Workshop #1 Solution

Question:

Sketch the parabola $y = x^2$ and the line $y = 2x - 1$.

a) Show that $(1, 1)$ is the only point where the parabola and the line intersect.

b) From the sketch, find another line that passes through $(1, 1)$ and intersects the parabola only at one point $(1, 1)$.

c) Show that any line containing $(1, 1)$, other than $y = 2x - 1$ and the line in (b), must intersect the parabola in some point besides $(1, 1)$.
Suggestion What condition guarantees that the line $y = mx + b$ contains the point $(1, 1)$? What condition guarantees that the quadratic equation $x^2 = mx + b$ has only one root?

Solution:

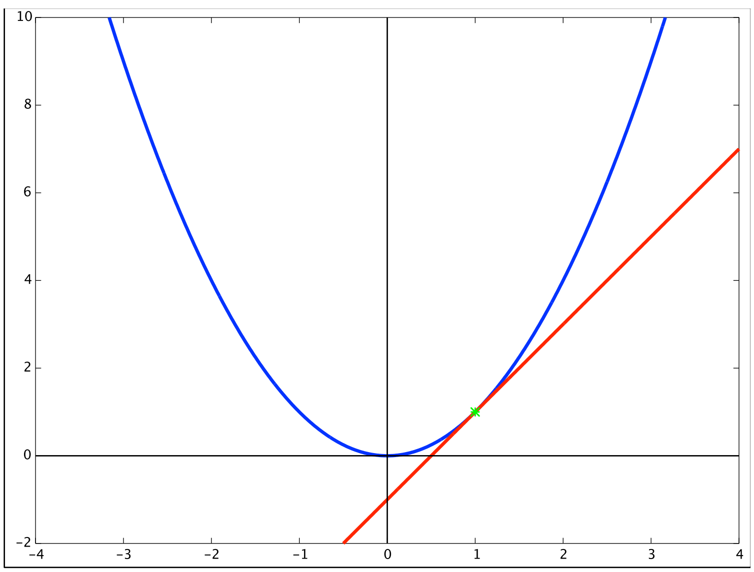

Figure 1: The curve $y = x^2$ is shown in blue. The line $y = 2x - 1$ is shown in red. The intersection point $(1, 1)$ is shown in green.

a) To see that $(1, 1)$ is the only point where these two curves intersect we equate them. So we set $x^2 = 2x - 1$ and solve for x. Rearranging we find $0 = x^2 - 2x + 1$. We can factor the right hand side of this equation to obtain: $0 = (x - 1)^2$. So the only solution is when $x = 1$ and in this case we find $y = 1^2 = 1$.

b) The vertical line $x = 1$ intersects the parabola at $(1, 1)$. Since $y = x^2$ is a function of x it cannot intersect a vertical line in more than one point.

c) To obtain a general equation of a line passing through $(1, 1)$ we use point-slope form. Let m be the slope of the line. Then the equation of a line with slppe m passing through $(1, 1)$

is $y - 1 = m(x - 1)$. Note that this excludes the vertical line $x = 1$, however we already considered this line in part (b). We can rearrange terms to write this line as $y = mx-(m-1)$. To find the points of intersection between this line and the parabola, we set $x^2 = mx-(m-1)$. We can rearrange this as $x^2 - mx + (m - 1) = 0$. We know $x = 1$ is a solution so we can factor out an $x - 1$ term. So we find $(x - 1)(x - (m - 1)) = 0$. Thus we have the solutions $x = 1$ and $x = m - 1$. Now if $m \neq 2$ we have $m - 1 \neq 1$ and thus these are two distinct solutions. So any line other than $y = 2x - 1$ and $x = 1$ will have real slope not equal to 2 and thus intersect the parabola $y = x^2$ at two distinct points.