Partial Hamiltonian formalism, multi-time dynamics and singular theories

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Abstract

(with formulas)

We formulate singular (with degenerate Lagrangians) classical theories (for clarity, in local coordinates) *without involving constraints*. First, we recall the standard action principle (for pedagogical reasons and in order to establish notation). Then, applying it to the action (27)

$$S=\int\left(p_i dq^i-H_\alpha dq^\alpha-H_0 dt\right),$$

we develop a *partial* (in the sense that *not all* velocities are transformed to momenta) Hamiltonian formalism in an initially reduced phase space (with canonical coordinates q_i,p_i, where the number $n_p\leq n$ of momenta, see (17),

$$p_i=\frac{\partial L}{\partial\dot{q}^i},\quad i=1,\ldots,n_p,$$

is *arbitrary*, where n is the dimension of the configuration space) in terms of a partial Hamiltonian $H_0(q_i,p_i,q^\alpha,\dot{q}^\alpha)$, see (18)

$$H_0=p_i\dot{q}^i+\frac{\partial L}{\partial\dot{q}^\alpha}\dot{q}^\alpha-L,$$

and $(n-n_p)$ additional Hamiltonians $H_\alpha(q_i,p_i,q^\alpha,\dot{q}^\alpha)$, $\alpha=n_p+1,\ldots,n$, see (20)

$$H_\alpha=-\frac{\partial L}{\partial\dot{q}^\alpha},\quad \alpha=n_p+1,\ldots n$$

(*instead* of the remaining momenta p_α defined in the standard full Hamiltonian formalism (6)). In this way we obtain $(n-n_p+1)$ Hamilton-Jacobi equations (25)-(26)

$$\frac{\partial S}{\partial t}+H_0\left(t,q^i,\frac{\partial S}{\partial q^i},q^\alpha,\dot{q}^\alpha\right)=0,$$
$$\frac{\partial S}{\partial q^\alpha}+H_\alpha\left(t,q^i,\frac{\partial S}{\partial q^i},q^\alpha,\dot{q}^\alpha\right)=0,$$

which fully determine the dynamics. The equations of motion are first-order differential equations (33)-(34)

$$\dot{q}^i=\left\{q^i,H_0\right\}+\left\{q^i,H_\beta\right\}\dot{q}^\beta,$$
$$\dot{p}_i=\left\{p_i,H_0\right\}+\left\{p_i,H_\beta\right\}\dot{q}^\beta,$$
$$\{A,B\}=\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i}-\frac{\partial B}{\partial q^i}\frac{\partial A}{\partial p_i}$$

with respect to the canonical coordinates q_i, p_i and second-order differential equations (35)

$$\frac{\partial H_\alpha}{\partial \dot{q}^\beta}\ddot{q}^\beta + \frac{d}{dt}\left(\frac{\partial H_0}{\partial \dot{q}^\alpha} + \frac{\partial H_\beta}{\partial \dot{q}^\alpha}\dot{q}^\beta\right) = \left(\frac{\partial H_\beta}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial q^\beta} + \{H_\beta, H_\alpha\}\right)\dot{q}^\beta + \left(\frac{\partial H_0}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial t} + \{H_0, H_\alpha\}\right)$$

in the noncanonical coordinates q_α (which *have no* corresponding momenta). In the partial Hamiltonian formalism (which describes the same dynamics as the Lagrange equations of motion (3)), the number of momenta $n_p \leq n$ is arbitrary. The limit cases $n_p = n$ and $n_p = 0$ correspond to the standard Hamiltonian and Lagrangian dynamics (discussed in (37)-(41)), respectively.

If the Hamiltonians $H_0(q_i, p_i, q^\alpha)$, $H_\alpha(q_i, p_i, q^\alpha)$ do not depend of the noncanonical velocities $\dot{q}_\alpha$, conditions (42)

$$\frac{\partial H_0}{\partial \dot{q}^\beta} = 0, \qquad \frac{\partial H_\alpha}{\partial \dot{q}^\beta} = 0, \qquad \alpha, \beta = n_p + 1, \ldots, n,$$

then the second-order differential equations (35) become *purely algebraic* equations (43)

$$\left(\frac{\partial H_\beta}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial q^\beta} + \{H_\beta, H_\alpha\}\right)\dot{q}^\beta = -\left(\frac{\partial H_0}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial t} + \{H_0, H_\alpha\}\right),$$

with respect to $\dot{q}_\alpha$. In this case we can interpret the noncanonical coordinates q_α as additional times by (45)

$$\begin{aligned} &\tau^\mu = t, \qquad \mathsf{H}_\mu = H_0, \qquad \mu = 0, \\ &\tau^\mu = q^{\mu + n_p}, \qquad \mathsf{H}_\mu = H_{\mu + n_p}, \qquad \mu = 1, \ldots, (n - n_p), \end{aligned}$$

such that the partial Hamiltonian formalism becomes equivalent to multi-time dynamics with action (51)

$$\mathsf{S} = \int \left(p_i dq^i - \mathsf{H}_\mu d\tau^\mu\right)$$

and equations of motion (53)-(54)

$$\begin{aligned} dq^i &= \left\{q^i, \mathsf{H}_\mu\right\} d\tau^\mu, \\ dp_i &= \left\{p_i, \mathsf{H}_\mu\right\} d\tau^\mu, \end{aligned}$$

with additional (integrability) conditions for the Hamiltonians (55)

$$\left(\frac{\partial \mathsf{H}_\mu}{\partial \tau^\nu} - \frac{\partial \mathsf{H}_\nu}{\partial \tau^\mu} + \{\mathsf{H}_\mu, \mathsf{H}_\nu\}\right) d\tau^\nu = 0, \qquad \mu, \nu = 0, \ldots, (n - n_p).$$

The independence of the Hamiltonians $H_0(q_i, p_i, q^\alpha)$, $H_\alpha(q_i, p_i, q^\alpha)$ of the noncanonical velocities $\dot{q}_\alpha$ (conditions (42)) is satisfied in singular theories

$$\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta} = 0, \qquad \alpha, \beta = n_p + 1, \ldots, n$$

with degenerate Lagrangians, in the sense that the determinant of the Hessian matrix

$$W_{AB} = \left\| \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right\|, \qquad A, B = 1, \ldots, n$$

is zero $\det W_{AB} = 0$, in which case the rank r_W of the Hessian is less than or equal to the number of momenta: $r_W \le n_p$, see (58). If we choose $n_p > r_W$, then we obtain $(n_p - r_W)$ primary (and perhaps, higher-level) constraints (as in the Dirac theory), but if we assume the equality $n_p = r_W$, then there will be *no constraints at all*. The $(n - r_W)$ equations for *the same number* of $(n - r_W)$ noncanonical velocities $\dot{q}_\alpha$ (62) $F_{\alpha\beta}\dot{q}^\beta = G_\alpha$ constitute a standard *system of linear algebraic equations*, but *not constraints*, because we do not define the "extra" momenta p_α, and the dynamics is fully described *without them* by the equations of motion (60)-(62)

$$\begin{aligned} \dot{q}^i &= \left\{q^i, H_0\right\} + \left\{q^i, H_\beta\right\} \dot{q}^\beta, \\ \dot{p}_i &= \left\{p_i, H_0\right\} + \left\{p_i, H_\beta\right\} \dot{q}^\beta, \\ F_{\alpha\beta}\dot{q}^\beta &= G_\alpha, \end{aligned}$$

where $i = 1, \ldots, r_W$, $\alpha, \beta = r_W + 1, \ldots, n$, and

$$\begin{aligned} F_{\alpha\beta} &= \frac{\partial H_\alpha}{\partial q^\beta} - \frac{\partial H_\beta}{\partial q^\alpha} + \left\{H_\alpha, H_\beta\right\}, \\ G_\alpha = D_\alpha H_0 &= \frac{\partial H_0}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial t} + \left\{H_0, H_\alpha\right\}. \end{aligned}$$

Classification of singular theories can be made by the analysis of the linear algebraic system (62) in terms of the rank of the tensor $F_{\alpha\beta}$ (63). If its rank is full, i.e., equal to $(n - r_W)$, then we can solve the system (62) by

$$\begin{aligned} \dot{q}^\alpha &= \bar{F}^{\alpha\beta} G_\beta, \\ \bar{F}^{\alpha\beta} F_{\beta\gamma} &= F_{\gamma\beta} \bar{F}^{\beta\alpha} = \delta^\alpha_\gamma, \end{aligned}$$

and there will be no arbitrary parameters (gauge degrees of freedom) in the theory; in other cases, some of the noncanonical velocities $\dot{q}_\alpha$ will remain arbitrary, which is a sign of gauge theory. In both cases we define new antisymmetric brackets (69)

$$\{A, B\}_{nongauge} = \{A, B\} + D_\alpha A \cdot \bar{F}^{\alpha\beta} \cdot D_\beta B,$$

and (80)

$$\begin{aligned} \{A, B\}_{gauge} &= \{A, B\} + D_{\alpha_1} A \cdot \bar{F}^{\alpha_1\beta_1} \cdot D_{\beta_1} B, \\ \alpha_1, \beta_1 &= r_W + 1, \ldots, r_F, \\ r_F &= \operatorname{rank} F_{\alpha\beta}, \end{aligned}$$

which govern time evolution of physical variables (70)and (83) and present the equations of motion in the Hamilton-like form, (67)-(68)

$$\begin{aligned} \dot{q}^i &= \left\{q^i, H_0\right\}_{nongauge}, \\ \dot{p}_i &= \left\{p_i, H_0\right\}_{nongauge}, \end{aligned}$$

and (81)-(82)

$$\begin{aligned} \dot{q}^i &= \left\{q^i, H_0\right\}_{gauge}, \\ \dot{p}_i &= \left\{p_i, H_0\right\}_{gauge}. \end{aligned}$$

Finally, we clarify the origin of the Dirac constraints in our framework: if we define $(n - r_W)$ "extra" dynamical variables, that is to say, the momenta p_α by (86)

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}, \qquad \alpha = r_W + 1, \ldots, n,$$

then we obtain the standard primary constraints (87)

$$\Phi_\alpha = p_\alpha + H_\alpha = 0, \qquad \alpha = r_W + 1, \ldots, n,$$

and the equations of motion (90)-(91)

$$\begin{aligned} dq^A &= \left\{ q^A, H_{total} \right\}_{full} \ dt, \\ dp_A &= \left\{ p_A, H_{total} \right\}_{full} \ dt, \quad A = 1, \ldots, n, \\ \{A, B\}_{full} &= \frac{\partial A}{\partial q^A} \frac{\partial B}{\partial p_A} - \frac{\partial B}{\partial q^A} \frac{\partial A}{\partial p_A}, \end{aligned}$$

in terms of the Dirac total Hamiltonian (88)

$$H_{total} = p_i \dot{q}^i + p_\alpha \dot{q}^\alpha - L = H_0 + \dot{q}^\alpha \Phi_\alpha .$$

In this case our new brackets (69) and (80) will transform into the Dirac bracket.

At the end of the paper, quantization is discussed briefly.

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