Minimal models and quantum cohomology of toric varieties

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arXiv:1207.3253 Quantum cohomology and toric minimal model programs

arXiv:1208.1727 A wall-crossing formula for Gromov-Witten invariants under variation of git quotient

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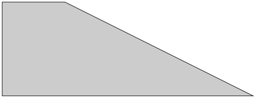
Toric varieties

T complex torus acting on

X normal variety with ample line bundle $L \to X$

X is a toric variety iff T has a dense orbit

moment polytope $P = \text{hull}(\text{weights}(H^0(X, L))$

Example: X Hirzebruch surface $P =$  .

P is the image of X under the *moment map* $\Phi : X \to \text{Lie}(T)^\vee$.

Cohomology of toric varieties

For each facet Q of P there is a codimension one subvariety $X(Q) \subset X$.

Assume that X is rationally smooth. The cohomology classes $[X_Q]$ generate $H(X, \mathbb{Q})$.

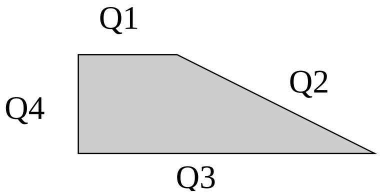
Danilov-Jurkiewicz: There are two kinds of relations:

sums of divisor classes with coefficients given by pairing normal vectors to facets with vectors in Lie(T);

products of divisor classes for divisors that do not intersect.

Cohomology of a Hirzebruch surface

Example: X Hirzebruch surface

$P =$ 

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$X(Q3) - 2X(Q2) - X(Q1) = X(Q4) - X(Q2) = 0$ and
$X(Q_i)X(Q_j)X(Q_k) = 0$ any i, j, k distinct.

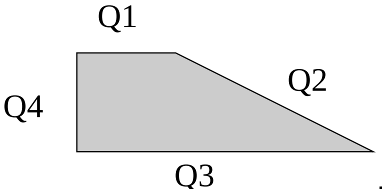
This implies that $X(Q1)^2 = -2$, that is, $X(Q1)$ is a -2-sphere.

Another description of the cohomology

$V : H^2_+(X) \to \mathbb{R}$ volume of the moment polytope of a symplectic class

SR = symbols of operators that annihilate V

For example, for the Hirzebruch surface moving both $Q4$ and $Q2$ at opposite rates does not change the volume, hence $X(Q4) = X(Q2)$ in cohomology.



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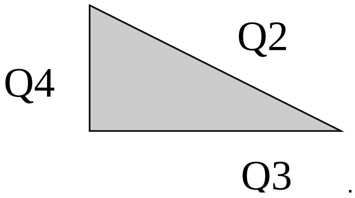
Toric orbifolds

Blowing down the -2-sphere in the Hirzebruch surface above gives a *weighted projective plane* $\mathbb{P}(1,1,2)$



Relations
$X(Q3) - 2X(Q2) = X(Q4) - X(Q2) = X(Q2)X(Q3)X(Q4) = 0.$

The toric variety $\mathbb{P}(1,1,2)$ is the coarse moduli space for a *toric orbifold*: smooth Deligne-Mumford toric stack.

Interpretation via Stanley-Reisner

Cox: Any projective toric variety X is a git quotient of a G-vector space $\mathbb{C}^k$ where G is a torus.

$H_G(\mathbb{C}^k) = \mathrm{Sym}(\mathrm{Lie}(G)^\vee)$ equivariant cohomology

If SR_a resp. SR_m is the ideal of additive resp. multiplicative Danilov-Jurkiewicz relations then there are *Stanley-Reisner exact sequences*

$$0 \to SR_a \to H_{(\mathbb{C}^\times)^k}(\mathbb{C}^k) \to H_G(\mathbb{C}^k) \to 0$$

$$0 \to SR_m \to H_G(\mathbb{C}^k) \to H(X) \to 0.$$

Orbifold quantum cohomology

$QH(X)$ orbifold quantum cohomology of X (Chen-Ruan, Abramovich-Graber-Vistoli)

As a vector space $QH(X) = H(I_X) \otimes$ Novikov ring

I_X = loops on X = the *inertia orbifold* of X

components of I_X correspond to points on X with automorphisms.

Example $X = \mathbb{P}(1,1,2) \implies I_X = \mathbb{P}(1,1,2) \cup \mathrm{pt}$.

Product structure on $T_a QH(X)$ counts *orbifold stable maps* to X with additional insertions of a.

$T_0 QH(X)$ is the *small quantum cohomology* of X.

Quantum Stanley-Reisner exact sequence

Say $X = \mathbb{C}^k /\!\!/ G$, weights of G are $\mu_1, \ldots, \mu_k$

There is a map $QK : QH_G(\mathbb{C}^k) \to QH(X)$ which *quantizes* the usual Kirwan map.

Main Theorem: There is a quantum version of the Stanley-Reisner exact sequence after formal completion

$$0 \to QSR_m \to T_a QH_G(\mathbb{C}^k) \to T_{QK(a)} QH(X) \to 0.$$

QSR_m is the *Batyrev* ideal generated by products for $d \in H_2^G(\mathbb{C}^k, \mathbb{Q})$

$$\prod_{\mu_i(d) \geq 0} \mu_i^{\mu_i(d)} - q^d \prod_{\mu_i(d) \leq 0} \mu_i^{-\mu_i(d)}$$

Generalizes Batyrev, Givental, Iritani, McDuff-Tolman, Fukaya-Oh-Ohta-Ono, Cieliebak-Salamon

Example: quantum cohomology of projective space

Say $X = \mathbb{C}^k /\!\!/ G, \quad G = \mathbb{C}^\times$.

$T_0 QH_G(\mathbb{C}^k) = \mathbb{C}[u, q]$

$D_0 QK : T_0 QH_G(\mathbb{C}^k) \to T_0 QH(X)$ is given by $u^i \mapsto v^j q^r$

where $i = rk + j$, that is, j is the remainder.

v is the hyperplane class in $QH(X)$

so $QSR_m = \langle u^k - q \rangle$

so $T_0 QH(X) \cong \mathbb{C}[u, q]/\langle u^k - q \rangle$

Jacobian ring of the Givental potential

The quotient $QH_G(\mathbb{C}^k)/QSR_m$ is a *Jacobian ring*:

$T^\vee$ torus dual to T

$P = \{\lambda \mid \langle \lambda, \nu_j \rangle + c_j \geq 0,\ j = 1, \dots k\}$ moment polytope

$W : T^\vee \to$ Novikov ring, $\quad y \mapsto \sum_{j=1}^{k} q^{c_j} y^{\nu_j}$

is the *Givental potential* (re-appears in Hori-Vafa)

$\mathrm{val}_q : \mathrm{Crit}(W) \to \mathrm{Lie}(T)^\vee$ valuation in q

Jacobian ring of the Givental potential, ctd.

$\mathrm{Crit}_+(W)$ preimage of $\mathrm{int}(P)$: part of the critical locus living over the interior of the moment polytope.

$\mathrm{Jac}_+(W)$ functions on $\mathrm{Crit}_+(W)$

"Combinatorial mirror symmetry": $QH_G(\mathbb{C}^k)/QSR_m \cong \mathrm{Jac}_+(W)$ (Givental, Iritani etc.)

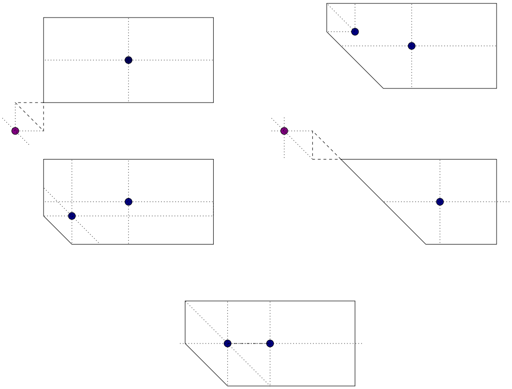
The points in $\mathrm{val}_q(\mathrm{Jac}_+(W))$ have (generically) the simple interpretation as points such that the "closest facets" have linearly dependent normal vectors (recursively if these do not span $\mathrm{Lie}(T)$).

Behavior of Jacobian ring under variation of git quotient

Vary the moment polytope.

Some of the points in $\text{Crit}_+(W)$ can disappear.

Example: varying the polytope for a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$.



Reid's minimal model program

There is a cute explanation of $\mathrm{val}_q(\mathrm{Crit}_+(W))$ in terms of Reid's minimal model program.

Goal of mmp: given X, obtain a birational variety Y whose first Chern class $c_1(Y)$ has a definite sign.

Reid: this works in any dimension for toric varieties, with three types of transitions:

divisorial contractions (blow-downs)

flips (birational transformations with exceptional loci of dimension ≥ 2 with positive relative anticanonical class)

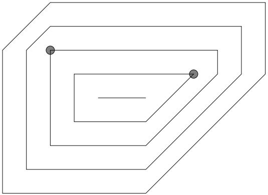
the final stage is a fibration with a fiber with $c_1 \geq 0$-fiber

Reid's minimal model program via variation of git

Construction of Reid's mmp by variation of git:

$P_t = \{\lambda \mid \langle \lambda, \nu_j \rangle + c_j \geq t, \ j = 1, \dots k\}$ "shrinking moment polytope":

Here is an example with a thrice-blow-up of $\mathbb{P}^2$:



Star Wars trash compactor scene

The walls are closing in!

In our case, some of the walls disappear before the others. These are the source of the "transitions" in the mmp.

Jacobian ring and Reid's minimal model program

X_t corresponding sequence of toric varieties with moment maps Φ_t

X_t^{sing} points with infinitely many automorphisms (where quotient is not an orbifold)

Then $\text{val}_q(\text{Crit}_+(W)) = \cup_t X_t^{\text{sing}}$.

$\text{Crit}_+(W)$ is finite-to-one over the locations of transitions in the minimal model program.

A curious connection between Hamiltonian dynamics and birational geometry

(Cho-Oh, Biran-Entov-Polterovich, Fukaya-Oh-Ohta-Ono, Woodward) : $\Phi^{-1}(\lambda), \lambda \in \mathrm{val}_q(\mathrm{Crit}_+(W))$ are Hamiltonian non-displaceable

$\implies$ #{ non-displaceable Lagrangian tori/isotopy} $\geq$ #{ of transitions in any mmp}

Question: Is this just a toric phenomenon or is it more general? Can it be related to Kahler-Ricci flow? (Conjectural geometric interpretation of mmp.)

Proof of exactness at the first step

QSR_m = symbols of differential operators annihilating the *fundamental solution*

In Givental's language, the fundamental solution is the J-function.

Woodward: extension of this result without any semipositivity conditions on c_1:

the I-function and J-function intertwined by QK.

$\implies QK =$ "mirror map" in the papers of Givental, Lian-Liu-Yau etc in the case $c_1 \geq 0$.

But the quantum Kirwan map is more general.

Proof of exactness at the middle step via Reid's toric minimal model program

Given exactness at the first step and last step it remains to show

$\dim QH_G(\mathbb{C}^k)/QSR_m = \dim QH(X)$.

For this we can use the toric mmp.

In the semi-Fano case $\mathrm{Crit}(W) = \mathrm{Crit}_+(W)$ and the claim follows from Kouchnirenko's theorem.

The mmp allows to deduce to the semi-Fano case, once we have checked that the quantities $\dim(\mathrm{Jac}_+(W))$ and $\dim(QH(X))$ change in the same way under the transitions in the mmp.

This is just combinatorics plus Kouchnirenko's theorem again.

Invariance under flops

Simple explanation of why flops have the same orbifold quantum cohomology: they have combinatorially equivalent minimal model programs so variation of git does not change quantum cohomology.

More general result for the non-toric case (Gonzalez-W): there is a wall-crossing formula for Gromov-Witten invariants under variation of git.

In the flop case Gromov-Witten potentials are invariant under the wall-crossing by invariance of the wall-crossing terms under an action of the Picard group of the curve.

Towards a quantum Kirwan map for Fukaya categories

Conjecture: the equivariant Fukaya category and the Fukaya category of a git quotient are related by an "open version" of the quantum Kirwan map

Would-be-consequence: Givental-(Hori-Vafa) potential and the "actual" (Fukaya-Oh-Ohta-Ono potential) are related by a canonical change of coordinates, see Chan-Lau-Leung-Tseng, Gonzalez-Iritani for discussion of the semipositive case.