Math 250, Quiz #2 Solutions, February 11, 2015

For this quiz $A = \begin{bmatrix} 1 & 1 & 4 \\ 3 & 3 & 12 \\ 2 & 0 & 2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}$.

1. Calculate the reduced row echelon form of the matrix given below for the systen of linear equations $A\vec{x} = \vec{b}$. Indicate each elementary row operation that you use.
Solution:

$$\begin{bmatrix} 1 & 1 & 4 & 1 \\ 3 & 3 & 12 & t \\ 2 & 0 & 2 & 0 \end{bmatrix} \xrightarrow[-2\vec{r_1}+\vec{r_3}\to\vec{r_3}]{-3\vec{r_1}+\vec{r_2}\to\vec{r_2}} \begin{bmatrix} 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & t-3 \\ 0 & -2 & -6 & -2 \end{bmatrix} \xrightarrow{\vec{r_1}\leftrightarrow\vec{r_2}} \begin{bmatrix} 1 & 1 & 4 & 1 \\ 0 & -2 & -6 & -2 \\ 0 & 0 & 0 & t-3 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}\vec{r_1}\to\vec{r_1}} \begin{bmatrix} 1 & 1 & 4 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & t-3 \end{bmatrix} \xrightarrow{\vec{r_1}-\vec{r_2}\to\vec{r_1}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & t-3 \end{bmatrix}$$

Now if $t = 3$ then the matrix above is in reduced row echelon form: $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

However, if $t \neq 3$, then we can perform the row operation $\frac{1}{t-3}\vec{r_3} \to \vec{r_3}$ and obtain

$$\cdots \xrightarrow{\frac{1}{t-3}\vec{r_3}\to\vec{r_3}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\vec{r_2}-\vec{r_3}\to\vec{r_3}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is in rref (and not consistent!).

2. The equation $A\vec{x} = \vec{b}$ is consistent when $t =$ <u>**3**</u>. For this value of t the general solution has <u>**1**</u> free variables. (Fill in the blanks with the correct numbers.

3. For the value of t in (2) find the general solution $\vec{x}$ to $A\vec{x} = \vec{b}$. Write the solution in *vector form.*
Solution: x_3 is a free variable. The first equation is $x_1 + 2x_3 = 0$. So $x_1 = -2x_3$. Similarly, $x_2 = 1 - 3x_3$. Thus the general solution is

$$\begin{bmatrix} -2x_3 \\ 1-3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}.$$

Either of these two is an acceptable answer.

4. What is rank(A)? What is the nullity of A?
Solution: Note that in computing the rref of $[A \ \vec{b}]$ we have computed the rref of A:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the rank of A is **2**. So the nulity is $3 - 2 = \mathbf{1}$. Note that we had **2** basic variable in our solution above and **1** free variable.