Answers to Review Problems for Midterm 1

Please note that this set of problems does *not* necessarily cover all topics that may appear on your exam. Below are answers to these problems with sketches of some of the solutions. These were hastily typed, so please alert me to any mistakes/typos. Thanks and good luck!

1. Let $f(x) = e^{\sin(2x+3)}$. Find $f'(x)$. **Answer:** $f'(x) = e^{\sin(2x+3)} \cos(2x+3) \cdot 2$.

2. Let $f(x) = \tan(2\pi x)$. Find $f'(1/8)$. **Answer:** $f'(x) = 2\pi \cdot \sec^2(2\pi x)$. So $f'(1/8) = 2\pi \sec^2(\pi/4) = 2\pi(\sqrt{2})^2 = 4\pi$.

3. Let $y = \sin(x^3)\cos(2x)$. Find $\dfrac{dy}{dx}$. **Answer:** $\dfrac{dy}{dx} = 3x^2\cos(x^3)\cos(2x) - 2\sin(x^3)\cos(2x)$.

4. Find the tangent line to $y = e^x + 5x^2$ at $x = 1$. **Answer:** $y' = e^x + 10x$. So the slope of the line is $y'(1) = e + 10$. The line passes through the point $(1, y(1)) = (1, (e+5))$. So using point-slope form the equation of the tangent line is $y - (e+5) = (e+10)(x-1)$.

5. Solve $\log(x+2) - \log(x-1) = \log(2)$ for x. **Answer:** The base of the log does *not* matter. Exponentiating both sides to the base of the log we find $\dfrac{x+2}{x-1} = 2$. Thus $x + 2 = 2x - 2$. Solving, $x = 4$. Notice that if we plug in $x = 4$ to the left hand side we get $\log(6) - \log(3) = \log(6/3) = \log(2)$.

6. Evaluate (or state that the limit does not exist): $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^3 - 3x^2 + 5}}{x^2 + x - 10}$. **Answer:** The limit is 0. If we factor out an x^3 from the radical in the numerator and a x^2 from the denominator we will find our limit grows as $\dfrac{x^{3/2}}{x^2} \to 0$ since $3/2 < 2$.

7. Evaluate (or state that the limit does not exist): $\lim\limits_{x\to\infty} \sin x$. **Answer:** The limit does not exist. The function $\sin(x)$ osscilates between 1 and -1 infinitely. It never settles near either value.

8. Evaluate (or state that the limit does not exist): $\lim\limits_{x\to\infty} \dfrac{x^3 + 2}{x^7 - x}$. **Answer:** The limit is 0. The power in the denominator is 7 which is larger than the power in the numerator, 3.

9. Let $f(x) = \dfrac{x^3 - x^2 + 2}{x^3 - 1}$. Find all vertical and horizontal asymptotes. **Answer:** We see that $x^3 - 1 = (x-1)(x^2 + x + 1)$. So the denominator is zero when $x = 1$. Using the quadratic formula we can see that there is no real x such that $x^2 + x + 1 = 0$. So this is the only root of the denominator. Since the numerator equals 2 when $x = 1$ we see that we have an asymptote at $x = 1$ (had the numerator equaled 0 at $x = 1$ we would have had a hole at $x = 1$). Next, to find the horizontal asymptotes we take the limit as $x \to \infty$ and $x \to -\infty$ Since the highest powers in the numeartor and denominator are 3 and they both have coefficient 1 these limits tend to $\frac{1}{1} = 1$. So we have a horizontal limit on both sides at $y = 1$.

10. Use the limit definition of the derivative to compute the derivative of $y = 2x^2 + 5x - 6$. **Answer:** $y' = \lim_{h\to 0} \dfrac{2(x+h)^2 + 5(x+h) - 6 - (2x^2 + 5x - 6)}{h} = \lim_{h\to 0} \dfrac{h^2 + 4xh + 5h}{h} = \lim_{h\to 0} \dfrac{h}{+}4x + 5 = 4x + 5$. Note that this is indeed the correct answer.

11. Evaluate (or state that the limit does not exist): $\lim_{x\to 0} \dfrac{\sin(x)\cos(x)}{x}$. **Answer:** Recall that $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$. So $\lim_{x\to 0} \dfrac{\sin(x)\cos(x)}{x} = \lim_{x\to 0} 1 \cdot \cos(x) = \cos(0) = 1$. Alternatively, recall that $\sin(2x) = 2\sin(x)\cos(x)$, so $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$. Thus, $\lim_{x\to 0} \dfrac{\sin(x)\cos(x)}{x} = \lim_{x\to 0} \dfrac{\frac{1}{2}\sin(2x)}{x} = \lim_{x\to 0} \dfrac{\sin(2x)}{2x} = 1$.

12. Show that there is some x such that $x^2 = \sin(x) + 5$. **Answer:** Set this equation equal to 0 and call the function f. So $f(x) = x^2 - \sin(x) - 5$. We see that if f has a root then this gives a solution to the equation. So we want to show that f has a root. We will use the intermediate value theorem (IVT). Note, that $f(x)$ is continuous since it is the sum of continuous functions. Consider the interval $[0, 2\pi]$. We see that $f(0) = 0 - 0 - 5 = -5$. Next $f(2\pi) = (2\pi)^2 - 5$. Since $2\pi > 6$, we see that $(2\pi)^2 > 36$ and so $f(2\pi) > 31$. In particular we have $f(0) < 0$ and $f(2\pi) > 0$. So as 0 lies in the interval between $f(0)$ and $f(2\pi)$, there exists some c such that $0 < c < 2\pi$ such that $f(c) = 0$. Then $f(c) = c^2 - \sin(c) - 5 = 0$ so $c^2 = \sin(c) + 5$ so we have found a solution to this equation on the interval $[0, 2\pi]$.