

KUREPA TREES AND SPECTRA OF $\mathcal{L}_{\omega_1, \omega}$ -SENTENCES

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ABSTRACT. We use set-theoretic tools to make a model-theoretic contribution. In particular, we construct a *single* $\mathcal{L}_{\omega_1, \omega}$ -sentence ψ that codes Kurepa trees to prove the following statements:

- (1) The spectrum of ψ is consistently equal to $[\aleph_0, \aleph_{\omega_1}]$ and also consistently equal to $[\aleph_0, 2^{\aleph_1})$, where 2^{\aleph_1} is weakly inaccessible.
- (2) The amalgamation spectrum of ψ is consistently equal to $[\aleph_1, \aleph_{\omega_1}]$ and $[\aleph_1, 2^{\aleph_1})$, where again 2^{\aleph_1} is weakly inaccessible.
- (3) Consistently, ψ has maximal models in finite, countable, and uncountable many cardinalities. This complements the examples given in [2] and [3] of sentences with maximal models in countably many cardinalities.
- (4) Consistently, $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and there exists an $\mathcal{L}_{\omega_1, \omega}$ -sentence with models in \aleph_{ω_1} , but no models in 2^{\aleph_1} .

This is the first example of an $\mathcal{L}_{\omega_1, \omega}$ -sentence whose spectrum and amalgamation spectrum are consistently both right-open and right-closed. It also provides a positive answer to a question in [19].

This relates to a conjecture by Shelah that if $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1, \omega}$ -sentence with a model of size \aleph_{ω_1} also has a model of size 2^{\aleph_0} . Our result proves that 2^{\aleph_0} can not be replaced by 2^{\aleph_1} , even if $2^{\aleph_0} < \aleph_{\omega_1}$.

1. INTRODUCTION

Definition 1.1. For an $\mathcal{L}_{\omega_1, \omega}$ -sentence ϕ , the (model-existence) *spectrum* of ϕ is the class

$$\text{Spec}(\phi) = \{\kappa \mid \exists M \models \phi \text{ and } |M| = \kappa\}.$$

If $\text{Spec}(\phi) = [\aleph_0, \kappa]$, we say that ϕ *characterizes* κ .

The *amalgamation spectrum* of ϕ , $\text{AP-Spec}(\phi)$ for short, is the set of all cardinals κ so that (i) ϕ has at least one model of size κ and (ii) the models of ϕ of size κ satisfy the amalgamation property¹ Similarly define $\text{JEP-Spec}(\phi)$ the *joint embedding spectrum* of ϕ .

².

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¹For the definition of the amalgamation property and the joint embedding property, see:

- (1) https://en.wikipedia.org/wiki/Joint_embedding_property (accessed January 2020)
- (2) https://en.wikipedia.org/wiki/Amalgamation_property (accessed January 2020)

² The amalgamation property, the joint embedding property and the property of being maximal are not unique. Rather they depend on the embeddings used. In this paper we consider these properties under the substructure relation. See definition of $(\mathbf{K}, \prec_{\mathbf{K}})$ before Lemma 2.5

The *maximal models spectrum* of ϕ is the class

$$\text{MM-Spec}(\phi) = \{\kappa \mid \exists M \models \phi \text{ and } M \text{ is maximal}\}.$$

Morley and López-Escobar independently established that all cardinals that are characterized by an $\mathcal{L}_{\omega_1, \omega}$ -sentence are smaller than \beth_{ω_1} .³

Theorem 1.2 (Morley, López-Escobar). *Let Γ be a countable set of sentences of $\mathcal{L}_{\omega_1, \omega}$. If Γ has models of cardinality \beth_α for all $\alpha < \omega_1$, then it has models of all infinite cardinalities.*

In 2002, Hjorth ([7]) proved the following.

Theorem 1.3 (Hjorth). *For all $\alpha < \omega_1$, \aleph_α is characterized by a complete $\mathcal{L}_{\omega_1, \omega}$ -sentence.*

Combining Theorems 1.2 and 1.3 we get under GCH that \aleph_α is characterized by a complete $\mathcal{L}_{\omega_1, \omega}$ -sentence if and only if $\alpha < \omega_1$.

Given these results, one might ask if it is consistent under the failure of GCH that \aleph_{ω_1} is characterizable. The answer is positive because one can simply force $\aleph_{\omega_1} = 2^{\aleph_0}$ and by [15], 2^{\aleph_0} can be characterized by a complete $\mathcal{L}_{\omega_1, \omega}$ -sentence. In order to block the possibility of characterizing \aleph_{ω_1} by manipulating the value of e.g. 2^{\aleph_0} , in [19] the question was raised if any cardinal outside the set C which is defined to be the smallest set that contains \aleph_0 and is closed under successors, countable unions, countable products and powerset, can be characterized by an $\mathcal{L}_{\omega_1, \omega}$ -sentence. Notice that C contains all cardinals of the form \aleph_α , for countable α , and it follows by Theorems 1.2 and 1.3 that under GCH, $\aleph_\alpha \in C$ if and only if α is countable. Therefore, \aleph_{ω_1} is the smallest cardinal that consistently does not belong to C . With this in mind, we consider the possibility of \aleph_{ω_1} being a characterizable cardinal not in C . In Lemma 3.2 we prove that this can be consistently true, thus providing a positive answer to the question from [19].

The importance of \aleph_{ω_1} is also emphasized by a conjecture of Shelah that if $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1, \omega}$ -sentence with a model of size \aleph_{ω_1} also has a model of size 2^{\aleph_0} . In [18], Shelah proves the consistency of the conjecture. In Theorem 2.10 (see also Corollary 2.15) we prove that there is a sentence that consistently characterizes the maximum of 2^{\aleph_0} and \aleph_{ω_1} . It follows that consistently $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and there is an $\mathcal{L}_{\omega_1, \omega}$ -sentence that characterizes \aleph_{ω_1} . This proves that 2^{\aleph_0} can not be replaced by 2^{\aleph_1} in the conjecture, even if $2^{\aleph_0} < \aleph_{\omega_1}$. Our example can not be used to refute Shelah's conjecture because our sentence always has a model of size 2^{\aleph_0} .

Another contribution of the current paper is in providing an $\mathcal{L}_{\omega_1, \omega}$ -sentence whose spectrum (and amalgamation spectrum) exhibits quite peculiar behavior when we manipulate the underlying set theory. In particular, we provide the first example of an $\mathcal{L}_{\omega_1, \omega}$ -sentence whose spectrum can consistently be right-open and right-closed. If $\kappa = \sup_{n \in \omega} \kappa_n$ and ϕ_n characterizes κ_n , then $\bigvee_n \phi_n$ is a (necessarily incomplete) sentence that has spectrum $[\aleph_0, \kappa)$. All the previously known examples of $\mathcal{L}_{\omega_1, \omega}$ -sentences with a right-open spectrum are variations of this example and have the form $[\aleph_0, \kappa)$, with κ of countable cofinality. In Corollary 2.14 we prove that it is consistent that 2^{\aleph_1} is a weakly inaccessible cardinal and at the same time, our example has spectrum $\text{Spec}(\psi) = [\aleph_0, 2^{\aleph_1})$. By Lemma 2.2, our methods can not be used to establish the consistency of $\text{Spec}(\psi) = [\aleph_0, \aleph_{\omega_1})$, which remains

³See [11] for more details.

open. It also open to find a *complete* sentence with a right-open spectrum, much less a complete sentence with a spectrum which is consistently right-open and right-closed.

In Theorem 2.16 we prove that the amalgamation spectrum contains exactly the uncountable cardinals in the model-existence spectrum. So, the same comments about the spectrum also apply to the amalgamation spectrum. Moreover, by manipulating the size of 2^{\aleph_1} , Corollary 2.17 implies that for κ regular with $\aleph_2 \leq \kappa \leq 2^{\aleph_1}$, κ -amalgamation for $\mathcal{L}_{\omega_1, \omega}$ -sentences is non-absolute for models of ZFC. This complements the result in [17] that under GCH, \aleph_α -amalgamation (under substructure again) is not absolute for $1 < \alpha < \omega$. It is interesting to notice that the examples presented both in this paper and in [17] can not settle the absoluteness question for \aleph_1 -amalgamation, which remains open. We also notice that the amalgamation spectra in this paper are always non-empty, as they always contain \aleph_1 . This is in contrast with the examples in [17], which maybe consistently empty or non-empty.

Investigating the maximal models spectrum of our example, $\text{MM-Spec}(\psi)$, we prove in Theorem 2.10 that $\text{MM-Spec}(\psi)$ is the set of all cardinals κ such that κ is either equal to \aleph_1 , or equal to 2^{\aleph_0} , or there exists a Kurepa tree with exactly κ -many branches. Manipulating the cardinalities on which there exist Kurepa trees, we can produce maximal models in finite, countable, and uncountable many cardinalities, plus we can carefully control which these cardinalities are. Results concerning the maximal models spectrum can also be found in [2] and [3]. From [2] we know that if $\bar{\lambda} = \langle \lambda_i : i \leq \alpha < \aleph_1 \rangle$ is a strictly increasing sequence of characterizable cardinals, there is an $\mathcal{L}_{\omega_1, \omega}$ -sentence $\phi_{\bar{\lambda}}$, which has the maximum number (that is $2^{\lambda_i^+}$) non-isomorphic maximal models in cardinalities λ_i^+ , for all $i \leq \alpha$, and no maximal models in any other cardinality. The sentence $\phi_{\bar{\lambda}}$ is incomplete. In contrast, the examples produced in [3] are complete sentences. The example in the current paper is also an incomplete sentence.

In [6], Grossberg conjectured that for every Abstract Elementary Class (AEC)⁴ \mathbf{K} there exists a cardinal $\mu(LS(\mathbf{K}))$ such that if \mathbf{K} has the $\mu(LS(\mathbf{K}))$ -amalgamation property then \mathbf{K} has the λ -amalgamation property for all $\lambda \geq \mu(LS(\mathbf{K}))$. This cardinal $\mu(LS(\mathbf{K}))$ is called the Hanf number for amalgamation. Baldwin and Boney in [21] proved the existence of a Hanf number for amalgamation, but their definition of a Hanf number is different than the one that Grossberg conjectured. Their result states that if μ is a strongly compact cardinal, \mathbf{K} is an AEC with $LS(\mathbf{K}) < \mu$, and \mathbf{K} satisfies amalgamation cofinally below μ , then \mathbf{K} satisfies amalgamation in all cardinals $\geq \mu$. The question was raised if the strongly compact upper bound was optimal for the Hanf number for amalgamation.

Far from being optimal, a lower bound for the Hanf number for amalgamation is given by the examples in [12]. There it is proved that for every cardinal κ and every α with $\kappa \leq \alpha < \kappa^+$ there exists an AEC W_α with countable $LS(W_\alpha)$, the vocabulary is of size κ , W_α satisfies amalgamation up to \beth_α , but fails amalgamation in \beth_{κ^+} .

Although in this paper we do not claim to provide any lower bounds for the Hanf number for amalgamation, we believe that the idea of coding κ -Kurepa trees, for $\kappa > \aleph_1$, can be pursued further with the goal of producing amalgamation spectra that exhibit similar behavior to the ones in this paper, but in higher cardinalities.

⁴A good reference about Abstract Elementary Classes is [1].

Finally, we want to emphasize again the fact that in this paper we produce a *single* $\mathcal{L}_{\omega_1, \omega}$ -sentence ψ and all the results mentioned above follow by considering the various spectra of this sentence ψ and how set-theory affects these spectra. In particular, we prove the consistency of the following.

- (1) $2^{\aleph_0} < \aleph_{\omega_1}$ and $\text{Spec}(\psi) = [\aleph_0, \aleph_{\omega_1}]$.
- (2) $2^{\aleph_0} < 2^{\aleph_1}$, 2^{\aleph_1} is weakly inaccessible and $\text{Spec}(\psi) = [\aleph_0, 2^{\aleph_1}]$.
- (3) $2^{\aleph_0} < \aleph_{\omega_1}$ and $\text{AP-Spec}(\psi) = [\aleph_1, \aleph_{\omega_1}]$.
- (4) $2^{\aleph_0} < 2^{\aleph_1}$, 2^{\aleph_1} is weakly inaccessible and $\text{AP-Spec}(\psi) = [\aleph_1, 2^{\aleph_1}]$.
- (5) $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and there exists an $\mathcal{L}_{\omega_1, \omega}$ -sentence with models in \aleph_{ω_1} , but no models in 2^{\aleph_1} .

Section 2 contains the description of the sentence ψ and the results about the model-theoretic properties of ψ . Section 3 contains the consistency results.

2. KUREPA TREES AND $\mathcal{L}_{\omega_1, \omega}$

The reader can consult [8] about trees. The following definition summarizes all that we will use in this paper.

Definition 2.1. Assume κ is an infinite cardinal. A κ -tree has height κ and each level has at most $< \kappa$ elements. A κ -Kurepa tree is a κ -tree with at least κ^+ branches of height κ . Kurepa trees, with no κ specified, refer to \aleph_1 -Kurepa trees. For this paper we will assume that κ -Kurepa trees are *pruned*, i.e. every node is contained in a maximal branch of order-type κ .

If $\lambda \geq \kappa^+$, a (κ, λ) -Kurepa tree is a κ -Kurepa tree with exactly λ branches of height κ . $KH(\kappa, \lambda)$ is the statement that there exists a (κ, λ) -Kurepa tree.

Define $\mathcal{B}(\kappa) = \sup\{\lambda \mid KH(\kappa, \lambda) \text{ holds}\}$ and $\mathcal{B} = \mathcal{B}(\aleph_1)$.

If κ -Kurepa trees exist, it is immediate that $\mathcal{B}(\kappa)$ is always between κ^+ and 2^κ . We are interested in the case where \mathcal{B} is a supremum but not a maximum. The next lemma proves some restrictions when \mathcal{B} is not a maximum. In Section 3 we prove that it is consistent with ZFC that \mathcal{B} is not a maximum.

Lemma 2.2. *If $\mathcal{B}(\kappa)$ is not a maximum, then $cf(\mathcal{B}(\kappa)) \geq \kappa^+$.*

Proof. Towards contradiction assume that $cf(\mathcal{B}(\kappa)) = \mu \leq \kappa$. Let $\mathcal{B}(\kappa) = \sup_{i \in \mu} \kappa_i$ and let $(T_i)_{i \in \mu}$ be a collection of κ -Kurepa trees, where each T_i has exactly κ_i -many cofinal branches.

Create new κ -Kurepa trees S_i by induction on $i \leq \mu$: S_0 equals T_0 ; S_{i+1} equals the disjoint union of S_i together with a copy of T_{i+1} , arranged so that the j^{th} level T_{i+1} coincides with the $(j+i)^{\text{th}}$ level of S_i . In particular, no new nodes are added in S_{i+1} below the i^{th} level. At limit stages take unions.

We leave the verification to the reader that S_μ is a κ -Kurepa tree with exactly $\sup_{i \in \mu} \kappa_i = \kappa$ cofinal branches, contradicting the fact that $\mathcal{B}(\kappa)$ is not a maximum. \square

Definition 2.3. Let $\kappa \leq \lambda$ be infinite cardinals. A sentence σ in a language with a unary predicate P admits (λ, κ) , if σ has a model M such that $|M| = \lambda$ and $|P^M| = \kappa$. In this case, we will say that M is of type (λ, κ) .

From [4], theorem 7.2.13, we know

Theorem 2.4. *There is a (first-order) sentence σ such that for all infinite cardinals κ , σ admits (κ^{++}, κ) iff $KH(\kappa^+, \kappa^{++})$.*

The idea of the proof of Theorem 2.4 is also given in [4], but the details are left to the reader. Since the amalgamation property is heavily dependent on the particular formulation⁵, we give a full description of sentence σ , including the vocabulary, before we proceed.

The vocabulary τ consists of the constant 0, the unary symbols P, L , the binary symbols $S, V, T, <, \prec, H$, and the ternary symbols F, G . The idea is to build a κ -tree. L is a set that corresponds to the “levels” of the tree. L is linearly ordered by $<$ and 0 is its minimum element. L may or may not have a maximum element. Every element $a \in L$ that is not the maximum element (if any) has a successor b that satisfies $S(a, b)$. We will freely denote the successor of a by $S(a)$. The maximum element is not a successor. For every $a \in L$, $V(a, \cdot)$ is the set of nodes at level a and we assume that $V(a, \cdot)$ is disjoint from L . If $V(a, x)$, we will say that x is at level a and we may write $x \in V(a)$.

T is a tree ordering on $V = \bigcup_{a \in L} V(a)$. If $T(x, y)$, then x is at some level strictly less than the level of y . If $y \in V(a)$ and $b < a$, there is some x so that $x \in V(b)$ and $T(x, y)$. If a is a limit, that is neither a successor nor 0, then two distinct elements in $V(a)$ can not have the same predecessors. If m is the maximum element of L , $V(m)$ is the set of maximal branches through the tree. Both “the height of T ” and “the height of L ” refer to the order type of $(L, <)$. Although it is not necessary for Theorem 2.4, we can stipulate that the Kurepa tree is pruned.

There is an old trick to bound the size of a linear order by κ^+ by bounding the size of each initial segment by κ . We use this trick to bound the size of L by κ^+ and the size of $V(m)$ by κ^{++} .

For every $a \in L$ that is not the maximum element of L (if any), there is a surjection $F(a, \cdot, \cdot)$ from the predicate P to $L_{\leq a} = \{b \in L \mid b \leq a\}$ and another surjection $G(a, \cdot, \cdot)$ from P to $V(a)$. This bounds the size of every initial segment $L_{\leq a}$ and the size every $V(a)$ by $|P|$, and in return the size of L by $|P|^+$.

If L has a maximum element m , we linearly order the set of maximal branches $V(m)$ by \prec so that there is no maximum element and require that H is a surjection from L to each initial segment. It follows that the size of $V(m)$ is bounded by $|L|^+$.

The disjoint union of $L, P, \cup_{a \in L} V(a)$ gives the domain of the model.

Call σ the (first-order) sentence that stipulates all the above. If \mathcal{M} is a model of σ with $P^{\mathcal{M}}$ of size κ and \mathcal{M} has size κ^{++} , then L has a maximum element m , $|L^{\mathcal{M}}| = \kappa^+$ and $|V^{\mathcal{M}}(m)| = \kappa^{++}$. In fact, the cofinality of $L^{\mathcal{M}}$ in this case will also be κ^+ , although $L^{\mathcal{M}}$ may not be well-order. So, \mathcal{M} codes a $|P|$ -Kurepa tree. Conversely, from every κ -Kurepa tree, we can define a model of σ with $|P| = \kappa$. This proves Theorem 2.4.

We want to emphasize here the fact that since well-orderings can not be characterized by an $\mathcal{L}_{\omega_1, \omega}$ -sentence, it is unavoidable that we will be working with non-well-ordered trees. However, using an $\mathcal{L}_{\omega_1, \omega}$ -sentence we can express the fact that P is countably infinite. There are many ways to do this. The simplest way is to introduce countably many new constants $(c_n)_{n \in \omega}$ and require that $\forall x, P(x) \leftrightarrow \bigvee_n x = c_n$. Let ϕ be the conjunction of σ together with the requirement that P is countably infinite. Then ϕ has models of size \aleph_2 iff there exist a Kurepa tree of size \aleph_2 iff $KH(\aleph_1, \aleph_2)$.

⁵This means it is possible to come up with two different sentences σ_1, σ_2 that both satisfy Theorem 2.4, but which exhibit different amalgamation spectra.

Fix some $n \geq 2$. Then the above construction of σ (and ϕ) can be modified to produce a first-order sentence σ_n and the corresponding $\mathcal{L}_{\omega_1, \omega}$ -sentence ϕ_n so that ϕ_n has a model of size \aleph_n iff there exist a Kurepa tree of size \aleph_n iff $KH(\aleph_1, \aleph_n)$. The argument breaks down at \aleph_ω . Since we will be dealing with Kurepa trees of size potentially larger than \aleph_ω , we must make some modifications.

Let τ' be equal to τ with the symbols \prec, H removed. Let σ' be equal to σ with all requirements that refer to \prec, H removed. Let ψ be the conjunction of σ' and the requirement that P is countably infinite as formulated above⁶. Notice that both ϕ and ψ are not complete sentences.

For any $\lambda \geq \aleph_2$, any (\aleph_1, λ) -Kurepa tree gives rise to a model of ψ , but unfortunately, there are models of ψ of size 2^{\aleph_0} , that do not code a Kurepa tree. These trees have countable height. For instance consider the tree $(\omega^{\leq \omega}, \subset)$ which contains 2^{\aleph_0} many maximal branches.

The dividing line for models of ψ to code Kurepa trees is the size of L . By definition L is \aleph_1 -like, i.e. every initial segment has countable size. If in addition L is uncountable, then we can embed ω_1 cofinally into L .⁷ Hence, every model of ψ of size $\geq \aleph_2$ and for which L is uncountable, codes a Kurepa tree. Otherwise, the model does not code a Kurepa tree.

Let \mathbf{K} be the collection of all models of ψ , equipped with the substructure relation. I.e. for $M, N \in \mathbf{K}$, $M \prec_{\mathbf{K}} N$ if $M \subset N$.

Lemma 2.5. *If $M \prec_{\mathbf{K}} N$, then*

- (1) L^M is an initial segment of L^N ;
- (2) for every non-maximal $a \in L^M$, $V^M(a, \cdot)$ equals $V^N(a, \cdot)$.
- (3) the tree ordering is preserved.

We will express (1) – (3) by saying that “(the tree defined by) M is an initial segment of (the tree defined by) N ”.

Proof. For part (1), first recall that if $M \prec_{\mathbf{K}} N$, then $P^M = P^N$. Towards contradiction, assume that L^N contains some point $x \notin L^M$ and there exists some $y \in L^M$ such that $x < y$. Then the function $F^M(y, \cdot, \cdot)$ defined on M disagrees with the function $F^N(y, \cdot, \cdot)$ defined on N . Contradiction.

For part (2), the argument is similar to the argument for part (1), using the functions $G^M(y, \cdot, \cdot)$ and $G^N(y, \cdot, \cdot)$ this time.

Part (3) is immediate from the definition. \square

Corollary 2.6. *Assume M is an initial segment of N . Then:*

- (1) If $L^N = L^M$, then N differs from M only in the maximal branches it contains.
- (2) If L^M is uncountable and L^N is a strict end-extension of L^M , then L^M does not have a maximum element and L^N is a one-point end extension of L^M .
- (3) If L^M has a maximum element and L^N is a strict end-extension of L^M , then M must be countable.

Proof. Part (1) is immediate from Lemma 2.5 (2).

⁶Our formulation has the property that that if $M \subset N$ are two models of ψ , then $P^M = P^N$. That is, P does not grow.

⁷The embedding is not necessarily continuous, i.e. it may not respect limits.

For part (2), observe that if L^M had a maximum element, then L^N would not be \aleph_1 -like. We derive a similar contradiction if we assume that L^N contains at least two new points not in L^M .

For part (3) first observe that L^M is countable. Otherwise we get a contradiction from part (2). Then the statement follows from the requirement that in all models of ψ all non-maximal levels are countable. \square

Corollary 2.7. *Assume κ is an uncountable cardinal and $M \in \mathbf{K}$ codes a tree together with all its κ many cofinal branches. Then M is maximal.*

Proof. Towards contradiction, assume that M is not maximal. Then there exists some $N \in \mathbf{K}$ with $M \prec_{\mathbf{K}} N$. Since L^M has a maximum element and M is uncountable, it follows by Corollary 2.6(3) that L^N can not be a strict end-extension of L^M . Therefore, $L^M = L^N$. By Corollary 2.6(1), M, N differ only on the maximal branches they contain. But M contains all the cofinal branches through the tree. Contradiction. \square

Lemma 2.8. *$(\mathbf{K}, \prec_{\mathbf{K}})$ is an AEC with countable Lowenheim-Skolem number.*

Proof. We verify closure under unions of chains. The other properties for being an AEC are immediate. The requirements set forth by σ' can be written using a $\forall \exists$ sentence. Therefore σ' is preserved under unions of chains. The requirement that P is countable was formulated in such a way that does not allow P to gain new elements. Therefore, this requirement is also preserved under unions of chains.

To prove that $LS(\mathbf{K})$ is countable, consider N a model of ψ and some countable $X \subset N$. We need some countable model M of ψ that contains X and $M \prec_{\mathbf{K}} N$. If L^N is countable, then take $M \subset N$ so that the domain of M is the union of P^N, L^N , every non-maximal level of V^N , and all maximal branches from V^N that belong to X (if any). If L^N is uncountable, then without loss of generality assume that it has a maximum element m . Let L_0 be a strict initial segment of L^N that does not have a maximum element and such that if x is a non-maximal element in X , then x belongs to a level in L_0 , and if x, y are two maximal elements in $X \cap V^N$, then x, y differ at some level in L_0 . Take $M \subset N$ so that the domain of M is the union of $P^N, L_0 \cup \{m\}$, the elements of V^N that belong to a level in L_0 , and all maximal branches from V^N that belong to X (if any). The reader can verify that in either case M is a model in \mathbf{K} and contains X . \square

Convention 2.9. For the rest of the paper, when we talk about “the models of ψ ” we will mean $(\mathbf{K}, \prec_{\mathbf{K}})$.

The next theorem characterizes the spectrum and the maximal models spectrum of ψ .

Theorem 2.10. *The spectrum of ψ is characterized by the following properties:*

- (1) $[\aleph_0, 2^{\aleph_0}] \subset \text{Spec}(\psi)$;
- (2) if there exists a Kurepa tree with κ many branches, then $[\aleph_0, \kappa] \subset \text{Spec}(\psi)$;
- (3) no cardinal belongs to $\text{Spec}(\psi)$ except those required by (1) and (2). I.e. if ψ has a model of size κ , then either $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$, or there exists a Kurepa tree which κ many branches.

The maximal models spectrum of ψ is characterized by the following:

- (4) ψ has maximal models in cardinalities 2^{\aleph_0} and \aleph_1 ;

- (5) if there exists a Kurepa tree with exactly κ many branches, then ψ has a maximal model in κ ;
(6) ψ has maximal models only in those cardinalities required by (4) and (5).

Proof. For (1) and (2), we observed already that $(\omega^{\leq\omega}, \subset)$ is a model of ψ and that every Kurepa tree gives rise to a model of ψ .

For (3), let N be a model of ψ of size κ . If L^N is countable, then N has size $\leq 2^{\aleph_0}$. If L^N is uncountable, then N codes a κ -tree with $|N|$ -many branches. By the proof of theorem 2.4, this is a Kurepa tree, assuming that $|N| \geq \aleph_2$. Otherwise, $|N| \leq \aleph_1 \leq 2^{\aleph_0}$.

To prove (4), notice that $(\omega^{\leq\omega}, \subset)$ is a maximal model of ψ , and it is easy to construct trees with height ω_1 and $\leq \aleph_1$ many branches. Maximality in either case follows from Corollary 2.7.

Now, let N code a Kurepa tree with exactly κ many branches. Using Corollary 2.7 again, N is maximal. This proves (5).

For (6), let N be a maximal model. If N was countable, we could end-extend it and it would not be maximal. So, N must be uncountable. We split into two cases depending on the size of L^N .

If L^N is countable, assume without loss of generality, that it has height ω (otherwise consider a cofinal subset of order type ω). By Corollary 2.6(1) and maximality of N , N must contain all its maximal branches. So, (V^N, T^N) is a pruned tree which is a subtree of $\omega^{\leq\omega}$. The set of maximal branches through (V^N, T^N) is a closed subset of the Baire space (cf. [10], Proposition 2.4). Since closed subsets of ω^ω have size either \aleph_0 or 2^{\aleph_0} , we conclude that N has size 2^{\aleph_0} .

The second case is when L^N is uncountable. If N has size \aleph_1 , we are done. If $|N| \geq \aleph_2$, by Corollary 2.6 again and maximality of N , the tree defined by N contains exactly $|N|$ many maximal branches. Therefore, N defines a Kurepa tree. \square

Recall that $\mathcal{B} = \mathcal{B}(\aleph_1)$ is the supremum of the size of cofinal branches of Kurepa trees.

- Corollary 2.11.** (1) If there are no Kurepa trees, then $\text{Spec}(\psi)$ equals $[\aleph_0, 2^{\aleph_0}]$ and $\text{MM-Spec}(\psi)$ equals $\{\aleph_1, 2^{\aleph_0}\}$.
(2) If \mathcal{B} is a maximum, i.e. there is a Kurepa tree of size \mathcal{B} , then ψ characterizes $\max\{2^{\aleph_0}, \mathcal{B}\}$.
(3) If \mathcal{B} is not a maximum, then $\text{Spec}(\psi)$ equals either $[\aleph_0, 2^{\aleph_0}]$ or $[\aleph_0, \mathcal{B})$, whichever is greater. Moreover ψ has maximal models in \aleph_1 , 2^{\aleph_0} and in cofinally many cardinalities below \mathcal{B} .

Proof. (1) and (2) follow immediately from Theorem 2.10. We only establish (3). If \mathcal{B} is not a maximum, then $[\aleph_0, \mathcal{B}) \subset \text{Spec}(\psi)$ and $\text{Spec}(\psi)$ equals either $[\aleph_0, 2^{\aleph_0}]$ or $[\aleph_0, \mathcal{B})$, whichever is greater. For the the last assertion, assume $\mathcal{B} = \sup_i \kappa_i$ and for each i , there is a Kurepa tree with κ_i many branches. Then each κ_i is in the $\text{MM-Spec}(\psi)$ by Theorem 2.10 (5). \square

In Section 3 we prove the following consistency results.

Theorem 2.12. *The following are consistent with ZFC:*

- (i) $(2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1}) + \text{“}\mathcal{B} \text{ is a maximum“}$, i.e. there exists a Kurepa tree of size \aleph_{ω_1} .

- (ii) $(\aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_0}) +$ “ \mathcal{B} is a maximum”.
- (iii) $(2^{\aleph_0} < \mathcal{B} = 2^{\aleph_1}) +$ “ 2^{\aleph_1} is weakly inaccessible + “for every $\kappa < 2^{\aleph_1}$ there is a Kurepa tree with (at least) κ -many maximal branches, but no Kurepa tree has exactly 2^{\aleph_1} -many branches.”

For part (iii) we assume the existence of a Mahlo cardinal. Moreover, in parts (i) and (ii) we can replace \aleph_{ω_1} by most cardinals below or equal to 2^{\aleph_1} and 2^{\aleph_0} respectively.

From [9] we know the consistency of the following:

Theorem 2.13 (R. Jin). *Assume the existence of two strongly inaccessible cardinals. It is consistent with CH (or $\neg CH$) plus $2^{\aleph_1} > \aleph_2$ that there exists a Kurepa tree with 2^{\aleph_1} many branches and no ω_1 -trees have λ -many branches for some λ strictly between \aleph_1 and 2^{\aleph_1} . In particular, no Kurepa trees have less than 2^{\aleph_1} many branches.*

It follows from the proof of Theorem 2.13 that if λ is regular cardinal above \aleph_2 , we can force the size of 2^{\aleph_1} to equal λ .

Corollary 2.14. *There exists an $\mathcal{L}_{\omega_1, \omega}$ -sentence ψ such that it is consistent with ZFC that*

- (1) ψ characterizes 2^{\aleph_0} ;
- (2) CH (or $\neg CH$), 2^{\aleph_1} is a regular cardinal greater than \aleph_2 and ψ characterizes 2^{\aleph_1} ;
- (3) $2^{\aleph_0} < \aleph_{\omega_1}$ and ψ characterizes \aleph_{ω_1} ; and
- (4) $2^{\aleph_0} < 2^{\aleph_1}$, 2^{\aleph_1} is weakly inaccessible and $\text{Spec}(\psi) = [\aleph_0, 2^{\aleph_1})$.

For the same ψ it is consistent with ZFC that

- (5) $\text{MM-Spec}(\psi) = \{\aleph_1, 2^{\aleph_0}\}$
- (6) $\text{MM-Spec}(\psi) = \{\aleph_1, 2^{\aleph_0}, 2^{\aleph_1}\}$
- (7) $2^{\aleph_0} < 2^{\aleph_1}$, 2^{\aleph_1} is weakly inaccessible and $\text{MM-Spec}(\psi)$ is a cofinal subset of $[\aleph_1, 2^{\aleph_1})$.

For parts (4) and (7) we assume the existence of a Mahlo cardinal.

Proof. The result follows from Theorem 2.10 and Corollary 2.11 using the appropriate model of ZFC each time. For (1) consider a model with no Kurepa trees, or a model where $\mathcal{B} < 2^{\aleph_0}$, e.g. case (ii) of Theorem 2.12. For (2) and (6) use Theorem 2.13. For (3),(4), use Theorem 2.12 cases (i),(iii) respectively. For (5) consider a model of ZFC with no Kurepa trees. For (7) use Theorem 2.12 case (iii) again. \square

Corollary 2.15. *It is consistent with ZFC that $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and there exists an $\mathcal{L}_{\omega_1, \omega}$ -sentence with models in \aleph_{ω_1} , but no models in 2^{\aleph_1} .*

2.1. Amalgamation and Joint Embedding Spectra.

In this section we provide the amalgamation and joint embedding spectrum of models of ψ .

The following characterizes JEP- $\text{Spec}(\psi)$ and AP- $\text{Spec}(\psi)$.

- Theorem 2.16.**
- (1) $(\mathbf{K}, \prec_{\mathbf{K}})$ fails JEP in all cardinals;
 - (2) $(\mathbf{K}, \prec_{\mathbf{K}})$ satisfies AP for all uncountable cardinals that belong to $\text{Spec}(\psi)$, but fails AP in \aleph_0 .

Proof. The first observation is that in all cardinalities there exists two linear orders L^M, L^N none of which is an initial segment of the other. By Lemma 2.5(1), M, N can not be jointly embedded to some larger structure in \mathbf{K} . So, JEP fails in all cardinalities.

A similar argument to JEP proves that there exist three countable linear orders $L^{M_0}, L^{M_1}, L^{M_2}$ such that L^{M_0} is an initial segment of both L^{M_1} and L^{M_2} , and the triple $(L^{M_0}, L^{M_1}, L^{M_2})$ can not be amalgamated. This proves that amalgamation fails in \aleph_0 .

Now, assume that M, N are uncountable models of ψ and $M \prec_{\mathbf{K}} N$. By Corollary 2.6, either $L^M = L^N$ or L^N is a one-point extension of L^M . In either case M, N agree on all non-maximal levels, but N may contain more maximal branches. We use this observation to prove amalgamation.

Let M_0, M_1, M_2 be uncountable models in \mathbf{K} with $M_0 \prec_{\mathbf{K}} M_1, M_2$. Define the amalgam N of (M_0, M_1, M_2) to be the union of M_0 together with all maximal branches in M_1 and M_2 (if any). If two maximal branches have exactly the same predecessors, we identify them. It follows that N is a structure in \mathbf{K} and $M_i \prec_{\mathbf{K}} N$, $i = 1, 2, 3$. \square

Notice that, in general, the amalgamation is not disjoint, since both M_1 and M_2 may contain the same maximal branch.

Corollary 2.17. *The following are consistent:*

- (1) $AP\text{-Spec}(\psi) = [\aleph_1, 2^{\aleph_0}]$;
- (2) $AP\text{-Spec}(\psi) = [\aleph_1, 2^{\aleph_1}]$;
- (3) $2^{\aleph_0} < \aleph_{\omega_1}$ and $AP\text{-Spec}(\psi) = [\aleph_1, \aleph_{\omega_1}]$; and
- (4) $2^{\aleph_0} < 2^{\aleph_1}$, 2^{\aleph_1} is weakly inaccessible and $AP\text{-Spec}(\psi) = [\aleph_1, 2^{\aleph_1}]$.

Proof. The result follows from Theorem 2.16 and Corollary 2.14. \square

It follows from Corollary 2.17 that if $\aleph_2 \leq \kappa \leq 2^{\aleph_1}$ is a regular cardinal, then the κ -amalgamation property for $\mathcal{L}_{\omega_1, \omega}$ -sentences is not absolute for models of ZFC. The result is useful especially under the failure of GCH, since GCH implies that $2^{\aleph_1} = \aleph_2$.

In addition, it is an easy application of Shoenfield's absoluteness that \aleph_0 -amalgamation is an absolute property for models of ZFC. The question for \aleph_1 -amalgamation remains open. Recall that by [5], model-existence in \aleph_1 for $\mathcal{L}_{\omega_1, \omega}$ -sentences is an absolute property for models of ZFC and by [17], if we assume GCH, \aleph_α -amalgamation is non-absolute for $2 \leq \alpha < \omega_1$.

Open Question 2.18. Is \aleph_1 -amalgamation for $\mathcal{L}_{\omega_1, \omega}$ -sentences absolute for models of ZFC?

3. CONSISTENCY RESULTS

In this section we prove the consistency results announced by Theorem 2.12. Recall that \mathcal{B} is the supremum of $\{\lambda \mid \text{there exists a Kurepa tree with } \lambda\text{-many branches}\}$.

Theorem 3.1. *It is consistent with ZFC that:*

- (1) $2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1}$ and there exists a Kurepa tree with \aleph_{ω_1} -many cofinal branches.
- (2) $\aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_0}$ and there exists a Kurepa tree of size \aleph_{ω_1} .

We start with a model V_0 of ZFC+GCH. Let \mathbb{P} be the standard σ -closed poset for adding a Kurepa tree K with \aleph_{ω_1} -many ω_1 -branches. More precisely, conditions in \mathbb{P} are of the form (t, f) , where:

- t is a tree of height $\beta + 1$ for some $\beta < \omega_1$ and countable levels;
- f is a function with $\text{dom}(f) \subset \aleph_{\omega_1}$, $|\text{dom}(f)| = \omega$, and $\text{ran}(f) = t_\beta$, where t_β is the β -th level of t .

Intuitively, t is an initial segment of the generically added tree, and each $f(\delta)$ determines where the δ -th branch intersects the tree at level β . The order is defined as follows: $(u, g) \leq (t, f)$ if,

- t is an initial segment of u , $\text{dom}(f) \subset \text{dom}(g)$,
- for every $\delta \in \text{dom}(f)$, either $f(\delta) = g(\delta)$ (if t and u have the same height) or $f(\delta) <_u g(\delta)$.

We have that \mathbb{P} is countably closed and has the \aleph_2 -chain condition. Suppose that H is \mathbb{P} -generic over V_0 . Then $\bigcup_{(t,f) \in H} t$ is a Kurepa tree with \aleph_{ω_1} -many branches, where for $\delta < \aleph_{\omega_1}$, the δ -th branch is given by $\bigcup_{(t,f) \in H, \delta \in \text{dom}(f)} f(\delta)$. These branches are distinct by standards density arguments. Also, note that since $|B|$ cannot exceed 2^{\aleph_1} and in this model $\aleph_{\omega_1} = |B|$, we have that $\aleph_{\omega_1} < 2^{\aleph_1}$.

The model $V_0[H]$ proves part (1) of Theorem 3.1. The same model also answers positively a question raised in [19]. The question was whether any cardinal outside the smallest set which contains \aleph_0 and which is closed under successors, countable unions, countable products and powerset, can be characterized by an $\mathcal{L}_{\omega_1, \omega}$ -sentence. \aleph_{ω_1} is consistently such an example.

Lemma 3.2. *Let C be the smallest set of cardinals that contains \aleph_0 and is closed under successors, countable unions, countable products and powerset. In $V_0[H]$, the set C does not contain \aleph_{ω_1} .*

Proof. Since $2^\omega < \aleph_{\omega_1} < 2^{\omega_1}$, it is enough to show that \aleph_{ω_1} is not the countable product of \aleph_α 's with $\alpha < \omega_1$. Suppose $\langle \alpha_n \mid n < \omega \rangle$ is an increasing sequence of countable ordinals and let $\alpha = \sup_n \alpha_n + 1$. Then $\prod_n \aleph_{\alpha_n} = (\aleph_\alpha)^\omega = \aleph_{\alpha+1}$. Here we use that GCH holds in V_0 and \mathbb{P} is countably closed in that model. \square

Let $\mathbb{C} = \text{Add}(\omega, \aleph_{\omega_1+1})$ denote the standard poset for adding \aleph_{ω_1+1} -many Cohen reals. Suppose G is \mathbb{C} -generic over $V := V_0[H]$. (Note that \mathbb{C} is interpreted the same in V_0 and in V and that actually genericity over V_0 implies genericity over V by the ccc of \mathbb{C} and the countable closure of \mathbb{P} .)

We claim that the forcing extension $V_0[H \times G] = V[G]$ satisfies $2^{\aleph_0} > \aleph_{\omega_1} = \mathcal{B}$. Let T be a Kurepa tree in $V[G]$. Denote $\mathbb{C}_{\omega_1} := \text{Add}(\omega, \omega_1)$, i.e. the Cohen poset for adding ω_1 many reals. The following fact is standard and can be found in [8], but we give the proof for completeness.

Lemma 3.3. *There is a generic \bar{G} for \mathbb{C}_{ω_1} , such that $V[\bar{G}] \subset V[G]$ and $T \in V[\bar{G}]$.*

Proof. Since T is a tree of height ω_1 and countable levels, we can index the nodes of T by $\langle \alpha, n \rangle$, for $\alpha < \omega_1$ and $n < \omega$, where the first coordinate denotes the level of the node. In particular each level $T_\alpha = \{\alpha\} \times \omega$. Note that this is in the ground model (although of course the relation $<_T$ may not be). Working in $V[G]$, for every $\alpha < \beta < \omega_1$ and n, m , let $p_{\alpha, \beta, m, n} \in G$ decide the statement $\langle \alpha, m \rangle <_T \langle \beta, n \rangle$. Let $d_{\alpha, \beta, m, n} = \text{dom}(p_{\alpha, \beta, m, n})$; this is a finite subset of $\aleph_{\omega_1+1} \times \omega$.

Now let $d^* = \bigcup_{\alpha,\beta,m,n} d_{\alpha,\beta,m,n}$ and $d = \{i < \aleph_{\omega_1+1} \mid (\exists k)\langle i, k \rangle \in d^*\}$. Then d has size at most ω_1 . By increasing d if necessary, assume that $|d| = \omega_1$.

Write G as $\prod_{i \in \aleph_{\omega_1+1}} G_i$, where every G_i is $\text{Add}(\omega, 1)$ -generic, and let $\bar{G} = \prod_{i \in d} G_i$. Then \bar{G} is \mathbb{C}_{ω_1} -generic, containing every $p_{\alpha,\beta,m,n}$. And so $T \in V[\bar{G}] \subset V[G]$. \square

Fix \bar{G} as in the above lemma. Next we need an abstract lemma due to Mitchell. We briefly go over the proof for completeness.

Lemma 3.4. *Suppose that \mathbb{R} is a forcing, such that $\mathbb{R} \times \mathbb{R}$ is ccc. Then \mathbb{R} cannot add a new cofinal branch to a tree of height ω_1 .*

Proof. Suppose otherwise, i.e. t is tree in the ground model, and $1_{\mathbb{R}}$ forces that \dot{b} is a name for a new cofinal branch. Build $\langle (p_\eta^0, p_\eta^1), \gamma_\eta \mid \eta < \omega_1 \rangle$, such that for each η and $i \in \{0, 1\}$, $p_\eta^i \in \mathbb{R}$, $\langle \gamma_\eta \mid \eta < \omega_1 \rangle$ is a strictly increasing sequence of points in ω_1 , and:

- (1) p_η^0, p_η^1 both force that $x_\eta \in \dot{b} \cap t_{\gamma_\eta}$ (here t_{γ_η} is the γ_η -th level of the tree).
- (2) for $i = 0, 1$, $p_\eta^i \Vdash y_\eta^i \in \dot{b} \cap t_{\gamma_\eta+1}$, with $y_\eta^0 \neq y_\eta^1$.

By the ccc of $\mathbb{R} \times \mathbb{R}$, for some $\eta < \delta$, (p_η^0, p_η^1) and (p_δ^0, p_δ^1) are compatible. But then, for $i = 0, 1$, $y_\eta^i <_t x_\delta$. Contradiction with t being a tree. \square

Lemma 3.5. *All ω_1 -branches of T in $V[G]$ are already in the forcing extension $V[\bar{G}]$.*

Proof. Since \mathbb{C}/\bar{G} , i.e. the forcing to get from $V[\bar{G}]$ to $V[G]$ is Knaster, we have that $\mathbb{C}/\bar{G} \times \mathbb{C}/\bar{G}$ is ccc. Then the result follows from the above lemma. \square

For every $\alpha < \aleph_{\omega_1}$, let $\mathbb{P}_\alpha := \{(t, f \upharpoonright \aleph_\alpha) \mid (t, f) \in \mathbb{P}\}$. Then the poset \mathbb{P} is the union of the sequence $\langle \mathbb{P}_\alpha \mid \alpha < \omega_1 \rangle$ and each \mathbb{P}_α is a regular \aleph_2 -cc subordering of \mathbb{P} as witnessed by the projection $\pi : \mathbb{P} \rightarrow \mathbb{P}_\alpha$ given by $\pi(t, f) = (t, f \upharpoonright \aleph_\alpha)$. Clearly, π is order-preserving. Now, if $(u, g) \leq \pi(t, f)$, let $\beta + 1$ be the height of u and for all $\gamma \in \text{dom}(f)$, define $f'(\gamma) = g(\gamma)$ if $\gamma < \aleph_\alpha$ and otherwise set $f'(\gamma)$ to be some node in u_β above $f(\gamma)$ in the u -ordering. Then $(u, f') \leq (t, f)$ and $\pi(u, f') = (u, g)$.

For each α , \mathbb{P}_α adds \aleph_α many branches to the generic Kurepa tree. Let H_α be the generic filter for \mathbb{P}_α obtained from H . Also, for a condition $p \in \mathbb{P}$, we use the notation $p = (t^p, f^p)$.

Claim 3.6. For every ω_1 branch b of T , there is some $\alpha < \omega_1$, such that $b \in V_0[H_\alpha \times \bar{G}]$.

Proof. As before, we index the nodes of T by $\langle \alpha, n \rangle$, for $\alpha < \omega_1$ and $n < \omega$, where the first coordinate denotes the level of the node. Similarly to the arguments in Lemma 3.3, we can find a set $d \subset \aleph_{\omega_1}$ of size ω_1 , such that T is in $V_0[\bar{G} \times \bar{H}]$, where $\bar{H} = \{(t, f \upharpoonright d) \mid (t, f) \in H\}$. Then \bar{H} is actually a generic filter for \mathbb{P}_1 , and we view $V_0[\bar{G} \times H] = V_0[\bar{G} \times \bar{H}][H']$, where H' is $\mathbb{P}/\bar{H} := \{(t, f) \in \mathbb{P} \mid (t, f \upharpoonright \aleph_1) \in \bar{H}\}$ -generic.

Suppose \dot{b} is a \mathbb{P} -name for a cofinal branch through T , which is not in $V_0[\bar{G} \times \bar{H}]$. We say $p \Vdash \dot{b}(\alpha) = n$ to mean that p forces that $\dot{b} \cap \dot{T}_\alpha = \{\langle \alpha, n \rangle\}$. In $V_0[\bar{G} \times \bar{H}]$ let $\mathbb{Q} := \{(t, f \upharpoonright \aleph_1 + 1) \mid (t, f) \in \mathbb{P}/\bar{H}\}$. Define a \mathbb{Q} -name τ , by setting

$$\tau = \{(\langle \alpha, n \rangle, q) \mid \alpha < \omega_1, n < \omega, f^q(\omega_1) = n, \exists p \in \mathbb{P}/\bar{H}, t^p \upharpoonright (\alpha+1) = t^q, p \Vdash \dot{b}(\alpha) = n\}.$$

Also let $K = \{(t, f) \mid \exists \alpha < \omega_1, \exists p \in H, t^p \upharpoonright \alpha + 1 = t, \omega_1 \in \text{dom}(f), p \Vdash \dot{b}(\alpha) = f(\omega_1)\}$. Since b is not in $V_0[\bar{G} \times \bar{H}]$, it is straightforward to check that K is \mathbb{Q} -generic, and also that $\dot{b}_H = \tau_K$.

Finally, since $\bar{H} * K \in V_0[\bar{G} \times H]$ is generic for the suborder $\{(t, f \upharpoonright \aleph_1 + 1) \mid (t, f) \in \mathbb{P}\}$, K must be in $V_0[\bar{G} \times H_\alpha]$, for some $\alpha < \omega_1$. Then $b \in V_0[\bar{G} \times H_\alpha]$. \square

Lemma 3.7. *If $\mathbb{P} \times \mathbb{C}_{\omega_1}$ adds more than \aleph_{ω_1} -many ω_1 -branches to T then there is some $\alpha < \omega_1$, so that $\mathbb{P}_\alpha \times \mathbb{C}_{\omega_1}$ adds more than \aleph_{ω_1} many ω_1 -branches to T .*

Proof. For every $\alpha < \omega_1$, let H_α be \mathbb{P}_α -generic over V_0 , induced by H . Then $V_0[H_\alpha \times \bar{G}] \subset V_0[H \times \bar{G}] = V[\bar{G}]$.

Suppose that for some $\lambda > \aleph_{\omega_1}$, T has λ -many branches, enumerate them by $\langle b_i \mid i < \lambda \rangle$. For every $i < \lambda$, let $\alpha_i < \omega_1$ be such that $b_i \in V_0[H_{\alpha_i} \times \bar{G}]$ given by Claim 3.6. Then for some $\alpha < \omega_1$ there is an unbounded $I \subset \lambda$, such that for all $i \in I$, $\alpha_i = \alpha$.

But that implies that in $V_0[H_\alpha \times \bar{G}]$, $2^{\omega_1} > \aleph_{\omega_1}$, which is a contradiction since we started with $V_0 \models GCH$. So the forcing extension of $\mathbb{P} \times \mathbb{C}$ has at most \aleph_{ω_1} many ω_1 -branches of T , i.e $\mathcal{B} = \aleph_{\omega_1}$ in this forcing extension. \square

$V[G] = V_0[H][G]$ proves part (2) of Theorem 3.1.

Theorem 3.8. *From a Mahlo cardinal, it is consistent with ZFC that $2^{\aleph_0} < \mathcal{B} = 2^{\aleph_1}$, for every $\kappa < 2^{\aleph_1}$ there is a Kurepa tree with (at least) κ -many maximal branches, but no Kurepa tree has 2^{\aleph_1} -many maximal branches.*

Proof. The proof uses the forcing axiom principle, GMA , defined by Shelah and a maximality principle from [14] that generalizes GMA .

We start with some definitions. Let κ be a regular cardinal. A poset \mathbb{P} is *stationary κ^+ -linked*, if for every sequence of conditions $\langle p_\gamma \mid \gamma < \kappa^+ \rangle$, there is a regressive function $f : \kappa^+ \rightarrow \kappa^+$, such that for some club $C \subset \kappa^+$, for all $\gamma, \delta \in C$ with cofinality κ , $f(\gamma) = f(\delta)$ implies p_γ and p_δ are compatible. Let Γ_κ be the class of κ -closed, stationary κ^+ -linked, well met poset \mathbb{P} with greatest lower bounds.

Definition 3.9. For a regular κ , GMA_κ states that for every $\mathbb{P} \in \Gamma_\kappa$, and for every collection of less than 2^κ many dense sets there is a filter for \mathbb{P} meeting them.

Definition 3.10. For a regular κ , $SMP_n(\kappa)$ says the following:

- $\kappa^{<\kappa} = \kappa$;
- for any Σ_n statement ϕ with parameters in $H(2^\kappa)$ and any $\mathbb{P} \in \Gamma_\kappa$, if for all \mathbb{P} -generic G , and all κ closed, κ^+ -c.c. posets $\mathbb{Q} \in V[G]$, we have that $V[G] \models 1_{\mathbb{Q}} \Vdash \phi$, then ϕ is true in V .

$SMP(\kappa)$ is the statement that $SMP_n(\kappa)$ holds for all n .

Fact 3.11. ([14]) If κ satisfies $\kappa = \kappa^{<\kappa}$, then a model of $SMP(\kappa)$ can be forced starting from a Mahlo cardinal $\theta > \kappa$.

The following is also due to [14]. We include the proof for completeness.

Proposition 3.12. *Let κ be a regular cardinal.*

- (1) *If for all $\tau < 2^\kappa$, $\tau^{<\kappa} < 2^\kappa$, then $SMP_1(\kappa)$ implies that GMA_κ .*
- (2) *$SMP_2(\kappa)$ implies that 2^κ is weakly inaccessible and that for all $\tau < 2^\kappa$, $\tau^{<\kappa} < 2^\kappa$.*

(3) $SMP_2(\kappa)$ implies that every Σ_1^1 -subset of κ^κ of cardinality 2^κ contains a perfect set.⁸

Proof. For the first item, suppose that $\tau < 2^\kappa \rightarrow \tau^{<\kappa} < 2^\kappa$ and $SMP_1(\kappa)$ holds. Let $\mathbb{P} \in \Gamma_\kappa$ and let \mathcal{D} be a family of less than 2^κ many dense sets. By our cardinal arithmetic assumptions, we may assume that the underlying set of \mathbb{P} is an ordinal of size less than 2^κ . Let ϕ be the statement that there exists a filter for \mathbb{P} meeting every set in \mathcal{D} . Note that ϕ is Σ_1 and uses the parameters \mathbb{P} and \mathcal{D} , which are in $H(2^\kappa)$. If G is a \mathbb{P} -generic filter, then ϕ will hold in any forcing extension of $V[G]$, so by $SMP_1(\kappa)$, we have that ϕ is true in V .

For the second item, suppose that $SMP_2(\kappa)$ holds, and let $\tau < 2^\kappa$. Let ϕ be the statement that there is no surjection from $\tau^{<\kappa}$ to $\mathcal{P}(\kappa)$. Then ϕ is Σ_2 with parameters in $H(2^\kappa)$. Denote $\lambda := \tau^{<\kappa}$. Let \dot{Q} be an $Add(\kappa, \lambda^+)$ -name for a κ -closed forcing with κ^+ c.c. Note that since this forcing is κ -closed, $\tau^{<\kappa}$ is computed the same as in V . Then ϕ is true after forcing by $Add(\kappa, \lambda^+) * \dot{Q}$. By $SMP_2(\kappa)$, ϕ is true in V . I.e. $\tau^{<\kappa} < 2^\kappa$.

Next, we show that 2^κ is a limit cardinal. Suppose that $\kappa \leq \mu < 2^\kappa$, and let ψ be the statement $\exists x(\mu < |x| < 2^\kappa)$. This is Σ_2 with parameters in $H(2^\kappa)$. Let $\nu := 2^\kappa$. Then if \dot{Q} is an $Add(\kappa, \nu^+)$ -name for a κ -closed forcing with κ^+ c.c., after forcing with $Add(\kappa, \nu^+) * \dot{Q}$, ψ holds as witnessed by taking x to be ν . So by $SMP_2(\kappa)$, ψ is true in V . So, 2^κ is a limit cardinal.

By a similar argument, $SMP_2(\kappa)$ implies that for all $\kappa \leq \mu < 2^\kappa$, $2^\mu = 2^\kappa$. Therefore 2^κ is regular (since by standard cardinal arithmetic if θ is singular and 2^μ is constant for all large $\mu < \theta$, then $2^\theta = 2^\mu$ for all large $\mu < \theta$).

Finally, we show the third item. Again, suppose that $SMP_2(\kappa)$ holds and T is a tree of $\kappa^{<\kappa} \times \kappa^{<\kappa}$. We want to show that $p[T]$ contains a perfect set or has cardinality less than 2^κ .⁹ As above, denote $\nu := 2^\kappa$. Let \dot{Q} be an $Add(\kappa, \nu^+)$ -name for a κ -closed forcing with κ^+ c.c., and let $G * H$ be generic for $Add(\kappa, \nu^+) * \dot{Q}$. Note that V and $V[G][H]$ have the same cardinals. We have two cases:

- (1) $(p[T])^V \subsetneq (p[T])^{V[G][H]}$,
- (2) $V[G][H] \models |p[T]| < 2^\kappa$.

Suppose the first case holds. Then by Lemma 7.5 in [13], one can construct a continuous order-embedding $i : 2^{<\kappa} \rightarrow \kappa^{<\kappa} \times \kappa^{<\kappa}$ with range contained in T , which witnesses that the projection of all cofinal branches, $p[T]$, contains a perfect set. The idea is to take a new branch that is forced not to belong to V and at the inductive step, pick conditions that force splitting along that branch; the witnessing splitting nodes are used to define i , so that if $s, s' \in 2^{<\kappa}$ are incompatible, then so are $i(s)$ and $i(s')$. For the full details, see Lemma 7.5 in [13].

It follows that the statement that there is such an embedding or $|p[T]| < 2^\kappa$ is true in $V[G][H]$. This is Σ_2 with parameters in $H(2^\kappa)$ (since $\kappa^{<\kappa} = \kappa$), so by $SMP_2(\kappa)$, it is true in V . This concludes the proof of the last item. \square

Remark 3.13. Regarding the first item of the above proposition, actually the converse also holds. I.e. assuming for all $\tau < 2^\kappa$, $\tau^{<\kappa} < 2^\kappa$, we have $SMP_1(\kappa)$ if and only if GMA_κ and $\kappa^{<\kappa} = \kappa$.

⁸Recall that $A \subset \kappa^\kappa$ contains a perfect set if there is a continuous injection $2^\kappa \rightarrow \kappa^\kappa$ with range contained in A . We assume the discrete topology on 2 and κ respectively.

⁹Recall that $A \subset \kappa^\kappa$ is Σ_1^1 iff $A = p[T]$ for some tree $T \subset \kappa^{<\kappa} \times \kappa^{<\kappa}$.

Now, towards the proof of Theorem 3.8, let V be a model of $SMP_2(\omega_1)$. By the above proposition, in V we have:

- (1) GMA_{ω_1} ,
- (2) CH ,
- (3) 2^{ω_1} is weakly inaccessible,
- (4) every Σ_1^1 -subset of $\omega_1^{\omega_1}$ of cardinality 2^{ω_1} contains a perfect set.

We claim that this is the desired model. Let $\omega_1 < \lambda < 2^{\omega_1}$. To see that there is a Kurepa tree with (at least) λ -many maximal branches, let \mathbb{P} be the standard poset to add such a tree (i.e. we take the poset from earlier but with λ in place of \aleph_{ω_1}). Namely, conditions are pairs (t, f) , where:

- t is a tree of height $\beta + 1$ for some $\beta < \omega_1$ and countable levels;
- f is a function with $\text{dom}(f) \subset \lambda$, $|\text{dom}(f)| = \omega$, and $\text{ran}(f) = t_\beta$, where t_β is the β -th level of t .

The order is the same as in the poset from Theorem 3.1.

Then \mathbb{P} satisfies the hypothesis of GMA_{ω_1} ¹⁰, and there are only λ -many dense sets to meet in order to get a Kurepa tree with (at least) λ -many branches. So, by GMA_{ω_1} , there is a Kurepa tree with (at least) λ many branches.

Finally, we use the last item of the properties listed above to argue that there are no Kurepa trees with 2^{ω_1} -many branches. Suppose that T is a Kurepa tree. We claim that the set of branches $[T]$ is a closed set that does not contain a perfect set. For suppose that $g : 2^{\omega_1} \rightarrow \omega_1^{\omega_1}$ is a continuous injection with range contained in $[T]$. Then, build sequences $\langle p_s \mid s \in 2^{<\omega} \rangle$ and $\langle \alpha_n \mid n < \omega \rangle$, such that:

- if $s' \supset s$, then $p_{s'} \prec_T p_s$,
- if $|s| = n$, then $\alpha_n = \text{dom } p_s$,
- for each s , $p_{s \smallfrown 0} \neq p_{s \smallfrown 1}$.

We do this by induction on $|s|$, using continuity. Then let $\alpha = \sup_n \alpha_n$ and for all $\eta \in 2^\omega$, let $p_\eta := \bigcup_n p_{\eta \upharpoonright n}$. But then if $\eta \neq \delta$, $p_\eta \neq p_\delta$. So, the α -th level of the tree has 2^ω many elements. Contradiction with T being a Kurepa tree. So $[T]$ does not contain a perfect set. By our assumption in item (4), it follows that $|[T]| < 2^{\omega_1}$. \square

Remarks 3.14. (i) There is a similar argument to the above proof in Section IV.2 in [20]. The idea of using Kurepa tree to get counterexamples of the perfect set property goes back to [16].

- (ii) For simplicity we used a Mahlo cardinal, which suffices to provide us a model of $SMP_2(\omega_1)$. A slightly weaker hypothesis would be enough. For precise upper and lower consistency strength bounds on $SMP(\omega_1)$, see Theorem 4.5 in [14].

Finally we want to note that the results presented here can be extended to κ -Kurepa trees with $\kappa \geq \aleph_2$.

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¹⁰Note that by CH there are only \aleph_1 -many possibilities for the first coordinate and we can apply a Δ -system lemma for the second coordinate to prove that \mathbb{P} is stationary κ^+ -linked.

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