GROUP EXTENSION HOMOMORPHISM MATRICES

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Abstract. If

$$1 \to N_1 \xrightarrow{i_1} G_1 \xrightarrow{j_1} H_1 \to 1$$
$$1 \to N_2 \xrightarrow{i_2} G_2 \xrightarrow{j_2} H_2 \to 1$$

are short exact sequences of groups, then we associate to each homomorphism $G_1 \xrightarrow{f} G_2$ a 2×2 matrix of functions, and recover f from this matrix. We also investigate the dependence of the matrix on the cross-sections s_i of j_i, $i = 1, 2$. Matrix multiplication is defined and interpreted in terms of homomorphism composition. Formulas are also given for coordinate change. All the formulas reduce to the standard ones in case G_1, G_2 are vector spaces and f a linear map.

I. Introduction

If we have a direct sum $A_1 \oplus A_2$ of abelian groups, its endomorphisms f are known to be described by 2×2 matrices $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$, where $f_{ij} \colon V_j \to V_i$ are homomorphisms for $1 \le i, j \le 2$. For example, see [W].

It is possible to do the same thing if we only have an extension

$$1 \to N \xrightarrow{i} G \xrightarrow{j} H \to 1$$

of groups. This can be quite useful if we want to study, e.g., the automorphisms of a group. In case N is characteristic, we may describe the automorphisms as upper triangular matrices, with a particularly simple law of multiplication.

It is rather surprising that such a technique has apparently never been carried out in the generality treated here. It is true that the functional equations satisfied by the entries in the matrix are quite imposing, and are probably impossible to solve in general. However, before we can begin to conceptualize what is involved in computing homomorphisms between groups which have been coordinatized by means of normal subgroups and quotient groups,

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it certainly behooves us to at least write down the appropriate equations. This we have done, and shown how to begin to solve them in the simplest nonabelian cases in [D1],[D2].

This technique can obviously applied in the case of lie and associative algebras, for derivations as well as for homomorphisms. These will be the objects of study in forthcoming papers.

2. Group Extension Coordinates

Suppose we have an extension of groups given by a short exact sequence

$$1 \to N \xrightarrow{i} G \xrightarrow{j} H \to 1 \ . \tag{2.1}$$

Suppose also that we are given a *section* $s\colon H \to G$ of j, i.e., $j \circ s = 1_H$, which has been *normalized* so that $s(e) = e$, and a *projection* $t\colon G \to N$ *along* s, i.e., $t \circ i = 1_N$, $t \circ s = e$, where e is the constant map equal to e everywhere. All of this may be illustrated by a diagram

$$1 \to N \underset{t}{\overset{i}{\rightleftarrows}} G \underset{s}{\overset{j}{\rightleftarrows}} H \to 1 \ .$$

Lemma 2.1.

$$1_G = (i \circ t)(s \circ j) \ . \tag{2.3}$$

Proof.

This is just a restatement of the fact that, in our situation, every element $g \in G$ can be written uniquely in the form $i(x)s(y)$ for $x \in N$, $y \in H$.

Definition 2.1. *For $x \in N$, $y \in H$, let*

$$\varphi(y, x) = x^y = t(s(y)i(x)s(y)^{-1}) \ . \tag{2.4}$$

This almost gives an operation of H on N, whose failure to be so is measured by a function $\sigma\colon H \times H \to N$.

Definition 2.2. *For $y_1, y_2 \in H$ define*

$$\sigma(y_1, y_2) = t(s(y_1)s(y_2)) \ . \tag{2.5}$$

Lemma 2.2. *σ also satisfies*

$$s(y_1)s(y_2) = i(\sigma(y_1, y_2))s(y_1 y_2) \ .$$

Proof.

Apply t to both sides.

Lemma2.3. *σ is called the 2-cocycle associated to s and satisfies*

$$\sigma(y_1, y_2)\sigma(y_1y_2, y_3) = (\sigma(y_2, y_3))^{y_1}\sigma(y_1, y_2y_3) \ , \tag{2.6}$$

for $y_i \in H$.

Proof.

This is well known. See, e.g., [B].

Lemma 2.4. *For $y_i \in H, x \in N$, we have*

$$\varphi(y_1, \varphi(y_2, x)) = (\varphi(y_1y_2, x))^{\sigma(y_1, y_2)} \ , \tag{2.7}$$

where, for $x_1, x_2 \in N$, $x_1^{x_2} = x_2x_1x_2^{-1}$.

Proof.

$$\begin{aligned} t(s(y_1)(i \circ t)(s(y_2)i(x)s(y_2)^{-1})s(y_1)^{-1}) &= t(s(y_1)s(y_2)i(x)(s(y_2))^{-1}(s(y_1))^{-1}) \\ &= t(i(\sigma(y_1, y_2))s(y_1y_2)i(x)(s(y_1y_2))^{-1}(i(\sigma(y_1, y_2)))^{-1}) \\ &= \sigma(y_1, y_2)t(s(y_1y_2)i(x)(s(y_1y_2))^{-1})(\sigma(y_1, y_2))^{-1} \ , \end{aligned}$$

since t is a homomorphism when restricted to $i(N)$.

We also need

Lemma 2.5. *For $g_1, g_2 \in G$,*

$$t(g_1g_2) = t(g_1)[t(g_2)]^{j(g_1)}\sigma(j(g_1), j(g_2)) \ . \tag{2.8}$$

Proof.

$$\begin{aligned} t(g_1g_2) &= t((i \circ t)(s \circ j)(g_1)(i \circ t)(s \circ j)(g_2)) \\ &= t(i(t(g_1))s(j(g_1))i(t(g_2))s(j(g_2))) \\ &= t(i(t(g_1))s(j(g_1))i(t(g_2))(s(j(g_1)))^{-1}s(j(g_1))s(j(g_2)) \\ &= t(i(t(g_1))s(j(g_1))i(t(g_2))(s(j(g_1)))^{-1}i(\sigma(j(g_1), j(g_2)))s(j(g_1))s(j(g_2)) \\ &= t(g_1)t(s(j(g_1))i(t(g_2))(s(j(g_1)))^{-1})\sigma(j(g_1), j(g_2)) \ , \\ &= t(g_1)[t(g_2)]^{j(g_1)}\sigma(j(g_1), j(g_2)) \ . \end{aligned}$$

Now suppose that another section s' of j is given, and another projection t' along s'. Then define

Definition 2.3.

$$\lambda(y) = t((s's^{-1})(y)) \ ,$$

where, for $f, g \colon X \to G$, $(fg)(x) = f(x)g(x)$, and $f^{-1}(x) = (f(x))^{-1}$, and use $f^{(-1)}$ for the functional inverse of f.

Definition 2.4. *We define*

$$\sigma'(y_1, y_2) = t'(s'(y_1)s'(y_2)) \quad and \quad (x^y)' = t'(s'(y)i(x)s'(y)^{-1}) \ .$$

Lemma 2.6. $\sigma(y^{-1}, y)^y = \sigma(y, y^{-1})$.

Proof.

Apply Lemma 2.3 for $y_1, y_2, y_3 = y, y^{-1}, y$.

Lemma 2.7.

$$t(g^{-1}) = \left[t(g)^{j(g^{-1})}\sigma(j(g^{-1}), j(g))\right]^{-1} \ . \tag{2.8}$$

Proof.

Use Lemma 2.5 applied to $g_1, g_2 = g^{-1}, g$, get

$$t(gg^{-1}) = t(g^{-1}) \left[t(g)^{j(g^{-1})}\sigma(j(g^{-1}), j(g))\right]$$

and solve for $t(g^{-1})$.

Corollary 2.1.

$$t \circ s^{-1} = \left[\sigma(j \circ s^{-1}, j \circ s)\right]^{-1} \ . \tag{2.9}$$

Proof.

Apply both sides to $y \in H$.

Lemma 2.8.

$$\lambda = t \circ s' \ . \tag{2.10}$$

Proof.

$$\begin{aligned}
\lambda(y) &= t(s'(y)s'(y)^{-1}) \\
&= t(s'(y)) \left[t(s(y)^{-1})\right]^{(j \circ s')(y)} \sigma((j \circ s')(y), (j \circ s)^{-1}(y)) \\
&= (t \circ s')(y) \left[\sigma(y^{-1}, y)^{-1}\right]^y \sigma(y, y^{-1}) \\
&= (t \circ s')(y) \left[\sigma(y^{-1}, y)^{-1}\right] \sigma(y^{-1}, y)^y \\
&= (t \circ s')(y) \ .
\end{aligned}$$

Lemma 2.9.

$$s' = (i \circ \lambda)s \tag{2.11}$$
$$t' = t(\lambda \circ j)^{-1} \text{ or } \lambda \circ j = (t')^{-1}t \tag{2.12}$$

Proof.

Apply i to both sides of $\lambda = t \circ s' s^{-1}$ for the first equation. For the second equation,

$$\begin{aligned} t &= t \circ (i \circ t')(s' \circ j) \\ &= (t \circ i \circ t')\,[t \circ s' \circ j]^{j \circ i \circ t'}\, \sigma(j \circ i \circ t', j \circ s' \circ j) \\ &= t'[\lambda \circ j]^e \sigma(e, j) = t'(\lambda \circ j) \end{aligned}$$

The next theorem indicates how the 2-cocycle and "action" transform upon changing from s to s'.

Theorem 2.1.

$$\sigma'(y_1, y_2) = \lambda(y_1)\lambda(y_2)^{y_1}\sigma(y_1, y_2)\lambda(y_1 y_2)^{-1} \tag{2.13}$$
$$(x^y)' = \lambda(y) x^y \lambda(y)^{-1} \tag{2.14}$$

Proof.

$$\begin{aligned} \sigma'(y_1, y_2) &= t'(s'(y_1)s'(y_2)) = t(\lambda \circ j)^{-1}(s'(y_1)s'(y_2)) \\ &= t(s'(y_1))\,[t(s'(y_2))]^{(j \circ s')(y_1)}\, \sigma((j \circ s')(y_1), (j \circ s')(y_2))(\lambda \circ j)^{-1}(s'(y_1)s'(y_2)) \\ &= \lambda(y_1)[\lambda(y_2)]^{y_1}\sigma(y_1, y_2)\lambda(y_1, y_2)^{-1} \\ i((x^y)') &= i(t'(s'(y)i(x)s'(y)^{-1})) \\ &= (i \circ \lambda)(y)s(y)i(x)s(y)^{-1}(i \circ \lambda)^{-1}(y) \\ &= (i \circ \lambda)(y)i(x^y)(i \circ \lambda)^{-1}(y) \\ &= i(\lambda(y)x^y\lambda(y)^{-1})\ , \end{aligned}$$

Definition 2.5. *If g is an extension as above, the N- and H-coordinates of $g \in G$ are $t(g), j(g)$ respectively. We usually write them as a column matrix* $\begin{pmatrix} t(g) \\ j(g) \end{pmatrix}$.

3. Homomorphism Matrices

Now suppose we have two group extensions G, G' of N, N' with quotients H, H', together with the attendant sections s, s' and projections t, t' along s, s', and a homomorphism $f: G \to G'$. Then we make

Definition 3.1. *Define the coordinates f_{ij} or matrix entries* $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ *of* $f\colon G_1 \to G_2$ *as follows:*

$$\begin{aligned} f_{11} &= t' \circ f \circ i\colon N \to N' \\ f_{21} &= j' \circ f \circ i\colon N \to H' \\ f_{12} &= t' \circ f \circ s\colon H \to N' \\ f_{22} &= j' \circ f \circ s\colon H \to H' \end{aligned}$$

Since the proofs of the next several theorems are minor modifications of the proofs of the corresponding theorems in [D1], we omit them.

Theorem 3.1. *The f_{ij}, $1 \le i,j \le 2$ satisfy the following functional equations:*

$$f_{11}(x_1x_2) = f_{11}(x_1)\,[f_{11}(x_2)]^{f_{21}(x_1)}\,\sigma'(f_{21}(x_1), f_{21}(x_2)) \tag{3.1}$$

$$f_{21}(x_1x_2) = f_{21}(x_1)f_{21}(x_2) \tag{3.2}$$

$$\begin{aligned} f_{12}(y_1y_2) = {} & f_{11}(\sigma(y_1,y_2)^{-1})\cdot \\ & \left[f_{12}(y_1)[f_{12}(y_2)]^{f_{22}(y_1)}\sigma'(f_{22}(y_1), f_{22}(y_2))\right]^{f_{21}(\sigma(y_1,y_2)^{-1})} \cdot \\ & \sigma'(f_{21}(\sigma(y_1,y_2)^{-1}), f_{22}(y_1)f_{22}(y_2)) \end{aligned} \tag{3.3}$$

$$f_{22}(y_1y_2) = f_{21}(\sigma(y_1,y_2)^{-1})f_{22}(y_1)f_{22}(y_2)\ , \tag{3.4}$$

where the primes have been omitted from $(x^y)'$ where the intention is unambiguous.

Theorem 3.2.

$$f = (i' \circ f_{11} \circ t)(s' \circ f_{21} \circ t)(i' \circ f_{12} \circ j)(s' \circ f_{22} \circ j)\ .$$

Theorem 3.3. *If we define f as in Theorem 3.3, where the f_{ij} satisfy the equations (3.1)–(3.4), then f is a homomorphism from G_1 to G_2.*

Write the matrix we have associated to f and the coordinates

$$C_1 = \begin{pmatrix} t(g_1) \\ j(g_1) \end{pmatrix}_{C_1}$$

$$C_2 = \begin{pmatrix} t'(G_2) \\ j(g_2) \end{pmatrix}_{C_2}$$

for g_1, g_2 in G_1, G_2 respectively as $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}_{C_1,C_2}$, and we may define a product of this matrix and the column of coordinates as

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}_{C_1,C_2} \begin{pmatrix} t(g_1) \\ j(g_1) \end{pmatrix}_{C_1} = \begin{pmatrix} [(f_{11} \circ t)(g_1)][(f_{12} \circ j)(g_1)] \\ [(f_{21} \circ t)(g_1)][(f_{22} \circ j)(g_1)] \end{pmatrix}_{C_2}$$

Theorem 3.4. *If $G_1 \xrightarrow{g} G_2 \xrightarrow{f} G_3$, where each G_i is a group extension, with accompanying sections s_i, projections t_i and cocycles σ_i, for $i = 1, 2, 3$, then if to (f, C_2, C_3) is associated the matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}_{C_2,C_3}$, and to (g, C_1, C_2) is associated $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}_{C_1,C_2}$, then to $(f \circ g, C_1, C_3)$ is associated the matrix*

$$\begin{pmatrix} (f_{11}\circ g_{11})\,[f_{12}\circ g_{21}]^{f_{21}\circ g_{11}}\sigma_3(f_{21}\circ g_{11}, f_{22}\circ g_{21}) & (f_{11}\circ g_{12})\,[f_{12}\circ g_{22}]^{f_{21}\circ g_{12}}\sigma_3(f_{21}\circ g_{12}, f_{22}\circ g_{22}) \\ (f_{21}\circ g_{11})(f_{22}\circ g_{21}) & (f_{21}\circ g_{12})(f_{22}\circ g_{22}) \end{pmatrix}_{C_1,C_3}$$

where the $x_3^{y_3}$ appearing for $x_3 \in N_3,\ y_3 \in H_3$ is actually $(x_3^{y_3})_3$.

Proof.

$$\begin{aligned}
(f\circ g)_{11} =& t_3\circ f\circ g\circ i = t_3\circ f\circ(i_2\circ t_2)(s_2\circ j_2)\circ g\circ i \\
=& t_3\circ(f\circ i_2\circ t_2\circ g\circ i)(f\circ s_2\circ j_2\circ g\circ i) \\
=& (t_3\circ f\circ i_2\circ t_2\circ g\circ i)\,[t_3\circ f\circ s_2\circ j_2\circ g\circ i]^{j_3\circ f\circ i_2\circ t_2\circ g\circ i}\,\sigma_3(j_3\circ f\circ i_2\circ t_2\circ g\circ i, j_2\circ f\circ s_2\circ j_2\circ g\circ i) \\
=& (f_{11}\circ g_{11})\,[f_{12}\circ g_{21}]^{f_{21}\circ g_{11}}\,\sigma_3(f_{21}\circ g_{11}, f_{22}\circ g_{21}) \\
(f\circ g)_{21} =& j_3\circ f\circ g\circ i = j_2\circ f\circ(i_2\circ t_2)(s_2\circ j_2)\circ g\circ i \\
=& (j_3\circ f\circ i_2\circ t_2\circ g\circ i)(j_3\circ f\circ s_2\circ j_2\circ g\circ i) \\
=& (f_{21}\circ g_{11})(f_{22}\circ g_{21}) \\
(f\circ g)_{12} =& t_3\circ f\circ g\circ s = t_3\circ f\circ(i_2\circ t_2)(s_2\circ j_2)\circ g\circ s \\
=& t_3\circ(f\circ i_2\circ t_2\circ g\circ s)(f\circ s_2\circ j_2\circ g\circ s) \\
=& (t_3\circ f\circ i_2\circ t_2\circ g\circ s)\,[t_3\circ f\circ s_2\circ j_2\circ g\circ s]^{j_3\circ f\circ i_2\circ t_2\circ g\circ s}\,\sigma_3(j_3\circ f\circ i_2\circ t_2\circ g\circ s, j_3\circ f\circ s_2\circ j_2\circ g\circ s) \\
=& (f_{11}\circ g_{12})\,[f_{12}\circ g_{22}]^{f_{21}\circ g_{12}}\,\sigma_3(f_{21}\circ g_{12}, f_{22}\circ g_{22}) \\
(f\circ g)_{22} =& j_3\circ f\circ g\circ s = j_3\circ f\circ(i_2\circ t_2)(s_2\circ j_2)\circ g\circ s \\
=& (j_3\circ f\circ i_2\circ t_2\circ g\circ s)(j_2\circ f\circ s_2\circ j_2\circ g\circ s) \\
=& (f_{21}\circ g_{12})(f_{22}\circ g_{22})
\end{aligned}$$

These formulas clearly reduce to the classical formulas in case the G_i are abelian groups which are direct sums $N_i \oplus H_i$. If we now let $G_i = G,\ N_i = N,\ H_i = H$ for $1 \le i \le 4$, choose bases $C_1 = C_4 = \begin{pmatrix} t'(g) \\ j(g) \end{pmatrix}_{C_1}$, $C_2 = C_3 = \begin{pmatrix} t(g) \\ j(g) \end{pmatrix}_{C_2}$, and the maps

$$G \xrightarrow{1_G} G \xrightarrow{f} G \xrightarrow{1_G} G\ ,$$

then we have

Corollary 3.1. *For the data described above, the matrix $\begin{pmatrix} f'_{11} & f'_{12} \\ f'_{21} & f'_{22} \end{pmatrix}_{C',C'}$ of f with respect to the coordinates $C' = \begin{pmatrix} t'(g) \\ j(g) \end{pmatrix}_{C'}$ written in terms of the matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}_{C,C}$ of f with respect to the coordinates $C = \begin{pmatrix} t(g) \\ j(g) \end{pmatrix}_C$ is*

$$\begin{pmatrix} f_{21}(\lambda\circ f_{21})^{-1} & (f_{11}\circ\lambda)\,[f_{12}]^{f_{21}\circ\lambda}\,\sigma(f_{21}\circ\lambda, f_{22})(\lambda^{-1}\circ(f_{21}\circ\lambda)f_{22}) \\ f_{21} & (f_{21}\circ\lambda)f_{22} \end{pmatrix}_{C',C'} \tag{3.6}$$

Proof.

In order to illustrate the techniques of calculation, we give two proofs. First, from theorem 2.5,

$$\begin{pmatrix} f'_{11} & f'_{12} \\ f'_{21} & f'_{22} \end{pmatrix}_{C',C'} = \begin{pmatrix} 1_N & \lambda^{-1} \\ e & 1_H \end{pmatrix}_{C,C'} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}_{C,C} \begin{pmatrix} 1_N & \lambda \\ e & 1_H \end{pmatrix}_{C',C} ,$$

where $\begin{pmatrix} 1_N & \lambda^{-1} \\ e & 1_H \end{pmatrix}_{C,C'}$ is the matrix of 1_G with respect to C, C'. To see this, write $t' = t(\lambda \circ j)^{-1}$. Then

$$\begin{pmatrix} 1_N & \lambda^{-1} \\ e & 1_H \end{pmatrix}_{C,C'} \begin{pmatrix} t(g) \\ j(g) \end{pmatrix}_{C} = \begin{pmatrix} t(g)(\lambda^{-1} \circ j)(g) \\ j(g) \end{pmatrix}_{C'} = \begin{pmatrix} t'(g) \\ j(g) \end{pmatrix}_{C'} .$$

First compute the product

$$\begin{pmatrix} 1_N & \lambda^{-1} \\ e & 1_H \end{pmatrix}_{C,C'} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}_{C,C} = \begin{pmatrix} f_{11}(\lambda^{-1} \circ f_{21}) & f_{12}(\lambda^{-1} \circ f_{22}) \\ f_{21} & f_{22} \end{pmatrix}_{C,C'} ,$$

and multiply this on the right by $\begin{pmatrix} 1_N & \lambda \\ e & 1_H \end{pmatrix}_{C',C}$, obtaining

$$\begin{pmatrix} f_{11}(\lambda^{-1} \circ f_{21})\sigma'(f_{21}, e) & [f_{11}(\lambda^{-1} \circ f_{21}) \circ \lambda] \left([f_{12}(\lambda^{-1} \circ f_{22})]^{f_{21} \circ \lambda} \right)' \sigma'(f_{21} \circ \lambda, f_{22}) \\ f_{21} & (f_{21} \circ \lambda) f_{22} \end{pmatrix}_{C',C'}$$

We must now express $(x^y)'$ in terms of x^y and σ' in terms of σ, via λ. Thus we have

$$\left([f_{12}(\lambda^{-1} \circ f_{21})]^{f_{21} \circ \lambda} \right)' = (\lambda \circ f_{21} \circ \lambda) \, [f_{12}(\lambda^{-1} \circ f_{21})]^{f_{21} \circ \lambda} \, (\lambda^{-1} \circ f_{21} \circ \lambda)$$

and

$$\sigma'(f_{21} \circ \lambda, f_{22}) = (\lambda \circ f_{21} \circ \lambda)(\lambda \circ f_{22})^{f_{21} \circ \lambda} \sigma(f_{21} \circ \lambda, f_{22})(\lambda^{-1} \circ (f_{21} \circ \lambda) f_{22}) .$$

Since f'_{12} is the only term involving $(x^y)', \sigma'$, we only have to compute

$$\begin{aligned} f'_{12} =& (f_{11} \circ \lambda)(\lambda^{-1} \circ f_{21} \circ \lambda)(\lambda \circ f_{21} \circ \lambda) \, [f_{12}]^{f_{21} \circ \lambda} \left[\lambda^{-1} \circ f_{22}\right]^{f_{21} \circ \lambda} (\lambda^{-1} \circ f_{21} \circ \lambda) \cdot \\ & (\lambda \circ f_{21} \circ \lambda) \, [\lambda \circ f_{22}]^{f_{21} \circ \lambda} \sigma(f_{21} \circ \lambda, f_{22})(\lambda^{-1} \circ (f_{21} \circ \lambda) f_{22}) \\ =& (f_{11} \circ \lambda) \, [f_{12}]^{f_{21} \circ \lambda} \sigma(f_{21} \circ \lambda, f_{22})(\lambda^{-1} \circ (f_{21} \circ \lambda) f_{22}) . \end{aligned}$$

For the second proof, we may calculate the f'_{ij} directly as follows:

$$\begin{aligned} f'_{11} =& t' \circ f \circ i = t(\lambda \circ j)^{-1} \circ f \circ i \\ =& (t \circ f \circ i)(\lambda \circ j \circ f \circ i)^{-1} = f_{21}(\lambda \circ f_{21})^{-1} \\ f'_{21} =& j \circ f \circ i = f_{21} \\ f'_{12} =& t' \circ f \circ s = t(\lambda \circ j)^{-1} \circ f \circ (i \circ \lambda) s \\ =& (t \circ f \circ (i \circ \lambda) s)(\lambda \circ j \circ f \circ (i \circ \lambda) s)^{-1} \\ =& [t \circ (f \circ i \circ \lambda)(f \circ s)] \, [\lambda \circ (j \circ f \circ i \circ \lambda)(j \circ f \circ s)]^{-1} \\ =& (t \circ f \circ i \circ \lambda) \, [t \circ f \circ s]^{j \circ f \circ i \circ \lambda} \sigma(j \circ f \circ i \circ \lambda, j \circ f \circ s) \, (\lambda \circ (f_{21} \circ \lambda) f_{22})^{-1} \\ =& (f_{11} \circ \lambda) \, [f_{12}]^{f_{21} \circ \lambda} \sigma(f_{21} \circ \lambda, f_{22}) \, (\lambda \circ (f_{21} \circ \lambda) f_{22})^{-1} \\ f'_{22} =& j \circ f \circ s' = j \circ f \circ (i \circ \lambda) s \\ =& (j \circ f \circ i \circ \lambda)(j \circ f \circ s) = (f_{21} \circ \lambda) f_{22} \end{aligned}$$

If we are used to dealing with the special case of vector spaces, we might be inclined to interpret $\begin{pmatrix} 1_N & e \\ \lambda^{-1} & 1_H \end{pmatrix}$ as an automorphism of G. But this will not work, because λ^{-1} does not satisfy the right functional equation. Equation (3.3) reduces to

$$f_{12}(y_1y_2) = \sigma^{-1}(y_1,y_2)f_{12}(y_1)\left[f_{12}(y_2)\right]^{f_{22}(y_1)}\sigma(y_1,y_2) \tag{3.7}$$

and the equation satisfied by λ^{-1} is

$$\lambda^{-1}(y_1y_2) = \sigma^{-1}(y_1,y_2)\left[\lambda^{-1}(y_2)\right]^{y_1}\lambda^{-1}(y_1)\sigma'(y_1,y_2)\ . \tag{3.8}$$

It seems as if the choosing of a different section s' should somehow generate an automorphism of G, but this does not seem to be the case. On the other hand, if $\lambda\colon H \to N$ is a function satisfying

$$\lambda(y_1,y_2) = \sigma^{-1}(y_1,y_2)\lambda(y_1)\left[\lambda(y_2)\right]^{y_1}\sigma(y_1,y_2)\ ,$$

which we might call a *self-equivalence* of σ, then the matrix $\begin{pmatrix} 1_N & e \\ \lambda & 1_H \end{pmatrix}_{C,C}$ *will* give an automorphism of G with inverse $\begin{pmatrix} 1_N & e \\ \lambda^{-1} & 1_H \end{pmatrix}_{C,C}$. The set of self equivalences of any 2-cocycle σ' defining the same extension form a group isomorphic to the group of self-equivalences of σ, and this is effected by a λ joining σ to σ'. Thus all the λ's form a groupoid under pointwise product, acting on the set of 2-cocycles associated to the extension.

4. The Endomorphism Monoid

Now we come to a curious phenomenon. As is well known, the inner automorphisms of a group form a normal subgroup of the group of all its automorphisms. Since N is normal, and is fixed setwise by all inner automorphisms of G, we see that the matrices of each of the inner automorphisms are all upper triangular, i.e.,

$$\begin{pmatrix} f_{11} & f_{12} \\ e & f_{22} \end{pmatrix}\ .$$

with respect to any basis, with the multiplication

$$\begin{pmatrix} f_{11} & f_{12} \\ e & f_{22} \end{pmatrix}\begin{pmatrix} g_{11} & f_{12} \\ e & g_{22} \end{pmatrix} = \begin{pmatrix} f_{11}\circ g_{11} & (f_{11}\circ g_{12})(f_{12}\circ g_{22}) \\ e & f_{22}\circ g_{22} \end{pmatrix}\ .$$

Now in the case of G a direct sum of vector spaces, there are no inner automorphisms, so the upper triangular matrices do not represent inner automorphisms. Of course, since we have not mentioned topology, we see that the linear maps of finite-dimensional vector spaces over $\mathbb{R}$ are just the continuous homomorphisms of the associated topological group.

Note also

Theorem 3.5. *A matrix* $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ *associated to an endomorphism* f *of* G *represents an automorphism iff there is an endomorphism* g *with matrix* $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ *satisfying*

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1_N & e \\ e & 1_H \end{pmatrix} .$$

Proof.

Clear.

In equational terms this says that we must find four functions g_{ij} satisfying (3.1)−−(3.4), which, in addition satisfy the following set of equations

$$(3.10) \qquad (f_{11}\circ g_{11})\,[f_{12}\circ g_{21}]^{f_{21}\circ g_{12}}\,\sigma(f_{21}\circ g_{11}, f_{22}\circ g_{21}) = 1_N$$

$$(3.11) \qquad (f_{11}\circ g_{12})\,[f_{12}\circ g_{22}]^{f_{21}\circ g_{12}} = e$$

$$(3.12) \qquad (f_{21}\circ g_{11})(f_{22}\circ g_{21}) = e$$

$$(3.13) \qquad (f_{21}\circ g_{12})(f_{22}\circ g_{22}) = 1_H .$$

It is not likely that the g_{ij} can be solved for explicitly in terms of the f_{ij} in general. In particular cases it *is* possible to solve for them, and the matrix representation is quite a convenient form in which to multiply automorphisms. In the case of the direct sum of abelian groups, it is possible to use the newly-created theory of noncommutative determinants in [GR] to solve them.

4. Conclusion

It is possible to extend all the results of this paper both to locally compact polish groups and to lie and associative algebras. In [D1], the endomorphism monoid of the real Heisenberg group is determined. Its automorphism group was determined in [W] without using the matrices associated to their elements.

Theoretically, if there is a normal tower $N_1 \lhd N_2 \lhd \cdots \lhd N_k \lhd G$ of subgroups of a group G, it is possible to associate $(k+1)\times(k+1)$ matrices of functions with an endomorphism of G, in the manner done here, using iterated extensions, cocycles, projections, and sections. In the two by two case, we obtain a generalization of the theory of noncommutative determinants of [GR], which is the subject of a forthcoming paper. It will also be the object of future research to extend the techniques introduced here and apply them to the case of local fields and adeles.

After this paper had been written, I became aware of the paper of Hsu [H], in which similar techniques are introduced for the case of a semidirect product of groups. In that paper, the automorphism group of a holomorph of a perfect group is computed.

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