Workshop Problems–September 28

1. If a pizza is in a perfect circular shape, we know one can divide the pizza into halves in a precise way: cut along the diameter. Now the question is: Can we cut any pizza, not necessarily a disk, into halves precisely? Well, you can actually prove it's possible, using the material you learned in calculus.

   Here we model the pizza as some region Ω contained in the unit square $S = \{(x, y) : |x| \leq 1, |y| \leq 1\}$ on the plane, just like a pizza in a square box. And assume the boundary of the pizza is "nice", which means the area is well defined. For simlicity we assume the pizza is of equal thickness everywhere, so that we need only consider dividing the area. Let $|\Omega|$ be the area of Ω.

   (a) Now we can define a function $f(z)$ as the area of $\Omega \cap \{(x, y) \in S : x < z\}$, i.e. one of the two pieces you get while cutting along the line $x = z$. Choose a proper domain for $f(z)$. Can you guess its range?

   (b) Prove that the function $f(z)$ is continuous on its domain, by using the $\epsilon - \delta$ definition of limit. Hint: $|f(z) - f(w)| \leq |z - w|$.

   (c) Use the Intermediate Value Theorem to prove that there exists some c in the domain of $f(z)$, such that $f(c) = |\Omega|/2$.

   (d) How do you prove that your guess in part (a) for the range of f is correct?

2. Some lines which are tangent to the parabola $y = x^2$ also pass through the point $(2, 3)$. Find all of these lines. Graph the parabola and the tangent lines which were found on the same coordinate axes.

3. Let $f(x) = \min(x, x^2)$, where $\min(a, b)$ is the minimum of a and b (e.g. $\min(4, 2) = 2$).

   (a) Show that $\lim_{x \to 1} f(x)$ exists. (You do not have to use the ϵ-δ definition of a limit in this problem but it is more fun if you do use it.)

   (b) Is the function f continuous at $x = 1$?

   (c) Using the definition of the derivative at a point, show that f is not differentiable at $x = 1$. (Hint: For the derivative to exist at $x = 1$, a particular limit must exist. Analyze the left and right limits as x approaches 1.)

4. Suppose that $f(x)$ and $g(x)$ are differentiable functions, and the following information is known about them:

   $$f(2) = -3 \qquad f'(2) = 5 \qquad g(2) = 1 \qquad g'(2) = 2 \qquad g(0) = 2 \qquad g'(0) = 4$$

   (a) If $F(x) = \dfrac{f(x)}{g(x)}$, compute $F(2)$ and $F'(2)$.

   (b) If $G(x) = x^3 f(x) - 7g(x)$, compute $G(2)$ and $G'(2)$.

   (c) If $H(x) = \dfrac{3 + e^x}{g(x)}$, compute $H(0)$ and $H'(0)$.