Workshop Problems–November 16

1. Consider line segments which are tangent to a point on the right half ($x > 0$) of the curve $y = x^2 + 1$ and connect the tangent point to the x-axis. Several are displayed in Figure 1 below. If the tangent point is close to the y-axis, the line segment is long. If the tangent point is far from the y-axis, the line segment is also very long. Which tangent point has the shortest line segment?

   How to get started Suppose C is a *positive* number. What point on the curve has first coordinate equal to C? What is the slope of the tangent line at that point? Find the x-intercept of the resulting line. Compute the distance between the point on the curve and the x-intercept, and find the minimum of the *square* of that distance (minimizing the square of a positive quantity gets the same answer as minimizing the quantity, and here we get rid of a square root).

2. Define $S_6(n) = 1^6 + 2^6 + 3^6 + \cdots + n^6$. (In summation notation, $S_6(n) = \sum_{k=1}^{n} k^6$.) An explicit formula for $S_6(n)$ is known, and it is:

$$S_6(n) = \frac{1}{42}\left(6n^7 + 21n^6 + 21n^5 - 7n^3 + n\right) .$$

   (You can check the formula for small values of n.)

   Problem. Find some area and some approximating sum for this area which knowledge of this formula will allow you to evaluate exactly. Write the approximating sums, and evaluate the limit of these sums as $n \to \infty$ to compute the area.

3. Suppose $f(x) = \sqrt{3x + 6x^4}$.

   a) Prove that f is increasing on the interval $[0, 1]$.

   b) Write down a finite sum which will be within 10^{-10} of the true value of the area enclosed by the x-axis, $y = f(x)$, and $x = 1$. You are *not* asked to actually compute the sum, just describe it in any convenient fashion.

   Hint Your reasoning and your explanation may be guided by the picture in Figure 2 below. Note that the horizontal and vertical axes have different scales. The shaded rectangles represent the difference between right- and left-endpoint approximations.

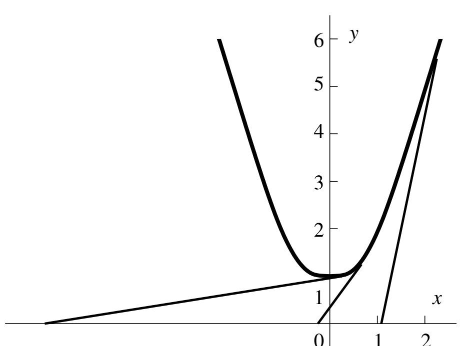

Figure 1: tangents to $y = x^2 + 1$

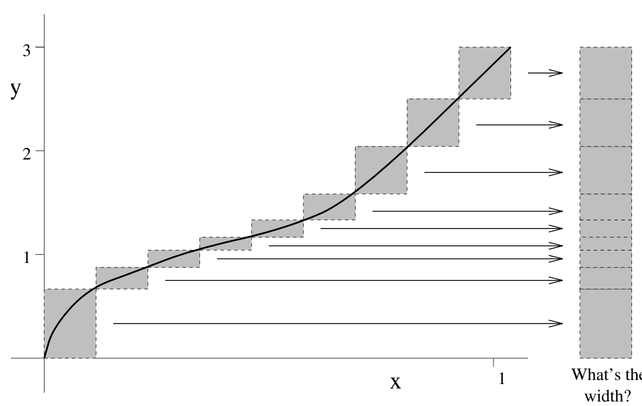

Figure 2: Area under $f(x) = \sqrt{3x + 6x^4}$