Disk counting for tropical Lagrangians

2024 Yamabe Symposium

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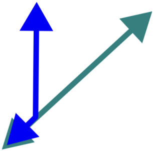
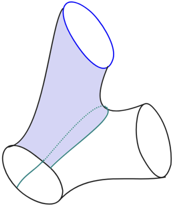
Abstract: In continuation of a work with Sushmita Venugopalan, I will describe a calculus for counting disks bounding tropical Lagrangians. In particular, I will describe a new kind of vertex called "tropical trouser", corresponding to a half-pair-of-pants bounding a Lagrangian.





Sample Theorem

Setup: Let $L \subset X$ be a compact spin monotone tropical Lagrangian in a del Pezzo surface.

$W_L : \mathrm{Hom}(\pi_1(L), \mathbb{C}^\times) \to \mathbb{C}^\times$ the *disk potential* of L counting Maslov index two disks passing through a generic point, weighted by holonomies.

If L is a Lagrangian sphere, W_L is just an integer.

Sample Theorem Ctd

Thm: W_L is a sum over tropical graphs Γ, with the contribution of each Γ given as a product over vertices v of some multiplicities $m(v)$ by a Mikhalkin-type formula

$$m(Y) = \sum_{\Gamma} \#\operatorname{Aut}(\Gamma)^{-1} \prod_{v \in \operatorname{Vert}(\Gamma)} m(v)$$

assuming “general position”. I will describe the multiplicities $m(v)$.

Sample application: Fukaya categories of del Pezzos

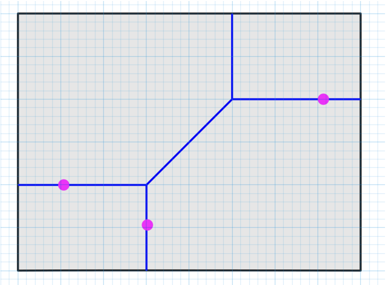
Manin observed that Lagrangian homology classes in del Pezzos corresponding to simply-laced root systems.

I will explain how to understand the numbers in the following table which can be computed purely from tropical geometry.

X	Manin Root System	$\approx \mathrm{Spec}(c_1\star \in \mathrm{End}(QH(X)))$
$\mathbb{P}^2$		$3\alpha, \alpha^3 = 1$
$\mathbb{P}^1 \times \mathbb{P}^1$	A_1	$4, 0^{\oplus 2}, -4$
$\mathrm{Bl}^1\,\mathbb{P}^2$		$-0.33, 3.8, -2.23 \pm 1.94I$
$\mathrm{Bl}^2\,\mathbb{P}^2$	A_1	$(-1)^{\oplus 2}, 4.73, -2.86 \pm 0.94I$
$\mathrm{Bl}^3\,\mathbb{P}^2$	$A_1 \oplus A_2$	$(-2)^{\oplus 3}, (-3)^{\oplus 2}, 6$
$\mathrm{Bl}^4\,\mathbb{P}^2$	A_4	$(-3)^{\oplus 5}, 8.09, -3.09$
$\mathrm{Bl}^5\,\mathbb{P}^2$	D_5	$(-4)^{\oplus 7}, 12$
$\mathrm{Bl}^6\,\mathbb{P}^2$	E_6	$(-6)^{\oplus 8}, 21$
$\mathrm{Bl}^7\,\mathbb{P}^2$	E_7	$(-12)^{\oplus 9}, 52$
$\mathrm{Bl}^8\,\mathbb{P}^2$	E_8	$(-60)^{\oplus 10}, 372.$

Mikhalkin formula

Mikhalkin introduced a formula for counting holomorphic spheres in toric surfaces X as a sum over tropical graphs. For example, here is the unique tropical graph contributing to the count of spheres of degree $(1,1)$ in $X = \mathbb{P}^1 \times \mathbb{P}^1$ passing through three generic points (drawn in pink.)



"There is a unique automorphism of $\mathbb{P}^1$ with specified values at three points. "

Mikhalkin formula

Let $m(Y)$ be the number of holomorphic genus zero maps with the given constraints Y. $m(Y)$ is a sum over tropical graphs Γ, with the contribution of each Γ given as a product over vertices v of some multiplicities $m(v)$.

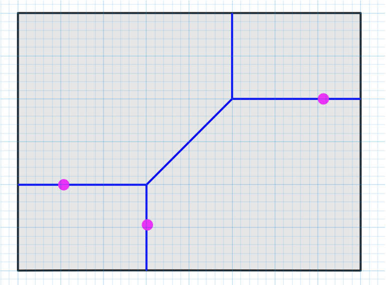
$$m(Y) = \sum_{\Gamma} \# \operatorname{Aut}(\Gamma)^{-1} \prod_{v \in \operatorname{Vert}(\Gamma)} m(v).$$

This formula was generalized to toric varieties by Nishinou-Siebert.

Mikhalkin multiplicities, toric case

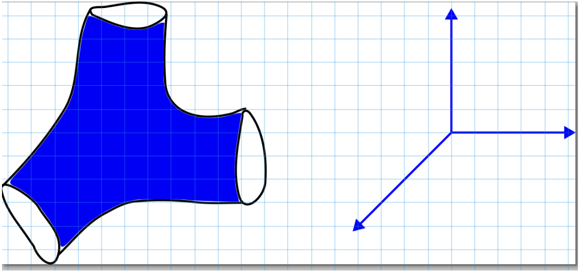
The tropical graphs Γ live in the *dual complex* $B = \cup_{P \in \mathcal{P}} P^\vee$ for some particular polyhedral decomposition of the moment polytope $\Phi(X) = \cup_{P \in \mathcal{P}} P$ (and it's also necessary to take perturbation).

It's convenient to draw the graphs on top of the moment polytope.



Mikhalkin multiplicities, toric case

In Mikhalkin's case for spheres there are only two multiplicities needed: At a trivalent vertex v, $m(v)$ is the area parallelogram spanned by the directions if the outgoing directions of the edges at v sum to zero, and vanishes otherwise.



At a univalent vertex v mapping to the boundary of the moment polytope, $m(v) = 1$ and the direction must be normal to the boundary.

Proof of Mikhalkin formula

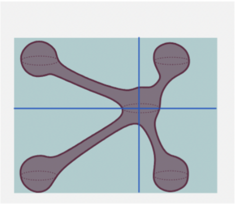
The original proof was rather explicit.

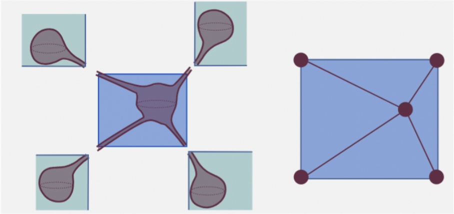
Brett Parker gave an sft-style proof uses a degeneration argument under multi-directional symplectic cutting.

Venugopalan and I gave a different sft-style version that also works for holomorphic disks bounding Lagrangians, also in almost toric manifolds.

SFT-STYLE PROOF OF MIKHALKIN FORMULA

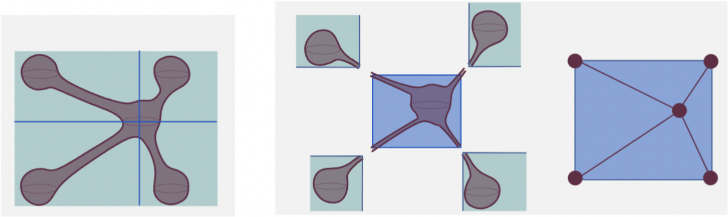
The proof uses the fact that any almost toric manifold admits a toric degeneration: Pick a polyhedral decomposition $\mathcal{P} = \{P\}$ of the base diagram so that each cut space X_P contains at most one constraint, face of minimal dimension, or focus-focus singularity. Each sequence u of maps contributing to the count has a broken limit u_∞ in $\cup_P X_P$.





Proof of Mikhalkin formula

Define a tropical graph Γ by assigning to each component $\approxeq$ a vertex v, and each node an edges. For each Γ we wish to count the subset of $\Pi_{v \in \mathrm{Vert}(\Gamma)} M(X_{P(v)})$ of tuples satisfying matching conditions at the nodes.



Proof of Mikhalkin formula

A special feature in the genus zero $\dim(X) = 4$ case:

Each fiber product $M = M_1 \times_{M_{12}} M_2$ is actually a Cartesian product after adding constraints.

Since $dim(M) = 0$, without loss of generality $dim(M_1) = 0$, and the image of M_{12} in M_2 provides a constraint Y_2 so that the constrained moduli space $dim(M_2(Y_2)) = 0$. So

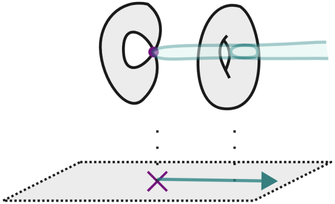
$$M(X) = \bigcup_{\Gamma} \prod_{v \in \mathrm{Vert}(\Gamma)} M(X_{P(v)}, Y_v)$$

There is something like this in higher dimension, but it's more complicated (Fulton-Sturmfels splitting of diagonal.)

Mikhalkin multiplicities, almost toric case

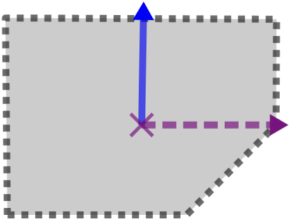
Venugopalan and I worked out the generalization of Mikhalkin's formula to the almost toric case, which in some sense is "known" by results of Brett Parker and others. The case of K3 surfaces had been done by Lin.

An almost toric moment map is like a toric moment map, except that fibers that are nodal tori are allowed. The nodes are called *focus-focus values*.



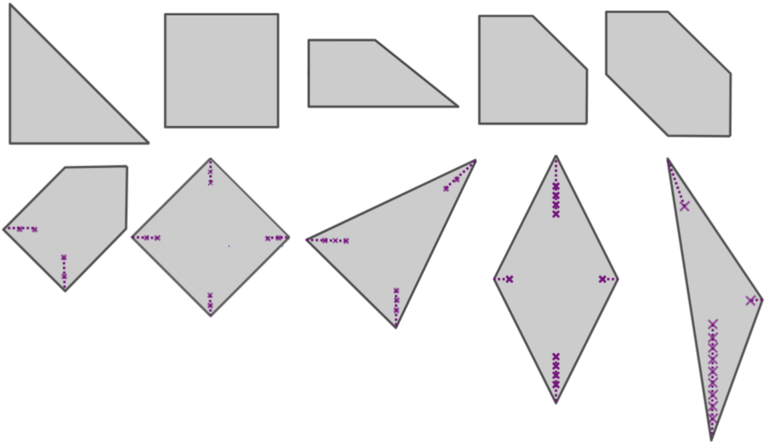
There is just one additional vertex for a sphere with direction going to a focus-focus singularity perpendicular to a branch cut direction.

The multiplicity is the Bryan-Pandharipande formula $m(v) = (-1)^{d-1}/d^2$, where d is the lattice length.



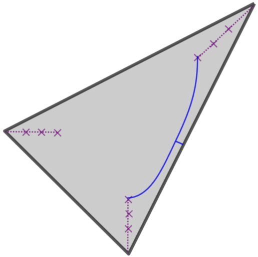
VIANNA'S DIAGRAMS FOR DEL PEZZOS

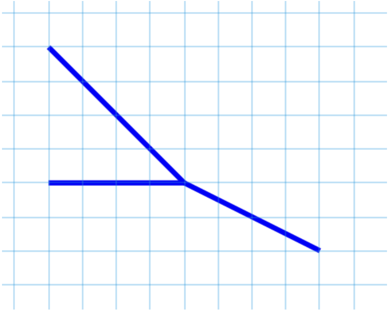
Vianna has shown that every del Pezzo admits an almost toric structure a la Symington.



HOLOMORPHIC SPHERES IN DEL PEZZOS

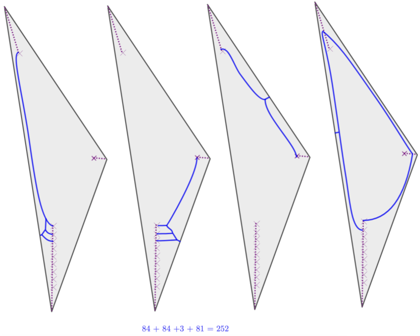
Example: A famous formula of Cayley-Salmon is the count of twenty seven lines (lowest degree curves) in a cubic surface. The 27 lines in the cubic are all variations on the following graph:





HOLOMORPHIC SPHERES FOR DEL PEZZOS

There is a less well-known count of 252 degree one curves in the del Pezzo of degree one:



Of these 240 are embedded (and correspond to the roots of E_8). The extra 12 are half of the 24 nodal fibers in a $K3$.

The potential of a Lagrangian

X symplectic manifold L compact oriented spin Lagrangian

We now want to describe how to count holomorphic disks in X bounding L using tropical curve counting.

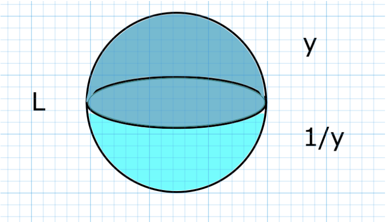
Although in general disk counts depend on the choice of almost complex structure, some counts are independent.

For example, the *disk potential* W_L of a monotone Lagrangian L is the number of Maslov-index-two holomorphic disks passing through a generic point, and is independent of choices by a result of Oh.

Example: Potential for the two-sphere

Typically one counts disks weighted by the holonomies of a local system $y : \pi_1(L) \to \mathbb{C}^\times$ to obtain a function $W_L(y)$.

Example: The potential of the equator in the two-sphere is $W_L(y) = y + 1/y$, with the two terms coming from the two hemispheres.



Motivation: mirror symmetry/classification of Fano's

The function W_L is expected to be a chart on the mirror manifold (sometimes the whole mirror.)

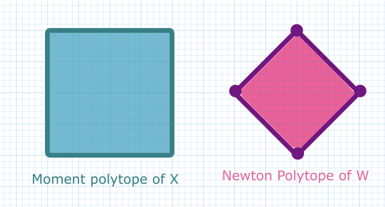
By a conjecture of Coates et al, there is a classification of Fano varieties by functions W_L with certain properties.

By results of Vianna et al, there are infinitely many Hamiltonian-isotopy classes of monotone Lagrangian tori in del Pezzos, by counting the number of terms in W_L.

Example: Potential of $S^2 \times S^2$

Potentials behave in the expected way with respect to products. So, for example, if $L = S^1 \times S^1 \subset X = S^2 \times S^2$ then $W_L(y_1, y_2) = (y_1 + 1/y_1) + (y_2 + 1/y_2)$.

Its Newton polytope is *dual* to the moment polytope.



Multiplicities for almost toric moment fibers

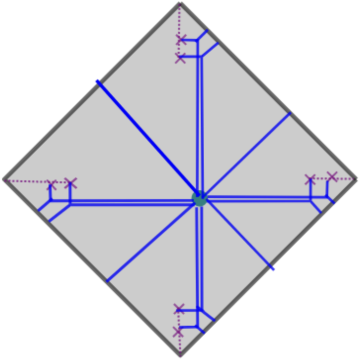
The Mikahlkin-style formula for Lagrangians fibers in almost-toric manifolds just one additional possible vertex v, representing the boundary on the Lagrangian, which is univalent and has $m(v) = 1$.

Using this one can computed the potentials of monotone del Pezzos in a straight-forward way. (Was deduced earlier by Pascaleff-Tonkonog using "structural properties".)

Similar results using Gross-Siebert by Bardwell-Evans-Cheung-Hong-Lin

EXAMPLE

For the degree four del Pezzo $W_L(y_1, y_2) = (1 + y_1)^2(1 + y_2)^2/y_1y_2 - 4$, arising from the tropical graphs

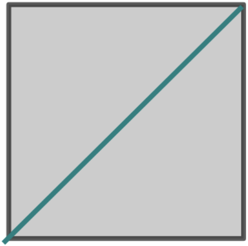


The critical value $W_L = 12$ is easy to see from this picture, since $y = 1$ is a critical value by symmetry. What about the other critical value -4?

Tropical Lagrangians

Def: A tropical Lagrangian L of a symplectic manifold X with a Hamiltonian torus action is a sequence of Lagrangians whose moment image approximates some piecewise linear complex Λ in the moment polytope $\Phi(X)$ of X.

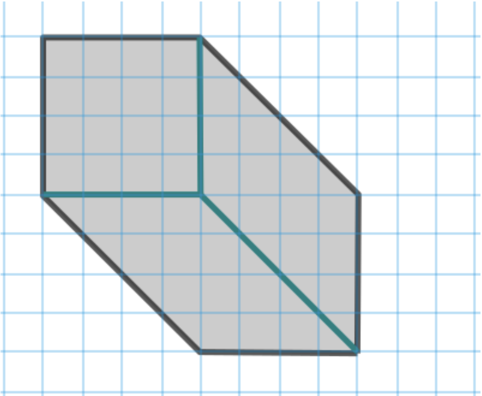
Example: The diagonal in $X = (S^2)^- \times S^2$ (that is, the graph of the identity as a symplectomorphism) is a tropical Lagrangian with graph the diagonal in $\Phi(X) = [-1, 1]^2$.



Tropical realizability theorems

Hicks, Matessi, Mikhalkin have proved various tropical realizibility theorems, describing which tropical graphs or in higher dimensions, which polyhedral complexes can be approximated by Lagrangians.

Example: A tropical Lagrangian in $\mathrm{Bl}^3\,\mathbb{P}^2$:



Tropical realizability theorems

Generalization of Mikhalkin's Realizibility Thm for Almost Toric Four-Manifolds: Any tropical graph $\Lambda \subset B$ whose univalent vertices v appear either

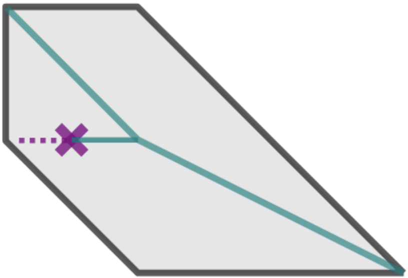
-at the vertices of B (in which case they must bisect the angle in the lattice sense) or

- at the focus-focus values B^{foc} (in which case they must be shear directions) and

- have primitive lattice directions $\mu(\epsilon), \epsilon \in \mathrm{Edge}(\Lambda)$ satisfying the Mikhalkin balancing condition at any trivalent vertex, is realizable.

An undergraduate Annie Wei working with me pointed out that there are a lot more graphs that are realizable.

A picture of a tropical Lagrangian in an almost toric manifold



Manin configurations

The diagonal is the first case of a *Manin system*: An ADE configuration of tropical Lagrangians in a del Pezzo surface X.

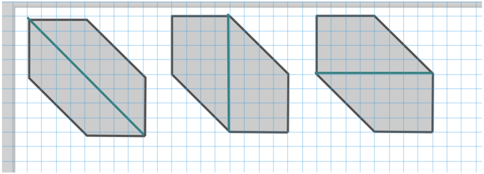
A system of Lagrangian two spheres whose intersection diagram is one of the simply-laced root systems.

On the homology level these were discovered by Manin (as generators for the second cohomology perpendicular to the canonical class).

MANIN CONFIGURATIONS

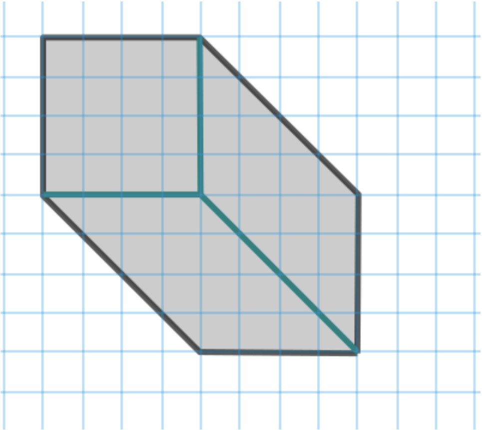
I'll focus on the thrice-blown-up projective plane $X = \mathrm{Bl}^3(\mathbb{P}^2)$, whose associated Manin root system is $A_1 \oplus A_2$.

To get the positive roots of the A_2 system, just take the inverse images of the diagonal in any of the three blow-down-to $\mathbb{P}^1 \times \mathbb{P}^1$ maps.



The Lagrangian pair of pants

To get the A_1 system in $\mathrm{Bl}^3\,\mathbb{P}^2$ one needs the Lagrangian pair of pants. Near a vertex, this Lagrangian is defined by taking a holomorphic pair of pants and taking a hyperKahler rotation.

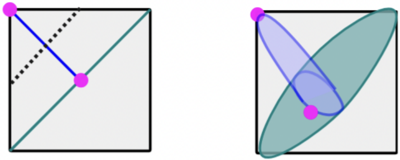


Potentials for Manin configurations

Each Lagrangian in a Manin configuration is a sphere, and so W_L is an integer.

Main Thm: There is a Mikhalkin-type formula for counting disks bounding these Lagrangians.

Two new types of vertices: univalent v starting on an edge e of lattice length $\ell(e) \in \mathbb{Z}_{\geq 0}$ of Lagrangian, with $m(v) = (-1)^{\ell(e)}$, corresponding to a half-cylinder, perpendiular to the tropical Lagrangian.



Anti-symplectic involutions

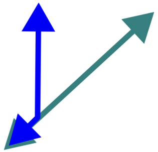
The multiplicity of a half-cylinder can be computed using anti-symplectic involutions. Recall (see e.g. Fukaya-Oh-Ohta-Ono) if $L \subset X$ is the fixed point set of an anti-symplectic involution then disks bounding L correspond to spheres in X.

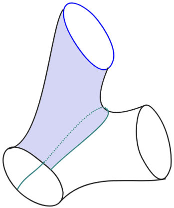
Example: Let L be the diagonal in $S^2 \times S^2$. Then disks bounding L with an interior and boundary constraint correspond to spheres in $S^2 \times S^2$ with three interior constraints.

The count of the spheres is 1, but the orientation is reversed at one of the interior constraints in the "double" so the count of the disks is -1.

The holomorphic trouser

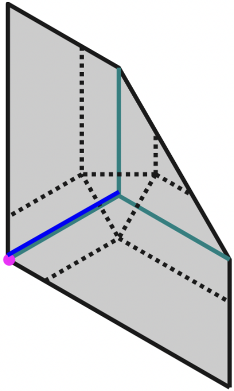
Trivalent vertex v corresponding to the holomorphic pant (half a pair of pants) with $m(v) = |\det(\mu_\circ \mu_\bullet)| \in \mathbb{Q}$ where $\mu_\circ$ resp. $\mu_\bullet$ is the boundary puncture resp. interior puncture direction . Proof is a reduction to the case of real lines by a multiple cover argument.

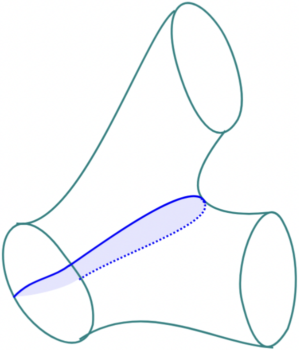




Disks bounding the Lagrangian pair of pants

$m(v) = -1, 1, 1$ for disks bounding the Lagrangian pair of pants and hitting one resp. two resp. three legs.



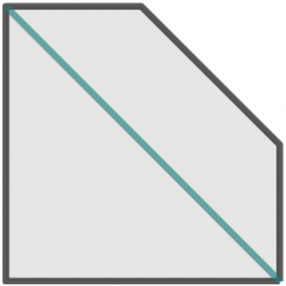


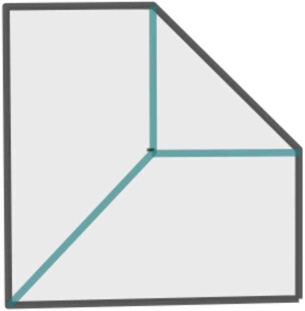
Proof of multiplicities for Lagrangian pants

The proof of the multiplicities for Lagrangian pants is harder.

Hind, Evans: The Lagrangian pants L in $\mathrm{Bl}^2(\mathbb{P}^2)$ is Hamiltonian isotopic to the inverse image L' of the diagonal (which has tropical graph an interval).

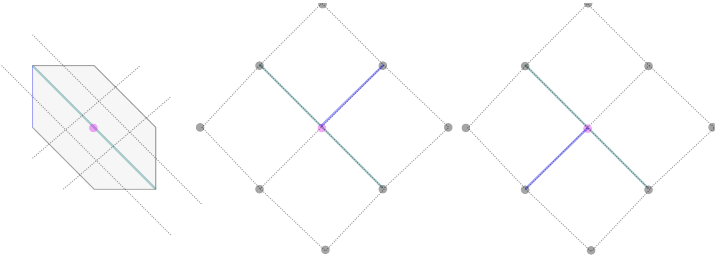
So $W_L = W_{L'} = -1$.





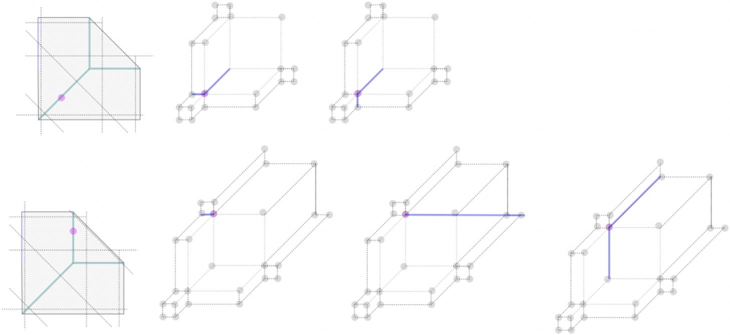
Example for $X = \mathrm{Bl}^3(\mathbb{P}^2)$.

For the A_2-system Lagrangians $W_L = -2$ by the degeneration argument in the following picture: The tropical graphs of the holomorphic curves are drawn on the right.

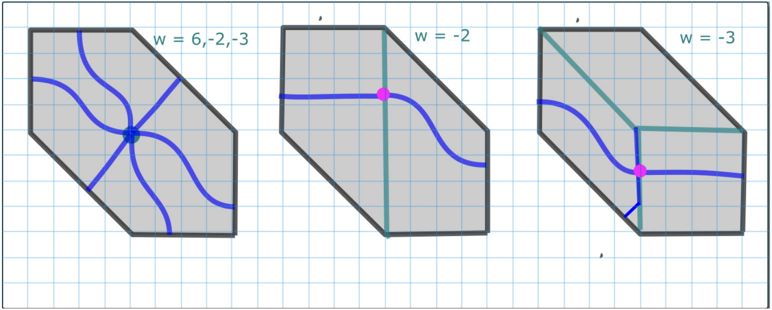


Example for $X = \mathrm{Bl}^3(\mathbb{P}^2)$.

For the A_1-system Lagrangians $W_L = -3$.



The cartoon picture for the Fukaya category of $\mathrm{Bl}^3(\mathbb{P}^2)$



Fukaya categories of del Pezzos

Under the open-closed map $HH_*(Fuk(X)) \to QH(X)$ classes of Lagrangians with potential W_L map to generalized eigenvectors of quantum multiplication by $c_1(X)$. (Dubrovin-Kontsevich spectrum)

Def: A Manin summand in $QH(X)$ is a generalized eigenspace with W_L integer and $|W_L|$ non-maximal.

Thm: Each Manin summand in $QH(X)$ is generated by the corresponding Manin configurations of Lagrangians in X under the open-closed map. By Abouzaid generation criterion, the Manin configurations split-generate the corresponding Fukaya eigencategories.

THE CASE OF LOW DEGREE

In the case of low degree, there are generalized eigenvectors of $c_1\star$ that are not eigenvectors. These cannot be generated by Hochschild chains of length one; one has to explicitly construct Hochschild chains of higher length.

One can find such chains that involve combinations of the Lagrangians in the Manin system. But one never needs non-trivial Maurer-Cartan solutions in these cases.

Final remarks

Analogy:

Cohomology	Quantum cohomology
Mayer-Vietoris	Relative Gromov-Witten theory
Čech cohomology	Multi-directional sft/tropical geometry

Generally speaking not any "decomposition" is allowed, only those corresponding to "sympectic cutting" a la Lerman.

Final remarks

Tropical curve counting does not "reduce pseudoholomorphic curve counting to combinatorics".

Instead, one sometimes finds that using formal properties of holomorphic curves one is reduced to computing a few examples, which can then be computed e.g. using tropical curve counts.

Final remarks

The techniques work best in, but are not limited to, dimension four.

For example, one can computing potentials and split-generation for moduli spaces of flat bundles on Riemann surfaces (my student Sumeet Khandelwal is finishing a thesis about this; he proves split-generation in some cases).

To give another example, the Chiang Lagrangian is tropical in $\mathbb{P}^3$, but curve counts reduce to a tropical Lagrangian of an orbifold projective plane which can be performed tropically.