Question:

Here are some approximate values:

$$\ln 2 \approx .7 \text{ and } \ln 10 \approx 2.3\,.$$

Use these approximations and an indirect argument using only arithmetic to explain how many decimal digits to expect in the number 2^{40}. Are you correct?

Solution:

Let x be an integer with d decimal digits. We note that the smallest integer with d decimal digits is 10^{d-1} as this is written as a 1 followed by $d-1$ 0's. The largest integer with d decimal digits is $10^d - 1$. This is one less than the smallest $d+1$ digit number. Alternatively, we can note that when written out, $10^d - 1$, is written as d 9's. So we have the inequality

$$10^{d-1} \leq x < 10^d.$$

Note that log is an increasing function so if we take the log of this equation (to any base) it will preserve the inequality. Since the exponents on the left and right side of the inequality are to the base 10, it is natural to take the $\log_{10}$ of this equation:

$$d - 1 \leq \log_{10}(x) < d. \qquad (*)$$

So we see that if x has d digits then $\log_{10}(x)$ will lie on the interval $[d-1, d)$.

Here is an alternative demonstration of the same fact. Write x in scientific notation as $x = c \times 10^{d-1}$. Note that $1 \leq c < 10$. Note also the $d-1$ in the exponent. For example, the six digit number 654321 is written in scientific notation as 6.54321×10^5. Now we can compute $\log_{10}(x) = \log_{10}(c) + d - 1$. Since $1 \leq c < 10$ we have $0 \leq \log_{10}(c) < 1$. So $d-1] \leq \log_{10}(x) < d$ as in equation $(*)$.

What this equation tells us is that if we want to know how many digits a number has we take the log base 10 of this number and we will get a number smaller than the number of digits by at most one. So we compute the log base 10 of 2^{40}:

$$\log_{10}\left(2^{40}\right) = 40 \log_{10}(2).$$

To complete this computation we need to estimate the value of $\log_{10}(2)$. We are given $\ln 2 \approx 0.7$ and $\ln 10 \approx 2.3$. We can use the change of base formula to estimate this log:

$$\log_{10}\left(2^{40}\right) = 40 \log_{10}(2) = 40 \frac{\ln 2}{\ln 10} \approx 40 \frac{0.7}{2.3} \approx 12.17.$$

From equation $(*)$ we can then see that 2^{40} has **13** decimal digits.

We can check our answer by computing 2^{40} on a calculator. On a scientific calculator I typed "$2 \wedge 40$" and it gave the result "1.0995116×10^{12}" which is a 13 digit number.