ADVANCES IN MATHEMATICS 106, 96–117 (1994)

Combinatorial Extension Cohomology I. Groups

A. M. DuPré

Rutgers University, Newark, New Jersey

A new method is proposed for calculating the measurable, continuous, or differentiable cohomology of a group extension, which involves deriving functional equations for the restrictions of cocycles to certain well-behaved subsets of its domain and showing that the cocycle can be written as a certain sum of such restrictions. This technique is capable of determining how the quotients of the filtration given by the spectral sequence fit together, and is applied to the case of the Heisenberg group H_n to yield extremely explicit cocycle representatives, culminating in a stability theorem regarding multilinearizability of cocycles. One of the main tools for doing this is the derivation of a formula for trivializing the product of two alternating multilinear functions, one of which is the nondegenerate bilinear one defining the Heisenberg group, which has interesting connections with Hodge theory.

I. Introduction

This is the first in a series of three projected papers. The second will treat Lie algebras and the third associative algebras.

Whenever the subject of cohomology of group extensions arises, it almost automatically invokes the concept of a spectral sequence in response, and the two are uniformly conceived of as inseparable. This was not always so. Just as Eilenberg and MacLane were entering into a longlasting and fruitful collaboration on a series of papers, and before even the first of this series saw print, an algorithmically inclined Ph.D. student of MacLane's at Harvard, Roger Lyndon, was working on a dissertation which he completed on May 18, 1946 [Ln1].

The subject of Lyndon's thesis was the problem of computing the cohomology groups $H^n(G, A)$, when G was given as an extension

$$1 \rightarrow N \xrightarrow{\ i\ } G \underset{s}{\overset{j}{\rightleftarrows}} H \rightarrow 1.$$

Lyndon was only able to completely solve this problem for G the direct product of the finitely generated abelian groups H and N operating trivially on the coefficient group $\mathbf{Z}$ of integers. This part of the thesis was published

0001-8708/94 $9.00

in [Ln2], but the most important part of his thesis was not, and furthermore, all indications are that it was never assimilated by anyone in the mathematical community. It fact, it evidences at least a misunderstanding of the purpose and thrust of his thesis to append his name to the Hochschild–Serre spectral sequence, even though a particular application of the much stronger methods of Lyndon yields the main result of spectral sequences.

After Leray in [Le1, Le2] published his theory of spectral sequences for approximating the computation of the cohomology of a fiber space in terms of the cohomology of the base and fiber, Hochschild and Serre saw that it could similarly be applied to computing an approximation to the cohomology of a group extension and published their results in [HS]. Now, although the titles of [Ln1] and [HS] are almost identical, and although some of the results are quite close, the two approaches are radically different. The difference in the two approaches is this: the spectral sequence has as its aim the computation of a set of successive quotient groups associated to a filtration of $H^n(G, A)$, which turn out to be subquotients of the terms E_2^{ij} of the spectral sequence, and is not capable of approaching the problem of how these quotients fit together to form $H^n(G, A)$. On the other hand, Lyndon had begun to develop an apparatus for completely computing this group. There were a few more loose ends to tie up, and for $H^2(G, A)$ this was done by Mackey in [Ma1]. He used an approach involving the solution of a set of functional equations, which originally stimulated my interest in the problem as a graduate student. At that time, I extended Mackey's approach to $H^2(G, A)$ for general extensions and then to $H^3(G, A)$, but never published the results because it was not clear to me that the approach had much interest or merit, due to the ostensible intransigency of the functional equations involved.

Returning to this problem a few years ago, I found that I was able to solve Mackey's functional equations and was then able to untangle the procedure for the higher groups. Immediately after doing this, I read Lyndon's original unpublished thesis [Ln1], a copy of which was kindly sent to me by MacLane, and then I discovered that my approach had been partially anticipated by Lyndon some 46 years previously. There is a significant difference in our approaches and I feel that mine is easier to understand. Also, I supply a previously missing link in order to carry through Lyndon's approach to its logical conclusion: the explicit construction of a set of functional equations for low dimensions, the proof of the effective calculability of such a construction for any dimension, and finally, the solving of this set of equations for the cohomology groups $H^m(H_n, \mathbb{R})$ of the Heisenberg groups H_n.

Kleppner in [Kl] showed that, for almost all locally compact abelian groups, every measurable 2-cocycle with values in $\mathbf{T}$, the circle group, was cohomologous to an alternating bilinear function. In [Du1] I showed that, for all locally compact abelian groups, every measurable n-cocycle with values in the reals $\mathbb{R}$ was cohomologous to an alternating multilinear function. Four years ago, Moskowitz in [Mos] claimed to have shown that for a certain class of nonabelian groups which included the $(2n+1)$-dimensional Heisenberg group H_n every continuous 2-cocycle on H_n with values in $\mathbb{R}$ or $\mathbf{T}$ was cohomologous to an alternating bilinear function. There was an error in this paper, so I set about discovering just what the true story was, [Du2] being the result of this.

This paper tells an interesting story with regard to the cohomology of the Heisenberg group H_n. For fixed m there is what I call the cohomological multilinearizability stability theorem, which says that $H^m(H_n, \mathbb{R})$, the measurable cohomology, is computable with alternating multilinear functions as soon as $n \geqslant m$ and otherwise is computable with polynomial cocycles which are sums of terms of degree two in one variable and linear in all the others, and these suffice. I should also be able to eventually describe explicitly what sort of polynomial cocycles suffice to compute the cohomology for a general n step nilpotent group, either continuous or discrete.

The computations in this paper may be conceived of as effecting the construction of a particularly apposite and efficient resolution for computing the cohomology of a group extension.

II. The Lyndon Resolution

We have an extension of groups

$$1 \to N \xrightarrow{\ i\ } G \underset{s}{\overset{j}{\rightleftarrows}} H \to 1,$$

where s is a cross section for j and $\alpha\colon H \times H \to N$ is the 2-cocycle associated to s. We also suppose that H operates on N via s as $\beta(y)(s) = s(y)\,x(s(y))^{-1} = x^{s(y)}$.

We also have an action of $\varphi\colon G \to \mathrm{Aut}(A)$ of G on the abelian group A. Following [McL], we define nonhomogeneous n-cochains $C^n(G, A)$, coboundary operators $\delta_n\colon C^n \to C^{n+1}$, cocycles Z^n, coboundaries B^n and cohomology groups $H^n = Z^n/B^n$. We also assume that our cochains are normalized, i.e., $\sigma(g_1, ..., g_n) = 0$ if $g_i = e$ for some $1 \leqslant i \leqslant n$.

A starting point for defining the spectral sequences associated to a group extension is to first define the double complex $C^{i,j}(G, A)$ as

$$C^{i,j}(G, A) = \{f\colon H^i \times N^j \to A\}.$$

The most important theorem in Lyndon's thesis is that $H^n(G, A)$ may be computed using cocycles which are determined by their values on the sets $(s(H))^i \times N^j$, $i+j=n$. The exact nature of this dependence can be proved to be computable and is computed in this paper for $n=4$.

THEOREM 2.1. *If $\sigma \in C^n(G, A)$ then σ is cohomologous to a cochain $\bar{\sigma}$ which vanishes on all sets of the form $X_1 \times \cdots \times X_n$, where each X_i is either $s(H)$ or N and $X_1 \times \cdots \times X_n$ is not of the form $s(H)^i \times N^j$.*

Proof. If $\iota \in \{1, 2\}^n$, let $\sigma_\iota = \sigma \circ (\varepsilon_1 \times \cdots \times \varepsilon_n)$, where $\varepsilon_j = i$ or s according as $\iota(j) = 1$ or 2. Further, let ι have the form stated in the theorem. Then define

$$\gamma_\iota(x_1 s(y_1), ..., x_{n-1} s(y_{n-1})) = \sigma(g_1, ..., g_n),$$

where $g_j = x_{p_j}$ or $s(y_{q_j})$ for some $1 \leqslant p_j,\ q_j \leqslant n$, and such that if $i<j$ then $p_i < q_j$, and finally so that $g_n =$ either x_{n-1} or $s(y_{n-1})$, and also that every $1 \leqslant j \leqslant n-1$ is represented.

LEMMA 2.1. *There is exactly one $1 \leqslant k \leqslant n$ so that $g_k = x_{p_k}$ and $g_{k+1} = s(y_{q_{k+1}})$ and $p_k = q_{k+1}$*

Proof. Equivalent to having the form stated above is that some 1 must be followed by a 2. The pigeonhole principle gives the remainder of the lemma, if we think of how we define γ_i. Consider having to take n steps starting from a sequence of stepping stones alternately labelled 1 and 2, starting with 1, and suppose that there are $2(n-1)$ stepping stones altogether. A further restriction is given by the sequence ι which is of length n. It is clear that we must step on precisely one pair of a 1 followed by a 2, since we may not skip entirely any pair 1, 2.

LEMMA 2.2. *Only one term in $(\delta\gamma_\iota)(g_1, ..., g_n)$ is nonzero when we write out each term in terms of σ, and this one nonzero term is $(-1)^k \sigma_\iota$, where k is as in Lemma 2.1.*

Proof. It is clear that all terms other than that referred to have the requisite x_i or $s(y_i)$, and hence will have an e in the place of σ in terms of which it is expressed.

LEMMA 2.3. *If we order the ι lexically, where $1<2$, then by defining, in succession,*

$$\sigma' = \sigma + (-1)^{k_{\iota_1}} \delta(\gamma_{\iota_1}),$$

$$\sigma'' = \sigma' + (-1)^{k_{\iota_2}} \delta(\gamma_{\iota_2}), ..., \sigma^{(2^n-n-1)} = \sigma^{(2^n-n-2)} + (-1)^{k_{\iota_{2^n-n-1}}} \delta(\gamma_{\iota_{2^n-n-1}}),$$

we have $\delta(\gamma_{\iota_i})(g_1, ..., g_n) = 0$ if $(g_1, ..., g_n)$ is a sequence associated to a ι_j, where $j < i$, and the association is that determined in the beginning of the proof of Theorem 2.1.

Proof. Every term in $\delta(\gamma_{\iota_i})(g_1, ..., g_n)$ will have an e in it when expressed in terms of σ if the $(g_1, ..., g_n)$ is associated to a ι_j with $j < i$. The theorem is proved by chaining together the string of cohomologies to obtain $\sigma^{(2^n - n - 1)}$, which satisfies the conditions in the theorem for $\bar{\sigma}$.

THEOREM 2.2. *$\sigma \in C^n$ is cohomologous to a $\bar{\sigma}$ for which $\bar{\sigma}(g_1, ..., g_n) = 0$ whenever the sequence $(g_1, ..., g_n)$ ends in a sequence $(g_k, ..., g_n)$ where $k \leqslant n-2$ and $g_j =$ either an x_j or an $s(y_j)$ for $k \leqslant j \leqslant n$, and there is at least one x_j in this sequence which is followed by an $s(y_k)$.*

Proof. If $\iota \in \{1, 2, x\}$ write $\sigma_\iota = \sigma \circ (\varepsilon_1 \times \cdots \times \varepsilon_n)$, where $\varepsilon_j = i,\ s,\ 1_G$ according to whether $\iota(j) = 1, 2, x$. Then order the ι lexically, assuming $1 < 2 < x$. Define

$$\gamma_\iota(x_1 s(y_1), ..., x_{n-1} s(y_{n-1})) = \sigma(g_1, ..., g_n),$$

where $g_j = x_j,\ s(y_j)$ or $x_j s(y_j)$ according to whether $\iota(j) = 1, 2$, or x. We are concerned here with those ι which start off with a string of x's, and are then followed by a string of 1's and 2's in which at least one 1 is followed by a 2. Reasoning similarly to the proofs of lemmas 2.1, 2.2, and 2.3, we arrive at a $\bar{\sigma}$ satisfying the theorem.

DEFINITION. A cochain $\sigma \in C^n$ is called *Lyndon* if it satisfies the conditions of Theorem 2.2 for $\bar{\sigma}$.

THEOREM 2.3. *If $\sigma \in C^n$ is a Lyndon cocycle then it can be written as a sum of n-cocycles τ_ι, where $\iota \in \{1, 2\}^n$ and ι consists of a string of 2's followed by a string of 1's for some cochains τ.*

Proof. The proof is preceded by several lemmas.

We introduce the following string rewriting system: for $\iota \in \{1, 2, x\}^n$, if there is no x in ι, we do not rewrite it; otherwise replace the rightmost x by $1 \mid 2$ and use the n-cocycle identity to rewrite ι as follows:

$$\begin{aligned}\iota \to & \sum_{j=1}^{k-2} (-1)^{j+k+1} (\alpha_1, ..., \alpha_j\alpha_{j+1}, ..., \alpha_{k-1}, 1, 2, \alpha_{k+1}, ..., \alpha_n) \\ & + (\alpha_1, ..., \alpha_{k-1}, 1, 2\alpha_{k+1}, \alpha_{k+2}, ..., \alpha_n) \\ & + \sum_{j=k+2}^{n} (-1)^{j+k+1} (\alpha_1, ..., \alpha_{k-1}, 1, 2, \alpha_{k+1}, ..., \alpha_j\alpha_{j+1}, ..., \alpha_n).\end{aligned}$$

Next replace $\alpha_j\alpha_{j+1}$, $\alpha_{k-1}1$, $2\alpha_{k+1}$ according to the rule: $x1$, $1x$, $2x$, $x2$, 22, 12, 21 all go to x and 11 goes to 1. Finally, because of Theorem 2.2, we assume that the part of the string ι past the rightmost x consists of a string of the form $2^a 1^b$, and if any term in the sum above has its tail end after its rightmost x not of the form $2^a 1^b$, we simply delete the term. The following properties of this rewrite system are clear.

(1) In any rewrite $\iota \to \sum \beta$, no term β has more x's than ι.

(2) In any rewrite $\iota \to \sum \beta$, if β has the same number of x's as ι, then if an x in ι moves to the left past a 2, the 2 does not change. If an x moves to the right past a string $2^a 1$, the string $2^a 1$ becomes 12^a.

(3) No x of ι can move to the left past a 1.

LEMMA 2.4. *Using the rewrite rules as stated above, every ι can be rewritten as a sum of terms with no x's.*

Proof. Because of the way that the x's move past strings involving 2's, no cycling is possible, and so the rewrite system eventually eliminates all the x's by rightmost derivation.

Returning to the proof of the theorem, we have only to apply our rewrite system to the string $(x, ..., x)$, which represents the n-cocycle $\sigma(x_1 s(y_1), ..., x_n s(y_n))$.

THEOREM 2.4. *If an n-cocycle σ is determined, then it vanishes on any set of the form $G^k \times X^l \times G^m$, where the last X is N, $l \geq 3$, and in X^{l-1}, some $X = N$ is followed by an $X = s(H)$.*

Proof. This follows by a simple application of the rewrite rules.

As an example of the reduction procedure described above, we show how this is actually carried out for $n = 5$ by reducing (x, x, x, x, x).

$$(x, x, x, x, 1 \mid 2) \to (x, x, x, x, 1) + (x, x, x, x, 2)$$

$$(x, x, x, 1 \mid 2, 1) \to (x, x, x, 1, x) + (x, x, x, 2, 1)$$
$$(x, x, x, 1 \mid 2, 2) \to (x, x, x, 1, x) + (x, x, x, 2, 2)$$

$$(x, x, x, 1, 1 \mid 2) \to (x, x, x, 1, 1)$$
$$(x, x, 1 \mid 2, 2, 1) \to (x, x, 2, 2, 1) + (x, x, 1, x, 1) - (x, x, 1, 2, x)$$
$$(x, x, 1 \mid 2, 2, 2) \to (x, x, 2, 2, 2) + (x, x, 1, x, 2) - (x, x, 1, 2, x)$$

$$(x, x, 1 \mid 2, 1, 1) \to (x, x, 2, 1, 1) + (x, x, 1, x, 1)$$
$$(x, 1 \mid 2, 2, 2, 1) \to (x, 2, 2, 2, 1) + (x, 1, x, 2, 1) - (x, 1, 2, x, 1)$$
$$(x, x, 1, 1 \mid 2, 1) \to (x, x, 1, 1, x)$$
$$(x, x, 1, 2, 1 \mid 2) \to (x, x, 1, x, 1)$$

$$(x, 1 \mid 2, 2, 2, 2) \to (x, 2, 2, 2, 2) + (x, 1, x, 2, 2)$$
$$(x, x, 1, 1 \mid 2, 2) \to (x, x, 1, 1, x)$$
$$(x, x, 1, 2, 1 \mid 2) \to (x, x, 1, x, 2)$$

$$(x, 1 \mid 2, 2, 1, 1) \to (x, 2, 2, 1, 1) + (x, 1, x, 1, 1) - (x, 1, 2, x, 1)$$
$$(1 \mid 2, 2, 2, 2, 1) \to (2, 2, 2, 2, 1) + (1, x, 2, 2, 1) - (1, 2, x, 2, 1)$$
$$(x, 1, 1 \mid 2, 2, 1) \to (x, 1, 1, x, 1)$$
$$(x, 1, 2, 1 \mid 2, 1) \to (x, 1, x, 2, 1)$$
$$(x, x, 1, 1, 1 \mid 2) \to (x, x, 1, 1, 1)$$
$$(1 \mid 2, 2, 2, 2, 2) \to (2, 2, 2, 2, 2) + (1, x, 2, 2, 2) - (1, 2, x, 2, 2)$$
$$(x, 1, 1 \mid 2, 2, 2) \to (x, 1, 1, x, 2)$$

$$(1 \mid 2, 2, 2, 1, 1) \to (2, 2, 2, 1, 1) + (1, x, 2, 1, 1) - (1, 2, x, 1, 1)$$
$$(x, 1, 1 \mid 2, 1, 1) \to (x, 1, 1, x, 1)$$
$$(1, 1 \mid 2, 2, 2, 1) \to (1, 1, x, 2, 1)$$
$$(1, 2, 1 \mid 2, 2, 1) \to (1, x, 2, 2, 1)$$
$$(x, 1 \mid 2, 1, 1, 1) \to (x, 2, 1, 1, 1) + (x, 1, x, 1, 1)$$
$$(1, 1 \mid 2, 2, 2, 2) \to (1, 1, x, 2, 2)$$
$$(1, 2, 1 \mid 2, 2, 2) \to (1, x, 2, 2, 2)$$
$$(x, 1, 1, 1 \mid 2, 2) \to (x, 1, 1, 1, x)$$

$$(1, 1 \mid 2, 2, 1, 1) \to (1, 1, x, 1, 1)$$
$$(1, 2, 1 \mid 2, 1, 1) \to (1, x, 2, 1, 1)$$
$$(x, 1, 1, 1 \mid 2, 1) \to (x, 1, 1, 1, x)$$
$$(1, 1, 1 \mid 2, 2, 1) \to (1, 1, 1, x, 1)$$
$$(1 \mid 2, 2, 1, 1, 1) \to (2, 2, 1, 1, 1) + (1, x, 1, 1, 1) - (1, 2, x, 1, 1)$$
$$(1, 1, 1 \mid 2, 2, 2) \to (1, 1, 1, x, 2)$$
$$(x, 1, 1, 1, 1 \mid 2) \to (x, 1, 1, 1, 1)$$

$$(1, 1, 1 \mid 2, 1, 1) \to (1, 1, 1, x, 1)$$
$$(1, 1, 1, 1 \mid 2, 1) \to (1, 1, 1, 1, x)$$

$(1, 1 \mid 2, 1, 1, 1) \to (1, 1, 1 \mid 2, 1, 1)$

$(1, 1, 1, 1 \mid 2, 2) \to (1, 1, 1, 1, x)$

$(1 \mid 2, 1, 1, 1, 1) \to (1, x, 1, 1, 1) - (1, 2, x, 1, 1)$

$(1, 1, 1, 1, 1 \mid 2) \to (1, 1, 1, 1, 1)$

$(1, 1 \mid 2, 1, 1, 1) \to (1, 1, x, 1, 1)$

THEOREM 2.5. *Each of the τ_ι's in Theorem 2.3 appears in at least two functional equations except τ_{2^n} and τ_{1^n}.*

Proof. Suppose $\iota = 2^a 1^b$, where neither a nor b are zero. Then apply the reduction process to both sides of each of the following two equations:

$$(2, 2, ..., 2, 2 \mid 2, 1, ..., 1) = (2, ..., 2, 1 \mid 2, 1, ..., 1)$$
$$(2, 2, ..., 2, 1 \mid 1, ..., 1) = \sum \beta_\iota,$$

where there are $(a-1)$ 2's on the left side of the first equation before $2 \mid 2$ and b 1's after it, and the same on the right-hand side with respect to $1 \mid 2$. In the second equation, there are a 2's to the left of $1 \mid 1$, and $(a-1)$ 1's to its right, and the right-hand side is merely the expansion of the left according to the cocycle identity.

THEOREM 2.6. *σ_{1^n} is just an n-cocycle by restriction, and σ_{2^n} satisfies a functional equation obtained by reducing both sides of $(2, ..., 2, 2 \mid 2) = \sum \beta_\iota$, where the right-hand side is obtained by applying the cocycle identity.*

Proof. Calculational.

THEOREM 2.7. *If we have a collection of τ_ι's satisfying the functional equations according to Theorems 2.4 and 2.5, then if we add them according to Theorem 2.3, we obtain an n-cocycle on G, and every n-cocycle on G is obtained in this way.*

Proof. All the above rewrite rules are reversible.

DEFINITION. If σ satisfies the conclusions of Theorem 2.3, then σ is called *determined.*

THEOREM 2.8. *If σ and $\delta\sigma$ are both Lyndon, then σ is determined.*

Proof. Instead of the usual initial application of the rewrite rule using the cocyle identity, we use a modified version where we add in a string corresponding to $\delta\sigma$, which, being a Lyndon cocycle, is determined.

THEOREM 2.9. *If a determined n-cocycle σ is a coboundary, it is the coboundary of a determined $(n-1)$-cochain.*

Proof. Suppose $\sigma = \delta b$. Then b is cohomologous to a Lyndon $(n-1)$-cochain $\bar{b}$. But then we may apply the previous theorem to $\bar{b}$ and $\delta\bar{b} = \sigma$, and obtain that $\bar{b}$ is a determined $(n-1)$-cochain.

Thus the problem of computing $H^n(G, A)$ has been significantly reduced to computing with Lyndon cochains, and we make the

DEFINITION. The resolution of A using determined cochains is called the *Lyndon resolution of A corresponding to the given group extension.*

We thus see that we may compute $H^n(G, A)$ with cochains which are determined by their restrictions to sets of the form $(s(H))^i \times N^j$, which does not follow from the theory of spectral sequences. Furthermore, these restrictions satisfy certain functional equations which may be solved in simple cases and we build up a knowledge of how to solve these equations in successively more complicated situations. Let us now investigate how these functional equations fashion themselves in the lowest dimensions.

II. FUNCTIONAL EQUATIONS

If we want to describe $H^1(G, A)$, utilizing the fact that G is an extension as earlier, it is easy enough to see that 1-cocycles, i.e., crossed homomorphisms from G to A, are determined, since $\sigma(xs(y)) = x(\sigma(s(y))) + \sigma(x)$, $\sigma_1 = \sigma \circ i$ is just a crossed homomorphism from N to A, and σ_2 satisfies the functional equation

$$\alpha(y_1, y_2)(\sigma_2(y_1 y_2)) + \sigma_1(\alpha(y_1, y_2)) = \sigma_2(y_2)^{s(y_1)} + \sigma_2(y_1).$$

If we assume that N operates trivially on A, this just says that $\sigma_1(\alpha(y_1, y_2))$ is a trivial 2-cocycle on H and is the coboundary of the 1-cochain σ_2. From spectral sequence theory, we have an exact sequence $1 \to E_2^{1,0} \to H^1(G, A) \to E_2^{0,1} \to E_2^{2,0}$, and if we write $E_2^{1,0}$ as $H^1(H, A^N)$ we see the reason for assuming trivial action of N on A. Also, writing $E_2^{0,1}$ as $H^1(N, A)^H$, we see that the only members of $H^1(N, A)^H$ in which we are interested are those which yield trivial 2-cocycles in $E_2^{2,0} = H^2(H, A^N)$. A function σ_2 as above actually gives us a cross section for the map $H^1(G, A) \to (H^1(N, A)^H)_0$, where the right-hand side is the image of $H^1(G, A) \to H^1(N, A)^H$.

Moving along to $H^2(G, A)$, we remark that first Mackey in [Ma1] and then Tahara in [Ta] had each derived similar sets of functional equations for the case of a semidirect product. Mackey solved these equations for the case $N = H = \mathbb{R}$ and Tahara for $H = \mathbf{Z}_m$, $N = \mathbf{Z}_n$, two finite cyclic groups. In [Du2] I derived the functional equations for H^2 for the case of a general group extension, solving them in the case of the Heisenberg group H_n. In [Du3] I derived the equations for H^3 and again solved them in the case of the Heisenberg groups H_n.

THEOREM 3.1. *The functional equations for H^4 are as listed below. We assume for convenience that G operates trivially on A. We write $j^{\bar{i}} = s(y_i)\,x_j(s(y_i))^{-1}$ and $i^{\bar{j}\cdot\bar{k}} = (x_i^{y_j})^{y_k}$. Also, $\bar{i}\cdot\bar{j} = \alpha(y_i, y_j)$, $((1\bar{1}), ..., (n\bar{n})) = \sigma(x_1 s(y_1), ..., x_n s(y_n))$, and $(\ )_\iota = \sigma_\iota(\ )$.*

(1) $$(2, 3, 4, 5)_{1^4} - (12, 3, 4, 5)_{1^4} + (1, 23, 4, 5)_{1^4} - (1, 2, 34, 5)_{1^4} + (1, 2, 3, 45)_{1^4} - (1, 2, 3, 4)_{1^4} = 0$$

(2.a) $$\begin{aligned}&(\bar{2}, 1, 2, 3)_{21^3} - (\bar{1}\bar{2}, 1, 2, 3)_{21^3} + (\bar{1}, 1^{\bar{2}}, 2^{\bar{2}}, 3^{\bar{2}})_{21^3}\\ &\quad + (\bar{1}, \bar{2}, 2, 3)_{2^21^2} - (\bar{1}, \bar{2}, 12, 3)_{2^21^2}\\ &\quad + (\bar{1}, \bar{2}, 1, 23)_{2^21^2} - (\bar{1}, \bar{2}, 1, 2)_{2^21^2}\\ &= (\bar{1}, \bar{2}, 1^{\bar{1}\bar{2}}, 2^{\bar{1}\bar{2}}, 3^{\bar{1}\bar{2}})_{1^4} - (1^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, \bar{2}^{\bar{1}\bar{2}}, 3^{\bar{1}\bar{2}})_{1^4}\\ &\quad + (1^{\bar{1}\cdot\bar{2}}, 2^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{3}, 3^{\bar{1}\bar{2}})_{1^4} - (1^{\bar{1}\cdot\bar{2}}, 2^{\bar{1}\cdot\bar{2}}, 3^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2})_{1^4}\end{aligned}$$

(2.b) $$\begin{aligned}&(\bar{1}, 2, 3, 4)_{21^3} - (\bar{1}, 12, 3, 4)_{21^3} + (\bar{1}, 1, 23, 4)_{21^3}\\ &\quad - (\bar{1}, 1, 2, 34)_{21}{}^3 + (\bar{1}, 1, 2, 3)_{21^3}\\ &= (1^{\bar{1}}, 2^{\bar{1}}, 3^{\bar{1}}, 4^{\bar{1}})_{1^4} - (1, 2, 3, 4)_{1^4}\end{aligned}$$

(3.a) $$\begin{aligned}&(\bar{2}, \bar{3}, 1, 2)_{2^21^2} - (\bar{1}\bar{2}, \bar{3}, \bar{1}, 2)_{2^21^2} + (\bar{1}, \bar{2}\bar{3}, 1, 2)_{2^21^2} - (\bar{1}, \bar{2}, 1, 2)_{2^21^2}\\ &\quad - [(\bar{1}, \bar{2}, \bar{3}, 1)_{2^31} - (\bar{1}, \bar{2}, \bar{3}, 12)_{2^31} + (\bar{1}, \bar{2}, \bar{3}, 1)_{2^31}]\\ &\quad + (\bar{1}, 1^{\bar{2}\cdot\bar{3}}, 2^{\bar{2}\cdot\bar{3}}, \bar{2}\cdot\bar{3})_{21^3} - (\bar{1}, 1^{\bar{2}\cdot\bar{3}}, \bar{2}\cdot\bar{3}, 2^{\bar{2}\bar{3}})_{21^3}\\ &\quad + (\bar{1}, \bar{2}\cdot\bar{3}, 1^{\bar{2}\bar{3}}, 2^{\bar{2}\bar{3}})_{21^3}\\ &= ((\bar{2}\cdot\bar{3})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}, 1^{\bar{1}\bar{2}\bar{3}}, 2^{\bar{1}\bar{2}\bar{3}})_{1^4} - ((\bar{2}\cdot\bar{3})^{\bar{1}}, 1^{\bar{1}\cdot\bar{2}\bar{3}}, \bar{1}\cdot\bar{2}\bar{3}, 2^{\bar{1}\bar{2}\bar{3}})_{1^4}\\ &\quad + (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, (\bar{2}\cdot\bar{3})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}, 2^{\bar{1}\bar{2}\bar{3}})_{1^4}\\ &\quad + (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, 2^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3})_{1^4} - (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, \bar{1}\cdot\bar{2}, 2^{\bar{1}\bar{2}\cdot\bar{3}}, \bar{1}\bar{2}\cdot\bar{3})_{1^4}\\ &\quad + (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}, 2^{\bar{1}\bar{2}\bar{3}})_{1^4}\\ &\quad - (\bar{1}\cdot\bar{2}, 1^{\bar{1}\bar{2}\cdot\bar{3}}, \bar{1}\bar{2}\cdot\bar{3}, 2^{\bar{1}\bar{2}\bar{3}})_{1^4} + (\bar{1}\cdot\bar{2}, 1^{\bar{1}\bar{2}\cdot\bar{3}}, 2^{\bar{1}\bar{2}\cdot\bar{3}}, \bar{1}\bar{2}\cdot\bar{3})_{1^4}\\ &\quad + (\bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}, 1^{\bar{1}\bar{2}\bar{3}}, 2^{\bar{1}\bar{2}\bar{3}})_{1^4}\end{aligned}$$

(3.b) $$\begin{aligned}&(\bar{1}, \bar{2}, 2, 3)_{2^21^2} - (\bar{1}, \bar{2}, 12, 3)_{2^21^2} + (\bar{1}, \bar{2}, 1, 23)_{2^21^2} - (\bar{1}, \bar{2}, 1, 2)_{2^21^2}\\ &\quad - [(\bar{2}, 1, 2, 3)_{21^3} - (\bar{1}\bar{2}, 1, 2, 3)_{21^3} + (\bar{1}, 1^{\bar{2}}, 2^{\bar{2}}, 3^{\bar{2}})_{21^3}]\\ &= (\bar{1}\cdot\bar{2}, 1^{\bar{1}\bar{2}}, 2^{\bar{1}\bar{2}}, 3^{\bar{1}\bar{2}})_{1^4} - (1^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, 2^{\bar{1}\bar{2}}, 3^{\bar{1}\bar{2}})_{1^4}\\ &\quad + (1^{\bar{1}\cdot\bar{2}}, 2^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, 3^{\bar{1}\bar{2}})_{1^4} - (1^{\bar{1}\cdot\bar{2}}, 2^{\bar{1}\cdot\bar{2}}, 3^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2})_{1^4}\end{aligned}$$

$$
\begin{aligned}
(4.\mathrm{a})\quad & [(\bar{2}, \bar{3}, \bar{4}, 1)_{2^31} - (\bar{1}\bar{2}, \bar{3}, \bar{4}, 1)_{2^31} + (\bar{1}, \bar{2}\bar{3}, \bar{4}, 1)_{2^31} \\
& \quad - (\bar{1}, \bar{2}, \bar{3}\bar{4}, 1)_{2^31} + (\bar{1}\cdot\bar{2}, \bar{3}, \bar{4}, 1^{\bar{4}})_{2^31}] \\
& \quad + [(\bar{1}\cdot\bar{2}, \bar{3}, 1^{\bar{3}\cdot\bar{4}}, \bar{3}\cdot\bar{4})_{2^21^2} - (\bar{1}, \bar{2}, \bar{3}\cdot\bar{4}, 1^{\bar{3}\bar{4}})_{2^21^2}] \\
& \quad + [(\bar{1}, 1^{\bar{2}\cdot\bar{3}\cdot\bar{4}}, \bar{2}\cdot\bar{3}, \bar{2}\bar{3}\cdot\bar{4})_{21^3} - (\bar{1}, \bar{2}\cdot\bar{3}, 1^{\bar{2}\bar{3}\cdot\bar{4}}, \bar{2}\bar{3}\cdot\bar{4})_{21^3} \\
& \quad + (\bar{1}, \bar{2}\cdot\bar{3}, \bar{2}\bar{3}\cdot\bar{4}, 1^{\bar{2}\bar{3}\bar{4}})_{21^3} \\
& \quad - [(\bar{1}, 1^{\bar{2}\cdot\bar{3}\cdot\bar{4}}, (\bar{3}\cdot\bar{4})^{\bar{2}}, \bar{2}\cdot\bar{3}\bar{4})_{21^3} - (\bar{1}, (\bar{3}\cdot\bar{4})^{\bar{2}}, 1^{\bar{2}\cdot\bar{3}\bar{4}}, \bar{2}\cdot\bar{3}\bar{4})_{21^3} \\
& \quad + (\bar{1}, (\bar{3}\cdot\bar{4})^{\bar{2}}, \bar{2}\cdot\bar{3}\bar{4}, 1^{\bar{2}\bar{3}\bar{4}})_{21^3}]] \\
= & (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, \bar{1}\cdot\bar{2}, (\bar{3}\cdot\bar{4})^{\bar{1}\bar{2}}, \bar{1}\bar{2}\cdot\bar{3}\bar{4})_{1^4} \\
& \quad - (\bar{1}\cdot\bar{2}, 1^{\bar{1}\bar{2}\cdot\bar{3}\cdot\bar{4}}, (\bar{3}\cdot\bar{4})^{\bar{1}\bar{2}}, \bar{1}\bar{2}\cdot\bar{3}\bar{4})_{1^4} \\
& \quad + (\bar{1}\cdot\bar{2}, (\bar{3}\cdot\bar{4})^{\bar{1}\bar{2}}, 1^{\bar{1}\bar{2}\cdot\bar{3}\cdot\bar{4}}, \bar{1}\bar{2}\cdot\bar{3}\bar{4})_{1^4} \\
& \quad - (\bar{1}\cdot\bar{2}, (\bar{3}\cdot\bar{4})^{\bar{1}\bar{2}}, \bar{1}\bar{2}\cdot\bar{3}\bar{4}, 1^{\bar{1}\bar{2}\bar{3}\bar{4}})_{1^4} \\
& \quad - [(1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, (\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}\bar{4})_{1^4} \\
& \quad - ((\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, 1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}\bar{4})_{1^4}] \\
& \quad - [((\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, 1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\bar{4}}, \bar{1}\bar{2}\cdot\bar{3}\bar{4})_{1^4} \\
& \quad - ((\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}\bar{4}, 1^{\bar{1}\bar{2}\bar{3}\bar{4}})_{1^4}] \\
& \quad + (\bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4}, 1^{\bar{1}\bar{2}\bar{3}\bar{4}})_{1^4} - (\bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}, 1^{\bar{1}\bar{2}\bar{3}\cdot\bar{4}}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4} \\
& \quad + (\bar{1}\cdot\bar{2}, 1^{\bar{1}\bar{2}\bar{3}\cdot\bar{4}}, \bar{1}\bar{2}\cdot\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4} \\
& \quad - (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4} \\
& \quad - [((\bar{2}\cdot\bar{3})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4}, 1^{\bar{1}\bar{2}\bar{3}\bar{4}})_{1^4} \\
& \quad - ((\bar{2}\cdot\bar{3})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}, x^{\bar{1}\bar{2}\bar{3}\cdot\bar{4}}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4}] \\
& \quad - [((\bar{2}\cdot\bar{3})^{\bar{1}}, 1^{\bar{1}\cdot\bar{2}\bar{3}\cdot\bar{4}}, \bar{1}\cdot\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4} \\
& \quad - (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, \bar{2}\cdot\bar{3}, \bar{1}\bar{2}\cdot\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4}] \\
& \quad + (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, (\bar{3}\cdot\bar{4})^{\bar{1}\bar{2}}, (\bar{2}\cdot\bar{3}\bar{4})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}\bar{4})_{1^4} \\
& \quad - ((\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, 1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, (\bar{2}\cdot\bar{3}\bar{4})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}\bar{4})_{1^4} \\
& \quad + ((\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, (\bar{2}\cdot\bar{3}\bar{4})^{\bar{1}}, 1^{\bar{1}\cdot\bar{2}\bar{3}\bar{4}}, \bar{1}\cdot\bar{2}\bar{3}\bar{4})_{1^4} \\
& \quad - ((\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, (\bar{2}\cdot\bar{3}\bar{4})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}\bar{4}, 1^{\bar{1}\bar{2}\bar{3}\bar{4}})_{1^4} \\
& \quad - [(1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, \bar{2}\cdot\bar{3}, (\bar{2}\cdot\bar{3}\bar{4})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}\bar{4})_{1^4} \\
& \quad - ((\bar{2}\cdot\bar{3})^{\bar{1}}, 1^{\bar{1}\cdot\bar{2}\bar{3}\cdot\bar{4}}, (\bar{2}\bar{3}\cdot\bar{4})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}\bar{4})_{1^4}] \\
& \quad - [((\bar{2}\cdot\bar{3})^{\bar{1}}, 1^{\bar{1}\cdot\bar{2}\bar{3}\cdot\bar{4}}, \bar{1}\cdot\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4} \\
& \quad - (1^{\bar{1}\cdot\bar{2}\cdot\bar{3}\cdot\bar{4}}, \bar{2}\cdot\bar{3}, \bar{1}\cdot\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4}]
\end{aligned}
$$

(4.b)
$$\begin{aligned}
&(\bar 1, \bar 2, \bar 3, 2)_{2^31} - (\bar 1, \bar 2, \bar 3, 12)_{2^31} + (\bar 1, \bar 2, \bar 3, 1)_{2^31}\\
&- [(\bar 2, \bar 3, 1, 2)_{2^21^2} - (\bar 1\bar 2, \bar 3, 1, 2)_{2^21^2}\\
&+ (\bar 1, \bar 2\bar 3, 1, 2)_{2^21^2} - (\bar 1, \bar 2, 1^{\bar 3}, 2^{\bar 3})_{2^21^2}]\\
&- [(\bar 1, 1^{\bar 2\cdot\bar 3}, 2^{\bar 2\cdot\bar 3}, \bar 2\cdot\bar 3)_{21^3} - (\bar 1, 1^{\bar 2\cdot\bar 3}, \bar 2\cdot\bar 3, 2^{\bar 2\bar 3})_{21^3}\\
&+ (\bar 1, \bar 2\cdot\bar 3, 1^{\bar 2\bar 3}, 2^{\bar 2\bar 3})_{21^3}]\\
&= [(1^{\bar 1\cdot\bar 2\cdot\bar 3}, 2^{\bar 1\cdot\bar 2\cdot\bar 3}, (\bar 2\cdot\bar 3)^{\bar 1}, \bar 1\cdot\bar 2\bar 3)_{1^4}\\
&- (1^{\bar 1\cdot\bar 2\cdot\bar 3}, (\bar 2\cdot\bar 3)^{\bar 1}, 2^{\bar 1\cdot\bar 2\bar 3}, \bar 1\cdot\bar 2\bar 3)_{1^4}\\
&+ (1^{\bar 1\cdot\bar 2\cdot\bar 3}, (\bar 2\cdot\bar 3)^{\bar 1}, \bar 1\cdot\bar 2\bar 3, 2^{\bar 1\bar 2\bar 3})_{1^4} - ((\bar 2\cdot\bar 3)^{\bar 1}, 1^{\bar 1\cdot\bar 2\bar 3}, \bar 1\cdot\bar 2\bar 3, 2^{\bar 1\bar 2\bar 3})_{1^4}\\
&+ ((\bar 2\cdot\bar 3)^{\bar 1}, 1^{\bar 1\cdot\bar 2\bar 3}, 2^{\bar 1\cdot\bar 2\bar 3}, \bar 1\cdot\bar 2\bar 3)_{1^4} + ((\bar 2\cdot\bar 3)^{\bar 1}, \bar 1\cdot\bar 2\bar 3, 1^{\bar 1\bar 2\bar 3}, 2^{\bar 1\bar 2\bar 3})_{1^4}]\\
&- [(1^{\bar 1\cdot\bar 2\cdot\bar 3}, 2^{\bar 1\cdot\bar 2\cdot\bar 3}, \bar 1\cdot\bar 2, \bar 1\bar 2\cdot\bar 3)_{1^4} - (1^{\bar 1\cdot\bar 2\cdot\bar 3}, \bar 1\cdot\bar 2, 2^{\bar 1\bar 2\cdot\bar 3}, \bar 1\bar 2\cdot\bar 3)_{1^4}\\
&+ (1^{\bar 1\cdot\bar 2\cdot\bar 3}, \bar 1\cdot\bar 2, \bar 1\bar 2\cdot\bar 3, 2^{\bar 1\bar 2\bar 3})_{1^4} - (\bar 1\cdot\bar 2, 1^{\bar 1\bar 2\cdot\bar 3}, \bar 1\bar 2\cdot\bar 3, 2^{\bar 1\bar 2\bar 3})_{1^4}\\
&+ (\bar 1\cdot\bar 2, 1^{\bar 1\bar 2\cdot\bar 3}, 2^{\bar 1\bar 2\cdot\bar 3}, \bar 1\bar 2\cdot\bar 3)_{1^4} + (\bar 1\cdot\bar 2, \bar 1\bar 2\cdot\bar 3, 1^{\bar 1\bar 2\bar 3}, 2^{\bar 1\bar 2\bar 3})_{1^4}]
\end{aligned}$$

(5)
$$\begin{aligned}
&[(\bar 2, \bar 3, \bar 4, \bar 5)_{2^4} - (\bar 1\bar 2, \bar 3, \bar 4, \bar 5)_{2^4} + (\bar 1, \bar 2\bar 3, \bar 4, \bar 5)_{2^4}\\
&- (\bar 1, \bar 2, \bar 3\bar 4, \bar 5)_{2^4} + (\bar 1, \bar 2, \bar 3, \bar 4\bar 5)_{2^4} - (\bar 1\cdot\bar 2, \bar 3, \bar 4)_{2^4}]\\
&+ [(\bar 1, \bar 2, \bar 3, \bar 4\cdot\bar 5)_{2^31}]\\
&+ [(\bar 1, \bar 2, (\bar 4, \bar 5)^{\bar 3}, \bar 3\cdot\bar 4\bar 5)_{2^21^2} - (\bar 1, \bar 2, \bar 3\cdot\bar 4, \bar 3\bar 4\cdot\bar 5)_{2^21^2}]\\
&+ [(\bar 1, (\bar 3\cdot\bar 4)^{\bar 2}, (\bar 3\bar 4\cdot\bar 5)^{\bar 2}, \bar 2\cdot\bar 3\bar 4\bar 5)_{21^3}\\
&- (\bar 1, (\bar 4\cdot\bar 5)^{\bar 2\cdot\bar 3}, (\bar 3\cdot\bar 4\bar 5)^{\bar 2}, \bar 2\cdot\bar 3\bar 4\bar 5)_{21^3}\\
&+ (\bar 1, (\bar 4\cdot\bar 5)^{\bar 2\cdot\bar 3}, \bar 2\cdot\bar 3, \bar 2\bar 3\cdot\bar 4\bar 5)_{21^3} - (\bar 1, \bar 2\cdot\bar 3, (\bar 4\cdot\bar 5)^{\bar 2\bar 3}, \bar 2\bar 3\cdot\bar 4\bar 5)_{21^3}\\
&+ (\bar 1, \bar 2\cdot\bar 3, \bar 2\bar 3\cdot\bar 4, \bar 2\bar 3\bar 4\cdot\bar 5)_{21^3} - (\bar 1, (\bar 3\cdot\bar 4)^{\bar 2}, \bar 2\cdot\bar 3\bar 4, \bar 2\bar 3\bar 4\cdot\bar 5)_{21^3}]\\
&= (\bar 1\cdot\bar 2, \bar 1\bar 2\cdot\bar 3, \bar 1\bar 2\bar 3\cdot\bar 4. \bar 1\bar 2\bar 3\bar 4\cdot\bar 5)_{1^4}\\
&- ((\bar 2\cdot\bar 3)^{\bar 1}, \bar 1\cdot\bar 2\bar 3, \bar 1\bar 2\bar 3\cdot\bar 4, \bar 1\bar 2\bar 3\bar 4\cdot\bar 5)_{1^4}\\
&- ((\bar 3\cdot\bar 4)^{\bar 1\cdot\bar 2}, (\bar 2\cdot\bar 3\bar 4)^{\bar 1}, \bar 1\cdot\bar 2\bar 3\bar 4, \bar 1\bar 2\bar 3\bar 4\cdot\bar 5)_{1^4}\\
&+ ((\bar 2\cdot\bar 3)^{\bar 1}, \bar 1\cdot\bar 2\bar 3, (\bar 4\cdot\bar 5)^{\bar 1\bar 2\bar 3}, \bar 1\bar 2\bar 3\cdot\bar 4\bar 5)_{1^4}\\
&- ((\bar 2\cdot\bar 3)^{\bar 1}, (\bar 4\cdot\bar 5)^{\bar 1\cdot\bar 2\bar 3}, \bar 1\cdot\bar 2\bar 3, \bar 1\bar 2\bar 3\cdot\bar 4\bar 5)_{1^4}\\
&+ (\bar 1\cdot\bar 2, (\bar 4\cdot\bar 5)^{\bar 1\bar 2\cdot\bar 3}, \bar 1\bar 2\cdot\bar 3, \bar 1\bar 2\bar 3\cdot\bar 4\bar 5)_{1^4}\\
&- ((\bar 4\cdot\bar 5)^{\bar 1\cdot\bar 2\cdot\bar 3}, \bar 1\cdot\bar 2, \bar 1\bar 2\cdot\bar 3, \bar 1\bar 2\bar 3\cdot\bar 4\bar 5)_{1^4}\\
&+ (\bar 1\cdot\bar 2, (\bar 3\cdot\bar 4)^{\bar 1\cdot\bar 2}, (\bar 3\cdot\bar 4\bar 5)^{\bar 1\bar 2}, \bar 1\bar 2\cdot\bar 3\bar 4\bar 5)_{1^4}\\
&- (\bar 1\cdot\bar 2, (\bar 4\cdot\bar 5)^{\bar 1\bar 2\cdot\bar 3}, (\bar 3\cdot\bar 4\bar 5)^{\bar 1\bar 2}, \bar 1\bar 2\cdot\bar 3\bar 4\bar 5)_{1^4}\\
&+ ((\bar 3\cdot\bar 4)^{\bar 1\cdot\bar 2}, (\bar 3\bar 4\cdot\bar 5)^{\bar 1\cdot\bar 2}, \bar 1\cdot\bar 2, \bar 1\bar 2\cdot\bar 3\bar 4\bar 5)_{1^4}\\
&- ((\bar 3\cdot\bar 4)^{\bar 1\cdot\bar 2}, \bar 1\cdot\bar 2, (\bar 3\bar 4\cdot\bar 5)^{\bar 1\bar 2}, \bar 1\bar 2\cdot\bar 3\bar 4\bar 5)_{1^4}\\
&+ ((\bar 4\cdot\bar 5)^{\bar 1\cdot\bar 2\cdot\bar 3}, \bar 1\cdot\bar 2, (\bar 3\cdot\bar 4\bar 5)^{\bar 1\bar 2}, \bar 1\bar 2\cdot\bar 3\bar 4\bar 5)_{1^4}\\
&- ((\bar 4\cdot\bar 5)^{\bar 1\cdot\bar 2\cdot\bar 3}, (\bar 3\cdot\bar 4\bar 5)^{\bar 1\cdot\bar 2}, \bar 1\cdot\bar 2, \bar 1\bar 2\cdot\bar 3\bar 4\bar 5)_{1^4}
\end{aligned}$$

$$
\begin{aligned}
&+((\bar4\cdot\bar5)^{\bar1\cdot\bar2\cdot\bar3}, (\bar3\cdot\bar4\bar5)^{\bar1\cdot\bar2}, (\bar2\cdot\bar3\bar4\bar5)^{\bar1}, \bar1\cdot\bar2\bar3\bar4\bar5)_{1^4}\\
&-((\bar3\cdot\bar4)^{\bar1\cdot\bar2}, (\bar3\cdot\bar4\bar5)^{\bar1\cdot\bar2}, (\bar2\cdot\bar3\bar4\bar5)^{\bar1}, \bar1\cdot\bar2\bar3\bar4\bar5)_{1^4}\\
&+((\bar2\cdot\bar3)^{\bar1}, (\bar4\cdot\bar5)^{\bar1\cdot\bar2\bar3}, (\bar2\bar3\cdot\bar4\bar5)^{\bar1}, \bar1\cdot\bar2\bar3\bar4\bar5)_{1^4}\\
&-((\bar4\cdot\bar5)^{\bar1\cdot\bar2\cdot\bar3}, (\bar2\cdot\bar3)^{\bar1}, (\bar2\bar3\cdot\bar4\bar5)^{\bar1}, \bar1\cdot\bar2\bar3\bar4\bar5)_{1^4}\\
&+((\bar3\cdot\bar4)^{\bar1\cdot\bar2}, (\bar2\cdot\bar3\bar4)^{\bar1}, (\bar2\bar3\bar4\cdot\bar5)^{\bar1}, \bar1\cdot\bar2\bar3\bar4\bar5)_{1^4}\\
&-((\bar2\cdot\bar3)^{\bar1}, (\bar2\bar3\cdot\bar4)^{\bar1}, (\bar2\bar3\bar4\cdot\bar5)^{\bar1}, \bar1\cdot\bar2\bar3\bar4\bar5)_{1^4}
\end{aligned}
$$

Proof. (1) is clear by restriction. For (2.a), reduce both sides of

$$\sigma(s(y_1) \mid s(y_2), x_1, x_2, x_3) - \sigma(\alpha(y_1, y_2) \mid s(y_1 y_2), x_1, x_2, x_3);$$

for (2.b), reduce $\sigma(s(y), x_1, x_2, x_3 \mid x_4)$; for (3.a), reduce both sides of

$$\sigma(s(y_1), s(y_2) \mid s(y_3), x_1, x_2) = \sigma(s(y_1), \alpha(y_2, y_3) \mid s(y_2 y_3), x_1, x_2);$$

for (3.b), reduce $\sigma(s(y_1), s(y_2), x_1, x_2 \mid x_3)$; for (4.a), reduce both sides of

$$\sigma(s(y_1), s(y_2) \mid s(y_3), s(y_4), x) = \sigma(s(y_1), \alpha(y_2, y_3) \mid s(y_2 y_3), s(y_4), x);$$

for (4.b), reduce $\sigma(s(y_1), s(y_2), s(y_3), x_1 \mid x_2)$; for (5), reduce both sides of

$$
\begin{aligned}
&\sigma(s(y_1), s(y_2), s(y_3), s(y_4) \mid s(y_5))\\
&\quad = \sigma(s(y_1), s(y_2), s(y_3), \alpha(y_4, y_5) \mid s(y_4 y_5)).
\end{aligned}
$$

THEOREM 3.2. *Every determined cocycle σ in $Z^4(G, A)$ can be written as a sum*

$$
\begin{aligned}
&\sigma(1\bar1, 2\bar2, 3\bar3, 4\bar4)\\
&\quad = [(\bar1, \bar2, \bar3, \bar4)_{2^4}] + [(\bar1, \bar2, \bar3, 4)_{2^3 1}] + [(1, 2, 3, 4^{\bar3})_{2^2 1^2}]\\
&\qquad + [(1, 23^{\bar2}4^{\bar2\cdot\bar3}, \bar2\cdot\bar3, \bar2\bar3\cdot\bar4)_{21^3} - (1, 23^{\bar2}4^{\bar2\cdot\bar3}, (\bar3\cdot\bar4)^{\bar2}, \bar2\cdot\bar3\bar4)_{21^3}\\
&\qquad + (1, 23^{\bar2}, \bar2\cdot\bar3, 4^{\bar2\bar3})_{21^3} - (1, 23^{\bar2}, 4^{\bar2\bar3}, \bar2\cdot\bar3)_{21^3} + (1, 2, 3^{\bar2}, 4^{\bar2\cdot\bar3})_{21^3}]\\
&\qquad + (1, 2^{\bar1}, 3^{\bar1\cdot\bar2}, 4^{\bar1\cdot\bar2\cdot\bar3})_{1^4} + (12^{\bar1}, 3^{\bar1\cdot\bar2}, 4^{\bar1\cdot\bar2\cdot\bar3}, \bar1\cdot\bar2)_{1^4}\\
&\qquad + (12^{\bar1}, 3^{\bar1\cdot\bar2}, \bar1\cdot\bar2, 4^{\bar1\bar2\cdot\bar3})_{1^4} + (12^{\bar1}, \bar1\cdot\bar2, 3^{\bar1\bar2}, 4^{\bar1\bar2\cdot\bar3})_{1^4}\\
&\qquad - (12^{\bar1}, 3^{\bar1\cdot\bar2}, 4^{\bar1\cdot\bar2\cdot\bar3}, (\bar2\cdot\bar3)^{\bar1})_{1^4} + (1, 2^{\bar1}3^{\bar1\cdot\bar2}, (\bar2\cdot\bar3)^{\bar1}, 4^{\bar1\bar2\cdot\bar3})_{1^4}\\
&\qquad + (1, 2^{\bar1}3^{\bar1\cdot\bar2}4^{\bar1\cdot\bar2\cdot\bar3}, (\bar2\cdot\bar3)^{\bar1}, (\bar2\bar3\cdot\bar4)^{\bar1})_{1^4}\\
&\qquad - (1, 2^{\bar1}3^{\bar1\cdot\bar2}4^{\bar1\cdot\bar2\cdot\bar3}, (\bar3\cdot\bar4)^{\bar1\cdot\bar2}, (\bar2\cdot\bar3\bar4)^{\bar1})_{1^4}\\
&\qquad + (12^{\bar1}3^{\bar1\cdot\bar2}, 4^{\bar1\cdot\bar2\cdot\bar3}, \bar1\cdot\bar2, \bar1\bar2\cdot\bar3)_{1^4} - (12^{\bar1}3^{\bar1\cdot\bar2}, 4^{\bar1\cdot\bar2\cdot\bar3}, (\bar2\cdot\bar3)^{\bar1}, \bar1\cdot\bar2\bar3)_{1^4}
\end{aligned}
$$

$$+(12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}, 4^{\bar{1}\bar{2}\bar{3}})_{1^4} - (12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}, (\bar{2}\cdot\bar{3})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}, 4^{\bar{1}\bar{2}\bar{3}})_{1^4}$$

$$-(12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, 4^{\bar{1}\bar{2}\cdot\bar{3}}, \bar{1}\bar{2}\cdot\bar{3})_{1^4}$$

$$+(12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}4^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4}$$

$$-(12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}4^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, (\bar{2}\cdot\bar{3})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}\cdot\bar{4})_{1^4}$$

$$+(12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}4^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, (\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, \bar{1}\cdot\bar{2}, \bar{1}\bar{2}\cdot\bar{3}\bar{4})_{1^4}$$

$$-(12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}4^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, \bar{1}\cdot\bar{2}, (\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, \bar{1}\bar{2}\cdot\bar{3}\bar{4})_{1^4}$$

$$+(12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}4^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, (\bar{2}\cdot\bar{3})^{\bar{1}}, (\bar{2}\bar{3}\cdot\bar{4})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}\bar{4})_{1^4}$$

$$-(12^{\bar{1}}3^{\bar{1}\cdot\bar{2}}4^{\bar{1}\cdot\bar{2}\cdot\bar{3}}, (\bar{3}\cdot\bar{4})^{\bar{1}\cdot\bar{2}}, (\bar{2}\cdot\bar{3}\bar{4})^{\bar{1}}, \bar{1}\cdot\bar{2}\bar{3}\bar{4})_{1^4}$$

Proof. Reduce $\sigma(x_1 s(y_1), x_2 s(y_2), x_3 s(y_3), x_4 \mid s(y_4))$.

THEOREM 3.3. *If we have functions* σ_{1111}, σ_{2111}, σ_{2211}, σ_{2221}, σ_{2222} *satisfying the functional equations in Theorem* 3.1 *and we define* σ *as in Theorem* 3.2, *then* σ *is a* 4*-cocycle on* G.

Proof. The cocycle equation for σ thus defined is equivalent to the satisfaction of the functional equations, as follows from a straightforward calculation.

It is not immediate how one might set about solving these equations in general. We concentrate our attentions on solving them for the Heisenberg group H_n, for which group the equations simplify considerably, and in fact sufficiently so that it is possible to obtain extremely explicit formulas for the cocycles.

IV. COHOMOLOGY OF H_n

If we now consider cohomology defined using Borel cochains, as in [Du1, Mo, Wg], we see that all our previous algorithms and constructions work when we assume measurable, continuous, differentiable, analytic, or even polynomial cochains. Since H_n is simply connected and is also continuously solvable of finite length, it follows from [Du1] or [Wg] that its Borel cohomology can be computed using continuous, or by [Mst] even, analytic cochains.

Matters simplify considerably in the case of a central extension $1 \to N \to G \to H \to 1$, where $H^n(N, A) = 0$ for some n. We also assume until further notice that G operates trivially on A.

LEMMA 4.1. *If* $H^n(N, A)=0$, *then in calculating* $H^n(G, A)$, *if* $\sigma \in Z^n(G, A)$, *we may assume that* $\sigma|_{N^n}=0$.

Proof. We have to show that if $\sigma|_{N^n}$ is a coboundary, then σ is cohomologous to $\bar{\sigma}$ with $\bar{\sigma}|_{N^n}=0$. Suppose that $\sigma|_{N^n}=\delta c$ with $c \in Z^n(N, A)$. Define

$$\bar{c}(x_1 s(y_1), ..., x_{n-1} s(y_{n-1}))=c(x_1, ..., x_{n-1}).$$

Then $\bar{\sigma}=\sigma-\delta\bar{c}$ works.

A similar argument yields

LEMMA 4.2. *If* $H^k(N, A)=0$, *then in calculating* $H^n(G, A)$ *for* $n \geqslant k$, *we may assume that* $\sigma|_{s(H)^{n-k} \times N^k}=0$.

Note that all this is compatible with the property of being a determined cocycle, so that we obtain a corresponding simplification in the appropriate functional equations and expression of σ in terms of its restrictions. Since for H_n its center $\mathbb{R}$ has $H^n(\mathbb{R}, \mathbb{R})=0$ for $n \geqslant 2$, we obtain

THEOREM 4.1. *If* $\sigma \in Z^m(H_n, \mathbb{R})$ *and* $m \leqslant n$, *then we may assume that* $\sigma_{2^k 1^{m-k}}=0$ *for* $m-k \geqslant 2$.

THEOREM 4.2. *If* $\sigma \in Z^m(H_n, \mathbb{R})$ *and* $2 \leqslant m \leqslant n$, *then the functional equations for* $\sigma|_{2^{m-1}1}$ *and* $\sigma|_{2^m}$ *are*

$$\sigma_{2^{m-1}1}(y_2, ..., y_m, x)+\sum_{i=1}^{m-1}(-1)^i \sigma(y_1, ..., y_1 y_{i+1}, ..., y_m, x)$$
$$+(-1)^m \sigma_{2^{m-1}1}(y_1, ..., y_{m-1}, x)=0$$

$$\sigma_{2^{m-1}1}(y_1, ..., y_{m-1}, x_2)-\sigma_{2^{m-1}1}(y_1, ..., y_{m-1}, x_1 x_2)$$
$$+\sigma_{2^{m-1}1}(y_1, ..., y_{m-1}, x_1)=0$$

$$\sigma_{2^m}(y_2, ..., y_{m+1})+\sum_{i=1}^{m}(-1)^m \sigma_{2^m}(y_1, ..., y_i y_{i+1}, ..., y_{m+1})$$
$$+(-1)^{m+1}\sigma_{2^m}(y_1, ..., y_m)$$
$$=\sigma_{2^{m-1}1}(y_1, ..., y_{m-1}, \alpha(y_m, y_{m+1}))$$

Proof. A straightforward calculation, using the assumption that σ is a Lyndon cocycle for which $\sigma_{2^k 1^{m-k}}=0$ for $m-k \geqslant 2$.

Because, by [Du1], $H^m(\mathbb{R}^{2n}, \mathbb{R})$ may be calculated using multilinear cocycles, we have

LEMMA 4.3. *σ is cohomologous to $\bar{\sigma}$ with $\bar{\sigma}_{2^m-1_1}$ multilinear on $s(\mathbb{R}^{2n})^m \times \mathbb{R}$.*

Proof. The proof is similar to that of Lemma 4.2.

We now see that, in order to construct an m-cocycle on H_n, we need to know how to trivialize the multilinear function $\sigma_{2^m-1_1}(y_1, ..., y_{m-1}, \alpha(y_m, y_{m+1}))$ on $(\mathbb{R}^{2n})^{m+1}$. The theory of trivializing general trilinear functions is treated in [Du2], and in [Du3] the particular quadrilinear function in question is trivialized, as part of the calculation of $H^3(H_n, \mathbb{R})$.

LEMMA 4.4. *We mau assume that our $(m+1)$-multilinear function in question is of the form $\beta(y_1, ..., y_{m-1})\,\alpha(y_m, y_{m+1})$, where α is the non-degenerate skew 2-form on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ defining H_n, and β is an $(m-1)$-multilinear alternating form on $\mathbb{R}^{2n}$.*

Proof. Clear.

From [Du1] we see that an $(m+1)$-multilinear function on $\mathbb{R}^{2n}$ is trivial as a cocycle iff it is annihilated by the alternating map A. But

$$A(\beta(y_1, ..., y_{m-1})\,\alpha(y_m, y_{m+1})) = (\beta \wedge \alpha)(y_1, ..., y_{m+1}),$$

thus leading us to

LEMMA 4.5. *$\beta \in \Lambda^{m-1}(\mathbb{R}^{2n})$ is such that $\beta\alpha$ is a trivial cocycle precisely when we have $\beta \in \operatorname{Ker}\{L\colon \Lambda^{m-1}(\mathbb{R}^{2n}) \to \Lambda^{m+1}(\mathbb{R}^{2n})\}$, where $L(\beta) = \beta \wedge \alpha$.*

It follows from [Ho, We] that $\operatorname{Ker}(L) = 0$ if $m \leqslant n+1$ and surjective if $m \geqslant n+1$.

THEOREM 4.3. *In computing $Z^m(H_n, \mathbb{R})$, if $m \leqslant n$ then an m-cocycle σ on H_n has its $\sigma_{2^m-1_1}$ component zero, and the cohomology of $H^m(H_n, \mathbb{R})$ can be computed using m-multilinear alternating forms on $\mathbb{R}^{2n}$. If, on the other hand, $m > n$, then since $\dim(\operatorname{Ker} L) = \binom{2n}{m} - \binom{2n}{m+2}$, there is a vector space of this many dimensions of $(m+1)$-cocycles of the form $\beta(y_1, ..., y_{m-1})$ $\alpha(y_m, y_{m+1})$ which may be trivialized by a vector space of this number of dimensions of cochains which may be used as σ_{2^m} components of $\sigma \in Z^m(H_n, \mathbb{R})$. In this case σ is not cohomologous to an m-multilinear function.*

Proof. All the ingredients for the statements in the theorem have already been proved and it is clear how to assemble them.

If we fix m and let n vary, we see that we obtain a dichotomy between the Heisenberg groups for which multilinear functions suffice to compute the cohomology. For the lower dimensional Heisenbergs, it is necessary to use cocycle representatives which are not multilinear, but eventually, for large enough n, namely $n-1 \geqslant m$, it is possible to compute the cohomology using multilinear functions.

We must now address the question of determining when a cocycle s is a coboundary. Theorem 2.9 indicates that σ, if trivial, is the coboundary of a determined $(m-1)$-cochain b, but we now have to ascertain in our case of H_n how b is determined.

LEMMA 4.6. *If* $\sigma \in Z^m(H_n, \mathbb{R})$ *is the coboundary of a cochain* $b \in C^{m-1}(H_n, \mathbb{R})$ *which is determined, then we may assume that*

$$b(x_1 s(y_1), ..., x_{m-1} s(y_{m-1})) \\ = b(s(y_1), ..., s(y_{m-2}), x_{m-1}) + b(s(y_1), ..., s(y_{m-1})).$$

Proof. Start by expanding $(\delta b)(x_1 s(y_1), ..., x_{m-2} s(y_{m-2}), x_{m-1}, s(y_{m-1}))$ and using the fact that $\delta b = \sigma$, which is a determined cocycle.

THEOREM 4.4. $\sigma \in Z^m(H_n, \mathbb{R})$ *is a coboundary iff*

$$\sigma = \sigma_{2^m} \text{ and } \sigma_{2^m}(y_1, ..., y_m) = b_{2^m\ 11}(y_1, ..., y_{m-2}, \alpha(y_{m-1}, y_m)),$$

where $b_{2^{m-2}1}$ *is alternating multilinear in* $y_1, ..., y_{m-2}$ *and linear in* x_{m-1}.

Proof. Straightforward cohomological calculations.

Now since $H^m(H_n, \mathbb{R})$ is a vector space, we know its structure if we know the quotient in a tower of subspaces, which is precisely what the spectral sequence furnishes. But with the method we are using we obtain an explicit set of representatives for the cohomology classes. The results of Theorems 4.3 and 4.4 are consistent with the following result from [McL, p. 355].

THEOREM 4.5. *Suppose* $n \geqslant 1$ *and* $H^m(N, A) = 0$ *for* $1 < m < n$. *Then for* $0 < m < n$ *we have the long exact sequence*

$$\to H^m(H, A^N) \xrightarrow{\text{inf}} H^m(G, A) \to H^{m-1}(H, H^1(N, A)) \\ \xrightarrow{\text{tg}} H^{m+1}(H, A^N) \to \ .$$

Here tg is the transgression map, which is given at the cocycle level as

$$\text{tg}(\sigma)(y_1, ..., y_{m+1}) = \sigma(s(y_1), ..., s(y_{m-1}), \alpha(y_m, y_{m+1})),$$

and inf is the inflation map which is just j^* for $j: G \to H$ the quotient map. Counting up all the dimensions, we get

THEOREM 4.6. *If* $0 \leqslant m \leqslant n$ *then* $\dim H^m(H_n, \mathbb{R}) = \binom{2n}{m} - \binom{2n}{m-2}$, *and if* $n+1 \leqslant m \leqslant 2n+1$ *then* $\dim H^m(H_n, \mathbb{R}) = \binom{2n}{m-1} - \binom{2n}{m+1}$.

Proof. Suppose $0 \leqslant m \leqslant n$. Then the map $\Lambda^{m-2}(\mathbb{R}^{2n}) \to \Lambda^m(\mathbb{R}^{2n})$ which takens β to $\beta \wedge \alpha$ is injective, and, as we have seen in Theorem 4.3, every m-cocycle on H_n is cohomologous to an m-multilinear alternating function, which must then be defined on the quotient of H_n by its center $\mathbb{R}$, which is $\mathbb{R}^{2n}$. But such a cocycle is a coboundary iff it is of the form $\beta(y_1, ..., y_{m-2})\,\alpha(y_{m-1}, y_m)$ for some $(m-2)$-alternating multilinear function β, hence the dimension above. On the other hand, if $n+1 \leqslant m \leqslant 2n+1$, the map $\Lambda^{m+1} \to \Lambda^{m+1}$ is onto and so every m-cocycle on the quotient group $\mathbb{R}^{2n}$ is trivial and we must look elsewhere for elements in $H^m(H_n, \mathbb{R})$. Now the wedge map

$$\Lambda^{m-1}(\mathbb{R}^{2n}) \to \Lambda^{m+1}(\mathbb{R}^{2n})$$

has a nonzero kernel whose dimension is $\binom{2n}{m-1} - \binom{2n}{m+1}$.

Now let us see how explicitly we may choose our cocycle representatives in $Z^m(H_n, \mathbb{R})$. We need to be able to explicitly trivialize $\sigma_{2^{m-1}1}(y_1, ..., y_{m-1}, \alpha(y_m, y_{m+1}))$, an $(m+1)$-cocycle on $\mathbb{R}^{2n}$. As in Lemma 4.4, we may assume $\sigma = \beta\alpha$. Noting our success in [Du2, Du3] in trivializing tri- and quadrilinear cocycles of a certain special form, namely those which were products of alternating multilinear functions, we look for a trivialization of the form

$$\gamma^m = \sum_{i<j} \lambda_{ij}(\gamma^m_{ij})_1 + \mu_{ij}(\gamma^m_{ij})_2,$$

where

$$(\gamma^m_{ij})_1\,(y_1, ..., y_m) = \beta(y_1, ..., y_{j-1}, y_{j+1}, ..., y_m)\,\alpha(y_i, y_j) \tag{4.1}$$

$$(\gamma^m_{ij})_2\,(y_1, ..., y_m) = \beta(y_1, ..., y_{i-1}, y_{i+1}, ..., y_m)\,\alpha(y_i, y_j). \tag{4.2}$$

In order to shorten notation even further, we write $(i, j)_1$, $(i, j)_2$ for the right sides of (4.1), (4.2), and (i, j) for

$$\beta(y_1, ..., y_{i-1}, y_{i+1}, ..., y_{j-1}, y_{j+1}, ..., y_{m+1}\alpha(y_i, y_j)). \tag{4.3}$$

A bit of calculation yields

LEMMA 4.7. *The action of* δ *on the basis* $\{(i, j)_k \mid 1 \leqslant i < j \leqslant m,\ k = 0, 1\}$ *is*

$$\delta((i, j)_1) = (-1)^i\,[(i+1, j+1) + (i, j+1)] \tag{4.4}$$

$$\delta((i, j)_2) = (-1)^{(i+j)j+1}\,[(i, j+1) + (i, j)]. \tag{4.5}$$

Now using the fact that $\beta \wedge \alpha = 0$, we get the equation

$$\sum_{1 \leqslant i < j \leqslant m} (-1)^{i+j} (i, j) = 0. \tag{4.6}$$

We determine the $m(m-1)$ constants λ_{ij}, μ_{ij} so that $\delta(\gamma^m) = \sigma_{2^{m-1}1}$. Using Eq. (4.6) to write $(1, 2)$ in terms of the $\binom{m+1}{2} - 1$ other (i, j), we arrive at a set of $\binom{m+1}{2} - 1$ linear equations in the λ's and μ's. These equations take the form

$$\begin{aligned}
&2\mu_{12} - \mu_{13} - \lambda_{12} = 0 \ (1, 3) \\
&\mu_{12} - \mu_{1j} - \mu_{1,j+1} + (-1)^{j+1} \lambda_{1j} = 0 \ (1, j), \qquad 4 \leqslant j \leqslant m \\
&\mu_{12} + \mu_{1m} + \lambda_{1m} = 0 \ (1, m+1) \\
&\vdots \\
&\mu_{12} + (-1)^i [\mu_{i,i+1} + \lambda_{i-1,i}] = 0 \ (i.i+1) \\
&\mu_{12} + (-1)^{i+1} [\mu_{i,j-1} - \mu_{ij}] + (-1)^{j+1} [\lambda_{i-1,j} - \lambda_{ij}] = 0 \ (i, j), \ 2 \leqslant i < j \leqslant m \\
&\mu_{12} + (-1)^{i+1} \mu_{im} + (-1)^{j+1} [\lambda_{i-1,m} - \lambda_{im}] = 0 \ (i, m+1) \\
&\vdots \\
&\mu_{12} - \lambda_{m-1,m} = -1 \ (m, m+1).
\end{aligned} \tag{4.7}$$

THEOREM 4.7. *The above system of equations* (4.7) *can be solved for the λ's and μ's and the answer is*

$$\begin{aligned}
\gamma^m = &\sum_{1 \leqslant i < j \leqslant m-1} \frac{2(-1)^i (j-i)(i, j)_1}{m(m+1)} - \sum_{1 \leqslant i < m} \frac{2(i, m)_1}{m(m+1)} \\
&- \sum_{1 \leqslant i < m-1} \frac{i(2m+1-i)}{m(m+1)} (i, m-1)_2 + \frac{(m-1)(m+2)}{m(m+1)} (m-1, m)_2.
\end{aligned} \tag{4.8}$$

Proof. We arrange the equations in the following order:

$(m, m+1), (1, 3), \ldots, (1, m), (2, 3, \ldots, (m-1, m), (1, m+1), \ldots,$
$(m-1, m+1).$

From our system of linear equations we see that we get a $((m+2)(m-1)/2) \times m(m-1)$ matrix. The first column consists of a 1 followed by a 2 and then all 1's. The columns $(i, j)_1$ for $3 \leqslant j \leqslant m-1$ have 2 nonzero entries: a -1 in the $(1, j)$ place and a 1 in the $(1, j+1)$ place. Then the $(1, m)_1$ column has a -1 in the $(1, m)$ place and a 1 in the $(1, m+1)$ place. Then columns $(2, j)_1$ for $3 \leqslant j \leqslant m-1$ have 1's in the $(2, j)$ place and a -1 in the $(2, j+1)$ place. Then $(2, m)_1$ has $1, -1$ at places $(2, m)$, $(2, m+1)$ respectively. Continuing in this fashion, the columns

$(i, j)_1$ for $3 \leqslant i < j \leqslant m-1$ have $(-1)^i$, $(-1)^{i+1}$ in the (i, j), $(i, m+1)$ places, respectively, and the $(i, m)_1$ columns have $(-1)^i$, $(-1)^{i+1}$ in the places (i, m), $(i, m+1)$, for $1 \leqslant i < m$. The columns $(i, j)_2$ for $1 \leqslant i < j \leqslant m$ have zero entries below the $(1, m+1)$ place except for the columns $(i, m)_2$, and except for $(m-1, m)$ these all have 1, -1 at the places $(i, m+1)$, $(i+1, m+1)$, respectively. $(m-1, m)$ has -1, 1 at the first place $(m, m+1)$ and last $(m-1, m+1)$ places. We are also interested in the entries in the columns $(i, m-1)_2$ for $1 \leqslant i \leqslant m-2$. $(i, m-1)$ has -1, 1 at (i, m), $(i+1, m)$.

We reduce this matrix in five steps.

(1) Multiply row $(m, m+1)$ by -1. Add row $(m, m+1)$ to row (i, j) for $1 \leqslant i < j \leqslant m+1$, $i \neq m$, $j \neq 3$ and add twice row $(m, m+1)$ to row $(1, 3)$.

(2) Add row (i, j) to row $(i, j+1)$ for $1 \leqslant i < j \leqslant m$.

(3) Add row (i, m) to row $(i, m+1)$ for $1 \leqslant i \leqslant m-1$.

(4) Add row $(i, m+1)$ to row $(i+1, m+1)$ for $1 \leqslant i \leqslant m-2$.

(5) Divide row $(m-1, m+1)$ by $\binom{m+1}{2}$ and subtract $j-i$ times row $(m-1, m+1)$ from row (i, j) for $1 \leqslant i < j \leqslant m$ and $i(2m+1-i)/2$ times row $(m-1, m+1)$ from row $(i, m+1)$ for $1 \leqslant i \leqslant m-1$, and finally add row $(m-1, m+1)$ to row $(m, m+1)$.

This puts our matrix in what we may call reduced quasi-echelon form; i.e., we take the variables represented by the columns $(i, j)_2$ for $1 \leqslant i < j \leqslant m-1$ as undetermined parameters and express all the other variables in terms of these. If we do this, we get

$$\lambda_{m, m+1} = 1\Big/\binom{m+1}{2}$$

$$\lambda_{i, j} = \sum_{k=i}^{j-1} (-1)^k \mu_{i-1, k} + \sum_{k=i+1}^{j-1} (-1)^k \mu_{i, k} + (j-i)\Big/\binom{m+1}{2}$$

$$\text{for} \quad 1 \leqslant i < j \leqslant m$$

$$\mu_{i, m+1} = \sum_{k=i+1}^{m-1} (-1)^k \mu_{i, k} + \frac{i(2m+1-i)}{m(m+1)}, \tag{4.8}$$

where all undefined symbols are zero.

If we set the parameters $\mu_{i, j}$ for $1 \leqslant i < j \leqslant m-1$ equal to zero, we get the result in the theorem.

The next obvious step in an application of these techniques is the general n-step nilpotent simply connected and connected Lie group. The problem of computing $H^m(H_n, \mathbf{Z})$, for $\mathbf{Z}$ the integers and H_n the integral Heisenberg group, is also approachable and involves studying the decomposition of the tensor spaces $\otimes^m \mathbf{Z}^{2n}$ under the action of the symmetric group S_m acting by permuting the coordinates, which is treated in [ABW]. These methods may help algebra to gain some foothold in an area dominated by topological techniques, namely the computation of the cohomology of discrete groups, in particular that of $Sl(n, \mathbf{Z})$.

The explicit calculations in this paper should make it clear that mathematicians have at least a fighting chance to compute the cohomology of a group extension directly and completely, without having to rely on the spectral sequence to merely furnish quotients of a filtration, which still leaves the very hard problem of describing how to put these quotients together to form the cohomology group of the extension.

I am now in the process of working out how this theory fashions itself in the case of Lie and associative algebras and am getting glimpses of a metatheorem which states that, whenever there is a spectral sequence connecting two cohomological objects, this is but a shadow of a much stronger and more explicit connection between these objects, which is just waiting for its bottle to be rubbed the right way to emerge and work its magic.

All the reductions which we proved we could carry out in the case of an extension can also be effected in the case $G = AB$, where $A \cap B = e$ and A, B are not necessarily normal subgroups of G. Instead of the cross section s and associated 2-cocycle α, we have to concern ourselves with two projections $G \to A$, B as in [Ma2], where the theory was first worked out for H^2.

As a first step toward crafting an example for the metatheorem in the topological case, there is the work by Hirsch in [Hi1, Hi2], the first one of which appeared one year after Leray's first paper [Le1] and was overshadowed by it. Hirsch appears to have developed a method similar to mine and Lyndon's for computing exactly the homology of a fiber space, and not just the quotients in a filtration.

References

[ABW] K. Akin, D. Buchsbaum, and J. Weyman, Schur functors and Schur complexes, *Adv. in Math.* **44** (1982), 207–278.

[Du1] A. M. DuPré, Real Borel cohomology of locally compact groups, *Trans. Amer. Math. Soc.* **134** (1968), 239–260.

[Du2] A. M. DuPré, Real Heisenberg group extension isomorphism classes, reprint, 1989.

[Du3] A. M. DuPré, Group extension functional equations, preprint 1989.

[Hi1] G. Hirsch, Sur les groupes d'homologie des espaces fibrés, *Bull. Soc. Math. Belg.* **1** (1947), 24–33.

[Hi2] G. Hirsch, Sur les groupes d'homologie des espaces fibrés, *Bull. Soc. Math. Belg.* **6** (1953), 79–96.

[HS] G. Hochschild and J. P. Serre, Cohomology of group extensions, *Trans. Amer. Math. Soc.* **74** (1953), 110–134.

[Ho] R. HOWE, Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* **313** (1989), 539–570.

[Kl] A. KLEPPNER, Multipliers on Abelian groups, *Math. Ann.* **158** (1965), 11–34.

[Le1] J. LERAY, L'anneau homologique d'une representation, *C.R. Acad. Sci. Paris* **222** (1946), 1366–68.

[Le2] J. LERAY, L'anneau spectral et l'anneau filtré d'homologie d'un space localement compact et d'une application continue, *J. Math. Pures Appl.* **29** (1950), 1–139.

[Ln1] R. C. LYNDON, "The Cohomology Theory of Group Extensions," Harvard Univ. Press, Cambridge, MA, 1946.

[Ln2] R. C. LYNDON, The cohomology theory of group extensions, *Duke Math. J.* **15** (1948), 271–292.

[Ma1] G. W. MACKEY, Unitary representations of group extensions, I, *Acta. Math.* **99** (1958), 265–311.

[Ma2] G. W. MACKEY, Products of subgroups and projective multipliers, *in* "Hilbert Space Operators and Operator Algebras," Colloq. Math. Soc. János Bolyai, Vol. 5, pp. 401–413, North-Holland, Amsterdam, 1972.

[McL] S. MACLANE, "Homology," Springer-Verlag, Berlin/New York, 1963.

[Mo] C. C. MOORE, Extensions and low-dimensional cohomology theory of locally compact groups, I–IV, *Trans. Amer. Math. Soc.* **113** (1964), 40–86; **221** (1976), 1–58.

[Mos] M. MOSKOWITZ, Bilinear forms and 2-dimensional cohomology, *J. Austral. Math. Soc.* **41** (1986), 165–179.

[Mst] G. D. MOSTOW, Cohomology of topological groups and solvmanifolds, *Ann. Math. (2)* **73** (1961), 20–49.

[Ta] K. I. TAHARA, On the second cohomology groups of semidirect products, *Math. Z.* **129** (1972), 365–379.

[We] ANDRÉ WEIL, "Introduction à l'étude des variétés kählériennes," Hermann, Paris, 1958.

[Wg] D. WIGNER, Algebraic cohomology of topological groups, *Trans. Amer. Math. Soc.* **178** (1973), 83–93.