Solutions to Review Problems for Math 250 Final

1. Let $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$. Compute A^{100}.

   Solution: We will diagnolize and to find P and D such that $A = PDP^{-1}$. Then $A = PD^{100}P^{-1}$. The characteristic polynomial is $(5-t)(-4-t)+18 = t^2 - t - 2 = (t+1)(t-2)$. So the eigenvalues are -1 and 2, thus $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. If we find corresponding eigenvectors we find $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{1}1$ respectively. Of course any multiple of these vectors will also work. We set $P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. We next compute $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Note that $D^{100} = \begin{bmatrix} 1 & 0 \\ 0 & 2^100 \end{bmatrix}$. So $A^{100} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^100 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. This is an acceptable form.

2. Find the distance from the point $(2,5)$ to the line $y = \frac{1}{3}x$.

   Solution: Let $\vec{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Let $\vec{w}$ be the orthogonal projection of $\vec{u}$ onto the line $y = \frac{1}{3}x$. Then $\vec{w}$ is the closest point on this line to $\vec{u}$. To project onto this line we take any vector that points in the direction of the line. A natural choice is $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Now we want to find the projection of $\vec{u}$ onto $\vec{v}$. We compute $\vec{w} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{6+5}{3^2+1^1}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{11}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3.3 \\ 1.1 \end{bmatrix}$.
   Our anwwer then is $\|u - w\| = \left\| \begin{bmatrix} -1.1 \\ 3.9 \end{bmatrix} \right\| = \sqrt{1.1^2 + 3.9^2} \approx 4.05$

3. Find a unit vector perpendicular to both $\vec{v} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}$.

   Solution: Any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that is perpendicular to $\vec{v}$ and $\vec{u}$ satisfies $\vec{x} \cdot \vec{v} = \vec{x} \cdot \vec{u} = 0$. Thus we obtain the system of equations

   $$\begin{aligned} 3x_1 + 3x_2 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

   So we form the augmented matrix $\begin{bmatrix} 3 & 3 & 0 & 1 & 0 \\ 2 & 1 & -1 & 2 & 0 \end{bmatrix}$. We can reduce this matrix to obtain $\begin{bmatrix} 1 & 0 & -1 & 5/3 \\ 0 & 1 & 1 & -4/3 \end{bmatrix}$. We have two free variables and so infinitely many solutions.

We just need to find one solution and scale it so for our free varialbes we choose $x_3 = 1$ and $x_4 = 0$. Then we obtain $x_1 = 1$ and $x_2 = -1$. So we obtain $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. We can double check that $\vec{x}$ is indeed orthogonal to both $\vec{u}$ and $\vec{v}$. The question asks for a unit vector so we need to scale. Our answer is then

$$\vec{y} = \frac{1}{\|x\|}\vec{x} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

4. Let $M = \begin{bmatrix} 1.5 & 2.5 \\ -0.5 & 4.5 \end{bmatrix}$. Compute the QR factorization of M. Use it to solve $M\vec{x} = \begin{bmatrix} 2 \\ -22 \end{bmatrix}$.

   Solution: I intended for the numbers to work out nicely for this problem and they do not seem to. I will try to post a problem with nicer numbers shortly.

5. Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 7 \end{bmatrix}$. Let $W = \text{Span}\{\vec{v}, \vec{u}\}$. Find a basis for $W^\perp$.

   Solution: Let $A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 2 & 2 & 7 \end{bmatrix}$. Note that $W = \text{Row}(A)$. We want $W^\perp = \text{Null}(A)$. To find the null space of A we compute the rref of the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 3 & 4 & 0 \\ 1 & 2 & 2 & 7 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix}.$$

   So x_3 and x_4 are free variables and we have the general vector solution $\begin{bmatrix} -4x_3 - x_4 \\ x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. So $\mathcal{B} = \left\{ \begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $W^\perp$.

6. Let $\{\vec{v}_1, \ldots \vec{v}_5\}$ be a basis for a subspace, V, of $\mathbb{R}^8$. What is $\dim(V^\perp)$?

   Solution: Note that $\dim V = 5$. We know that if V is a subspace of $\mathbb{R}^n$ then $\dim(V) + \dim(V^\perp) = \dim(R^n) = n$. In this case, $n = 8$ and we have $5 + \dim(V^\perp) = 8$ so $\dim(V^\perp) = 3$.

7. Consider the plane, $P = \text{Span}\left\{\begin{bmatrix}3\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\2\end{bmatrix}\right\}$. Show that $\vec{v} = \begin{bmatrix}-9\\4\\6\end{bmatrix}$ is *not* in P. Find the closest point in P to $\vec{v}$.

Solution: First we show that $\vec{v}$ is not in P. If it was then there are scalars a and b such that $a\begin{bmatrix}3\\1\\2\end{bmatrix} + b\begin{bmatrix}-1\\2\\2\end{bmatrix} = \begin{bmatrix}-9\\4\\6\end{bmatrix}$. So we have an augmented system of equations $\begin{bmatrix}3 & -1 & -9\\1 & 2 & 4\\2 & 2 & 6\end{bmatrix}$. The rref of this matrix is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$, so the last equation is $0 = 1$. So there are no solutions. Thus $\vec{v}$ is not in P.

To find the closest point in P to $\vec{v}$ we want to find the orthogonal projection of $\vec{v}$ onto P. Let $C = \begin{bmatrix}3 & -1\\1 & 2\\2 & 2\end{bmatrix}$. The orthogonal projection matrix is then given by $C(C^TC)^{-1}C^T$.

So the closest point in P is then $C(C^TC)^{-1}C^T\vec{v} \approx \begin{bmatrix}-8.5214\\5.9145\\4.3248\end{bmatrix}$. I promise the numbers will work out better on the final!

8. Let $x_1, \ldots x_n$ be a list of real numbers. Prove that

$$(x_1 + x_2 + \ldots + x_n)^2 \leq n\left(x_1^2 + x_2^2 + \ldots + x_n^2\right).$$

Hint: use the Cauchy-Schwarz Inequality.

Solution: Let $\vec{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$ and $\vec{u}$ be the all ones vector in $\mathbb{R}^n$. Recall that $\vec{x}\cdot\vec{u} \leq \|\vec{x}\|\|\vec{u}\|$ by the Cauchy-Schwarz inequality. We compute $\vec{x}\cdot\vec{u} = x_1 + x_2 + \ldots + x_n \leq \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\sqrt{n}$. We can assume that the left hand side is positive and thus can square both sides to obtain $(x_1 + x_2 + \ldots + x_n)^2 \leq n\,(x_1^2 + x_2^2 + \ldots + x_n^2)$.