Solutions to 151 Final Exam Practice Problems

Problem 1

Suppose from $t = 0$ to $t = \pi$ a particle moves along the x-axis according to the velocity function $v(t) = \sin(t) - \cos(t)$. The particle starts at the origin.

1. Give a function $x(t)$ that describes the particles position over time.
2. What is the furthest the particle travels from the origin?
3. What is the total distance travelled by the particle?

Solution:

We integrate $\int v(t)\ dt = \int \sin(t) - \cos(t)\ dt = -\sin(t) - \cos(t) + C$. SInce $x(0) = 0$ we have $0 = -0 - 1 + C$ and so $C = 1$. Thus $x(t) = -\sin(t) - \cos(t) + 1$. The furthest distance will occur either at the endpoints 0 or π or when the velocity is zero. Solving $\sin(t) - \cos(t) = 0$ on $[0, \pi]$ we have $\sin(t) = \cos(t)$ or $\tan(t) = 1$ which has solution $t = \pi/4$. Now we plug in $x(0) = 0$, $x(\pi/4) = 1 - \sqrt{2}$ and $x(\pi) = 2$. So the furthest the particle travels from the origin is 2 (note $2 > \sqrt{2} - 1$). Finally we are asked to find the distance travelled by the particle. Note that the *displacement* is 2. However, the distance is given by $\int_0^\pi |v(t)|\ dt$. To break this up we observe when $v(t)$ is positive versus negative. We solved for when it is zero above (when $t = \pi/4$). On $(0, \pi/4)$ we have $v(t) < 0$ (for example, plug in $\pi/6$) and it is positive on the rest of the interval. So we have $\int_0^\pi |v(t)| = \int_0^{\pi/4} cos(t) - sin(t)\ dt + \int_{\pi/4}^{\pi} \sin(t) - \cos(t)\ dt = (\sqrt{2} - 1) + (\sqrt{2} + 1) = 2\sqrt{2}$.

Problem 2

Suppose $F(x) = \int_{-1}^{x^2+3x} \cos(t^2)\ dt$. Find $F'(0)$.

Solution:

By the second fundamental theorem of calculus, $F'(x) = \cos((x^2 + 3x)^2)(2x + 3)$. Note that the factor $2x + 3$ comes from the chain rule. Plugging in, $F'(0) = \cos(0)(2 \cdot 0 + 3) = 3$.

Problem 3

Let $f(x)$ be given by $-2x+4$ when $x<2$ and by the upper half of a circle of radius 4 centered at $(6,0)$ for $2 \leq x \leq 10$.

1. Compute $\displaystyle\int_0^6 f(x)\ dx$.
2. Compute R_6, i.e. the right Riemann sum using six rectanges that estimates the integral given above.

Solution:

$\displaystyle\int_0^6 f(x)\ dx = \int_0^2 f(x)\ dx + \int_2^6 f(x)\ dx$. The first integral gives the area of a right triangle of height 4 and base 2 thus has area 4. The second integral is a quarter of a circle of radius 4. This has area $\frac{1}{4}\pi \cdot 4^2 = 4\pi$. Thus the area is $4+4\pi$. Next, the Riemann sum. If $N=6$ we have $\Delta x = \dfrac{6-0}{6} = 1$. The endpoints are $x_0 = 0, x_1 = 1, \ldots, x_6 = 6$. We now need to find the heights of our points. $f(1) = 2$ and $f(2) = 0$. Next we can parameterize the circle as $(x-6)^2+y^2 = 16$ and so the upper half is $y = \sqrt{16-(x-6)^2}$. Thus $f(3) = \sqrt{16-9} = \sqrt{7}$. Similarly, $f(4) = \sqrt{12} = 2\sqrt{3}$, $f(5) = \sqrt{15}$ and $f(6) = 4$. Thus

$$R_6 = 1 \cdot (f(1) + f(2) + \ldots + f(6)) = 2 + 0 + \sqrt{7} + 2\sqrt{3} + \sqrt{15} + 4 = 6 + \sqrt{7} + 2\sqrt{3} + \sqrt{15}.$$

Problem 4

Compute $\int x\sqrt{x+1}\ dx$

Solution:

Let $u = x+1$. Then $du = dx$ and $x = u-1$. Substituting we have

$$\begin{aligned}
\int (u-1)\sqrt{u}\ du &= \int u^{3/2} - u^{1/2}\ du \\
&= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C
\end{aligned}$$

Problem 5

Let $g(x) = \frac{e^x - x - 1}{x^2}$ when $x \neq 0$. What value should we define $f(0)$ for f to be continuous.

Solution:

We want $g(0) = \lim_{x \to 0} \dfrac{e^x - x - 1}{x^2}$. Note that the numeator and denominator both approach 0 so we can use L'Hopital's rule. Thus $\lim_{x \to 0} \dfrac{e^x - x - 1}{x^2} = \lim_{x \to 0} \dfrac{e^x - 1}{2x}$. We still have the indeterminant $\frac{0}{0}$ so we use L'H one more time $\ldots = \lim_{x \to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$. Thus we define $g(0) = \dfrac{1}{2}$.

Problem 6

A right triangle with a vertex fixed at the origin is growing. One leg is growing to the right along the x-axis at a rate of 8 m/s. Another leg is shrinking along the y-axis (moving from positive toward the origin) at a rate of 5 m/s. How fast is the area of the triangle changing when $x = 12$ and $y = 5$? Is it increasing or decreasing? How fast is the hypotenuse changing? Is it increasing or decreasing?

Solution:

We have $\dfrac{dx}{dt} = 8$ and $\dfrac{dy}{dt} = -5$ (note that $\dfrac{dy}{dt}$ is negative since the particle is moving downward). For area we have $A = xy$. Differentiating, $\dfrac{dA}{dt} = \dfrac{dx}{dt}y + \dfrac{dy}{dt}x$. Plugging in $\dfrac{dA}{dt} = 8 \cdot 5 + (-5) \cdot 12 = 40 - 60 = -20\ m^2/s^2$. The area is decreasing. Next the hypotenuse is given by $h^2 = x^2 + y^2$. Differntiating, $2h\dfrac{dh}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$. Simplifying, $h\dfrac{dh}{dt} = x\dfrac{dx}{dt} + y\dfrac{dy}{dt}$. At the instant of interest, the hypotenuse is 13 (it helps to remember your Pythagorean triples!). Plugging in, $13\dfrac{dh}{dt} = 12 \cdot 8 + \cdot 5 \cdot (-5) = 96 - 25 = 71$. Thus $\dfrac{dh}{dt} = \dfrac{71}{13}\ m/s$.

Problem 7

A long river runs North to South. You have 100 feet of fencing with which to make a rectangular enclosure on the East bank. The borders of this enclosure will be the river to the West and three lines of fencing. What is the maximal area that can be formed?

Solution:

Let x be the width in feet (East to West) and y be the height (North to South) of the enclosure. The area is given by $A = xy$. The 100 feet of total fencing gives the constraint: $2x + y = 100$. Solving, $y = 100 - 2x$. Thus $A = x(100 - 2x)$. Next, let us note the domain of A which we now consider a function of x. We have $x \geq 0$ and since we must have $y \geq 0$ we have $x \leq 50$. So the domain is $[0, 50]$. To find the maximum we differentiate $A' = 1 \cdot (100 - 2x) + x \cdot (-2) = 100 - 4x$. We set $A' = 0$ which gives $100 - 4x = 0$ and thus $x = 25$ as a critical point. Plugging in $A(25) = 25 \cdot 50 = 1250$. Checking the endpoints, $A(0) = 0$ and $A(50) = 0$. So the maximum is 1250 ft^2.

Problem 8

Find:

1. $\int_0^{2\pi} \cos^2(x) + \sin^2(x)\ dx$

2. $\int_0^{2\pi} \cos^2(x) - \sin^2(x)\ dx$

3. $\int x^2(\cos(x^3) - x)\ dx$

Solution:

1. Recall $\cos^2(x) + \sin^2(x) = 1$. Thus we have $\displaystyle\int_0^{2\pi} 1\ dx = 2\pi$.

2. One way to do this problem is to recall $\cos(2x) = \cos^2(x) - \sin^2(x)$. Then we compute $\displaystyle\int_0^{2\pi} \cos(2x)$ via the subtitution $u = 2x$ which gives $du = 2dx$ and the new bounds are 0 and π. We then have $\displaystyle\int_0^{4\pi} \cos(u)\frac{du}{2} = \sin(u)\Big|_0^{4\pi} = 0 - 0 = 0$. Another, cheeky way to dot his problem is to write $\displaystyle\int_0^{2\pi} \cos^2(x) - \sin^2(x)\ dx = \int_0^{2\pi} \cos^2(x) - \int_0^{2\pi} \sin^2(x)$. Since $\sin^2(x)$ and $\cos^2(x)$ have the same curve on $[0, 2\pi]$ just shifted by $\pi/2$ their integrals are the same. So the difference is zero. The brute force way to do the problem is to use the identities $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$ and $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$.

3. $\displaystyle\int x^2(\cos(x^3) - x)\ dx = \int x^2 \cos(x^3)\ dx - \int x^3\ dx$. The second integral is just $\frac{1}{4}x^4$ (plus a constant of integration). The first intergral can be done via substituting $u = x^3$ which gives $du = 3x^2\ dx$. We obtain $\displaystyle\int x^2 \cos(x^3)\ dx = \int x^2 \cos(u)\frac{du}{3x^2} = \frac{1}{3}\int \cos(u)du = \frac{1}{3}\sin(u) = \frac{1}{3}\sin(x^3)$. So the answer is $\frac{1}{3}\sin(x^3) - \frac{1}{4}x^4 + C$.

Problem 9

Consider the curve $x^2 + y^2 = 3xy - 1$. Show that the point $(1, 2)$ lies on the curve. Find the tangent line at this point.

Solution:

To see that $(1, 2)$ is on the curve we show that the equation is satisfied: $1^2 + 2^2 = 3 \cdot 1 \cdot 2 - 1$. Both sides equal 5. To find the tangent line we differentiate, $2x + 2y\frac{dy}{dx} = 3(y + x\frac{dy}{dx})$. Substituting, $2 \cdot 1 + 2 \cdot 2 \cdot \frac{dy}{dx} = 3(2 + 1 \cdot \frac{dy}{dx})$. Simplifying, $2 + 4\frac{dy}{dx} = 6 + 3\frac{dy}{dx}$. Thus, $\frac{dy}{dx} = 4$. So the tangent line is $y - 2 = 4(x - 1)$.

Problem 10

Consider the equation $x = \cos(x)$. Show that this has a root. Next, if $x_0 = 0$ is an initial guess of the root what would be the next estimate, x_1, using Newton's method.

Solution:

Let $f(x) = x - \cos(x)$. Note that f is a continuous function. Observe $f(0) = 0 - 1 = -1 < 0$ and $f(\pi) = \pi - (-1) = \pi + 1 > 0$. So by the intermediate value theorem, f has a root between 0 and π. Next, we have $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. In this case $f'(x) = 1 + \sin(x)$. Thus $x_1 = 0 - \frac{-1}{1} = 1$.

Problem 11

Use linear approximation to estimate $\sqrt[5]{33}$. Is this an underestimate or an overestimate? Explain.

Solution:

Let $f(x) = x^{1/5}$. We know $f(32) = 2$. Differentiating, $f'(x) = \dfrac{1}{5}x^{-4/5}$. Thus $f'(32) = \frac{1}{5} \cdot 2^{-4} = \frac{1}{5 \cdot 16} = \frac{1}{80}$. Thus our linearization is $L(x) = f(32) + f'(32)(x - 32) = 2 + \frac{1}{80}(x - 32)$. So $f(33) \approx L(33) = 2 + \frac{1}{80} \cdot 1 = 2 + 1/80 = 2.0125$. Next, note that the curve is concave down. To see this we can compute $f''(x) = \frac{-4}{25}x^{-9/5}$ which is negative for any $x > 0$. So the tangent line lies above the curve. Thus this is an overestimate. By the way, $(2.0125)^5 \approx 33.012578$, so this is indeed a pretty good estimate (and this confirms it is a slight overestimate).

Problem 12

1. $\lim_{x \to \pi/2} \dfrac{x - \pi/2}{\sin(x)}$

2. $\lim_{x \to 1} \dfrac{\ln(x)}{x - 1}$

3. $\lim_{x \to 0} x^{\sin(x)}$

Solution:

1. The numerator is approaching 0 and the denomiantor is approaching 1 Thus the limit is 0. Don't use L'Hopital's rule if it doesn't apply!

2. This is the indeterimant $\dfrac{0}{0}$. Thus we use L'Hopital's rule: $\lim_{x \to 1} \dfrac{\ln(x) - 1}{x - 1} = \lim_{x \to 1} \dfrac{1/x}{1} = 1$.

3. Let $y = x^{\sin(x)}$. Then $\ln(y) = \sin(x)\ln(x)$. Then, $\lim_{x \to 0} \sin(x)\ln(x)$ is the indeterminant $0 \cdot -\infty$. So we rearrange so we can use L'H. $\sin(x)\ln(x) = \frac{\ln(x)}{\csc(x)}$. Then we have $\lim_{x \to 0} \dfrac{\ln(x)}{\csc(x)} = \lim_{x \to 0} \dfrac{1/x}{-\csc(x)\cot(x)} = \lim_{x \to 0} \dfrac{-\sin(x)\tan(x)}{x}$. Now recall $\frac{\sin(x)}{x} \to 1$ as $x \to 0$. Since $\tan(x) \to 0$ this limit is 0. Now! Don't forget we are computing $\lim_{x \to 0} \ln(y)$. So we need to exponentiate. Our answer is $e^0 = 1$.