THE TREE PROPERTY AT THE FIRST AND DOUBLE SUCCESSORS OF A SINGULAR

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ABSTRACT. We show that given ω many supercompact cardinals and a weakly compact above them, there is a generic extension in which the tree property holds at the first and second successor of a strong limit singular cardinal.

1. INTRODUCTION

The tree property at κ states that every tree of height κ with levels of size less than κ has an unbounded branch; or, equivalently, there are no κ-Aronszajn trees. A long term project in set theory is to get the consistency of the tree property on larger and larger intervals of regular cardinals. One of the earliest positive results was in 1972 by Mitchell [6] that the tree property can hold at $\aleph_2$. Building on that, in 1983 Abraham [1] showed that the tree property can simultaneously hold at $\aleph_2$ and $\aleph_3$. Later, in 1998, Cummings and Foreman [2] constructed a sophisticated iteration of Abraham's forcing, obtaining the tree property at $\aleph_n$ for each $n > 1$. For a long time after that it remained open whether $\aleph_{\omega+1}$ can also be included. Then recently, Neeman [8] showed that it is indeed consistent to have the tree property at each $\aleph_n$ for $n > 1$ and at $\aleph_{\omega+1}$.

The next major open question is to get the tree property for both $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ for $\aleph_\omega$ strong limit[1]. This would require violating the Singular Cardinal Hypothesis (SCH) at $\aleph_\omega$. In the 1980's Woodin asked if it is consistent to have the failure of the SCH at $\aleph_\omega$ with the tree property at $\aleph_{\omega+1}$. While this is still open, in the last several years there have been some important progress. Gitik-Sharon [3] showed the consistency of the failure of SCH at a singular cardinal κ together with the non-existence of *special* κ^+-Aronszajn trees. They also pushed down their result to $\kappa = \aleph_{\omega^2}$. Then in 2009, Neeman [7] obtained the failure of the singular cardinal hypothesis at a singular cardinal κ, together with the full tree property at κ^+. This was pushed down to κ being $\aleph_{\omega^2}$ by the author in [9].

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[1]If one drops the requirement for $\aleph_\omega$ to be strong limit, the tree property at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ was obtained by Fontanella and Friedman in [4]. Independently, in [11] Unger shows the tree property for regulars below $\aleph_{\omega\cdot 2}$, again in the case of not strong limit singulars.

Regarding the tree property at the double successor of a singular, the first important result was by Cummings and Foreman [2] in the late 90's. Starting from a supercompact κ and a weakly compact above it, they produced a generic extension in which $\mathrm{cf}(\kappa) = \omega$ and the tree property holds at κ^{++}. Their construction incorporates Prikry forcing within the Mitchell poset. Then Unger showed that it is consistent to have the tree property at κ^{++} together with no special Aronszajn trees at κ^+ for a singular κ, [10]. His construction uses the diagonal supercompact Prikry from [3] within the Mitchell poset. He also showed that one can also use Neeman's version of the diagonal supercompact Prikry forcing from [7]. In this paper we show that by combining Neeman's diagonal Prikry with the Mitchell poset we can actually obtain the full tree property at κ^+ and κ^{++}.

Theorem 1.1. *Suppose that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals and that λ is a weakly compact cardinal with $\lambda > \sup_n \kappa_n$. Then there is a generic extension in which κ_0 is strong limit singular with $\mathrm{cf}(\kappa_0) = \omega$, $\lambda = \kappa_0^{++}$, and the tree property holds at both κ_0^+ and κ_0^{++}.*

In section 2, we describe all the relevant forcing notions. In section 3 we show the key branch lemma, and then use it to show the tree property at the first successor of the singular in the main model, completing the argument.

2. The forcing notions

Suppose in V, $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of indestructible supercompact cardinals. Let $\kappa_\omega = \sup_n \kappa_n$, $\kappa = \kappa_0$, and $\mu = \kappa_\omega^+$. Suppose that λ is a weakly compact cardinal above μ. Let $\mathbb{A} = Add(\kappa, \lambda)$. In $V^{\mathbb{A}}$, κ remains supercompact, so in that model for $n < \omega$ let U_n be a normal measure on $\mathcal{P}_\kappa(\kappa_n)$, such that the U_n's project to each other. Then let $\mathbb{I}$ denote Neeman's supercompact Prikry from [7], with respect to these measures. More precisely, conditions are of the form $p = \langle x_0, ..., x_{n-1}, A_n, A_{n+1}, ... \rangle$, where each $x_i \in \mathcal{P}_\kappa(\kappa_i)$, each $A_j \in U_j$, and for $i < l < n$, $x_i \subset x_l$ and $|x_i| < \kappa \cap x_l$. We say that the length of p is n, and the stem of p is $\langle x_0, ..., x_{n-1} \rangle$. For more on the properties of $\mathbb{I}$, see [3] or [7]. Let us just note that conditions with the same stem are compatible.

Let $\mathbb{P} = \mathbb{A} * \dot{\mathbb{I}}$. For $\alpha < \lambda$, let $\mathbb{P}_\alpha$ denote the restriction of $\mathbb{P}$ to α. As shown in Cummings-Foreman [2], Section 5 (and in Section 3 of [10]), there is a set $B \subset \lambda$ of Mahlo cardinals in the weakly compact filter, for which this makes sense. More precisely, for every $\alpha \in B$, the restriction of the measures U_n for $n < \omega$ to $V^{\mathbb{A} \restriction \alpha}$ gives normal measures. Let $\mathbb{I}_\alpha$ be the diagonal Prikry forcing obtained from these measures, then $\mathbb{P}_\alpha$ is $\mathbb{A} \restriction \alpha * \dot{\mathbb{I}}_\alpha$. Then a generic object for $\mathbb{P}$ induces a generic object for $\mathbb{P}_\alpha$. Below we will restrict ourself to these α's.

Definition 2.1. *Conditions in $\mathbb{R}$ are of the form $\langle f, \dot{p}, r \rangle$, where:*

(1) $\langle f, \dot{p} \rangle \in \mathbb{A} * \dot{\mathbb{I}}$,
(2) *r is a partial function with* $\mathrm{dom}(r) \subset B$, $|\mathrm{dom}(r)| < \mu$,

(3) *for each* $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ *is a* $\mathbb{P}_\alpha$ *name for a condition in* $Add(\mu, 1)^{V^{\mathbb{P}_\alpha}}$.

Note that for $\alpha \in \mathrm{dom}(r)$, $\langle f, \dot{p} \rangle \upharpoonright \alpha \in \mathbb{P}_\alpha$. *Next we define the ordering.* $\langle f_1, \dot{p}_1, r_1 \rangle \leq \langle f_2, \dot{p}_2, r_2 \rangle$ *iff:*

(1) $\langle f_1, \dot{p}_1 \rangle \leq_{\mathbb{P}} \langle f_2, \dot{p}_2 \rangle$, *and*

(2) $\mathrm{dom}(r_1) \supset \mathrm{dom}(r_2)$ *and for every* $\alpha \in \mathrm{dom}(r_2)$,

$$\langle f_1, \dot{p}_1 \rangle \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} r_1(\alpha) \leq r_2(\alpha).$$

In the second item of the definition of the order $\langle f, \dot{p} \rangle \upharpoonright \alpha$ denotes $\pi_\alpha(\langle f, p \rangle)$, where π_α is a projection from $\mathbb{P}$ to $R.O.(\mathbb{P}_\alpha)$. Also, given $\langle f, \dot{p}, r \rangle \in \mathbb{R}$, we will refer to $\dot{p}$ as the Prikry part of the condition.

Now define $\mathbb{Q}$ to consist of conditions of the form $\langle 0, 0, r \rangle \in \mathbb{R}$ with the induced ordering. Then $\mathbb{Q}$ is μ-directed closed in V. Set $\mathbb{R}^* = \mathbb{P} \times \mathbb{Q}$. By the directed closure of $\mathbb{Q}$ and the indestructibility of the κ_n's, in $V^{\mathbb{Q}}$ we have that $\mathbb{A} * \dot{\mathbb{I}}$ is the forcing construction in Neeman [7] for the appropriate measures; for more details see [10], Section 6. Then, by [7], it follows that the tree property at μ holds in the extension by $\mathbb{R}^*$. Also, as in [2],

$$\langle \langle f, \dot{p} \rangle, \langle 0, 0, r \rangle \rangle \mapsto \langle f, \dot{p}, r \rangle$$

is a projection from $\mathbb{R}^*$ to $\mathbb{R}$. For simplicity of notation we will write $\mathbb{Q} = \{r \mid \langle 0, 0, r \rangle \in \mathbb{R}\}$ and denote conditions in $\mathbb{R}^*$ in the form $\langle f, \dot{p}, r \rangle$ and use $\Vdash_{\mathbb{R}^*}$, $\leq_{\mathbb{R}^*}$ to avoid ambiguity.

From [2] and [10] we have that after forcing with $\mathbb{R}$, κ is a strong limit singular cardinal with $\mathrm{cf}(\kappa) = \omega$, $\mu = \kappa^+$, $2^\kappa = \lambda = \kappa^{++}$, and the tree property holds at κ^{++}. It remains to show that in $V^{\mathbb{R}}$ the tree property holds at μ.

Definition 2.2. *Let* $\dot{p}$ *be a name for a condition in* $\dot{\mathbb{I}}$. *Conditions in* $\mathbb{R}_{\dot{p}}$ *are of the form* $\langle a, \dot{q}, r \rangle \in \mathbb{R}$, *where* $\langle a_1, \dot{p}_1, r_1 \rangle \leq_{\mathbb{R}_{\dot{p}}} \langle a_2, \dot{p}_2, r_2 \rangle$ *iff:*

(1) $\langle a_1, \dot{p}_1 \rangle \leq_{\mathbb{P}} \langle a_2, \dot{p}_2 \rangle$, *and*

(2) $\mathrm{dom}(r_1) \supset \mathrm{dom}(r_2)$ *and for every* $\alpha \in \mathrm{dom}(r_2)$,

$$\langle a_1, \dot{p} \rangle \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} r_1(\alpha) \leq r_2(\alpha).$$

We will show that $\mathbb{R}^*$ projects to $\mathbb{R}_{\dot{p}}$ and that $\mathbb{R}_{\dot{p}}$ projects to $\mathbb{R}$ below any condition with Prikry part forced to extend $\dot{p}$.

Lemma 2.3. $\mathbb{R}^*$ *projects to* $\mathbb{R}_{\dot{p}}$, *as witnessed by the identity.*

Proof. Clearly the identity is order preserving.

Now suppose that $\langle a', \dot{q}', r' \rangle \leq_{\mathbb{R}_{\dot{p}}} \langle a, \dot{q}, r \rangle$. Define r'', so that $\mathrm{dom}(r'') = \mathrm{dom}(r')$, and for every $\alpha \in \mathrm{dom}(r'')$, $r''(\alpha) = \{\langle \sigma, b \rangle$ s.t.

- $b \leq \langle a', \dot{p} \rangle \upharpoonright \alpha$ and $b \Vdash_{\mathbb{P}_\alpha} \sigma \in r'(\alpha)$ or
- $b \perp \langle a', \dot{p} \rangle \upharpoonright \alpha$ and $b \Vdash_{\mathbb{P}_\alpha} \sigma \in r(\alpha)\}$.

Then $\langle a', \dot{q}', r'' \rangle \leq_{\mathbb{R}^*} \langle a, \dot{q}, r \rangle$ and $\langle a', \dot{q}', r'' \rangle \leq_{\mathbb{R}_p} \langle a', \dot{q}', r' \rangle$.

□

Lemma 2.4. *Let* $s^* = \langle 1, \dot{p}, 1 \rangle \in \mathbb{R}$. *Then* $\mathbb{R}_{\dot{p}}/s^* := \{s \in \mathbb{R} \mid s \leq s^*\}$ *projects to* $\mathbb{R}/s^* := \{s \in \mathbb{R} \mid s \leq s^*\}$ *as witnessed by the identity.*

Proof. For the proof, assume that all conditions are below s^*. Note that since the last coordinate is the empty condition, then $s \leq_{\mathbb{R}_{\dot{p}}} s^*$ iff $s \leq_{\mathbb{R}} s^*$.

As before, it is straightforward to see that the identity is order preserving.

Suppose that $\langle a', \dot{q}', r' \rangle \leq_{\mathbb{R}} \langle a, \dot{q}, r \rangle$. Define r'', so that $\mathrm{dom}(r'') = \mathrm{dom}(r')$, and for every $\alpha \in \mathrm{dom}(r'')$, $r''(\alpha) = \{\langle \sigma, b \rangle$ s.t.

- $b \leq \langle a', \dot{q}' \rangle \restriction \alpha$ and $b \Vdash_{\mathbb{P}_\alpha} \sigma \in r'(\alpha)$ or
- $b \perp \langle a', \dot{q}' \rangle \restriction \alpha$ and $b \Vdash_{\mathbb{P}_\alpha} \sigma \in r(\alpha)\}$.

Then $\langle a', \dot{q}', r'' \rangle \leq_{\mathbb{R}_{\dot{p}}} \langle a, \dot{q}, r \rangle$ and $\langle a', \dot{q}', r'' \rangle \leq_{\mathbb{R}} \langle a', \dot{q}', r' \rangle$. □

Now let $\mathcal{A}$ be $\mathbb{A}$-generic; we will define a poset $\mathbb{Q}_p$ in $V[\mathcal{A}]$, such that $\mathbb{R}_{\dot{p}}$ is isomorphic to $\mathbb{A} * (\dot{\mathbb{I}} \times \dot{\mathbb{Q}}_p)$.

Definition 2.5. *Work in $V[\mathcal{A}]$. Let $p = \dot{p}_{\mathcal{A}}$. Define $\mathbb{Q}_p$ to consist of conditions of the form $r \in \mathbb{Q}$, with the following ordering: $r_1 \leq_{\mathbb{Q}_p} r_2$ iff $\mathrm{dom}(r_1) \supset \mathrm{dom}(r_2)$ and there is $a \in \mathcal{A}$, such that for every $\alpha \in \mathrm{dom}(r_2)$, $\langle a, \dot{p} \rangle \restriction \alpha \Vdash_{\mathbb{P}_\alpha} r_1(\alpha) \leq r_2(\alpha)$.*

Proposition 2.6. *$\mathbb{Q}_p$ is κ-closed.*

Lemma 2.7. *$\mathbb{R}_{\dot{p}}$ is isomorphic to $\mathbb{A} * (\dot{\mathbb{I}} \times \dot{\mathbb{Q}}_p)$ (i.e. a generic for one induces a generic for the other and vice versa).*

Proof. Work in $V[\mathcal{A}]$, where $\mathcal{A}$ is a generic for $\mathbb{A}$. For an $\mathbb{A}$-name $\dot{q}$ for a condition in $\dot{\mathbb{I}}$, we write q to denote $\dot{q}_{\mathcal{A}}$. Let $\pi : \mathbb{R}_{\dot{p}}/\mathcal{A} \to \mathbb{I} \times \mathbb{Q}_p$ be given by $\pi(\langle a, \dot{q}, r \rangle) = \langle q, r \rangle$.

To show that it is order preserving, suppose that $\langle a', \dot{q}', r' \rangle \leq_{\mathbb{R}_{\dot{p}}/\mathcal{A}} \langle a, \dot{q}, r \rangle$. Then $q' \leq_{\mathbb{I}} q$, and for every $\alpha \in \mathrm{dom}(r) \subset \mathrm{dom}(r')$, we have that $\langle a', \dot{p} \rangle \restriction \alpha \Vdash_{\mathbb{P}_\alpha} r'(\alpha) \leq r(\alpha)$. Since $a' \in \mathcal{A}$, that means $r' \leq_{\mathbb{Q}_p} r$.

On the other hand, if $\langle q', r' \rangle \leq_{\mathbb{I} \times \mathbb{Q}_p} \langle q, r \rangle = \pi(\langle a, \dot{q}, r \rangle)$, let $a' \in \mathcal{A}$ witness that $r' \leq_{\mathbb{Q}_p} r$. By extending a' if necessary, also assume that $a' \leq a$ and $a' \Vdash \dot{q}' \leq \dot{q}$. Then $\langle a', \dot{q}', r' \rangle \leq_{\mathbb{R}_{\dot{p}}/\mathcal{A}} \langle a, \dot{q}, r \rangle$.

Since we are working over $\mathcal{A}$, that means that $s \perp s'$ iff $\pi(s) \perp \pi(s')$. And so since it is onto, π is a dense embedding. □

Let G be $\mathbb{R}$-generic and let $\mathcal{A}$ be $\mathbb{A}$-generic induced by G. As before, when working in $V[\mathcal{A}]$ and referring to conditions $\langle f, \dot{p}, r \rangle$, we will write p for $\dot{p}_{\mathcal{A}}$. Also, let $\mathcal{I}$ be the $\mathbb{I}$-generic over $V[\mathcal{A}]$ induced by G.

Let G^* be $\mathbb{R}^*/G$-generic. For every $q \in \mathcal{I}$, let G_q be $\mathbb{R}_{\dot{q}}/G$ generic induced by G^*. Also, let $\mathcal{Q}$ be $\mathbb{Q}$-generic induced by G^* and for $p \in \mathbb{I}$, let $\mathcal{Q}_p$ be the $\mathbb{Q}_p$-generic over $V[\mathcal{A}]$ induced by G^*.

Remark 2.8. If $q \leq p$, then $V[G_q] \subset V[G_p]$ and $G_p \subset G_q$. This follows by a similar argument as in Lemma 2.3.

Remark 2.9. If $q \leq p$, then $\mathbb{Q}_p$ projects to $\mathbb{Q}_q$ and $\mathcal{Q}_p \subset \mathcal{Q}_q$.

Let T be a μ - tree in $V[G]$. For $\alpha < \mu$, we may assume that the α-th level of T is $T_\alpha = \{\alpha\} \times \kappa$. Applying Neeman's arguments from [7], there is an unbounded branch b through T in $V[G^*]$. More precisely, in [7] the following is shown:

(†) In $V[\mathcal{A} \times \mathcal{Q}]$, there are an unbounded set $J \subset \mu$, a stem h^*, and $\langle u_\alpha, p_\alpha \mid \alpha \in J\rangle$, where each $p_\alpha \in \mathbb{I}$ is a condition with stem h^*, and each u_α is a node on the tree of level α, and:

(1) for all $\alpha < \beta$ from J, $p_\alpha \wedge p_\beta \Vdash_{\mathbb{I}} u_\alpha <_{\dot{T}} u_\beta$;
(2) $\{\alpha \mid p_\alpha \in \mathcal{I}\}$ is unbounded in μ (due to the chain condition of $\mathbb{I}$);

Then, the branch b in $V[G^*]$ is the downward closure of $\{u_\alpha \mid p_\alpha \in \mathcal{I}\}$. In particular, each $p_\alpha \Vdash_{\mathbb{I}} u_\alpha \in \dot{b}$. Here and below, for two conditions q_1, q_2 in $\mathbb{I}$ with the same stem, $q_1 \wedge q_2$ denotes the weakest common extension in $\mathbb{I}$ (which also has that stem). By thinning out J, one can arrange that for some fixed $\xi < \kappa$, $u_\alpha = \langle \alpha, \xi\rangle$. In the next section, we will use a branch lemma to show that this branch must already be in $V[G]$.

Remark 2.10. One can show that the branch b is actually in $V[G_p]$ for every $p \in \mathcal{I}$. The argument requires a similar, somewhat simpler branch lemma, as the one presented in the next section.

3. A branch in $V[G]$

In this section, we show that for densely many stems h, there is some α, such that above α, for every p with stem h, there is no more splitting in $\mathbb{R}^*/G_p$ in deciding nodes of the branch. Our branch lemma is motivated by the splitting arguments in Magidor-Shelah [5], except that here we consider multiple models at the same time. Throughout, we maintain that the Prikry stem is constant and use the full power of [7]. Then we will take the supremum over all stems and use that the generic objects $\langle G_p \mid p \in \mathcal{I}\rangle$ in a sense approximate G. More precisely, we will show that $G = \bigcup_{p\in\mathcal{I}} G_p$.

3.1. **On names.** Let $\tau \in V[\mathcal{A}]$ be an $\mathbb{R}/\mathcal{A}$-name for the tree, forced to be such by the empty condition. Now let $\dot{T} \in V[\mathcal{A}][\mathcal{Q}]$ be an $\mathbb{I}$-name for the tree, obtained from τ. I.e. for any two nodes u, v, and $q \in \mathbb{I}$, $q \Vdash_{\mathbb{I}} u <_{\dot{T}} v$ iff there is $a \in \mathcal{A}$ and $r \in \mathcal{Q}$, such that $\langle a, \dot{q}, r\rangle \Vdash_{\mathbb{R}/\mathcal{A}} u <_\tau v$.

Similarly, for every $p \in \mathbb{I}$, let $\dot{T}_p \in V[\mathcal{A}][\mathcal{Q}_p]$, be an $\mathbb{I}$-name for the tree, obtained from τ. I.e. for any two nodes u, v, and $q \le p$, $q \Vdash_{\mathbb{I}}^{V[\mathcal{A}][\mathcal{Q}_p]} u <_{\dot{T}_p} v$ iff there is $a \in \mathcal{A}$ and $r \in \mathcal{Q}_p$, such that $\langle a, \dot{q}, r\rangle \Vdash_{\mathbb{R}/\mathcal{A}} u <_\tau v$.

Note that the only formal difference is that τ, $\dot{T}$ and each $\dot{T}_p$ are defined in different ground models. But their evaluations by generic filters, projecting to one another in the right way, will be the same. In particular, we have the following.

Remark 3.1. If $q \le p$ are in $\mathbb{I}$, then $q \Vdash_{\mathbb{I}}^{V[\mathcal{A}][\mathcal{Q}]} u <_{\dot{T}} v$ iff $q \Vdash_{\mathbb{I}}^{V[\mathcal{A}][\mathcal{Q}_p]} u <_{\dot{T}_p} v$.

Let $\dot{b} \in V[\mathcal{A}]$ be a $\mathbb{R}^*/\mathcal{A}$-name for the branch as in [7]. For simplicity, assume

$$1 \Vdash_{\mathbb{I}\times\mathbb{Q}}^{V[\mathcal{A}]} \text{“}\dot{b} \text{ is a cofinal branch through } \tau.\text{”}$$

Then we also have that for every $p \in \mathbb{I}$:

- $V[\mathcal{A}][\mathcal{Q}_p] \models \langle p, 1\rangle \Vdash_{\mathbb{I}\times(\mathbb{Q}/\mathcal{Q}_p)} \dot{b}$ is a cofinal branch through $\dot{T}_p$",
- over $V[\mathcal{A}][\mathcal{Q}]$, $p \Vdash_{\mathbb{I}}$ "$\dot{b}$ is a cofinal branch through $\dot{T}$".

In the last item, we interpret $\dot{b}$ as a $\mathbb{I}$-name in $V[\mathcal{A}][\mathcal{Q}]$ by taking $\{\langle \sigma, q\rangle \mid (\exists r \in \mathcal{Q})(\langle \sigma, \langle q, r\rangle\rangle \in \dot{b})\}$.

3.2. **Splitting.**

Definition 3.2. *Let h be a stem. We say that there is an h-splitting at a node u, if there is a $p \in \mathbb{I}$ with* stem$(p) = h$ *and $r \in \mathcal{Q}$, such that $\langle p, r\rangle \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} u \in \dot{b}$, and nodes u_1, u_2 of higher levels and conditions r_1, r_2, such that for $k \in \{1, 2\}$,*

- *$r_k \leq_{\mathbb{Q}} r$, $r_k \in \mathcal{Q}_p$,*
- *$\langle p, r_k\rangle \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} u_k \in \dot{b}$, and*
- *$p \Vdash^{V[\mathcal{A}][\mathcal{Q}]}_{\mathbb{I}} u_1 \perp_{\dot{T}} u_2$.*

Note that by Remark 3.1, $\mathbb{I}$-forcing is the same over $V[\mathcal{A}][\mathcal{Q}]$ or $V[\mathcal{A}][\mathcal{Q}_p]$ for appropriate p. With this in mind, below we will often just write $\Vdash_{\mathbb{I}}$.

Definition 3.3. *For a stem h, we say that $\dagger_h$ holds, if in $V[\mathcal{A}][\mathcal{Q}]$ there are unbounded $J \subset \mu$, $\xi < \kappa$, and $\langle p_\alpha \mid \alpha \in J\rangle$, where each $p_\alpha \in \mathbb{I}$ is a condition with stem h, and setting $u_\alpha = \langle \alpha, \xi\rangle$, we have:*

1. *for all $\alpha < \beta$ from J, $p_\alpha \wedge p_\beta \Vdash_{\mathbb{I}} u_\alpha <_{\dot{T}} u_\beta$;*
2. *for all $\alpha \in J$, $p_\alpha \Vdash_{\mathbb{I}} u_\alpha \in \dot{b}$.*

By density and the argument in [7], any stem can be extended to a stem for which the above holds.

In $V[\mathcal{A}][\mathcal{Q}]$, for each h, such that $\dagger_h$ holds, let E_h be the set of nodes u forced to be in the branch by a condition of the form $\langle p, r\rangle \in \mathbb{I}\times\mathcal{Q}$ with stem$(p) = h$. Note that $E_h \cap T_\gamma \neq \emptyset$ for unboundedly many $\gamma < \mu$. Set $\alpha_h := \sup\{\gamma < \mu \mid (\exists u \in T_\gamma \cap E_h)(\text{ there is an } h-\text{splitting at } u)\}$.

Proposition 3.4. *Let h be a stem, such that $\dagger_h$ holds; then $\alpha_h < \mu$.*

Proof. Suppose otherwise. Let $\bar{r} \in \mathcal{Q}$ force that $\dot{\alpha}_h = \mu$ and that $\dot{J}, \xi$ and $\langle \dot{p}_\alpha \mid \alpha \in \dot{J}\rangle$ witness $(\dagger)_h$, where these are $\mathbb{Q}$-names in $V[\mathcal{A}]$.

Lemma 3.5. *(Splitting) Let $r \leq_{\mathbb{Q}} \bar{r}$ be such that $r \in \mathcal{Q}_q$ for some q with stem h. Then there are nodes $\langle v_i \mid i < \kappa_\omega\rangle$ and conditions $\langle\langle p_i, r_i\rangle \mid i < \kappa_\omega\rangle$ in $\mathbb{I}\times\mathbb{Q}$, such that:*

- *for every i,* stem$(p_i) = h$, *$p_i \leq q$, $r_i \leq_{\mathbb{Q}} r$, $r_i \in \mathcal{Q}_{p_i}$.*
- *for every i, $\langle p_i, r_i\rangle \Vdash_{\mathbb{I}\times\mathbb{Q}} v_i \in \dot{b}$ and*
- *for every $i < j$, $p_i \wedge p_j \Vdash_{\mathbb{I}} v_i \perp_{\dot{T}} v_j$.*

Proof. Let $\mathcal{Q}'$ be $\mathbb{Q}/\mathcal{Q}_q$-generic over $V[\mathcal{A}][\mathcal{Q}_q]$, such that $r \in \mathcal{Q}'$. Work for now in $V[\mathcal{A}][\mathcal{Q}']$ with E_h, J, $\langle p_\alpha \mid \alpha \in J\rangle$ denoting the interpretations of the respective names in $V[\mathcal{A}][\mathcal{Q}']$.

Claim 3.6. *For every $u \in E_h$, there is $p \in \mathbb{I}$ with stem h, $p \leq q$, and $r_1, r_2 \in \mathcal{Q}_p$ and nodes v_1, v_2 of higher levels, such that for $k \in \{1,2\}$, $\langle p, r_k \rangle \Vdash_{\mathbb{I}\times\mathbb{Q}} v_k \in \dot{b}$ and $p \Vdash_{\mathbb{I}}$ "$v_1 \perp_{\dot{T}} v_2, u <_{\dot{T}} v_1, u <_{\dot{T}} v_2$".*

Proof. Let $u \in E_h$. Then there is $\langle p, t \rangle \in \mathbb{I} \times \mathcal{Q}'$ with $\mathrm{stem}(p) = h$, forcing that u is in the branch. By our assumptions that $\alpha_h = \mu$, there must be a node $v \in E_h$ with level higher than u, such that there is h-splitting at v. Namely there is $\langle p', t' \rangle \in \mathbb{I} \times \mathcal{Q}'$ with $\mathrm{stem}(p') = h$, forcing that v is in the branch, and $r_1, r_2 \in \mathcal{Q}_{p'}$, and nodes v_1, v_2 of higher levels, such that:

- for $k \in \{1,2\}$, $r_k \leq_{\mathbb{Q}} t'$,
- for $k \in \{1,2\}$, $\langle p', r_k \rangle \Vdash_{\mathbb{I}\times\mathbb{Q}} v_k \in \dot{b}$, and
- $p' \Vdash_{\mathbb{I}}$ "$v_1 \perp_{\dot{T}} v_2, v <_{\dot{T}} v_1, v <_{\dot{T}} v_2$".

We may assume that $p' \leq q$ by Remark 2.9, and that $p' \leq p$ since the stem is the same. Since both $t, t' \in \mathcal{Q}'$, we have that in $V[\mathcal{A}][\mathcal{Q}']$, $p' \Vdash_{\mathbb{I}} u <_T v$. Note that by our earlier remark, this is also forced over $V[\mathcal{A}][\mathcal{Q}_{p'}]$, and so it is also forced over $V[\mathcal{A}][\mathcal{Q}]$.

Then p', v_1, v_2 are as desired. □

Strengthen r if necessary to force the conclusion of the above claim. Then in $V[\mathcal{A}][\mathcal{Q}_q]$ there is a club $C \subset \mu$, such that for every $\beta \in C$, for all $\gamma < \beta$, for every node $u \in T_\gamma$, if u is forced by a condition below r to be in $\dot{E}_h$, then there are $\gamma < \gamma_1 \leq \gamma_2 < \beta$, and nodes at levels γ_1 and γ_2 as in the conclusion of the claim applied to u.

Going over to $V[\mathcal{A}][\mathcal{Q}']$ again, we build a sequence $\langle p^i, \gamma_i, \beta_i \mid i < \kappa_\omega \rangle$, such that each $\gamma_i \in J$, $p^i = p_{\gamma_i}$ (and so it is a condition in $\mathbb{I}$ with stem h), $\beta_i \in C$, and $\gamma_i < \beta_i \leq \gamma_{i+1}$. For every $i < \kappa_\omega$, denote $u_i = \langle \gamma_i, \xi \rangle$. Also let $s_i \in \mathcal{Q}'$, $s_i \leq_{\mathbb{Q}} r$, be such that $s_i \Vdash_{\mathbb{Q}}^{V[\mathcal{A}]}$ "$\gamma_i \in \dot{J}$ and $p^i = \dot{p}_{\gamma_i}$".

Since $\mathbb{Q}$ is μ-closed in V and $\mathbb{A}$ has the κ^+-chain condition, by Easton's lemma, $\langle \gamma_i, \beta_i, p^i, s_i \mid i < \kappa_\omega \rangle$ is actually in $V[\mathcal{A}]$. Then:

- if $i < j$, then $p^i \wedge p^j \Vdash_{\mathbb{I}} u_i <_{\dot{T}} u_j$.
- for every i, $\langle p^i, s_i \rangle \Vdash_{\mathbb{I}\times\mathbb{Q}} u_i \in \dot{b}$, and so $s_i \Vdash_{\mathbb{Q}} u_i \in \dot{E}_h$.

For the rest of the proof, work in $V[\mathcal{A}][\mathcal{Q}_q]$. For every $i < \kappa_\omega$, using that $\gamma_i < \beta_i \leq \gamma_{i+1}$ and $\beta_i \in C$, let $q_i \in \mathbb{I}$ with stem h, $q_i \leq q$, and v_1^i, v_2^i be nodes such that there are $r_1^i, r_2^i \in \mathcal{Q}_{q_i}$, such that:

1. q_i forces that $u_i <_{\dot{T}} v_1^i$, $u_i <_{\dot{T}} v_2^i$, and $v_1^i \perp_{\dot{T}} v_2^i$;
2. $\langle q_i, r_1^i \rangle \Vdash_{\mathbb{I}\times\mathbb{Q}} v_1^i \in \dot{b}$;
3. $\langle q_i, r_2^i \rangle \Vdash_{\mathbb{I}\times\mathbb{Q}} v_2^i \in \dot{b}$;
4. $\mathit{level}(v_1^i), \mathit{level}(v_2^i) < \gamma_{i+1}$.

We also arrange that each $q_i \leq p_i$, using the fact that they have the same stem.

By the Prikry property, let $p_i \leq^* q_i \wedge p^{i+1}$ be such that p_i decides "$v_1^i <_{\dot{T}} u_{i+1}$" and "$v_2^i <_{\dot{T}} u_{i+1}$". At least one of them is decided negatively, say this is true for v_1^i. Let $r_i = r_1^i$ and $v_i = v_1^i$. Then if $i < j < \kappa_\omega$, the level of v_i is less that the level of u_{i+1}, and

$$p_i \wedge p_j \Vdash_{\mathbb{I}} v_i \not<_{\dot{T}} u_{i+1} \leq_{\dot{T}} u_j <_{\dot{T}} v_j.$$

Also, each $r_i \in \mathcal{Q}_{q_i} \subset \mathcal{Q}_{p_i}$. It follows that $\langle p_i, v_i, r_i \mid i < \kappa_\omega \rangle$ are as desired. $\square$

Now, working in $V[\mathcal{A}][\mathcal{Q}]$ construct $\langle \langle p_\sigma, r_\sigma \rangle \mid \sigma \in \kappa_\omega^{<\omega} \rangle$, of conditions in $\mathbb{I} \times \mathbb{Q}$ and nodes $\langle u_\sigma \mid \sigma \in \kappa_\omega^{<\omega} \rangle$ such that:

(1) for each σ, $\text{stem}(p_\sigma) = h$, $r_\sigma \in \mathcal{Q}_{p_\sigma}$, $r_\sigma \leq_{\mathbb{Q}} \bar{r}$.
(2) if $\sigma_2 \supset \sigma_1$, then $\langle p_{\sigma_2}, r_{\sigma_2} \rangle \leq_{\mathbb{I} \times \mathbb{Q}} \langle p_{\sigma_1}, r_{\sigma_1} \rangle$,
(3) for every $\sigma \in \kappa_\omega^{<\omega}$,

$$\langle p_\sigma, r_\sigma \rangle \Vdash^{V[\mathcal{A}]}_{\mathbb{I} \times \mathbb{Q}} u_\sigma \in \dot{b},$$

(4) for every σ and $i < j < \kappa_\omega$, over $V[\mathcal{A}][\mathcal{Q}]$,

$$p_{\sigma^\frown i} \wedge p_{\sigma^\frown j} \Vdash_{\mathbb{I}} u_{\sigma^\frown i} \perp_{\dot{T}} u_{\sigma^\frown j}.$$

We do this by induction on the length of σ, using the splitting lemma.

Let $\alpha = \sup_{\sigma \in \kappa_\omega^{<\omega}} \text{level}(u_\sigma)$; then $\alpha < \mu$. For every $f \in \kappa_\omega^\omega$, let $p_f \in \mathbb{I}$ with stem h be the weakest common extension of $p_{f \restriction n}$ for all n. Then let $r_f \in \mathcal{Q}_{p_f}$ be such that $\langle p_f, r_f \rangle \leq_{\mathbb{I} \times \mathbb{Q}} \langle p_{f \restriction n}, r_{f \restriction n} \rangle$ for all n. Here we use the fact that since every $r_{f \restriction n} \in \mathcal{Q}_{p_{f \restriction n}}$, then it is also in $\mathcal{Q}_{p_f}$. Then let $\langle p'_f, r'_f \rangle \leq_{\mathbb{I} \times \mathbb{Q}} \langle p_f, r_f \rangle$ be such that for some $\xi_f < \kappa$, $\langle p'_f, r'_f \rangle \Vdash_{\mathbb{I} \times \mathbb{Q}} \langle \alpha, \xi_f \rangle \in \dot{b}$ and $r'_f \in \mathcal{Q}_{p_f}$. We can obtain the latter by a density argument, since $\mathbb{Q}$ projects to $\mathbb{Q}_{p_f}$.

Let $f, g \in \kappa_\omega^\omega$ be distinct, such that $\xi = \xi_f = \xi_g$ and $\text{stem}(p'_f) = \text{stem}(p'_g) = h'$. Let n be such that $f \restriction n = g \restriction n = \sigma$, but $i = f(n) \neq g(n) = j$.

Let $p = p'_f \wedge p'_g$. Then $r'_f, r'_g \in \mathcal{Q}_p$ (since $\mathcal{Q}_{p_f} \subset \mathcal{Q}_p$ and $\mathcal{Q}_{p_g} \subset \mathcal{Q}_p$).

Let $\mathcal{I}'$ be $\mathbb{I}$-generic over $V' = V[\mathcal{A}][\mathcal{Q}_p]$ with $p \in \mathcal{I}'$. Note that $\mathcal{A} * \mathcal{I}' \times \mathcal{Q}_p$ induces a generic for $\mathbb{R}/\mathcal{A} * \mathcal{I}'$. Then, in $V'[\mathcal{I}']$, interpreting $\dot{b}$ as a $\mathbb{Q}/\mathcal{Q}_p$-name in the natural way and setting $T = (\dot{T}_p)_{\mathcal{I}'}$, we have:

- $r'_f \Vdash_{\mathbb{Q}/\mathcal{Q}_p} \langle \alpha, \xi \rangle \in \dot{b}$, $r'_f \Vdash_{\mathbb{Q}/\mathcal{Q}_p} u_{\sigma^\frown i} \in \dot{b}$;
- $r'_g \Vdash_{\mathbb{Q}/\mathcal{Q}_p} \langle \alpha, \xi \rangle \in \dot{b}$, $r'_g \Vdash_{\mathbb{Q}/\mathcal{Q}_p} u_{\sigma^\frown j} \in \dot{b}$;
- $u_{\sigma^\frown i} \perp_T u_{\sigma^\frown j}$.

But both $\text{level}(u_{\sigma^\frown i}) < \alpha$ and $\text{level}(u_{\sigma^\frown j}) < \alpha$, contradiction.

$\square$

Remark 3.7. We could have done the proof by working in $V[\mathcal{A}][\mathcal{Q}_{h^\frown 1}]$ instead, where $h^\frown 1$ is the weakest element in $\mathbb{I}$ with stem h. Note that for every q with stem h, $V[\mathcal{A}][\mathcal{Q}_q] \subset V[\mathcal{A}][\mathcal{Q}_{h^\frown 1}]$. When doing the splitting, we just have to rely on Easton's lemma to get $\langle \gamma_i, \beta_i, p^i \mid i < \kappa_\omega \rangle$ in $V[\mathcal{A}]$, and so in $V[\mathcal{A}][\mathcal{Q}_{h^\frown 1}]$.

3.3. **Defining the branch.** Let $\alpha := \sup\{\alpha_h \mid \dagger_h \text{ holds}\} < \mu$, and let $u \in T_\alpha$ and $s^* \in G^*$ be such that $s^* \Vdash_{\mathbb{R}^*/\mathcal{A}} u \in \dot{b}$. Then in $V[G]$, set

$$d = \{v \mid u <_T v, (\exists s \in G) s \leq_{\mathbb{R}^*} s^*, s \Vdash_{\mathbb{R}^*/\mathcal{A}} v \in \dot{b}\}.$$

To prove that this is a branch we use that there is no more splitting after α and the following lemma:

Lemma 3.8. *If $s \in \mathbb{R}^*/G$, then there is $p \in \mathcal{I}$, such that $s \in G_p$.*

Proof. In $V[G]$, define $D = \{s' \in \mathbb{R}^*/G \mid (\exists p \in \mathcal{I}) s' \leq_{\mathbb{R}_{\dot{p}}} s\}$. We claim that D is dense in $\mathbb{R}^*/G$.

Suppose $s' \in \mathbb{R}^*/G$. Denote $s = \langle a, \dot{q}, r\rangle, s' = \langle a', \dot{q}', r'\rangle$. Since both are in G, they have a common extension $s'' = \langle b, \dot{p}, r"\rangle$ in G. Then $s'' \leq_{\mathbb{R}_p/G} s$ and $s'' \leq_{\mathbb{R}_p/G} s'$. Finally, by the same argument as in Lemma 2.3, let $s^* \in \mathbb{R}^*/G$ be such that $s^* \leq_{\mathbb{R}^*/G} s'$ and $s^* \leq_{\mathbb{R}_{\dot{p}}} s''$. Then $s^* \in D$ and is below s'.

Then let $s^* \in D \cap G^*$, and let $p \in \mathcal{I}$ witness that $s^* \in D$. Then $s^* \in G_p$ (since G^* induces G_p), and so $s \in G_p$ by upwards closure.

□

Corollary 3.9. *d induces a branch through T in $V[G]$.*

Proof. Clearly $\{\gamma < \mu \mid (\exists v \in T_\gamma) v \in d\}$ is unbounded in μ, and actually it is a tail end of μ. Next we have to show that for every $\gamma \geq \alpha$, $|d \cap T_\gamma| = 1$.

Suppose for contradiction there are distinct $v_1, v_2 \in d \cap T_\gamma$ for some $\gamma \geq \alpha$. Let $s_1, s_2 \leq_{\mathbb{R}^*} s^*$, $s_1, s_2 \in G$ witness that $v_1, v_2 \in d$. Denote $s_1 = \langle a_1, p_1, r_1\rangle$ and $s_1 = \langle a_2, p_2, r_2\rangle$. Note that since the nodes are on the same level, $p_1 \wedge p_2 \Vdash_{\mathbb{I}} v_1 \perp_{\dot{T}} v_2$.

By Lemma 3.8, there is some $q \in \mathcal{I}$, such that both $s_1, s_2 \in G_q$. By extending q if necessary, we may assume that $q \leq p_1$, $q \leq p_2$, and $h := \text{stem}(q)$ is such that $\dagger_h$ holds. Note that since $q \in \mathcal{I}$, extending the Prikry part of s^*, we have that $u \in E_h$. Now let $s'_1 = \langle a_1, q, r_1\rangle$ and $s'_1 = \langle a_2, q, r_2\rangle$. Then $s'_1 \leq_{\mathbb{R}^*/G} s_1$ and $s'_1 \leq_{\mathbb{R}^*/G} s_1$ and s'_1, s'_2 are still in G_q. It follows that q, s'_1 and s'_2 witness an h-splitting at u, but $\alpha > \alpha_h$, contradiction. □

The following remains open:

Question 1. Can the result of Theorem 1.1 be obtained at $\kappa_0 = \aleph_\omega$, or even $\kappa_0 = \aleph_{\omega^2}$?

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