_{21}(\sigma(y_1,y_2)^{-1}))s(f_{22}(y_1)f_{22}(y_2)) \\ &= f_1(\sigma(y_1,y_2)^{-1})(i\circ f_{12})(y_1)(s\circ f_{22})(y_1)(i\circ f_{12})(y_2)\,((s\circ f_{22})(y_1))^{-1}\cdot \\ &\quad (i\circ\sigma)(f_{22}(y_1),f_{22}(y_2))\,((i\circ\sigma)(f_{22}(y_1),f_{22}(y_2)))^{-1}\,(s\circ f_{22})(y_1)(s\circ f_{22})(y_2) \\ &= f_1(\sigma(y_1,y_2)^{-1})f_2(y_1)f_2(y_2) \end{aligned}$$

We may combine lemmas 1.4 – 1.10 to prove theorem 1.3, since theorem 1.2 says that $f = (f_1\circ t)(f_2\circ j)$.

Now let us associate the 2×2 matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ of functions f_{ij} to an endomorphism f of G. We may let the above matrix operate on the set $N\times H$ as

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_{11}(x)f_{12}(y) \\ f_{21}(x)f_{22}(y) \end{pmatrix} .$$

If we have two endomorphisms f, g of G, how do we compute the matrix of $f\circ g$?

Theorem 1.4. *If* $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ *and* $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ *are the matrices of endomorphisms* f, g *of* G *respectively, then the matrix of* $f\circ g$ *is*

$$\begin{pmatrix} (f_{11}\circ g_{11})\,[f_{12}\circ g_{21}]^{f_{21}\circ g_{11}}\,\sigma(f_{21}\circ g_{11}, f_{22}\circ g_{21}) & (f_{11}\circ g_{12})\,[f_{12}\circ g_{22}]^{f_{21}\circ g_{12}}\,\sigma(f_{21}\circ g_{12}, f_{22}\circ g_{22}) \\ (f_{21}\circ g_{11})(f_{22}\circ g_{21}) & (f_{21}\circ g_{12})(f_{22}\circ g_{22}) \end{pmatrix} .$$

Proof.

$$\begin{aligned} (f\circ g)_{11} &= t\circ f\circ g\circ i = t\circ f\circ (i\circ t)(s\circ j)\circ g\circ i \\ &= t\circ (f\circ i\circ t\circ g\circ i)(f\circ s\circ j\circ g\circ i) \\ &= (t\circ f\circ i\circ t\circ g\circ i)[t\circ f\circ s\circ j\circ g\circ i]^{j\circ f\circ i\circ t\circ g\circ i}\sigma(j\circ f\circ i\circ t\circ g\circ i, j\circ f\circ s\circ j\circ g\circ i) \\ &= (f_{11}\circ g_{11})[f_{12}\circ g_{21}]^{f_{21}\circ g_{11}}\sigma(f_{21}\circ g_{11}, f_{21}\circ g_{21}) . \end{aligned}$$

The expressions for $(f\circ g)_{21}, (f\circ g)_{12}, (f\circ g)_{22}$ are calculated similarly.

One of the simplest special cases of theorem 1.4 is when we have a central extension and N is equal to the center of G. In this case an endomorphism f of G has an upper triangular matrix $\begin{pmatrix} f_{11} & f_{12} \\ e & f_{22} \end{pmatrix}$, where e stands for the constant map f_{21} equal to e, the identity element of H. Then the matrix of $f \circ g$ is just

$$\begin{pmatrix} f_{11} \circ g_{11} & (f_{11} \circ g_{12})(f_{12} \circ g_{22}) \\ e & f_{22} \circ g_{22} \end{pmatrix} . \tag{1.7}$$

Note also that f_{11}, f_{22} are endomorphisms of N, H respectively and f_{12} satisfies the functional equation

$$f_{12}(y_1 y_2) f_{12}(y_1)^{-1} f_{12}(y_2)^{-1} = \sigma(f_{22}(y_1), f_{22}(y_2)) f_{11}(\sigma(y_1, y_2))^{-1} . \tag{1.8}$$

In words, f_{11}, f_{22} are endomorphisms of N, H respectively, chosen in such a way that the 2−cocycle $\sigma(f_{22}(y_1), f_{22}(y_2)) f_{11}(\sigma(y_1, y_2))^{-1}$ is trivial from H to N and f_{12} is any one of these trivializing 1−cochains. These equations appear in [W], and are crucial in calculating the outer automorphism of the generalized Heisenberg groups.

If we assume that G is a semidirect product, then the equations reduce to those of [H], except that the fourth and fifth equations of [H] follow from the first four, and the last is just the obvious condition for an endomorphism to be an automorphism.

2. Topological Groups

There are many theories of extensions of topological groups, e.g., see [C],[D],[Ma],[Mo], [Ms]. We shall use the theory presented in [D] for locally compact separable metrizable groups, also known as polish locally compact groups. N is then a closed subgroup of G, i is a homeomorphism into and j a continuous open map. s, t are then borel functions, as is the 2−cocycle σ of the extension. The f_{ij} are borel functions, and f_{21} is continuous, since it is a homomorphism. The algebra used in the proofs of the previous lemmas and theorems respects the borel character of the maps involved, so they remain valid in the context of polish locally compact groups.

As an application of the results above, we determine the monoid of continuous endomorphisms and the group of continuous automorphisms of the real $2n+1$−dimensional Heisenberg group H_n. H_n is thought of as a central extension of $\mathbb{R}$ by $\mathbb{R}^{2n}$ by means of a nondegenerate bilinear $\mathbb{R}$−valued skew form σ on $\mathbb{R}^{2n}$, used as a 2−cocycle. By [K] we may assume σ continuous. $\mathbb{R}$ is then the center of H_n, and for f a continuous endomorphism of H_n, each of the f_{ij} are continuous, since we can take s, t to be continuous. Since $\mathbb{R}, \mathbb{R}^{2n}$ are abelian, we write their group operations additively. f_{11}, f_{22} are continuous endomorphisms of $\mathbb{R}, \mathbb{R}^{2n}$ respectively and f_{12} satisfies the additive version of (1.8), viz.

$$f_{12}(y_1 + y_2) - f_{12}(y_1) - f_{12}(y_2) = \sigma(f_{22}(y_1), f_{22}(y_2)) - f_{11}(\sigma(y_1, y_2)) . \tag{3.9}$$

But notice this says that

$$\sigma(f_{22}(y_1), f_{22}(y_2)) - f_{11}(\sigma(y_1, y_2))$$

is a trivial $\mathbb{R}$−valued 2−cocycle on $\mathbb{R}^{2n}$, and by [K], such a 2−cocycle must be zero, since it is skew. Thus f_{12} is a borel homomorphism from $\mathbb{R}^{2n}$ to $\mathbb{R}$, and thus continuous.

Theorem 2.1. *The endomorphism monoid of H_n consists of two disjoint pieces. One, a closed ideal. consisting of endomorphisms of the form* $\begin{pmatrix} 0 & f_{12} \\ 0 & 0 \end{pmatrix}$, *$f_{12}$ being an $\mathbb{R}-$linear form on $\mathbb{R}^{2n}$, and the composition of any two of these being the zero endomorphism. This ideal is isomorphic to $\mathbb{R}^{2n}$ with the zero multiplication. Two, a subgroup of automorphisms of the form*

$$\begin{pmatrix} \lambda & f_{12} \\ 0 & \tau^{\frac{\lambda-|\lambda|}{2\lambda}} \sqrt{|\lambda|} f'_{22} \end{pmatrix} ,$$

where $\lambda \neq 0$, f_{12} is an $\mathbb{R}-$linear form on $\mathbb{R}^{2n}$ and f'_{22} is a symplectic automorphism of $\mathbb{R}^{2n}$ with respect to the skew form σ, and $\sigma(\tau(x), \tau(y)) = -\sigma(x, y)$. The automorphism group is a semidirect product of the normal subgroup of inner automorphisms of the form $\begin{pmatrix} 1_{\mathbb{R}} & f_{12} \\ 0 & 1_{\mathbb{R}^{2n}} \end{pmatrix}$ *and isomorphic to $\mathbb{R}^{2n}$ with the group of outer automorphisms. These are of the form* $\begin{pmatrix} \lambda & 0 \\ 0 & \tau^{\frac{\lambda-|\lambda|}{2\lambda}} \sqrt{|\lambda|} f'_{22} \end{pmatrix}$, *and isomorphic to $\mathbf{Z}_2 \times \mathbb{R} \times Sp(n, \mathbb{R})$, and thus has two connected components. It is the open dense subgroup of invertible elements of the endomorphism monoid. The endomorphism monoid is homeomorphic to $\mathbb{R}^{2n+1} \times Sp(n, \mathbb{R})$ and is connected.*

Proof.

Suppose that $f_{11} = 0$. Then the nondegeneracy of σ forces $f_{22} = 0$ also, and the endomorphisms of this form are the first monoid described above.

Next, if $f_{11} \neq 0$, write $\lambda = f_{11}$. Then

$$\sigma(f_{22}(y_1), f_{22}(y_2)) = \lambda \sigma(y_1, y_2) ,$$

which says that if $\lambda > 0$ then f_{22} is some scalar multiple $\sqrt{\lambda} f'_{22}$ of a symplectic automorphism f'_{22} of $\mathbb{R}^{2n}$ with respect to the nondegenerate skew form σ. If $\lambda < 0$, then by following f_{22} by any automorphism τ of $\mathbb{R}^{2n}$ which replaces σ by $-\sigma$, say by choosing a symplectic basis x_i, y_i for s and letting $\tau(x_i) = y_i$, $\tau(y_i) = x_i$, $\tau \circ f_{22}$ replaces s by a positive multiple λ of itself. We may then write $\tau \circ f_{22} = \sqrt{\lambda} f'_{22}$, where f'_{22} is a symplectic map for s, as before. Since $\tau^2 = 1_{\mathbb{R}^{2n}}$, we have that the group of the f_{22} is isomorphic to $Z_2 \times Sp(n)$. Multiplication of the matrices associated to the various automorphisms and endomorphisms, according to the scheme (1.6), yields the remaining statements of the theorem. This same cohomological equation (3.9) was obtained by [W] in calculating the automorphism group of the Heisenberg group, though he did not associate such with a matrix.

Of course, when we defined the Heisenberg group, we could just as well have used the general definition in [W], and our calculations will yield a determination of the outer automorphism group of this generalized Heisenberg group, if we use the result in [K] that any measurable 2-cocyle on A with values in the circle group $\mathbb{T}$ is cohomologous to a continuous antisymmetric 2-cocycle as long as A^2, the set of squares of elements of A is dense in A. This is certainly true for local fields and if we avoid characteristic two, also for global function fields.

The difficulty, in general, in explicitly solving the obvious equations distinguishing the automorphisms from the endomorphisms, lies in the fact that, even in the case of a direct product of abelian groups, the elements in the associated matrix do not commute. Consider the problem, for example, in trying obtain a closed form for the inverse of an $n \times n$ matrix of real numbers which has been divided into four blocks, reflecting the decomposition of $\mathbb{R}^n$ into the direct product of two subspaces, in terms of the four matrices constituting the blocks.

After this paper had been written, I became aware of the paper of Hsu [H], in which similar techniques are introduced for the case of a semidirect product of groups. In that paper, the automorphism group of a holomorph of a perfect group is computed.

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