640:250 Introductory Linear Algebra Midterm Exam #2 April 4, 2015

Solutions

Instructions: This is a closed book exam. Show your answers and arguments for your answers in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want graded. No calculators, cell phones, or any other electronic devices may be used during the exam.

Have your photo ID card available for checking. Do not start the exam until instructed to do so.

DO NOT WRITE BELOW THIS LINE

Problem	# Points	Score
1	15	
2	25	
3	15	
4	25	
5	20	
Extra Credit	10	
total	100+10	

Problem 1 (15 points total)

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -4 \\ 0 & 10 & -6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 9 \\ -15 \\ -22 \end{bmatrix}$.

a. (8 points) Compute the LU-decomposition of A.

Solution: $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -4 \\ 0 & 10 & -6 \end{bmatrix} \xrightarrow{\vec{r_1}+\vec{r_2}\to\vec{r_2}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 10 & -6 \end{bmatrix} \xrightarrow{-5\vec{r_2}+\vec{r_3}\to\vec{r_3}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix} = U$. Next we know that L will be unit lower diagonal and $L_{2,1} - 1$, $L_{3,2} = 5$. So we find $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$. We can double check our answer by multiplying L times U and we do indeed get A.

b. (7 points) Use the LU-decomposition found in part (a) to solve $A\vec{x} = \vec{b}$.

Solution: We are trying to solve $LU\vec{x} = \vec{b}$. Let $\vec{y} = U\vec{x}$. First we solve $L\vec{y} = \vec{b}$. This is the equation $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \vec{y} = \begin{bmatrix} 9 \\ -15 \\ -22 \end{bmatrix}$. So we see that $y_1 = 9$, $-y_1 + y_2 = -15$ so $y_2 = -6$ and similarly that $5y_2 + y_3 = -22$ so $y_3 = 8$. Thus $\vec{y} = \begin{bmatrix} 9 \\ -6 \\ 8 \end{bmatrix}$. Next we solve $U\vec{x} = \vec{y}$. This is the equation $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 9 \\ -6 \\ 8 \end{bmatrix}$. Again, we employ back substitution and find $x_3 = 2$, $x_2 = -1$ and $x_1 = 6$. Thus $\vec{x} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$. We can double check that $A\vec{x}$ does indeed equal $\vec{b}$. Note that we only expected one solution as A is invertible as our LU-decomposition shows.

Problem 2 (25 points total)

a. (5 points) Let $M = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & 5 & 4 \end{bmatrix}$. Find $\det M$ by employing cofactor expansion along the first row.

Solution: $\det M = 3 \cdot (0 - 15) - 1 \cdot (-4 - (-6)) + 2 \cdot (-5 - 0) = -57$

b. (5 points) Let $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 5 & 5 & -1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$. Find $\det B$. Indicate clearly which row or column you are cofactor expanding along.

Solution: We employ cofactor expansion along the second row since there is only one nonzero term. We have $a_{2,3} = 1$ and $c_{2,3} = (-1)^{2+3} \det B_{2,3}$ and

$$\det(B_{2,3}) = \det\left(\begin{bmatrix} 1 & 2 & 4 \\ 5 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}\right) = 15.$$

Thus $\det(B) = (-1) \cdot 1 \cdot 15 = -15$.

c. (5 points) Let $C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 7 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$. Find $\det C$. Hint: you do not need to employ cofactor expansion. Explain your answer.

Solution: Observe that the second row is double the first. So the rows are linearly dependent and thus the matrix is *not* invertible. Thus $\det C = 0$.

d. (10 points total, each part worth 2 points) Let A be a 5×5 matrix with $\det A = 10$. Compute the following:

i. Find $\det(A^2)$.

Solution: $\det(A^2) = \det(A) \cdot \det(A) = 10 \cdot 10 = 100$.

ii. Find $\det(A^{-1})$.

Solution: $1 = \det(I_5) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. Thus $\det(A^{-1}) = 1/\det(A)$. So $\det(A^{-1}) = 1/10$.

iii. Let B be the matrix formed by swapping rows 2 and 4 of A. What is $\det B$?

Solution: Swapping two rows multiplies the determinant by -1. So $\det(B) = -10$.

iv. Find $\det(A + A)$.

Solution: $\det(A + A) = \det(2A) = \det((2I_5)A) = \det(2I_5)\det(A) = 32 \cdot 10 = 320$. We get $\det(2I_5)$ by multiplying the five 2's on the main diagonal. Alternatively, the matrix $2A$ can be obtained from A by multipying each row by 2. Each of these operations multiplies the determinant by 2 and so we obtain $2^5 \cdot 10 = 320$.

v. Let D be the matrix formed by multiplying the i-th row of A by i. So the first row is scaled by 1, the second row is scaled by 2, etc. Find $\det(D)$.

Solution: Multiplying any row by a scalar mutliplies the determinant by that same scalar. Thus $\det(D) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 10 = 1200$.

Problem 3 (15 points total)

a. (7 points) Let A and B be $n \times n$ matrices that are *not* invertible. Prove that AB is not inveritible.

Solution: This problem was on the first midterm. See solutions to midterm 1 for how we solved this problem without determinants. The solution is easier with determinants. If A and B are not invertible then $\det(A) = 0$ and $\det(B) = 0$. Now observe $\det(AB) = \det(A) \cdot \det(B) = 0 \cdot 0 = 0$. So the matrix AB has determinant 0 and is therefore not invertible.

b. (8 points) A square matrix, M, is said to be *nilpotent* if some power of the matrix equals the zero matrix. That is, there is some positive integer k such that $M^k = 0$. Let M be a nilpotent matrix. Explain why $\det M = 0$.

Solution: We have observed that $\det(A^2) = \det(A) \cdot \det(A) = \det(A)^2$. Similarly, $\det(A^3) = \det(A) \cdot \det(A) \cdot \det(A) = \det(A)^3$. In general, we have $\det(A^p) = \det(A)^p$.

If M is nilpotent and k is such that $M^k = 0$ then observe that $0 = \det(0) = \det(M^k) = \det(M)^k$. So when we raise $\det(M)$ to the k-th power we get 0. So we must have $\det(M) = 0$.

Problem 4 (25 points total)

Let $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 7 \\ 1 & 4 & 5 & 6 \end{bmatrix}$. Parts (a), (b) and (c) refer to this matrix.

a. (5 points) Find a basis for Col$A$. What is dim(Col$A$)?

Solution: The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. So we see that columns $1, 2$ and 4 are pivot columns. Thus a basis for Col$A$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix} \right\}$. We have dim(Col$A$) $= 3$ which matches the rank of A as expected.

b. (5 points) Find a basis for Row$A$. What is dim(Row$A$)?

Solution: The slow way to solve this problem is to take A^T, use Gaussian elimnation to put it in rref and then identify the pivot columns. However, we already showed in part (a) that the rank of A is 3 and we know that the row rank equals the column rank. That is dim(Col$A$) $=$ dim(Row$A$) $= 3$. So the three rows must be linearly independent. Thus a basis for Row$A$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right\}$.

Alternatively, having put the matrix in rref in part (a), we recall that elementary row operations do *not* change the rowspace of a matrix. So $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is also a basis for Row$A$.

c. (5 points) Find a basis for Null(A)? What is dim(Null$A$)?

Solution: We can employ the rref of A found in part (a): $R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. We solve $R\vec{x} = 0$ and see that we have one free variable: x_3. Our general vector solution is $\begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. Thus a basis for Null($A$) is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Thus, $\dim(\text{Null}A) = 1$. Note, that this is just the nullity and since the rank is 3, we have $\text{Nullity}A = 4 - 3 = 1$.

d. (10 points) Let B be an invertible, 5×5 matrix. Describe the following:

i. Col(B)

Solution: Since B is invertible its columns are linearly independent. Thus they span all of $\mathbb{R}^5$. So $\text{Col}(B) = \mathbb{R}^5$.

ii. Row(B)

Solution: As above, the rows are linearly independent and span R^5. So $\text{Row}(B) = \mathbb{R}^5$.

iii. Null(B)

Solution: The nullity of B is 0, thus the dimension of the null space is 0. The only solution to $B\vec{x} = \vec{0}$ is the zero vector. $\text{Null}(B) = \{\vec{0}\}$.

Problem 5 (20 points total)

Let $M = \begin{bmatrix} -7 & 18 \\ -3 & 8 \end{bmatrix}$. Parts (a) and (b) deal with this matrix.

a. (5 points) Find the characteristic polynomial of M and use it to find the eigenvalues of M.

Solution: $p_M(t) = \det(M - tI_2) = \det\left(\begin{bmatrix} -7-t & 18 \\ -3 & 8-t \end{bmatrix}\right) = (-7-t)(8-t) + 54 = t^2 - 8t + 7t - 56 + 54 = t^2 - t - 2 = (t+1)(t-2)$. The eigenvalues of M are theof $p_M(t) = 0$ which are -1 and 2.

b. (5 points) Find bases for the eigenspaces corresponding to each of the eigenvalues found in part (a).

Solution: We begin with $\lambda = -1$. Then $M - (-1)I_2 = M + I_2 = \begin{bmatrix} -6 & 18 \\ -3 & 9 \end{bmatrix}$. The rref of this matrix is $R = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ and solving $R\vec{x} = \vec{0}$ we find x_2 is free and the general solution is $\begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix}$. So the null space has basis $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$. We can double check that $M\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Next, we do a similar computation for $\lambda = 2$. We have $M - 2I_2 = \begin{bmatrix} -9 & 18 \\ -3 & 6 \end{bmatrix}$ which has rref $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. Thus a basis for the null space is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. We can double check that $M\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as expected.

Note, since each root of our characteristic polynomial had algebraic multiplicity one, we knew ahead of time that the dimensions of our eigenspaces had to be one.

c. (5 points) Let A be a 5×5 matrix with characteristic polynomial $-(t-5)^3(t^2+8)$. Identify the eigenvalues of A and for each eigenvalue state its algebraic multiplicity. Furthermore, for each eigenvalue λ state the possible values for $\dim(\mathrm{Null}(A - \lambda I_5))$.

Solution: The characteristic polynomial factors as $-(t-5)^3(t-2\sqrt{2}i)(t+2\sqrt{2}i)$. Thus the eigenvalues are $5, 2\sqrt{2}i$ and $-2\sqrt{2}i$ with algebraic multiplicities $3, 1$ and 1 respectively. Since the dimension of an eigenspace is less than or equal to the algebraic multiplicity of the eigenvalue we see that $\dim(\mathrm{Null}(A - 5I_5))$ is either $1, 2$ or 3. For the other two eigenvalues, the dimension must be one.

d. (5 points) Let B be an $n \times n$ matrix with eigenvalue λ. Show that λ^2 is an eigenvalue of the matrix B^2.

Solution: Let $\vec{v}$ be an eigenvector of B with eigenvalue λ. Then

$$B\vec{v} = \lambda\vec{v}.$$

Multiplying on the left by B we find

$$B^2\vec{v} = B(\lambda\vec{v}) = \lambda(B\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}.$$

So we see that

$$B^2\vec{v} = \lambda^2\vec{v}$$

so λ^2 is an eigenvalue of B^2.

Extra Credit Problem 1 (5 points)

As stated in problem three, a square matrix, M, is said to be *nilpotent* if some power of that matrix equals the zero matrix. That is, there is some positive integer k such that $M^k = 0$. Let M be a nilpotent matrix. Show that all its eigenvalues must be 0.

Solution: Generalizing the result from question 5 part (d) we see that if v is an eigenvector of A with eigenvalue λ, then $A^p\vec{v} = \lambda^p\vec{v}$ and so λ^p is an eigenvalue of A^p.

Now, let M be nilpotent matrix and suppose for the sake of contradiction that it has a nonzero eigenvalue λ. Let $\vec{v}$ be an eigenvector of M such that $M\vec{v} = \lambda v$. Now $M^k\vec{v} = \lambda^k\vec{v}$. However $M^k = 0$ so $M^k\vec{v} = \vec{0}$. So we have $\lambda^k\vec{v} = \vec{0}$. Since $\vec{v} \neq \vec{0}$ (since it is an eigenvector) we see that $\lambda^k = 0$ which means $\lambda = 0$. Contradiction.

Extra Credit Problem 2 (5 points)

Prove that for any square matrix A, $\dim(\text{Null}(A)) \leq \dim(\text{Null}(A^2))$. Give an example of a matrix A where $\dim(\text{Null}(A)) < \dim(\text{Null}(A^2))$.

Solution: We will show that $\text{Null}(A) \subseteq \text{Null}(A^2)$ and the inequality for the dimensions follows. Let $\vec{x} \in \text{Null}(A)$. Then $A\vec{x} = \vec{0}$. Observe $A^2\vec{x} = A(A\vec{x}) = A\vec{0} = \vec{0}$. So we see that $\vec{x} \in \text{Null}(A^2)$. So every vector in $\text{Null}(A)$ is in $\text{Null}(A^2)$. So $\text{Null}(A) \subseteq \text{Null}(A^2)$.

One example of a matrix A such that $\dim(\text{Null}(A)) < \dim(\text{Null}(A^2))$ is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $\dim(\text{Null}(A)) = 1$. Note that A is nilpotent. $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. And so $\dim(\text{Null}(A^2)) = 2$.