## Math 151, Extra Credit Problem, Due December 3, 2013

## Worth 5 points

You want to compute  $\int_{1}^{9} x^{3} dx$  by approximations using Rieman sums. Let N be the number of rectangles that you will use for your approximation. Estimate the area by computing left sums and right sums for N = 8 and then computing left sums and right sums for N = 16. Why does one type of sum overestimate and the other underestimate? Suppose that a friend computes a Riemann sum with N = 117 and obtains 1615.2. Did your friend use a left or right Riemann sum? Without adding up 117 numbers can you compute the other sum (left or right)?

**Solution:** If N = 8 then  $\Delta x = \frac{9-1}{8} = 1$ . So if  $R_L(8)$  is the left Riemann sum with 8 rectangles then we have  $R_L(8) = 1(1^3 + 2^3 + 3^3 + \ldots + 8^3) = 1296$ . The right sum,  $R_R(8)$  begins at x = 2 and ends at x = 9. So  $R_R(8) = 1(2^3 + 3^3 + \ldots + 8^3 + 9^3) = 2024$ .

Next, if 
$$N = 16$$
 then  $\Delta x = \frac{9-1}{16} = \frac{1}{2}$ . So  $R_L(16) = \frac{1}{2} \left( 1^3 + 1.5^3 + 2^3 + \ldots + 8^3 + 8.5^3 \right) = \frac{1}{2} \left( 1^3 + 1.5^3 + 2^3 + \ldots + 8^3 + 8.5^3 \right)$ 

1463. The right sum,  $R_R(16)$  begins at x = 1.5 and ends at x = 9. So  $R_R(16) = \frac{1}{2} \left( 1.5^3 + 2^3 + 2.5^3 + \ldots + 8^3 + 1827 \right)$ .

The left sums underestimate and the right sums overestimate. This is because  $y = x^3$  is an increasing function. So when we draw a rectangle from a left endpoint, the rectangle is contained within the curve. However, when we draw a rectangle from the right endpoint the height of the rectangle is above the curve on that region. For a decreasing function, the left sum overestimates and the right sum underestimates.

Now, let N = 117 and we are told that a left or right sum equals 1615.2. We note that  $\int_{1}^{9} x^{3} dx = \frac{x^{4}}{4} \Big|_{1}^{9} = 1640$ . Since 1615.2 < 1640 we know that this is a left sum (since it underestimated). Now to compute the right sum, note that when we computed the left and right sums above many of the terms in the sums where the same. In fact, terms 2, 3, ..., N of the left sum are the exact same as terms  $1, 2, \ldots, N - 1$  of the right sum. So we know that  $R_{L}(117) = \frac{8}{117} \left(1^{3} + (1 + 8/117)^{3} + \ldots + (9 - 8/117)^{3}\right) = 1615.2$ . So from 1615.2 we can subtract  $\frac{8}{117} \cdot 1^{3}$  and then add  $\frac{8}{117} \cdot 9^{3}$  and this will give us  $R_{R}(117)$ . So

$$R_R(117) = 1615 - \frac{8}{117} \cdot 1^3 + \frac{8}{117} \cdot 9^3 = 1665$$