Answers/Partial Solutions to Second Round of Review Problems for Math 151 Final

1. Arrange A, B and C in order from least to greatest. Explain why.

$$A = \int_{-100}^{100} x\sin^2(x)\ dx \qquad B = \int_0^{100} x\sin^2(x)\ dx \qquad C = \int_0^{100} x\ dx$$

Solution: First note that $x\sin^2(x)$ is an odd function. Since in A we are integrating over an interval centered at 0 the right side exactly cancels the left side. So $A = 0$. Next, $C > 0$ since we are integrating a positive function. Finally, since $x\sin^2(x) \le x$ we see that $B < C$. So $A < B < C$.

2. Compute $\int_{-2}^{4} |x|\sin(x^2)\ dx$. **Solution:** Recall, $|x| = -x$ if $x < 0$ and $|x| = x$ if $x \ge 0$. So

$$\begin{aligned}\int_{-2}^{4} |x|\sin(x^2)\ dx &= \int_{-2}^{0} |x|\sin(x^2)\ dx + \int_0^4 |x|\sin(x^2)\ dx\\ &= \int_{-2}^{0} -x\sin(x^2)\ dx + \int_0^4 x\sin(x^2)\ dx.\end{aligned}$$

Both integrals can be evaluated by the substitution $u = x^2$. The answer is approximately 4.74.

3. Find the area bounded by the curves $y = x^2 - 10$ and $y = 2x - 2$. **Solution:** We can find where the curves intersect by equating $x^2 - 10 = 2x - 2$. Rearranging: $x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0$. So $x = -2, 4$ are the intersection points. So we integrate $\int_{-2}^{4} (2x-2) - (x^2-10)\ dx = 36$.

4. Evaluate the integral $\int e^{5x}\sec(e^{5x})$. **Solution:** Let $u = e^{5x}$ then $du = 5e^{5x}dx$. Substituting, the integral becomes $\frac{1}{5}\int \sec(u)\ du = \ln|\sec(u) + \tan(u)| + C = \ln|\sec(e^{5x}) + \tan(e^{5x})| + C$.

5. Write the following expression in sum notation: $3 + 7 + 11 + 15 + 19 + \ldots + 83$. **Solution:** Note that the terms increase by 4 each time. So if we index the sum by $k = 1, 2, \ldots$ then our term should be $4k+$something. When $k = 1$ we see that we should add -1 to get 3. So we sum $4k - 1$. If we sum $k = 1, \ldots, n$ then $4n - 1 = 83$ so $n = 21$. Thus our sum is $\sum_{k=1}^{21} 4k - 1$.

6. The number of cars passing a parked cop car on a highway at a given time is given by $f(t) = (1 + \sin(t^2))t$ (in cars per hour). What quantity does the integral $\int_0^8$ represent? Evaluate this integral. **Solution:** The integral represents the number of cars that pass in the first eight hours. $\int_0^8 (1 + \sin(t^2)t\ dt = \int_0^8 t\ dt + \int_0^8 t\sin(t^2)\ dt$. The first integral is straight forward and the second is accomplished by making the substitution $u = t^2$. The answer is approximately 32.3 (perhaps not a very realisitc problem).

7. Compute $\sum_{n=1}^{100}\left(\frac{1}{n} - \frac{1}{n+1}\right)$. **Solution:** If we write out the first few terms (from $n = 1$ to 4) we notice a pattern:

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)$$

All but the $\frac{1}{1}$ and the $-\frac{1}{4}$ are cancelled. So the above sum equals $\frac{1}{1} - \frac{1}{100} = \frac{99}{100}$.

8. A dart randomly strikes the unit square, which has corners $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. What are the chances that it strikes a point of the form (a, b) where $a^2 > b$? **Solution:** Points of this form lie below the curve $y = x^2$. So we need to find the area below $y = x^2$ in the unit square. Its area is $\int_0^1 x^2\, dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{3}$. Since the area of the unit square is 1, the probability is $\frac{1}{3}$.

9. Compute $\lim_{x\to\infty} \frac{\int_0^x e^{\sqrt{y}}\, dy}{e^x}$. **Solution:** Note that the numerator and denominator in this expression are approaching infinity. So we can apply L'Hopitals rule. The derivative of the numerator is accomplished via the fundamental theorem of calculus. So we obtain $\lim_{x\to\infty} \frac{e^{\sqrt{x}}}{e^x} = \lim_{x\to\infty} e^{-\sqrt{x}} = 0$.

10. Find the area of the region lying to the right of $x = y^2 - 5$ and to the left of $x = 3 - y^2$. **Solution:** Equating these curves and solving we find $y = \pm 2$. So we integrate $\int_{-2}^{2} \left[(3 - y^2) - (y^2 - 5)\right]\, dy = \frac{64}{3}$.