Workshop Problems–November 21

1. a) Suppose $f(x)$ is defined on $0 \leq x \leq 1$ by the following rule:

$$f(x) \text{ is the first digit in the decimal expansion for } x.$$

For example, $f(1/2) = 5$ and $f(0.719) = 7$.

Sketch the graph of $y = f(x)$ on the unit interval with appropriate scales for x and for y. Use a graphical interpretation of the definite integral to compute $\int_0^1 f(x)\,dx$.

b) Suppose the function $g(x)$ is defined as follows:

$$g(x) \text{ is the second digit in the decimal expansion for } x.$$

For example, $g(0.437) = 3$. Compute $\int_0^1 g(x)\,dx$. Again, a graph may help.

2. Let $f(t)$ be a function defined on $[-1, 7]$ with the following properties

   (i) For $-1 \leq t \leq 0$, the graph of $f(t)$ is the upper-right quarter of a circle of radius 1 centered at $(-1, 0)$.

   (ii) For $0 < t < 2$, the graph of $f(t)$ is the bottom half of a circle of radius 1 centered at $(1, 0)$.

   (iii) For $2 \leq t < 4$, the graph of $f(t)$ is a straight line with left endpoint $(2, 1)$ and slope $3/2$.

   (iv) On the interval $[4, 7]$, the function $f(t)$ is continuous but its graph is an unknown curve.

   (v) Finally, it is known that

$$\int_3^5 f(t)\,dt = 4 \quad \text{and} \quad \int_5^7 f(t)\,dt = 5.$$

   Compute each of the following, **without using the fundamental theorem of calculus**.

   (a) $\displaystyle\int_2^{-1} 8f(t) + 1\,dt$ (b) $\displaystyle\int_4^7 f(t)\,dt$ (c) $\displaystyle\int_1^x f(t)\,dt$ where $2 \leq x \leq 4$

3. The goal of this workshop is to explore yet another way of computing integrals. It is safe to assume that we all can find the numerical value of

$$\int_{-1}^1 (1 - x^2)dx.$$

But what about using "horizontal Riemann sums"? Please be cautious that the method we will be exploring here is NOT the Riemann sums method. Consider the region bounded by $y = 1 - x^2$ and the x-axis. Answer each of the following parts carefully.

   (a) Sketch the region.

   (b) Estimate the area of the region by dividing the figure up *horizontally* and covering the figure with two rectangles. What is the estimated region.

   (c) *Calculator needed.* Now divide the figure up *horizontally* 10, 100, 1000, and 10,000 times. For each number of divisions, cover the region with the appropriate number of rectangles, and find the approximate area. Express the approximate area using summation notation.

   (d) Based on your estimates for (c), what do you think the area of the figure is? Now prove it by relating the "limit" to another integral. (Hint: think rotated.)

   (e) Does this agree with the explicit calculation for $\int_{-1}^1 (1 - x^2)dx$? Justify why these two numbers should be the same by comparing this method with the definition of the integral using Riemann sums.