640:250 Introductory Linear Algebra Midterm Exam #1 February 25, 2015

Name (PRINT):______________________________

ID # (last 4 digits):__________

Signature: ______________________________

Instructions: This is a closed book exam. Show your answers and arguments for your answers in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want graded. No calculators, cell phones, or any other electronic devices may be used during the exam.

Have your photo ID card available for checking. Do not start the exam until instructed to do so.

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Problem	Score
1	
2	
3	
4	
5	
Extra Credit	
total (out of 100)	

Problem 1 (20 points total)

In this problem $A = \begin{bmatrix} 4 & 5 & 2 \\ -1 & 0 & 3 \\ -2 & 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & -3 \\ 6 & 7 & -5 \\ -4 & 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 1 \\ -7 & 1 & 0 \\ 6 & 2 & 0 \\ 3 & 2 & -3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

a. (5 points) For each of $A, B, C, \vec{v}, \vec{w}$ write the size of the matrix. For example, your answer might take the form "Matrix ____ has size ____ by ____."

Answer:
A has size 3×3
B has size 3×3
C has size 4×3
$\vec{v}$ has size 2×1
$\vec{w}$ has size 3×1

b. (10 points) Compute each of the following or state that it does not exist (problem continues on next page):

$$A + B = \begin{bmatrix} 8 & 7 & -1 \\ 5 & 7 & -2 \\ -6 & 1 & 8 \end{bmatrix}.$$

$$C^T = \begin{bmatrix} 0 & -7 & 6 & 3 \\ 1 & 1 & 2 & 2 \\ 1 & 0 & 0 & -3 \end{bmatrix}.$$

$$A\vec{w} = \begin{bmatrix} 15 \\ 2 \\ 3 \end{bmatrix}.$$

$$A - B^T = \begin{bmatrix} 0 & -1 & 6 \\ -3 & -7 & 3 \\ 1 & 6 & 4 \end{bmatrix}$$

$B^T\vec{v} =$ Does not exist.

$C^T\vec{w} =$ Does not exist.

$$BA = \begin{bmatrix} 20 & 17 & -4 \\ 27 & 25 & 3 \\ -20 & -18 & 4 \end{bmatrix}.$$

c. (5 points) What is the (2-3) entry of the matrix product AC^T?

Answer: This is the dot product of the second row of A and the third column of C^T. The second row of A is $[-1 \quad 0 \quad 3]$ and the third column of C^T is $\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$. The dot product is then $-1 \cdot 6 + 0 \cdot 2 + 3 \cdot 0 = -6$.

Problem 2 (20 points total)

For the following system of equations write the system as a matrix equation $A\vec{x} = \vec{b}$. Indicate clearly the matrix A and the vector $\vec{b}$. Compute the reduced row echelon form of the augmented matrix $[A\ \vec{b}]$. Use this to find the general vector solution.

$$\begin{aligned} 2x_2 - x_3 &= 6 \\ x_1 + 6x_2 - 3x_3 + x_4 &= 20 \\ 4x_1 + 26x_2 - 13x_3 + 5x_4 &= 90 \end{aligned}$$

Solution: We want to solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 0 & 2 & -1 & 0 \\ 1 & 6 & -3 & 1 \\ 4 & 26 & -13 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 20 \\ 90 \end{bmatrix}$. We can reduce the augmented matrix as follows:

$$\begin{bmatrix} 0 & 2 & -1 & 0 & 6 \\ 1 & 6 & -3 & 1 & 20 \\ 4 & 26 & -13 & 5 & 90 \end{bmatrix} \xrightarrow{\vec{r_1} \leftrightarrow \vec{r_2}} \begin{bmatrix} 1 & 6 & -3 & 1 & 20 \\ 0 & 2 & -1 & 0 & 6 \\ 4 & 26 & -13 & 5 & 90 \end{bmatrix} \xrightarrow{-4\vec{r_1} + \vec{r_3} \to \vec{r_3}} \begin{bmatrix} 1 & 6 & -3 & 1 & 20 \\ 0 & 2 & -1 & 0 & 6 \\ 0 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}\vec{r_2} \to \vec{r_2}} \begin{bmatrix} 1 & 6 & -3 & 1 & 20 \\ 0 & 1 & -1/2 & 0 & 3 \\ 0 & 2 & -1 & 1 & 10 \end{bmatrix} \xrightarrow{-2\vec{r_2} + \vec{r_3} \to \vec{r_3}} \begin{bmatrix} 1 & 6 & -3 & 1 & 20 \\ 0 & 1 & -1/2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{-6\vec{r_2} + \vec{r_1} \to \vec{r_1}} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1/2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-\vec{r_3} + \vec{r_1} \to \vec{r_3}} \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & -1/2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

From this we can see that x_3 is a free variable. We have $x_1 = -2$ and $x_4 = 4$. Also, $x_2 - \frac{1}{2}x_3 = 3$ which we can rearrange as $x_2 = 3 + \frac{1}{2}x_3$. Thus the general vector solution is

$$\begin{bmatrix} -2 \\ 3 + \frac{1}{2}x_3 \\ x_3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}.$$

Either of these two answers is acceptable.

Problem 3 (20 points total)

A is a 4×5 matrix and $\vec{v} \in \mathbb{R}^4$ [Note there was a typo on the exam stating $v \in \mathbb{R}^5$, of course in this case $A\vec{x} = \vec{v}$ doesn't make sense]. In trying to solve the matrix equation $A\vec{x} = \vec{v}$ you form the augmented matrix $\begin{bmatrix} A & \vec{v} \end{bmatrix}$ and then compute the reduced row echelon form of this matrix and obtain $\begin{bmatrix} 1 & 4 & 0 & 0 & 5 & 7 \\ 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. (4 points) What is rank(A)?

Answer: Notice that although we do not know A we know that the rref of A is just the first five columns of the matrix given above. There are three nonzero rows so rank$(A) = 3$.

b. (4 points) What is the nullity of A?

Answer: The nullity of A is $5 - \text{rank}(A) = 5 - 3 = 2$. Note this is the number of free variables.

c. (4 points) Is $A\vec{x} = \vec{v}$ consistent?

Answer: Yes. The equation is consistent provided the rref has no rows of the form $[0 \; \cdots \; 0 \; 1]$ (which corresponds to the inconsistent equation $0 = 1$). There are no rows of the form above.

One solution is $\begin{bmatrix} 7 \\ 0 \\ 3 \\ 5 \\ 0 \end{bmatrix}$ which is found by setting both free variables, x_2 and x_5 to zero.

d. (4 points) How many solutions does $A\vec{x} = \vec{0}$ have? Explain.

Answer: Since the nullity is greater than 0 there are infinitely many solutions.

e. (4 points) Give an example of a matrix in reduced row echelon form that if it were to represent the rref of an augmented system of equations, the system would *not* be consistent.

Answer: There are many examples. One is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Problem 4 (20 points total)

For each of the following sets of vectors, determine whether or not they are linearly independent. Explain your answer.

a. (7 points) $\left\{ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 17 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$

Answer: We have four vectors in $\mathbb{R}^3$. Since $4 > 3$ they are linearly dependent. One way to think about this is to consider the first three vectors. Either they are linearly dependent or they are not. If they are linearly dependent then the whole set of four is certainly linearly independent. If the are linearly independent then they span $\mathbb{R}^3$. But then the fourth vector is in their span and so the whole set is linearly dependent.

b. (7 points) $\left\{ \begin{bmatrix} -5 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} -11 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix} \right\}$. Also: is the span of these vectors equal to $\mathbb{R}^3$? Why/why not.

Answer: We form the matrix $A = \begin{bmatrix} -5 & -11 & 3 \\ 6 & -12 & 4 \\ 8 & 20 & -4 \end{bmatrix}$ and test to see if the nullity is greater than 0. If one performs row reduction you obtain $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the identity matrix. So the vectors are linearly independent. Three linearly independent vectors in $\mathbb{R}^3$ do span all of $\mathbb{R}^3$.

c. (6 points) $\left\{ \begin{bmatrix} 14 \\ 3.7 \end{bmatrix}, \begin{bmatrix} 2 \\ 6.1 \end{bmatrix} \right\}$. Also determine the span of these vectors.

Answer: Two vectors are linearly dependent only if one is the multiple of the other. It is clear that is not the case here. So the vectors are linearly independent.

Problem 5 (20 points total)

a. (5 points) Determine if the matrix $\begin{bmatrix} 12 & -4 \\ 6 & -2 \end{bmatrix}$ is invertible.

Answer: Recall that an $n \times n$ matrix is invertible if and only if its columns are linearly independent. In the case of this matrix the first column is -3 times the second column. So the columns are linearly dependent and thus the matrix is *not* invertible.

Alternatively, one could compute the rref and obtain $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since this does not equal I_2 the matrix is *not* invertible.

b. (10 points) Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -3 & -3 \\ 0 & -1 & 2 \end{bmatrix}$. Compute A^{-1}. Use A^{-1} to solve the equation $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Answer: We compute A^{-1} by finding the rref of $[A\ I_3]$:

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -3 & -3 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2\vec{r_1}+\vec{r_2}\to\vec{r_2}} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\vec{r_2}+\vec{r_3}\to\vec{r_3}} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix} \xrightarrow{2\vec{r_2}+\vec{r_1}\to\vec{r_1}} \begin{bmatrix} 1 & 0 & -3 & -3 & 2 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{3\vec{r_3}+\vec{r_1}\to\vec{r_1}} \begin{bmatrix} 1 & 0 & 0 & -9 & 5 & 3 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix} \xrightarrow{\vec{r_3}+\vec{r_2}\to\vec{r_2}} \begin{bmatrix} 1 & 0 & 0 & -9 & 5 & 3 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix}$$

Note that the left 3×3 matrix in this expression is indeed I_3. We see that $A^{-1} = \begin{bmatrix} -9 & 5 & 3 \\ -4 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$. Finally se can solve $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ by multiplying both sides on the left by A^{-1}. We obtain $\vec{x} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -9 & 5 & 3 \\ -4 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$.

We can double check this answer by confirming $A \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

c. (5 points) Let A and B be $n \times n$ matrices that are *not* invertible. Prove that AB is not inveritible.

Answer: Recall that an $n \times n$ matrix, B is invertible if and only if $B\vec{x} = \vec{0}$ has only one solution (the trivial one). Thus, B is not invertible if and only if $B\vec{x} = \vec{0}$ has infinitely many solutions. Since B is not invertible we know there are infinitely many $\vec{x}$ such that $B\vec{x} = 0$. For any such $\vec{x}$ we have $AB\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$. So $AB\vec{x} = \vec{0}$ for infinitely many values of $\vec{x}$. Thus AB is not invertible.

Notice here we only used the fact that B was invertible. So we proved the stronger result: If A and B are $n \times n$ matrices and B is not invertible then AB is not invertible. What about BA?

Extra Credit 1 (6 points)

Let $A_1 = [1]$, $A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and in generaly A_n is the $n \times n$ matrix containing the entries $1, 2, 3, \ldots, n^2$ with the first row $[1, 2, \ldots, n]$, the second row $[n+1, n+2, \ldots, 2n]$, etc. Determine for which values of n the matrix A_n is invertible.

Answer: A good way to try to solve this problem is to test the first few cases. Clearly A_1 is invertible. So is A_2. However, A_3 is not. With some work we can see that A_4 is not either. Let's look at A_3. Notice that in each row the middle number is the average of the other two: 2 is midway between 1 and 3, 5 is midway between 4 and 6, etc. So if the columns of A_3 are $\vec{a_1}, \vec{a_2}$ and $\vec{a_3}$ then we see that $\vec{a_2} = \frac{1}{2}(\vec{a_1} + \vec{a_3})$. So we have a linear dependence: $\vec{a_1} - 2\vec{a_2} + \vec{a_3} = \vec{0}$. For any $n > 2$ we have at least three columns and in the first three columns we have that the middle number is the average of the other two. So the linear dependence $\vec{a_1} - 2\vec{a_2} + \vec{a_3} = \vec{0}$ holds for all $n > 2$. So for $n > 2$ the matrix has linearly dependent columns and thus is *not* invertible. Thus the only values of n for which A_n is invertible are 1 and 2.

Extra Credit 2 (4 points)

Recall that a $n \times n$ matrix A is said to be *diagonal* if $a_{i,j} = 0$ if $i \neq j$. That is each of the non-diagonal entries of A are zero. True or false: If $A^4 = I_5$ (the 5×5 identity matrix) then A must be the identity matrix.

Answer: We showed in class that if we multiply two diagonal matrices, A and B the resulting matrix is diagonal and its entries are formed by multiplying the corresponding entries of A and B. So the (i, j) entry of the product is $a_{ij}b_{ij}$. So if we raise a diagonal matrix to a power we just take the corresponding power of the entries. So the (i, j) entry of A^4 is a_{ij}^4.

We are given $A^4 = I_5$. If A is diagonal then we see that the fourth power of the diagonal entries is one. That is, $a_{ii}^4 = 1$. One solution is to set each $a_{ii} = 1$. But there is another root of the equation $x^4 = 1$. This root is -1. So we can set the diagonal entries to -1 also and their fourth power will be 1. So the statement is false. It suffices to give any counterexample. One such coutnerexample is $-I_5$ (the $n \times n$ diagonal matrices with -1 in each diagonal entry).