# PRIKRY SEQUENCES AT $\aleph_\omega$

DIMA SINAPOVA

Abstract. We analyze the effect of singularizing cardinals on square properties in the case when the singularized cardinal becomes $\aleph_\omega$ in the outer model. The main motivation is an old question of Woodin whether one can get failure of SCH at $\aleph_\omega$ together with failure of weak square at $\aleph_\omega$. We show a negative theorem: Suppose $V \subset W$ are models of set theory, $\kappa := (\aleph_\omega)^W$ is $V$-regular, $\mu = (\aleph_{\omega+1})^W$ was a successor of a singular in $V$, and $W$ is a Prikry type extension of $V$. Then weak square at $\aleph_\omega$ holds in $W$.

## 1. Introduction

Ever since the invention of forcing, a major trend in set theory is to understand ZFC-constraints on infinitary combinatorics versus what can be obtained by forcing. There are two types of infinite cardinals - regular and singular. Informally, just like finite union of finite sets is finite; countable union of countable sets is countable, regular cardinals is where the higher analogue of this property holds, and singular cardinals are where it breaks down. More precisely, $\kappa$ is *singular* if it is the union of less than $\kappa$ many sets of size less than $\kappa$. For example $\omega, \aleph_1, \aleph_2$ are regular, while $\aleph_\omega$ is singular, as it is the countable union of smaller sets: $\aleph_\omega = \bigcup_n \aleph_n$.

Combinatorial properties at singular cardinals often have more intricate constraints than in the context of regular cardinals. And forcing construction at singulars use large cardinal axioms. In this paper we focus on *singularizing cardinals* i.e. the situation where $V \subset W$ are two models of set theory, $\kappa$ is a singular cardinal in $W$ and regular in $V$. We are interested when $\kappa$ becomes $\aleph_\omega$ in $W$.

The standard way to singularize a regular cardinal is by *Prikry type forcing*. For example, one way to obtain the failure of the singular cardinal hypothesis (SCH) is by taking a large cardinal $\kappa$, using Cohen forcing to make its powerset have cardinality $\kappa^{++}$, and then use Prikry forcing to turn $\kappa$ into a singular cardinal. In the final model, $\kappa$ is a singular strong limit with $2^\kappa > \kappa^+$, i.e. SCH fails at $\kappa$.

Another key property at singular cardinals is *square*. Square and its weakening, weak square, were defined by Jensen in his fine structure analysis of $L$. A square sequence at $\kappa$ is a coherent sequence of clubs in $\alpha$ for $\alpha < \kappa^+$ of small order type. A weak square sequence is where one allows for up

The author is partially supported by the National Science Foundation under Grant No. DMS - 1954117.

to $\kappa$ many guesses for each of these clubs. Later Schimmerling introduced an intermediate hierarchy of square principles between these two. Square properties are a yardstick as to how $L$-like the universe is. The more square properties hold in some model, the closer that model is to a canonical inner model. Squares tend to fail above large cardinals and are at odds with reflection properties, like stationary reflection and the tree property.

There is a particular tension between the failure of weaker square properties and failure of SCH. Failures of squares are compactness type properties, that follow from large cardinals. On the other hand, the failure of SCH is an instance of incompacness, since it means that the powerset function below $\kappa$ is small, but it blows up at $\kappa$. Since not SCH is obtained via Prikry type forcings, often in the context of singularizing cardinals, we investigate this tension by analyzing the effect that singularizing cardinals has on square properties.

Let $V \subset W$, $\kappa$ be a singular cardinal in $W$ and regular in $V$, and $W \models \kappa^+ = \mu$. Suppose further that every $V$-regular cardinal in the interval $[\kappa, \mu)$ has countable cofinality in $W$. What can we say about square properties at $\kappa$ in $W$?

When $\mu$ is a successor of a regular cardinal in $V$, then by joint results with Magidor [9], we know that $\square_{\kappa,\omega}$ holds in $W$. When $V$ and $W$ have the same cardinals, this result was already implicit by works of Dzamonja-Shelah [5] and Gitik [6]. In the case when $\mu$ is inaccessible in $V$, one can construct a weak square sequence at $\kappa$ in $W$ by using clubs from $V$. The remaining case is when $\mu$ is a successor of a singular in $V$. By Gitik-Sharon [7], in that case it is possible to obtain failure of weak square of $\kappa$ in the outer model. They start with $\mu = \kappa^{+\omega+1}$ in $V$, and get failure of SCH at $\kappa$ together with failure of weak square at $\kappa$ in the generic extension. Then they push down their result to $\kappa = \aleph_{\omega^2}$ in the final model.

The natural question is if we can do this when $\kappa$ becomes $\aleph_\omega$; summed up in the following question of Woodin from the 80s:

**Question 1.** *Is it possible to have a model where SCH fails at $\aleph_\omega$ together with $\neg\square^*_{\aleph_\omega}$?*

The situation at $\aleph_\omega$ is more complicated than when the singular caridinal is high up, or even just $\aleph_{\omega^2}$. On one hand, it is possible to modify the Gitik Sharon forcing to have interleaved collapses, so that $\kappa$ becomes $\aleph_\omega$ and SCH fails at $\kappa$. Moreover, we can show that all the intermediate squares $\square_{\aleph_\omega,\aleph_n}$ fail in this generic extension. This was done by Unger and the author in [13] and [14]. In particular, we obtained the consistency of the failure of SCH at $\aleph_\omega$ together with failure of $\square_{\aleph_\omega,\aleph_n}$ for all $n$. So far this remains the closest positive partial answer to Woodin's question. However, the forcing in [13] adds a weak square at $\kappa$. Here, we prove a negative theorem, in the context of a standard abstract Prikry construction, when $\mu = \kappa^{+\omega+1}$ in $V$:

**Theorem 1.1.** *Suppose that $V \subset W$ are models of set theory, such that*

*(1) $V \models \kappa$ is a regular cardinal, $\nu$ is a singular cardinal with countable cofinality, $\mu = \nu^+$, $2^\kappa > \mu$,and $U$ is a normal measure on $\kappa$,;*
*(2) $W \models \kappa = \aleph_\omega, \mu = \aleph_{\omega+1}$;*
*(3) $W$ is a forcing extension of $V$ by a strong Prikry poset $\mathbb{P}$, such that $\mathbb{P}$ projects to $\mathbb{P}_U$, and $Ult(U, V) \models \mu = \nu^+$.*

*Suppose further that any collapses below $\kappa$ are taken from an inner model $\bar{V} \subset V$, such that in $\bar{V}$, $\nu$ is strong limit. Then $W \models \square^*_{\aleph_\omega}$.*

We make some remarks on why the assumptions are the natural ones:

**Remark 1.2.** *We assume that $2^\kappa > \mu$, since we are interested in cases where SCH fails in the outer model. We note that, assuming GCH, one can combine the Gitik-Sharon forcing with collapses, so that in the generic extension $\kappa = \aleph_\omega$ and the tree property holds at $\kappa^+$, and so weak square fails at $\kappa$* [12].

**Remark 1.3.** *We require that collapses are taken from an inner model of $V$ where $\nu$ is strong limit, for otherwise $\kappa$ will be collapsed.*

Before we prove the main theorem, we will show the representative case when $V \models \kappa^{+\omega+1} = \mu$.

The paper is organized as follows. In the next section we go over some basic definitions, abstract properties of Prikry forcings, and pcf facts that will be used in the proof of the main theorem. In section 3, we prove the main theorem. We conclude with some open problems.

Notation: Given a set $X$, $Lim(X)$ denotes the limit points of $X$; given a cardinal $\kappa$, $\mathrm{cf}(\kappa)$ denotes the cofinality of $\kappa$, and $\mathrm{cof}(\kappa)$ denotes points with cofinality $\kappa$.

## 2. Preliminaries

2.1. **Square properties.** We start by defining the square properties.

**Definition 2.1.** *Square at $\kappa$, $\square_\kappa$, holds if there is a sequence $\langle C_\alpha \mid \alpha \in Lim(\kappa^+)\rangle$, such that, for each $\alpha$, $C_\alpha$ is a club in $\alpha$ with order type at most $\kappa$, and whenever $\beta \in \lim(C_\alpha)$, then $C_\alpha \cap \beta = C_\beta$.*

This combinatorial property can be weakened by allowing multiple guesses for the clubs at each $\alpha$:

**Definition 2.2.** *Let $\tau \leq \kappa$; $\square_{\kappa,\tau}$ holds if there is a sequence $\langle \mathcal{C}_\alpha \mid \alpha \in Lim(\kappa^+)\rangle$, such that, for each $\alpha$, $1 \leq |\mathcal{C}_\alpha| \leq \tau$, each $C \in \mathcal{C}_\alpha$ is a club in $\alpha$ with order type at most $\kappa$, and whenever $\beta \in \lim(C)$, then $C \cap \beta \in \mathcal{C}_\beta$.*

Weak square at $\kappa$ is $\square_{\kappa,\kappa}$, commonly denoted by $\square^*_\kappa$.

Square sequences are canonical instances of incompactness. For example, note that a square sequence cannot be threaded, meaning that there is no club $C \subset \kappa$, such that for all $\alpha \in Lim(C)$, $C \cap \alpha \in \mathcal{C}_\alpha$. Squares imply various incompactness principles, such as approachability, the existence of a special Aronszajn tree, and failure of stationary reflection.

2.2. **Prikry forcing.** Next we turn our attention to Prikry type forcing, the main forcing for dealing with singular cardinals. Suppose that $U$ is a normal measure on $\kappa$.

**Definition 2.3.** *$\mathbb{P}_U$ denotes the standard Prikry forcing at $\kappa$ with respect to $U$, namely the forcing that adds a sequence $\langle \kappa_n \mid n < \omega \rangle$ through $\kappa$, such that for every $A \in U$, for all large $n$, $\kappa_n \in A$.*

Next we make a general definition of Prikry type forcings.

**Definition 2.4.** *A poset $\mathbb{P}$ is a* **strong Prikry forcing**, *if every condition $p$ has an associated length, $\mathrm{lh}(p) < \omega$, such that*

*(1) $p \leq q$ implies $\mathrm{lh}(p) \geq \mathrm{lh}(q)$;*
*(2) for all $p$, for all $n > \mathrm{lh}(p)$, there is nonempty a maximal antichain $A$ below $p$, such that*
  *(a) for all $q \in A$, $\mathrm{lh}(q) = n$,*
  *(b) if $r \leq p$ with $\mathrm{lh}(r) \geq n$, then there is (a necessarily unique) $q \in A$, with $r \leq q$*

  *We use the standard notation $q \leq^* p$ if $q \leq p$ and $\mathrm{lh}(q) = \mathrm{lh}(p)$. Also, $q$ is an $n$-step extension of $p$, if $q \leq p$ and $\mathrm{lh}(q) = \mathrm{lh}(p) + n$.*
*(3) (the strong Prikry property) for every dense open set $D$ and a condition $p$, there is $n < \omega$ and $q \leq^* p$, such that each $n$-step extension of $q$ is in $D$. In the special case when $D$ is the set of all conditions deciding a formula, $n = 0$*
*(4) (pseudo countable closure) there is $\bar{n} < \omega$, such that for all $p$ with length at least $\bar{n}$, for all sequences $\langle D_n \mid n < \omega \rangle$ of dense open sets, there is $q \leq^* p$, witnessing the Strong Prikry property simultaneously for each $D_n$. I.e. for each $n$, there is $k_n < \omega$, such that that each $k_n$-step extension of $q$ is in $D_n$.*

Suppose $\mathbb{P}$ is a Prikry poset projecting to $\mathbb{P}_U$ for some normal measure $U$ on $\kappa$, and let $\langle \dot{\kappa}_n \mid n < \omega \rangle$ be a name for the $\mathbb{P}_U$-generic sequence. We make the following remarks:

**Remark 2.5.** *By passing to a dense subset we may assume that each $p \in \mathbb{P}$ decides an initial segment of the Prikry sequence, and then the length function is defined so that $n < \mathrm{lh}(p)$ iff $p \parallel \dot{\kappa}_n$,*

**Remark 2.6.** *By definition of genericity, for any $p \in \mathbb{P}$, for all large $n$, $\{\delta < \kappa \mid \exists q \leq p (q \Vdash \delta = \dot{\kappa}_n)\} \in U$. Again, by passing to a dense subset of $\mathbb{P}$, we assume that this is true for all $n \geq \mathrm{lh}(p)$.*

We note that all of the usual Prikry type forcings satisfy the above definition and remarks. In the next sections we assume that all strong Prikry forcings satisfy the properties in the above two remarks.

2.3. **An interlude on pcf.** The results of this subsection were first triggered by in a footnote by Sharon-Viale [10] and are implicit in the Cardinal Arithmetic chapter in the Handbook of Set Theory [1]. For completeness, we will outline some of the arguments, following unpublished notes of James Cummings [4]. We will then combine them with a key result from Cummings [2].

For functions $f, g : \omega \to ON$, let $f <^* g$ denote that for all large $n$, $f(n) < g(n)$. The following is a pcf fact:

**Fact 2.7.** *Suppose $\langle f_i \mid i < \mu \rangle$ is a $<^*$-increasing sequence of functions from $\omega \to ON$, for some regular $\mu$, and let $\lambda$ be uncountable, regular with $\lambda^{+3} < \mu$. Assume that for every $\delta \in \mu \cap \mathrm{cof}(\lambda^{++})$, there is a club $E \subset \delta$, such that for all large $n$, $\sup_{i \in E} f_i(n) <^* f_\delta(n)$. Then for every $\alpha < \mu$ with $\mathrm{cf}(\alpha) > \lambda^{++}$, there is an eub $g$ for $\langle f_i \mid i < \alpha \rangle$, such that for all $n$, $\mathrm{cf}(g(n)) \geq \lambda$.*

As a corollary, we have:

**Theorem 2.8.** *Suppose that $\langle f_\alpha \mid \alpha < \mu \rangle$ is a scale at $\kappa^{+\omega}$, where $\mu = \kappa^{+\omega+1}$. Then almost all points of cofinality at least $\kappa^{+4}$ are good.*

*Proof.* Build a scale of length $\mu$, such that at every point $\alpha$ of cofinality $\kappa^{+3}$, there is a club $E \subset \alpha$ with $\sup_{\delta \in E} f_\delta <^* f_\alpha$. Then by the previous theorem applied to $\lambda = \kappa^+$, at each point of cofinality at least $\kappa^{+4}$, there is an eub $g$ such that for all $n$, $\mathrm{cf}(g(n)) > \kappa$. Note that in particular, this cofinality is uncountable.

Fix $\alpha$ and $g$ as above. Next we show that for all large $n$, $\mathrm{cf}(g(n)) = \mathrm{cf}(\alpha)$, and so $\alpha$ is good. This argument is originally due to Shelah, and can be found in Cummings, Notes on Singular Cardinal Combinatorics [3]. We repeat it for completeness.

Suppose for contradiction, that for unboundedly many $n$, $\mathrm{cf}(g(n)) \neq \mathrm{cf}(\alpha)$.

Case 1. For unboundedly many $n$, $\mathrm{cf}(g(n)) > \mathrm{cf}(\alpha)$. Let $I \subset \omega$ be a witness. Also, let $A \subset \alpha$ be unbounded with $o.t.(A) = \mathrm{cf}(\alpha)$. By thinning out $A$ if necessary, we can find $\bar{n}$, such that for all $\delta \in A$, for all $n > \bar{n}$, $f_\delta(n) < g(n)$. Define $h(n) = \sup_{\delta \in A} f_\delta(n)$ for $n \in I$ with $n > \bar{n}$, and $h(n) = 0$ otherwise. But then, by assumption, $h < g$, and so since $g$ is an eub, for some $\delta < \alpha$, $h <^* f_\delta$. Contradiction with the definition of $h$.

Case 2. For unboundedly many $n$, $\mathrm{cf}(g(n)) < \mathrm{cf}(\alpha)$. Since the gap between $\kappa$ and $\mathrm{cf}(\alpha)$ is finite, we can find $\kappa < \tau < \mathrm{cf}(\alpha)$ and an unbounded $I \subset \omega$, such that for all $n \in I$, $\mathrm{cf}(g(n)) = \tau$. For each $n \in I$, let $\langle \alpha_i^n \mid i < \tau \rangle$ be increasing and cofinal in $g(n)$ and define $g_i(n) = \alpha_i^n$. Then $\langle g_i \mid i < \tau \rangle$ is cofinally interleaved with $\langle f_\delta \restriction I \mid \delta < \alpha \rangle$ (here we use that $\tau$ is uncountable). Contradiction with $\tau \neq \mathrm{cf}(\alpha)$.

□

In particular, if $\langle f_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ is a scale at $\aleph_\omega$, almost all points $\alpha$ with $\mathrm{cf}(\alpha) \geq \omega_4$ are good. Combining the above theorem with a result from Cummings [2], we get the following

**Corollary 2.9.** *Suppose $V \subset W$, and $\kappa < \mu$ are such that:*

- $V \models$ "$\mu = \kappa^{+\omega+1}$",
- $W \models \kappa$ *is a regular cardinal and* $\mu = \kappa^{+n}$.

*Then* $n \leq 4$.

*Proof.* We may assume $n > 1$, otherwise we are done. Let $\tau = (\kappa^{+n-1})^V$. By [2], we have that there are stationary many bad points $\alpha < \mu$ of cofinality $\tau$ in $V$. So by the above theorem $\tau < \kappa^{+4}$, i.e. $n \leq 4$. $\square$

In particular, if $V \subset W$, and $\aleph^V_{\omega+1} = \aleph^W_n$, then $n \leq 4$.

Next, using Fact 2.7, we can get more abstract versions of Theorem 2.8 and Corollary 2.9.

**Theorem 2.10.** *Suppose* $V \subset W$, *and* $\kappa < \nu < \mu$ *are such that:*

- $V \models$ "$\nu$ *is a singular cardinal,* $\mathrm{cf}(\nu) = \omega, \mu = \nu^+$",
- $W \models \kappa$ *is a regular cardinal and* $\mu = \kappa^{+n}$.

*Then* $n \leq 4$.

*Proof.* Suppose for contradiction, that $W \models \mu > \kappa^{+4}$.

In $V$, let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals with limit $\nu$, and $\kappa = \kappa_0$. Continuing to work in $V$, let $\langle f_\alpha \mid \alpha < \mu \rangle \in V$ be a scale in $\prod \kappa_n$, such that for every $\alpha \in \mu \cap \mathrm{cof}^V(\geq \kappa^{+3})$, there is a club $E$ of $\alpha$, such that $\sup_{\beta \in E} f_\beta <^* f_\alpha$.

We will argue that in $W$, almost all points of cofinality at least $(\kappa^{+4})^W$ are good. To that end, in $W$, apply Fact 2.7, to $\lambda := (\kappa^+)^W$, to get that for each point $\alpha$ of cofinality at least $(\kappa^{+4})^W$, there is an eub $g$ such that for all $n$, $\mathrm{cf}(g(n)) > \kappa$. Note that in particular, this cofinality is uncountable. Since the gap between $\kappa$ and $\mathrm{cf}^W(\alpha)$ is finite, by the argument in Theorem 2.8, we have that for all large $n$, $\mathrm{cf}(g(n)) = \mathrm{cf}(\alpha)$, and so $W \models \alpha$ is good. I.e. there is an unbounded $A \subset \alpha$, $A \in W$, such that for all large $n$, $\langle f_\beta(n) \mid \beta \in A \rangle$ is strictly increasing. But that is a contradiction with Cummings [2].

$\square$

## 3. The main theorem

We will first prove a warm up theorem, where we start with $\mu = \kappa^{+\omega+1}$ in $V$. Then we prove the general case when $\mu$ is any successor of a singular cardinal in $V$.

**Theorem 3.1.** *Suppose that* $V \subset W$ *are models of set theory, such that*

1. $V \models \kappa$ *is a regular cardinal,* $U$ *is a normal measure on* $\kappa$, $\mu = \kappa^{+\omega+1}$, $2^\kappa > \mu$;
2. $W \models \kappa = \aleph_\omega, \mu = \aleph_{\omega+1}$;
3. $W$ *is a forcing extension of* $V$ *by a strong Prikry poset* $\mathbb{P}$, *such that* $\mathbb{P}$ *projects to* $\mathbb{P}_U$, *and* $Ult(U, V) \models \mu = \kappa^{+\omega+1}$.

*Suppose further that any collapses below* $\kappa$ *are taken from an inner model* $\bar{V} \subset V$, *such that in* $\bar{V}$, $\nu$ *is strong limit. Then* $W \models \square^*_{\aleph_\omega}$.

*Proof.* Let $j : V \to M$ be the elementary embedding given by $U$. Since $|j(\kappa)|^V = 2^\kappa > \mu$, there are functions $\langle f_\alpha \mid \alpha \leq \mu \rangle$ from $\kappa$ to $\kappa$, such that for each $\alpha \leq \mu$, $[f_\alpha]_U = \alpha$. Also let $\nu = (\kappa^{+\omega})^V$.

In $W$ let $\langle \kappa_n \mid n < \omega \rangle$ be the $\mathbb{P}_U$ generic Prikry sequence. For each $n$, define $\mu_n = f_\mu(\kappa_n)$, $\nu_n = f_\nu(\kappa_n)$. Note that for all large $n$, $\nu_n$ is a singular $V$-cardinal, $\nu_n = (\kappa_n^{+\omega})^V$ and $\mu_n = (\nu_n^+)^V$. By moving to a tail end of the Prikry sequence, we may assume that this is true for all $n$. Similarly we may assume that each $\kappa_n$ is an inaccessible cardinal in $V$.

There are two cases: when the $\mu_n$'s are collapsed in $W$ for all large $n$, and when unboundedly many of them are preserved. We will show that in the first case, a weak square sequence is added, and that the second case leads to a contradiction. So we must be in the first case.

Note that in either case for all $n$, $\nu_n$ must be collapsed, since in $W$, $\kappa$ is $\aleph_\omega$, so there can be no singular cardinals below it. We begin with a lemma about constraints on the values of $|\nu_n|^W$.

**Lemma 3.2.** *Suppose that $\tau$ is a $V$-regular cardinal with $\kappa \leq \tau < \nu$, such that for unboundedly many $n$, $W \models |\nu_n| = |\tau_n|$, where $\tau_n = f_\tau(\kappa_n)$. Then $\mu$ is collapsed, and so we have a contradiction.*

*Proof.* Let $I \subset \omega$ be the unbounded set witnessing the assumptions of the lemma. Note that by the Prikry property and the pseudo countable closure clause of Definition 2.4, $I \in V$. We claim that this adds a bijection between $(\prod_{n\in I} \nu/fin)^V$ and $(\prod_{n\in I} \tau/fin)^V$. This will use the fact that $\mathbb{P}$ has the strong Prikry property.

More precisely, for $n \in I$, let $c_n : \nu_n \to \tau_n$ be a bijection in $W$. Let $\vec{\gamma} = \langle \gamma_n \mid n \in I \rangle/fin \in V$ be a sequence of points in $\nu$. Then for each $n \in I$, $A_n = \{\alpha \mid f_{\gamma_n}(\alpha) \in f_\nu(\alpha)\} \in U$, and $A = \bigcap_{n\in I} A_n \in U$. Then for all large $n$, $\kappa_n \in A$, and so $f_{\gamma_n}(\kappa_n) \in \nu_n$ for all large $n \in I$.

For the rest of the proof of the lemma, we restrict ourselves to $n \in I$. (We will omit mentioning $I$ for simplicity.)

**Claim 3.3.** *There is a sequence $\langle \delta_n \mid n < \omega \rangle \in V$ of points in $\tau$, such that for all large $n$, $f_{\delta_n}(\kappa_n) = c_n(f_{\gamma_n}(\kappa_n))$.*

*Proof.* For each $n$, the set $D_n := \{p \mid p \parallel \dot{\kappa}_n \text{ and } p \parallel \dot{c}_n(f_{\gamma_n}(\dot{\kappa}_n))\}$ is dense open. So for any $p$, we can find a direct extension $p'$, such that for all $n$, there is $k_n$, such that each $k_n$-step extension of $p'$ is in $D_n$. By density we can find such a $p' \in G$.

Now, working in $V$, for each $n$, let $h_n$ be a function with domain $A_n^{p'} := \{\delta \mid \exists q \leq p', q \Vdash \delta = \dot{\kappa}_n\}$, defined as follows. For each $\delta \in A_n^{p'}$, let $q$ be the weakest extension of $p'$ deciding $\dot{\kappa}_n = \delta$ of length $\max(n, k_n)$. Then, since $q \in D_n$, for some $\alpha$, $q \Vdash \alpha = \dot{c}_n(f_{\gamma_n}(\dot{\kappa}_n))$. Set $h_n(\delta) = \alpha$. Then let $\delta_n < \nu$, be such that $[h_n]_U = \delta_n$. Then $\langle \delta_n \mid n < \omega \rangle$ is as desired.

$\square$

Let $\langle \delta_n \mid n < \omega \rangle$ be as in the above claim. Set $\phi(\vec{\gamma}) = \langle \delta_n \mid n < \omega \rangle/fin$. We claim that $\phi : (\prod_n \nu/fin)^V \to (\prod_n \tau/fin)^V$ is one-to-one: suppose that

$\phi(\langle\gamma_n \mid n<\omega\rangle/fin) = \phi(\langle\beta_n \mid n<\omega\rangle/fin) = \langle\delta_n \mid n<\omega\rangle/fin$. Then for all large $n$, $c_n(f_{\gamma_n}(\kappa_n)) = c_n(f_{\beta_n}(\kappa_n)) = \delta_n$, and since each $c_n$ is one-to-one, we have that for all large $n$, $f_{\gamma_n}(\kappa_n) = f_{\beta_n}(\kappa_n)$. So, for all large $n$, $\{\delta \mid f_{\gamma_n}(\delta) = f_{\beta_n}(\delta)\} \in U$, and so for all large $n$, $\gamma_n = [f_{\gamma_n}] = [f_{\beta_n}] = \beta_n$.

Then since $V \models |\prod_n \nu/fin| = \mu, |\prod_n \tau/fin| = \tau$, we have that in $W$, $\mu$ is collapsed. $\square$

**Lemma 3.4.** *For all large $n$, $\mu_n$ is collapsed.*

*Proof.* Suppose for contradiction, that for unboundedly many $n$, $\mu_n$ is preserved. If for unboundedly many $n$, $W \models |\nu_n| = |\kappa_n|$, then by Lemma 3.2, applied to $\tau = \kappa$, we have that $\mu$ is collapsed; contradiction.

So assume for all large $n$, $W \models |\nu_n| > |\kappa_n|$. Let $\tau_n = |\nu_n|^W$, and so for unboundedly many $n$, $W \models \mu_n = \tau_n^+$. By Theorem 2.9, if $\mu_n$ is preserved, the $V$-gap between $\mu_n$ and $|\kappa_n|^W$ cannot be more than 4 i.e. for some $k \leq 4$ $|\kappa_n|^W < \tau_n < (|\kappa_n|^{+k})^V$. By the pigeon hole principle, there is some $k \leq 4$, such that for unboundedly many $n$, $W \models \tau_n = (|\kappa_n|^{+k-1})^V$. But then by Lemma 3.2, applied to $\tau := \kappa^{+k-1}$, we have that $\mu$ is collapsed in $W$. Contradiction.

$\square$

**Lemma 3.5.** $W \models \square^*_\kappa$.

*Proof.* By the above lemma, for all large $n$, $\mu_n$ is collapsed. Set $\tau_n = |\mu_n|^W$. By Lemma 3.2, $W \models |\kappa_n| < \tau_n$.

For each $n$, let

$$1 \Vdash \dot{c}^*_n : \dot{\tau}_n \to \dot{\mu}_n \text{ is a one-to-one function whose range, } \dot{c}_n \text{ is a club in } \dot{\mu}_n.$$

Let $X = \{\alpha \in \mu \cap \operatorname{cof}(<\kappa) \mid \text{ for all large } n, f_\alpha(\kappa_n) \in \lim(c_n)\}$.

**Claim 3.6.** *$X$ is $>\omega$-club.*

*Proof.* Closed: Suppose that $\alpha \in \lim(X)$ of uncountable cofinality in $W$, and so in $V$, and let $\tau = \operatorname{cf}^V(\alpha)$. Note that $\omega < \tau < \kappa$ and $\tau$ is $V$-regular.

Let $\langle\alpha_i \mid i<\tau\rangle \in V$ be a continuous increasing sequence with limit $\alpha$. Since $\tau < \kappa$, by genericity of the $\kappa_n$'s, for some $\bar{n}$, for all $n > \bar{n}$, $\langle f_{\alpha_i}(\kappa_n) \mid i<\tau\rangle$ is strictly increasing with limit $f_\alpha(\kappa_n)$.

By assumption $X \cap \alpha$ is unbounded in $\alpha$, and since $\operatorname{cf}^W(\alpha) > \omega$, we also have that $\lim(X \cap \alpha)$ is unbounded in $\alpha$, and actually a club in $\alpha$. Then $C := \lim(X) \cap \{\alpha_i \mid i<\tau\}$ is also a club in $X \cap \alpha$. Enumerate it by $\langle\beta_i \mid i<\tau\rangle$, and note that since this is a subsequence of $\langle\alpha_i \mid i<\tau\rangle$, we still have that for all $n > \bar{n}$, $\langle f_{\beta_i}(\kappa_n) \mid i<\tau\rangle$ is strictly increasing with limit $f_\alpha(\kappa_n)$.

For each $i<\tau$, let $\bar{n} < n_i < \omega$ be such that for all $n > n_i$, $f_{\beta_i}(\kappa_n) \in \lim(c_n)$. Let $k$ and $I \subset \tau$ be unbounded such that for all $i \in I$, $k = n_i$. Then for all $n > k$, $f_\alpha(\kappa_n) \in \lim(c_n)$, and so $\alpha \in X$.

Unboundedness: Let $\alpha < \mu$. We have to find $\beta$ with $\alpha < \beta < \mu$, such that for all large $n$, $f_\beta(\kappa_n) \in \lim(c_n)$.

By the Prikry lemma, for any condition, we can find a direct extension such that for all large $n$, for some $k$, all of its $k$-step extensions are in the dense set $D_n = \{p \mid (\exists\delta)p \Vdash \delta = \min(\lim(\dot{c_n}) \setminus f_\alpha(\dot{\kappa_n}) + 1)\}$.

By density, we can find such a condition $p$ in $G$. Let $n_0$ and $\langle k_n \mid n > n_0\rangle$, be such that for all $n > n_0$, each $k_n$-step extension of $p$ is in $D_n$. By increasing if necessary, we may assume that $n_0 \geq \mathrm{lh}(p)$, and that $k_n + \mathrm{lh}(p) \geq n$ .

Now, for each $n > n_0$, construct a function $f : \kappa \to \kappa$, in $V$, as follows: let $\delta < \kappa$. If there is a $k_n$-step extension of $p$, forcing that $\delta = \dot{\kappa}_n$, let $q_\delta$ be the weakest such. Let $\delta'$ be the point such that $q_\delta \Vdash \delta' = \min(\lim(\dot{c_n}) \setminus f_\alpha(\dot{\kappa_n}) + 1)$ and set $f(\delta) = \delta'$. Note that then $\delta' > f_\alpha(\delta)$ because $q_\delta$ forces that $f_\alpha(\dot{\kappa}_n) = f_\alpha(\delta)$. Otherwise set $f(\delta) = 0$.

Since, by Remark 2.6, there are measure one many $\delta$'s that are possible values of $\dot{\kappa}_n$ below $p$, we have that $[f_\alpha] < [f]_U < \mu$. Let $\beta < \mu$ be such that $\beta = [f_\beta]_U = [f]_U > \alpha$. Let $n_1$ be such that for all $n > n_1$, $f_\beta(\kappa_n) = f(\kappa_n)$.

Then since $p \in G$, we have that for all $n > \max(n_0, n_1)$, $f_\beta(\kappa_n) \in \lim(c_n)$ and so $\beta \in X$.

□

Fix $\alpha < \mu$, and for each $n$ consider the dense open set $D_n := \{q \mid (\exists z \in \bar{V})q \Vdash \dot{c}_n \cap f_\alpha(\dot{\kappa}_n) = z\}$. By our assumptions, $D_n$ is open dense. In other words the initial segments of $\dot{c}_n$ are in $\bar{V}$.

By the Prikry lemma, for all $p$, there is $p' \leq^* p$, such that for all $n$, there is $k_n > n$, such that each $k_n$-step extension of $p'$ is in $D_n$. By density we can find such a condition in $G$.

For each $\alpha$, let $p_\alpha \in G$ be as above. For each $\delta < \kappa$, if possible, let $q_\delta$ be the weakest $k_n$ step extension of $p_\alpha$ forcing that $\delta = \dot{\kappa}_n$, if such exists. And then let $z_n^{\alpha,\delta}$ to be such that $q_\delta \Vdash \dot{c}_n \cap f_\alpha(\dot{\kappa}_n) = z_n^{\alpha,\delta}$. Note that this is defined for measure one many $\delta$'s. Set $Z_n^\alpha = [\delta \to z_n^{\alpha,\delta}]_U$.

Now for $\alpha \in X$, define $\mathcal{C}_\alpha = \{Y \subset \alpha \mid Y \in \bar{V}$ is a club , $o.t.(Y) < \kappa, \exists k(Y \subset \bigcap_{n\geq k} Z^n_\alpha)\}$. Note that $\mathcal{C}_\alpha$ is nonempty because each $\bigcap_{n\geq k} Z^n_\alpha)$, for $k < \omega$, is in it.

**Claim 3.7.** $|\mathcal{C}_\alpha| \leq \kappa$

*Proof.* Since for each $n$, $W \models |z_n^{\alpha,\kappa_n}| < \tau_n = |\nu_n|$, we have that for all large $n$, for $U$-many $\delta < \kappa$, $|z_n^{\alpha,\delta}| < f_\nu(\delta)$. So for all large $n$, $V \models |Z^n_\alpha| < \nu$.

Since in $\bar{V}$, $\nu$ is strong limit, $\mathcal{P}^{\bar{V}}(Z^n_\alpha)$ has size less than $\nu$. So, $W \models |\mathcal{P}^{\bar{V}}(Z^n_\alpha))| \leq |\nu| = \kappa$.

So $|\mathcal{C}_\alpha| = \kappa$.

□

**Claim 3.8.** *(Coherence) Let $\alpha < \beta < \mu$, $\beta \in X$, and suppose that for some $Y \in \mathcal{C}_\beta$, $\alpha \in \lim(Y)$. Then $\alpha \in X$ and $Y \cap \alpha \in \mathcal{C}_\alpha$.*

*Proof.* First we show that $\alpha \in X$. Since $\beta \in X$, we have that for all large $n$, $f_\beta(\kappa_n) \in \lim(c_n)$. Also since $\alpha \in \lim(Y)$, then for all large $n$, $\alpha \in \lim Z^n_\beta$. It follows that for all large $n$, $f_\alpha(\kappa_n) \in \lim(z_n^{\beta,\kappa_n}) \subset \lim(c_n)$.

Next we show that for all large $n$, $Z^n_\beta \cap \alpha = Z^n_\alpha$. We have that:

- $Z^n_\alpha = [\delta \mapsto z_n^{\alpha,\delta}]_U$
- $Z^n_\beta \cap \alpha = [\delta \mapsto z_n^{\beta,\delta}]_U \cap [f_\alpha]_U = [\delta \mapsto z_n^{\beta,\delta} \cap f_\alpha(\delta)]_U$.

So it is enough to show that for all large $n$, for $U$-many $\delta < \kappa$, $z_n^{\beta,\delta} \cap f_\alpha(\delta) = z_n^{\alpha,\delta}$. But that follows since, in $W$, for all large $n$, $z_n^{\beta,\kappa_n} \cap f_\alpha(\kappa_n) = (c_n \cap f_\beta(\kappa_n)) \cap f_\alpha(\kappa_n) = c_n \cap f_\alpha(\kappa_n) = z_n^{\alpha,\kappa_n}$.

□

For $\alpha \in \mu \cap (\mathrm{cof}^W(\omega))$, such that $\alpha \notin X$, let $Y \subset \alpha$ be an $\omega$-sequence (in $W$), and set $\mathcal{C}_\alpha = \{Y\}$. Then $\langle \mathcal{C}_\alpha \mid \alpha \in X \cup \mathrm{cof}(\omega)\rangle$ is a $\square^*_\kappa$-sequence.

□

**Remark 3.9.** *We note that the proof of the above lemma is similar to (and actually follows) the proof that $\prod_n Col^{\bar{V}}(\kappa^{+n}, \mu)/fin$ adds a weak square at $\nu$, which in turn gives a weak square in $W$ at $\kappa$.*

□

**Remark 3.10.** *The proof of Lemma 3.5 does not use that in $V$, $\nu = \kappa^{+\omega}$; only that $\nu$ is a singular cardinal of countable cofinality above $\kappa$.*

Now we prove the main theorem. Most of the arguments will follow the above proof of theorem 3.1.

**Theorem 3.11.** *Suppose that $V \subset W$ are models of set theory, such that*

*(1) $V \models \kappa$ is a regular cardinal, $\nu$ is a singular cardinal with countable cofinality, $U$ is a normal measure on $\kappa$, $\mu = \nu^+$, $2^\kappa > \mu$;*
*(2) $W \models \kappa = \aleph_\omega, \mu = \aleph_{\omega+1}$;*
*(3) $W$ is a forcing extension of $V$ by a strong Prikry poset $\mathbb{P}$, such that $\mathbb{P}$ projects to $\mathbb{P}_U$, and $Ult(U, V) \models \mu = \nu^+$.*

*Suppose further that any collapses below $\kappa$ are taken from an inner model $\bar{V} \subset V$, such that in $\bar{V}$, $\nu$ is strong limit. . Then $W \models \square^*_{\aleph_\omega}$.*

*Proof.* As in the proof of Theorem 3.1, fix $j : V \to M$ to be the elementary embedding given by $U$ and functions $\langle f_\alpha \mid \alpha \leq \mu\rangle$ from $\kappa$ to $\kappa$, such that for each $\alpha \leq \mu$, $[f_\alpha]_U = \alpha$.

And in $W$ let $\langle \kappa_n \mid n < \omega\rangle$ be the $\mathbb{P}_U$ generic Prikry sequence. For each $n$, define $\mu_n = f_\mu(\kappa_n)$, $\nu_n = f_\nu(\kappa_n)$. Then for all large $n$, $\nu_n$ is a singular $V$-cardinal and $\mu_n = (\nu_n^+)^V$. By moving to a tail end of the Prikry sequence, we may assume that this is true for all $n$. Similarly we may assume that each $\kappa_n$ is an inaccessible cardinal in $V$.

**Lemma 3.12.** *For all large $n$, $\mu_n$ is collapsed.*

*Proof.* First note that Lemma 3.2 still holds here, as the proof only used that in $V$, $\nu$ is singular and the predecessor of $\mu$. So, as before, for all large $n$, $W \models |\nu_n| > |\kappa_n|$. Let $\tau_n = |\nu_n|^W$. By Theorem 2.10, if $\mu_n$ is preserved, the $W$-gap between $\mu_n$ and $|\kappa_n|^W$ cannot be more than 4, i.e., $W \models \tau_n \leq |\kappa_n|^{+3}$.

Let $\delta^*$ be such that $V \models \kappa^{+\delta^*} = \nu$. (Note that $\mathrm{cf}(\delta^*) = \mathrm{cf}(\nu) = \omega$). Again, by Lemma 3.2, for all large $n$, for all $\delta < \delta^*$, $(\kappa_n^{+\delta})^V < \tau_n$. Otherwise, we can apply Lemma 3.2, to $(\kappa^{+\delta})^V$ and would get that $\mu$ is collapsed. Note that $[\alpha \mapsto \alpha^{+\delta}]_U = (\kappa^{+\delta})^{Ult(U,V)} \leq (\kappa^{+\delta})^V$.

Let $\delta_{n,0}$ be such that $V \models \kappa_n^{+\delta_{n,0}} = \tau_n$. Then, by the above remark, $\sup_n \delta_{n,0} = \delta^*$. (Of course the full sequence $\langle \delta_{n,0} \mid n < \omega \rangle$ may not be in $V$).

Now let $\tau_{n,1} \geq |\kappa_n|^W$ be such that $W \models \tau_{n,1}$ is a cardinal, $\tau_{n,1}^+ = \tau_n$. Such a cardinal exists, since in $W$, $\kappa = \aleph_\omega$. If $\tau_{n,1} = |\kappa_n|^W$, set $\delta_{n,1} = 0$. Otherwise let $0 < \delta_{n,1} < \delta_{n,0}$ be such that $V \models \kappa_n^{+\delta_{n,1}} = \tau_{n,1}$.

**Claim 3.13.** $\sup_n \delta_{n,1} = \delta^*$

*Proof.* Otherwise, let $\delta$ be such that $\sup_n \delta_{n,1} < \delta < \delta^*$. Then n $W$, we can define a bijection between $(\prod_n \kappa^{+\delta})^V$ and $(\prod_n \kappa^{+\delta_{n,0}})^V$. But, in $V$, the latter has cardinality $\mu$ and the former has cardinality less than $\nu$. So, this would collapse $\mu$ in $W$. Contradiction. □

Now, we repeat this process (at most) twice. For each $n$, such that $\delta_{n,1} > 0$, let $\tau_{n,2}$ to be the $W$-cardinal, such that $W \models \tau_{n,2}^+ = \tau_{n,1}$. If $\tau_{n,2} = |\kappa_n|^W$, set $\delta_{n,2} = 0$, otherwise let $0 < \delta_{n,2} < \delta_{n,1}$ be such that $V \models \kappa_n^{+\delta_{n,2}} = \tau_{n,2}$. By the same claim as above, we must have that $\sup_n \delta_{n,2} = \delta^*$.

In a similar fashion if $\delta_{n,2} > 0$, define $\tau_{n,3}, \delta_{n,3}$, so that $W \models \tau_{n,3}^+ = \tau_{n,2}$ and either $\tau_{n,3} = |\kappa_n|^W$ or $V \models \kappa_n^{+\delta_{n,3}} = \tau_{n,3}$. Then, by the claim above $\sup_n \delta_{n,3} = \delta^*$. But since $W \models \tau_n \leq |\kappa_n|^{+3}$, we have that for all $n$, $\tau_{n+3} = |\kappa_n|^W$. Contradiction.

□

So, we have that for all large $n$, $\mu_n$ is collapsed. Then by the same arguments as in Lemma 3.5, $W \models \square^*_\kappa$.

□

As a corollary, we have that a forcing like Gitik-Sharon, or any known diagonal Prikry forcing, cannot be adapted to answer Woodin's question positively.

We conclude with some open problems.

**Question 2.** *Can we replace the Prikry forcing assumption with just "$\kappa$ is singularized"?*

Answering that question will involve analyzing "pseudo Prikry sequences" as in [9]. Following this paper, Lambie Hanson showed the existence of such sequences in the diagonal setting, [8]. We make the following conjecture:

**Conjecture:** Suppose that $V \subset W$ are models of set theory, such that

(1) $V \models \kappa$ is a regular cardinal, $\mu$ is a successor of a singular cardinal of countable cofinality, $2^\kappa > \mu$;
(2) $W \models \kappa = \aleph_\omega, \mu = \aleph_{\omega+1}$;
(3) Every $V$-regular $\tau$, $\kappa < \tau < \mu$ has countable cofinality in $W$.

Then $W \models \square^*_{\aleph_\omega}$.

## References

[1] Abraham and Menachem Magidor, *Cardinal Arithmetic*, The Handbook of Set Theory.
[2] James Cummings, *Collapsing successors of singulars*, ***Proc. of the AMS***, 125:2703–2709, 1997.
[3] James Cummings, *Notes on singular cardinal combinatorics*, ***Notre Dame Journal of Formal Logic***, 46(3):251–282, 2005.
[4] James Cummings, Unpublished notes and personal communication.
[5] Mirna Dzamonja and Saharon Shelah, *On squares, outside guessing of clubs and* $I_{<f}[\lambda]$, ***Fund. Math.***, 148(2):165-198, 1995.
[6] Moti Gitik, *Some results on the nonstationary ideal II*, ***Israel J. Math.***, 99:175-188, 1997.
[7] Moti Gitik and Assaf Sharon, *On SCH and the approachability property*, ***Proc. of the AMS***, vol. 136 (2008), pp. 311-320.
[8] Chris Lambie-Hanson, *Pseudo-Prikry sequences*, ***Proc. AMS***, 146: 4905-4920, 2018.
[9] Menachem Magidor and Dima Sinapova, *Singular cardinals and square properties*, ***Proc. Amer. Math. Soc.***, 145:4971-4980, 2017.
[10] Assaf Sharon and Matteo Viale, *Some consequences of reflection on the approachability ideal.*, ***Trans. Amer. Math. Soc.***, 362(8):4201–4212, 2010.
[11] Saharon Shelah, ***Cardinal arithmetic.*** , Oxford Logic Guides, 29, OxfordUeniersity Press 1994.
[12] Dima Sinapova, *The Tree Property at* $\aleph_{\omega+1}$, ***J. Symbolic Logic***, 77(1):279-290, 2012.
[13] Dima Sinapova and Spencer Unger, *Combinatorics at* $\aleph_\omega$, ***Annals of Pure and Applied Logic*** 165: 996-1007, 2014.
[14] Dima Sinapova and Spencer Unger, *Scales at* $\aleph_\omega$, ***Israel Journal of Mathemtics***, 209:463-486, 2015.

## References

Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA
*URL*: https://sites.math.rutgers.edu/~ds2005/
*Email address*: dima.sinapova@rutgers.edu