Answers/Solutions to Review Problems for Midterm 2

Please note that this set of problems does *not* necessarily cover all topics that may appear on your exam.

1. Let $f(x) = x^3 + 3x + 2$. Let $g(x) = f^{-1}(x)$ be the inverse function. Find $g'(2)$ and $g'(6)$. **Answer:** We note that $f(0) = 2$ and $f(1) = 6$. Therefore, $g(2) = 0$ and $g(6) = 1$. The inverse function theorem tells us that $g'(x) = \frac{1}{f'(g(x))}$. Therefore, $g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(0)}$. So we compute $f'(x) = 3x^2 + 3$. So $f'(0) = 3$. Thus $g'(2) = \frac{1}{3}$. Next, $g'(6) = \frac{1}{f'(g(6))} = \frac{1}{f'(1)} = \frac{1}{6}$.

2. Let $y = e^{x+\sin(x^2)}$. Find $\frac{dy}{dx}$. **Answer:** $\dfrac{dy}{dx} = e^{x+\sin(x^2)}(1 + \cos(x^2)2x)$.

3. Let $f(x) = \ln(3 - 5x^2)^3$. Find $f'(x)$. **Answer:** $f'(x) = 3\ln(3 - 5x^2)^2 \dfrac{1}{3 - 5x^2} \cdot -10x = \dfrac{-30x\ln(3 - 5x^2)^2}{3 - 5x^2}$.

4. Let $g(x) = \sin^2(\sqrt{x^2 + 1}) + \cos^2(\sqrt{x^2 + 1})$. Find $g'(x)$. **Answer:** Recall that for any α we have $\sin^2 \alpha + \cos^2 \alpha = 1$. So $g(x) = 1$. Thus $g'(x) = 0$.

5. Let $e^x y + y^2 x + x^3 \cos(y) = 0$. Find $\frac{dy}{dx}$. Show that $(0, 0)$ is on the curve and that there is a vertical tangent at this point. **Answer:** We use implict differentiation: $e^x y + e^x \dfrac{dy}{dx} + 2y\dfrac{dy}{dx}x + y^2 + 3x^2\cos(y) - x^3\sin(y)\dfrac{dy}{dx} = 0$. Solving, $\dfrac{dy}{dx} = \dfrac{x^3\sin(y) - 2y - e^x}{e^x y + y^2 + 3x^2\cos(y)}$. To see that $(0, 0)$ is on the curve we plug in $x = 0$ and $y = 0$ into the curve. We find $0 + 0 + 0 = 0$ which is true. So $(0, 0)$ is on the curve. To see that we have a vertical tangent at this point we note that if we plug $x = 0$ and $y = 0$ into our expression for $\frac{dy}{dx}$ the numerator evaluates to -1 and the denominator to 0. So we have a vertical tangent at this point.

6. Consider the circle $(x - 2)^2 + (y - 4)^2 = 25$. Where on this circle is the slope of the tangent line equal to 1? **Answer:** Using implicit differntiation: $2(x - 2) + 2(y - 4)\dfrac{dy}{dx} = 0$. Solving, $\dfrac{dy}{dx} = \dfrac{-2(x - 2)}{2(y - 4)} = \dfrac{x - 2}{y - 4}$. If we set this to equal 1 we find $\frac{x-2}{y-4} = 1$ so $x - 2 = y - 4$ and thus $y = x + 2$. If we plug this into the equation of the circle we find $(x - 2)^2 + ((x + 2) - 4)^2 = 25$. We can simplify this to find $2x^2 - 8x - 17 = 0$. Using the quadratic formula we find $x = 2 \pm \frac{5}{2}\sqrt{2}$. Recall that we found $y = x + 2$. So we have the points $(2 + \frac{5}{2}\sqrt{2}, 4 + \frac{5}{2}\sqrt{2})$ and $(2 - \frac{5}{2}\sqrt{2}, 4 - \frac{5}{2}\sqrt{2})$.

7. Let $h(x) = \cos(x) + \sin(x)$. Find the minimum and maximum values attained by $h(x)$ on the interval $[0, 2\pi]$. **Answer:** First we find $h'(x) = -\sin(x) + \cos(x)$. If we set this equal to 0 we find $-\sin(x) + \cos(x) = 0$, so $\cos(x) = \sin(x)$ or equivalently, $\tan(x) = 1$. On $[0, 2\pi]$ this has solutions $x = \pi/4$ and $x = 5\pi/4$. Note that the endpoints of our interval are also critical points so we have four critical points $0, \pi/4, 5\pi/4, 2\pi$. Now we plug these point into h. We find $h(0) = 1 + 0 = 1$, $h(\pi/4) = \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2}$, $h(5\pi/4) = -\sqrt{2}/2 + -\sqrt{2}/2 = -\sqrt{2}$ and finally, $h(2\pi) = 1 + 0 = 1$. Thus the maximum value attained is $\sqrt{2}$ and the minimum is $-\sqrt{2}$.

8. Let $y = xe^{-x}$ be defined on $[0,2]$. Find the min and max of this function. **Answer:** We find $y' = e^{-x} - xe^{-x} = e^{-x}(1-x)$. So if we set $y' = 0$ we have $e^{-x}(1-x) = 0$ and thus a critical point at $x = 1$. Note that e^z is not zero for any value of z. We also have critical points at our endpoints so we have three critical points: $0, 1$ and 2. We compute $y(0) = 0$, $y(1) = e^{-1} = \dfrac{1}{e}$ and $y(2) = 2e^{-2} = \dfrac{2}{e^2}$. So the minimum is 0. Note that $\dfrac{1}{e} > \dfrac{2}{e^2}$ since cross-multiplying gives $e > 2$ which we know to be true. So the max of this function is $\frac{1}{e}$.

9. Let $f(x) = x^2 + 2x + 3$. Find the average rate of change of this function on the interval $[1,3]$. The mean value theorem says there is some point c on this interval at which $f'(c)$ attains this average rate of change. Find such a value c. **Answer:** The average rate of change is $\dfrac{f(3) - f(1)}{3-1} = \dfrac{18-6}{2} = 6$. So we want to find a value c on the interval where $f'(c) = 6$. We compute $f'(x) = 2x + 2$. So we set $2x + 2 = 6$ and solve to find $x = 2$. So at $c = 2$ we have $f'(c) = 6$.

10. Let $f(x) = \frac{1}{4}x^4 + 2x^3 - 3x^2 + 3x - 1$. Find all inflection points of $f(x)$. **Answer:** We first compute $f'(x) = x^3 + 6x^2 - 6x + 3$. Next we compute $f''(x) = 3x^2 + 12x - 6$. We want to find where this is zero. So we set $3x^2 + 12x - 6 = 0$. We can simplify this to $x^2 + 4x - 3 = 0$. This has roots $x = -2 \pm \sqrt{7}$. Note that $2 < \sqrt{7} < 3$ so we have $-5 < -2 - \sqrt{7} < 0 < 2 + \sqrt{7} < 1$. So we will use the points -5, 0 and 1 as intermediate points to test. We find $f''(-5) = 3{\cdot}25 + 12{\cdot}5 - 6 = 129$, $f''(0) = -6$ and $f''(1) = 3 + 12 - 6 = 9$. So since these points are positive, negative and positive respectively we see that both $-2 \pm \sqrt{7}$ are inflection points.

11. Two runners start running at the origin. One runs due North at 8 m/s. The second runs due East at 6m/s. How fast are they moving apart from each other when the first runner is 80 meters from the origin and the second is 60 meters from the origin? **Answer:** I highly recommend drawing a picture. Let y be the distance as a function of time the first runner is from the origin and x be the distance as a function of time the second runner is from the origin. We know that $\frac{dy}{dt} = 8$ and $\frac{dx}{dt} = 6$. The distance s between the runners is related to x and y by the Pythagorean theorem: $s^2 = x^2 + y^2$. So we differentiate this with respect to t. We find $2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$. We want to find $\frac{ds}{dt}$ and the only other vale we don't know is s. But when $y = 80$ and $x = 60$ we have $s = \sqrt{80^2 + 60^2} = 100$. So we can plug all our values in: $2 \cdot 100\dfrac{ds}{dt} = 2 \cdot 80 \cdot 8 + 2 \cdot 60 \cdot 6$. Solving we find $\frac{ds}{dt} = 10$ m/s.

12. An object moves along the curve $y = e^x$. At what point(s) is the object moving twice as fast in the y direction as it is in the x direction? **Answer:** We differentiate with respect to t. $\frac{dy}{dt} = e^x\frac{dx}{dt}$. If it is moving twice as fast in the y direction as in the x direction we have $\dfrac{dy}{dt}/\dfrac{dx}{dt} = 2$, and thus $e^x = 2$. So this occurs at the point $(\ln(2), 2)$. Note, another way to solve this problem is to see that the object is moving twice as fast in the y direction as in the x direction when the slope of the curve is 2. So we set $\dfrac{dy}{dx} = e^x = 2$. This gives $x = \ln(2)$ and thus the point $(\ln(2), 2)$ as before.

13. Let $g(x) = x\cos(x) + e^x + 3$. Using a linear approximation, estimate $g(0.05)$. **Answer:** We use an approximation around $x = 0$. We compute $g(0) = 0 + 1 + 3 = 4$. Next we find

$g'(x) = \cos(x) - x\sin(x) + e^x$. So $g'(0) = 1 - 0 + 1 = 2$. Thus we have the approximation $g(x) \approx g(0) + (x - 0)g'(0)$. Thus $g(0.05) \approx 4 + (0.05) \cdot 2 = 4.1$.

14. Let $h(x) = x^3 - 2x^2 + 3x - 4$. We want to find its roots using Newton's Method. Write down a recursive formula for the value of x_{n+1} in terms of x_n. If $x_0 = 1$, find the value of x_1? **Answer:** We compute $h'(x) = 3x^3 - 4x + 3$. Newton's method is expressed by the formula $x_{n+1} = x_n - \dfrac{h(x_n)}{h'(x_n)}$. So in this case we have $x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n^2 + 3x_n - 4}{3x_n^3 - 4x_n + 3}$. If $x_0 = 0$ we find $x_1 = 0 - \dfrac{-4}{3} = \dfrac{4}{3}$.

15. Let $f(x) = \sin^2(x)$. Find the critical points, local maxima and minima, global maximum and minimum and inflection points on the interval $[0, 2\pi]$. **Answer:** We compute $f'(x) = 2\sin(x)\cos(x)$. If we set $f'(x) = 0$ then we have either $\sin(x) = 0$ or $\cos(x) = 0$. So we have the following critical points $0, \pi/2, \pi, 3\pi/2, 2\pi$. Next we compute $f''(x)$. Note that $f'(x) = \sin(2x)$ (don't forget your double angle formulas!) so we find $f''(x) = 2\cos(2x)$. We can use the second derivative test to test our critical points: $f''(0) = 1$, $f''(\pi/2) = -2$, $f''(\pi) = 2$, $f''(3\pi/2) = -2$ and $f''(2\pi) = 2$. Thus $0, \pi$ and 2π are local minimums since the function is concave up at these points. The points $\pi/2$ and $3\pi/2$ are local maximums as the function is concave down at these points. Note that $f(0) = f(\pi) = f(2\pi) = 0$ so all three of these points are global minima. Similarly, $f(\pi/2) = f(3\pi/2) = 1$ so these two points are global maxima. Finally, the inflection points will occur when $f''(x)$ crosses the x-axis. Since $f''(x) = 2\cos(2x)$ we know that it crosses wherever $f''(x) = 0$. So we set $2\cos(2x) = 0$. Thus $x = \pi/4, 3\pi/4, 5\pi/4$ or $7\pi/4$ are inflection points.

16. Evaluate $\lim\limits_{x\to\infty} \dfrac{x^2\ln(x) + x}{e^x}$. **Answer:** Note that the numerator and denominator are both tending towards ∞ so we can apply L'Hôpital's rule. Thus the given limit equals

$$\lim_{x\to\infty} \frac{2x\ln(x) + x + 1}{e^x}.$$

The numerator and denominator are still both tending towards ∞ so we can apply L'Hôpital's rule again:

$$\lim_{x\to\infty} \frac{2(\ln(x) + 1) + 1}{e^x}.$$

Almost there... one more time:

$$\lim_{x\to\infty} \frac{2/x}{e^x}.$$

Now the denominator is approaching 0 and the numerator ∞. So the limit is 0.

17. Evaluate $\lim\limits_{x\to 0} \dfrac{\sin(x)\cos(x)}{e^x - 1}$. **Answer:** Note that the numerator and denominator are both tending towards 0 so we can apply L'Hôpital's rule. Thus the given limit equals

$$\lim_{x\to 0} \frac{\cos^2(x) - \sin^2(x)}{e^x} = \frac{1 - 0}{1} = 1.$$

18. Evaluate $\lim_{x\to\infty} \frac{x^{100}}{e^x}$. **Answer:** Note that the numerator and denominator are both tending towards ∞ so we can apply L'Hôpital's rule. Thus the given limit equals

$$\lim_{x\to\infty} \frac{100x^{99}}{e^x}.$$

Applying again:

$$\lim_{x\to\infty} \frac{100 \cdot 99x^{98}}{e^x}.$$

We see a pattern. Each time we apply L'Hôpital's rule the degree of the numerator decreases by one and the denominator remains unchanged. So after performing L'Hôpital's rule 100 times we get

$$\lim_{x\to\infty} \frac{100!}{e^x} = 0.$$

Note that the limit is zero because although 100! is a very large number it is a constant and the denomintor is going to ∞.

19. Evaluate $\lim_{x\to 0} x^2 \ln(1/x^2)$. **Answer:** Note that this approaches $0 \cdot \infty$. In these situations we can rewrite the expression so that it is in a correct form to use L'Hôpital's rule. So we write the limit as $\lim_{x\to 0} \frac{\ln(1/x^2)}{x^{-2}}$. Applying L'Hôpital's rule we obtain $\lim_{x\to 0} \frac{x^2 \cdot -2x^{-3}}{-2x^{-3}} = \lim_{x\to 0} x^2 = 0$.