TROPICAL FUHYA ALGEBRAS

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Abstract. We introduce a tropical version of the Fukaya algebra of a Lagrangian submanifold and use it to show that tropical Lagrangian tori are weakly unobstructed. Tropical graphs arise as large-scale behavior of pseudoholomorphic disks under a multiple cut operation on a symplectic manifold that produces a collection of cut spaces each containing relative normal crossing divisors, following works of Ionel [37] and Brett Parker [50, 51]. Given a Lagrangian submanifold in the complement of the relative divisors in one of the cut spaces, the structure maps of the broken Fukaya algebra count broken disks associated to rigid tropical graphs. We introduce a further degeneration of the matching conditions (similar in spirit to Bourgeois’ version of symplectic field theory [9]) which results in a tropical Fukaya algebra whose structure maps are, in good cases, sums of products over vertices of tropical graphs. We show the tropical Fukaya algebra is homotopy equivalent to the original Fukaya algebra. In the case of toric Lagrangians contained in a toric component of the degeneration, an invariance argument implies the existence of projective Maurer-Cartan solutions.

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1. Introduction

In this paper we study the behavior of a Fukaya algebra of a Lagrangian under a multiple symplectic cut. The multiple symplectic cut operation is a generalization of the simple cut operation of Lerman [40] in which multiple circle actions are used for degeneration. The input of the construction, in our framework, is a decomposition of a compact symplectic manifold $X$ into submanifolds-with-corners indexed by a collection $P$ of convex polytopes. The multiple cut produces a collection of symplectic manifolds

$$\mathcal{X} = \bigcup_{P \in P} X_P.$$ 

Each manifold $X_P$ is obtained by collapsing a submanifold-with-corners $X_P^\circ \subset X$ by a collection of torus actions. The cut spaces $X_P$ contain the lower dimensional manifolds $X_P^\circ, P' \subset P'$; and their union is a symplectic normal crossing divisor in each piece $X_P$, see Figure 5. One may think of $\mathcal{X}$ as the space resulting from stretching necks so that the singular almost complex structure on $X$ is obtained as a limit of a sequence of almost complex structures on $X$. The case of a normal-crossing, smooth toric degeneration is a special case when each of the limiting pieces is a toric variety. In the case of toric degeneration with discriminant locus, to apply the results of this paper it would be necessary to adjust the degeneration so that the singularities are transverse to the cutting hypersurfaces.

A toy example is the cubic surface. That is, consider the blow-up $X$ of the projective plane $\mathbb{P}^2$ at six general points $p_1, \ldots, p_6$, biholomorphic to a cubic surface in $\mathbb{P}^3$. The variety $X$ admits a continuous map to a triangle $P = \text{hull}\{(-1, -1), (2, 1), (1, 2)\}$ which is a moment map $\Phi$ for a Hamiltonian torus action on the complement of the fibers over the three vertices. The inverse image of each vertex $\mu \in P$ is a union $\Phi^{-1}(\mu) \cong S^2 \cup S^2$ of two Lagrangian spheres, that is, an $A_2$ singularity. One may then separate the singularities by either simple or multiple cuts. As we explain further below in Examples 1.1, 1.2, 1.26, the multiple cut operation allows tropical computations of the numbers of holomorphic spheres or disks of minimal Chern number resp. Maslov index.

![Figure 1. No cuts, three simple cuts, or multiple cuts](image)
The behavior of holomorphic curves in the limit of multiple-neck stretching has been studied in a number of papers. In the case of a single cut the resulting relative Gromov-Witten theory of Ionel-Parker [38] and J. Li [41] is a special case of symplectic field theory of Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [9]. The basic compactness result is that any bounded energy sequence of maps has a subsequence that converges to a holomorphic building. The domain of a holomorphic building is a nodal curve, and each curve component is mapped either to a cut space, or to a neck piece which is a projectivized line bundle over the separating divisor. A generalization of this result to the case of multiple neck stretchings was developed by Eleny Ionel [37] and Brett Parker [50, 51, 52, 53, 54, 55, 56, 57, 58] using a multi-dimensional version of holomorphic buildings. The target space of such a building consists of a variety of neck pieces, each of which is a toric bundle over a boundary divisor or an intersection of boundary divisors. The tropical approach is developed in Parker [50]. The corresponding Gromov-Witten invariants in algebraic geometry are studied in, for example, Abramovich-Chen-Gross-Siebert [2]. Tehrani gives an alternate compactification of holomorphic curves relative to a normal crossing symplectic divisor [23, 25] and uses it to give a degeneration formula [24] for Gromov-Witten invariants in the almost Kähler category.

Figure 2. A holomorphic curve and a tropical graph

In Parker [50] a renormalization of the neck parameters on each bubble in the limit gives rise to a tropical graph. On the intersection of the necks in $X$ one has multiple coordinates corresponding to the various neck directions, and bubbles that develop on these neck regions inherit a natural collection of tropical coordinates. The limits of holomorphic maps under neck-stretching, called broken maps, are equipped with tropical graphs, where each component of a limit broken map is assigned its tropical coordinates.

A broken map is modelled on a graph $\Gamma$ and its domain is the normalization of a nodal curve $C$ whose irreducible components $C_v, v \in \text{Vert}(\Gamma)$ correspond to vertices of $\Gamma$, and nodes correspond to edges of $\Gamma$. The broken map is a collection of holomorphic maps

$$u_v : C_v \rightarrow X_{\mathcal{T}(v)}, v \in \text{Vert}(\Gamma),$$
each of whose domain is an irreducible curve component $C_v \subset C$ (possibly with boundary) and target space

$$X_{P(v)} = X_{p(v) \times p'(v)}$$

is a thickening of one of the cut spaces $X_{p(v)}$; satisfying for each edge $e \in \text{Edge}(\Gamma)$ connecting vertices $v_\pm$ a matching condition (3.4), (3.5) on the lifts $w_\pm(e) \in C_{v_\pm}$ of the nodal points. For a nodal lift $w \in C_v$, mapping to a relative submanifold $D \subset X_{P(v)}$, the matching condition consists of a matching on $D$, and matchings of leading Taylor coefficients of the map $u_v$ in all directions normal to $D$. A broken map additionally consists of a tropical structure

$$\mathcal{T} : \text{Vert}(\Gamma) \to B^\vee := \bigcup_P P^\vee, \quad \mathcal{T}(e) \in t_{P(e),\mathbb{Z}}, \quad e \in \text{Edge}(\Gamma).$$

where

(a) $B^\vee$ is the dual complex for the neck-stretching, and consists of a complementary dimensional polytope $P^\vee$ for every element $P$ of $\mathcal{P}$, see Figure 6,

(b) $\mathcal{T}(v) \in P^\vee(v)$ for any vertex $v$,

(c) and $\mathcal{T}(e)$ is the slope of the line segment connecting $P^\vee(v_-)$ to $P^\vee(v_+)$, and $\mathcal{T}(e)$ is equal to the ratio of intersection multiplicities with boundary divisors at the nodal point $w(e) \in C_{v_+} \cap C_{v_-}$.

The image of $\mathcal{T}(\Gamma)$ is called a tropical graph.

The broken map, equipped with a tropical structure, differs from a holomorphic building in [9] in that it is not a continuous map into a topological space. In a broken map $u : C \to \mathcal{X}$, curve components $C_v$ of the normalization of $C$ are mapped to different spaces $X_{P(v)}$, and the matching conditions are based on identifications between regions of the manifolds $X_{P(v \pm)}$ whenever $v_\pm$ are joined by an edge $e \in \text{Vert}(\Gamma)$. The tropical graph $\text{Vert}(\Gamma)$ determines how the various pieces of the target space are positioned with respect to each other in the ether containing them. One of the main points of Parker’s work [50] is the description of a somewhat complicated topology on the target of the broken map so that the map becomes continuous.

The set of broken maps has a free action of a tropical symmetry group arising out of the torus action on neck pieces. The tropical symmetry group of a broken map is generated by symmetries of the tropical graph: these are ways of moving the vertex positions without changing edge slopes, see Figure 9. In particular, the symmetry group is finite if in the tropical graph vertex positions are uniquely determined by the edge slope and such tropical graphs called rigid. We prove a compactness theorem under this degeneration for stable maps with boundary on a Lagrangian that is disjoint from the cut locus.

Our main interest is a further degeneration of the matching condition in broken maps to a split form. The construction is motivated by, and has similarities to the Bourgeois version of symplectic field theory [9] where the degeneration is via the flow of a Morse function, as well as the Fulton-Sturmfels degeneration of the diagonal in a toric variety [28]. The paper [13] by Charest-Woodward uses Bourgeois’ degeneration to give a split form in case of a single cut. The idea in [13] is to deform the matching condition at node by a torus-invariant Morse function. Given
a deformation parameter \( t \in (0, \infty) \), a \( t \)-deformed map is defined by replacing the matching condition at a node by a condition that the lifts of the node are the end-points of a length \( t \) Morse trajectory. As \( t \to \infty \) the Morse trajectory at any node degenerates to a broken Morse trajectory. Thus in [13], the matching condition degenerates to the following condition: The lifts of the node lie on two transversely intersecting Morse cycles, namely the stable and unstable submanifold of some critical point of the Morse function. This approach does not work in the case of multiple cuts because some of the unstable and stable manifolds of torus-invariant Morse functions are contained in boundary divisors that neighbour other pieces of the broken manifold. As a result taking the limit of \( t \)-deformed maps, as \( t \to \infty \), will give objects, called split maps, whose domains tend to have many components even in the rigid case. In this paper we degenerate the matching condition of only those nodes that lie in toric regions of the broken manifold, leaving the general case for future work. The set of nodes that are degenerated correspond to a subset of edges, which we call split edges. The torus action on the region containing the node is used to deform the edge-matching condition to a split form; that is, the moduli space of split maps is a true product, rather than fiber product, of moduli spaces corresponding to the vertices.

The definition of split maps is similar to the definition of the Fulton-Sturmfels degeneration of the diagonal of a toric variety in [28], described in Section 11.5. Namely, one picks a generic vector and requires that the vector can be written uniquely as a sum of vectors corresponding to the components appearing in the split map. The case of a multiple cut of a tropical torus action, however, is substantially easier: In this case the matching condition takes place either in a projective line: in the case that the edge maps to a polytope of dimension one then the matching is of values while if the edge maps to a polytope of dimension zero then the matching is of derivatives. Split maps have fixed value or derivative on one side of the node and no constraint on the other side.

![Figure 3. Nine of the twenty-seven lines on the cubic surface, as broken maps, and their tropical graphs](image)
Example 1.1. (Twenty-seven lines on a cubic surface) Although we are mainly thinking of pseudoholomorphic disks with Lagrangian boundary condition, the same arguments produce formulas for Gromov-Witten invariants as well, as sums of products of vertices of tropical graphs. For example, consider the Gromov-Witten invariants of the cubic surface of the lowest degree. Any cubic surface is isomorphic to a general six-point blow up of the projective plane, and has a monotone symplectic form. The number of lines in the cubic surface was proved by Salmon and Cayley to equal twenty-seven, see Mumford [47, Section 8D], and is easily seen to be the Gromov-Witten invariant.

We explain how each of the twenty-seven lines corresponds to a tropical graph. By invariance under deformation, it suffices to compute the Gromov-Witten invariant of the toric surface with polytope shown in Figure 3. Multiple cutting may be used to cut the surface into Fano surfaces, so that the homology class of any lowest area sphere degenerates to the homology class of a union of spheres with self-intersection $-1$. We show that any non-self crossing path of boundary divisors consisting of one “long divisor” and an arbitrary number of “short divisors” is represented by a single rigid broken holomorphic sphere. We see that there is one configuration corresponding to each of the pictures in Figure 3. By symmetry twenty-seven such configurations in total. Under the deformation to split maps described above each of the broken sphere deforms to a split disk.

Example 1.2. (Twenty-one disks in the cubic surface) The Fukaya category of the cubic surface was computed by Sheridan [63, Appendix B] to have a summand corresponding to twenty-one as an eigenvalue of the first Chern class. Sheridan also conjectured the existence of a Lagrangian torus whose potential had twenty-one terms, which was confirmed by Pascaleff-Tonkonog [48] using mutations. We confirm this by tropical methods as follows. these computations can also be addressed using the method of Chan-Lau-Cho-Tseng [18]. In the case of no cuts, there are three disks intersecting the long divisors, and eighteen disks intersecting the short divisors. See Example 1.1 and Figure 3.

We use the split limit to obtain a model of the Fukaya category in which the structure maps are products over vertices of tropical graphs. To achieve regularity we assume the Lagrangian $L \subset X$ and the symplectic form $\omega \in \Omega^2(X)$ are compact, connected, and rational in the sense that $X$ admits a line bundle whose curvature is the symplectic form and some tensor power is flat over $L$. This allows us to use Cieliebak-Mohnke perturbations [16] to regularize the moduli spaces of broken disks. These perturbations are collections $P = \{P_\Gamma = (J_\Gamma, F_\Gamma)\}$ of domain-dependent perturbations $P_\Gamma$ for each type $\Gamma$ of disk, each consisting of a domain-dependent almost complex structure $J_\Gamma$ and Morse function $F_\Gamma$ on the Lagrangian $L$. For any generic perturbation datum $P$, denote by $\tilde{M}_\Gamma(L, P)$ the regularized moduli space of broken maps of type $\Gamma$. Here $CF(L)$ is the Fukaya algebra of the Lagrangian $L$ in the Morse model constructed in, for example, Seidel [62] in the exact case and Charest-Woodward [13] in the rational case. Using these perturbations one may construct the Fukaya algebra $CF(L)$ of the Lagrangian $L$ in the Morse model constructed in, for example, Seidel [62] in the exact case and Charest-Woodward [13] in the rational
case. The underlying vector space generated by critical points of the Morse function, with structure maps defined by counting holomorphic treed disks. In the tropical case, its quotient by the tropical symmetry group is called the \textit{reduced moduli space} and is denoted by

$$\tilde{\mathcal{M}}_\Gamma(L, P) := \mathcal{M}_\Gamma(L, P) / T_{\text{trop}}(\Gamma).$$

The dimensions of the moduli space $\mathcal{M}_\Gamma(L, P)$ resp. the reduced moduli space $\tilde{\mathcal{M}}_\Gamma(L, P)$ are given by a certain \textit{index} resp. \textit{reduced index}, see (5.15). The action of the tropical symmetry group does not have infinitesimal stabilizers, and so, for a zero or one-dimensional moduli space the symmetry group is necessarily trivial. For the case of a single cut, broken maps of index zero or one do not have any components in a neck piece. This fact is a special feature limited to the case of a single cut; in a multiply cut manifold, there are maps with index zero that have components mapping to the neck, see Figure 9 for an example. Counts of rigid elements of the moduli space of broken disks with boundary in $L$ leads to a broken Fukaya algebra denoted $\text{CF}_{\text{brok}}(L)$ with $A_\infty$ structure maps

$$m^\text{brok}_d : \text{CF}_{\text{brok}}(L)^{\otimes d} \to \text{CF}_{\text{brok}}(L), \quad d \geq 0,$$

see Section 9. The single cut version of this $A_\infty$ algebra has been constructed by Charest-Woodward [13]. Counts of orbits of split maps satisfy an $A_\infty$-relation and the resulting $A_\infty$-algebra $\text{CF}_{\text{trop}}(L)$ is called the \textit{tropical Fukaya algebra}. Our main result is the following, proved in Section 10 and Section 9:

\textbf{Theorem 1.3.} \textit{For a rational Lagrangian submanifold $L \subset X$ as above, the broken Fukaya algebra $\text{CF}_{\text{brok}}(L)$ and tropical Fukaya algebra $\text{CF}_{\text{trop}}(L)$ are homotopy invariant to the unbroken Fukaya algebra $\text{CF}(L)$.}

The ingredients of this result for broken maps, proved in Proposition 9.17 below, are a convergence result and its converse, which is a gluing result. The convergence result is a generalization of sft compactness ([9], [17]) for a single cut and is proved in Section 7. The statement is that given a sequence of maps $u_\nu : C \to X$ holomorphic
with respect to almost structures $J_\nu$ that are stretched along multiple necks, there is a subsequence of $u_\nu$ that converges to a broken map. The limit is unique up to the action of the tropical symmetry group. The gluing result in Section 8 is proved only for broken maps of index zero: an index zero regular broken map can be glued to produce a family of $J_\nu$-holomorphic maps $u_\nu : C \to X$. We remark that the convergence and gluing preserve indices of maps, see Proposition 5.18. Therefore to obtain the homotopy equivalence, the broken Fukaya algebra must be defined using counts of index zero broken maps, and not that of broken maps whose reduced index is zero. The proof of the Theorem for split maps is completed in Proposition 11.41 below. We conjecture that the matching condition can be degenerated at all nodes, including the nodes which map to divisors that are not toric. In that case the composition maps in the tropical Fukaya algebra can be expressed as a sum of products, where the sum is over all rigid tropical graphs, and the product is over all vertices in the graph. A limited version of such a factorization formula is given in Remark 11.33. We emphasize that the bijection involves broken maps, and not tropical symmetry orbits of broken maps. The distinction is significant even for rigid maps because rigid broken maps may have a finite non-trivial tropical symmetry group, see Example 3.28. This phenomenon has also been observed by Abramovich-Chen-Gross-Siebert [2] and Tehrani [24].

Using the tropical Fukaya algebra, we prove a basic fact about unobstructedness of Lagrangian tori: We call a Lagrangian $L \subset X$ a tropical torus if there exists a polytope $P \in P$ of maximal dimension such that $X_P$ is a toric manifold and $L$ is the pre-image of a Lagrangian torus orbit $L \subset X_P$. A tropical torus in our sense is a somewhat stronger requirement than the notion of moment fiber in a toric degeneration used in mirror symmetry, where “discriminant” singularities are allowed, see for example Gross [34].

Homotopy equivalence with the tropical Fukaya algebra is used to prove that tropical tori are weakly unobstructed in the following sense. Recall that the projective Maurer-Cartan equation for $b \in CF(L)$ with positive $q$-valuation is

$$
\exists W(b) \in \Lambda, \quad m_0(1) + m_1(b) + m_2(b, b) + \ldots \in W(b)1_L
$$

where $1_L \in CF(L)$ is the strict unit, see Fukaya-Oh-Ohta-Ono [29]. The space of odd solutions to the equation (1.4) is denoted $MC(L)$. The function $W : MC(L) \to \Lambda$ is called the potential of the curved $A_\infty$ algebra $CF(L)$.

**Corollary 1.4.** If $L \subset X_P$ is a tropical torus, then there exists a solution $b \in MC(L)$ to the projective Maurer-Cartan equation.

Corollary 1.4 generalizes the result in the case that $L$ is an toric moment fiber proved by Fukaya-Oh-Ohta-Ono [30].

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2. Broken symplectic manifolds

In this section we review the multiple cut construction and the associated degenerations of almost complex structures. Our approach is much less general than, for example, Parker [50], but we wish to be completely explicit.

2.1. Symplectic cuts and neck-stretching. We review the symplectic cut construction of Lerman [40] and Gompf [33]. The construction uses Hamiltonian circle actions on symplectic manifolds.

Definition 2.1. (Lerman’s symplectic cut construction)

(a) (Hamiltonian circle actions) Let \((X, \omega)\) be a compact symplectic manifold. Let 
\[ S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \]
denote the circle group; we identify its Lie algebra \(\text{Lie}(S^1)\) with \(\mathbb{R}\) by division by \(i\). A Hamiltonian action of the circle group \(S^1\) on \(X\) is an action \(S^1 \times X \to X\) generated by the Hamiltonian flow of a moment map 
\[ \Phi : X \to \mathbb{R}, \quad \omega(\xi_X, \cdot) = -d\Phi \]
where the generating vector field of an element \(\xi \in \mathbb{R}\)
\[ \xi_X \in \text{Vect}(X), \quad \xi_X(x) = \frac{d}{dt}|_{t=0}\exp(it\xi)x. \]
In particular the affine line \(\mathbb{C}\) has symplectic form 
\[ \omega_{\mathbb{C}} = -\frac{i}{2}dz \land d\bar{z} \in \Omega^2(\mathbb{C}). \]
The Hamiltonian action of \(S^1\) is given by scalar multiplication and has moment map 
\[ \Phi_{\mathbb{C}} : \mathbb{C} \to \mathbb{R}, z \mapsto -\frac{|z|^2}{2}. \]

(b) (Global symplectic cuts) The product \(\hat{X} = X \times \mathbb{C}\) has product symplectic form \(\tilde{\omega} = \pi_X^*\omega_X + \pi_{\mathbb{C}}^*\omega_{\mathbb{C}}\). The diagonal action of \(S^1\) has moment map 
\[ \tilde{\Phi} : \hat{X} \to \mathbb{R}, \quad (x, z) \mapsto \Phi(x) - \frac{|z|^2}{2}. \]
The zero level set is the union 
\[ \tilde{\Phi}^{-1}(0) = (\Phi^{-1}(0) \times \{0\}) \sqcup \left\{ (x, z) \mid \Phi(x) = \frac{|z|^2}{2} > 0 \right\} \]
of two pieces where both \(\Phi\) and \(z\) are zero and the piece where both \(\Phi\) and \(z\) are non-zero. The action on \(z \neq 0\) has a natural slice given by \(z \in \mathbb{R}_{>0}\) so that 
\[ \{(x, z) | \Phi(x) = |z|^2/2 > 0\} \cong \Phi^{-1}(\mathbb{R}_{>0}). \]
The symplectic quotient \(\tilde{\Phi}^{-1}(0)/S^1\) is called the symplectic cut
\[ X^+ := \hat{X} / S^1 = (X / S^1) \cup \Phi^{-1}(\mathbb{R}_{>0}). \]
One has a similar construction of a cut space \( X_- \) which is the union of the symplectic quotient \( X/\mathbb{S}^1 \) and \( \Phi^{-1}(\mathbb{R}_{>0}) \). The symplectic manifolds \( X_-, X_+ \) both contain a copy of \( X/\mathbb{S}^1 \) given by embeddings

\[ i_- : X/\mathbb{S}^1 \to X_-, \quad i_+ : X/\mathbb{S}^1 \to X_+ \]

with opposite normal bundles \( N_- \to X_- \), so that \( N_-^{-1} \cong N_+ \).

(c) (Local symplectic cuts) Given an open subset \( U \subset X \) with a Hamiltonian \( \mathbb{S}^1 \)-action with moment map \( \Phi : U \to \mathbb{R} \), gluing together the cut \( U_+ \cup U_- \) with \( X - \Phi^{-1}(0) \) produces cut spaces \( X_+, X_- \) and

\[ X_0 := \Phi^{-1}(0)/\mathbb{S}^1. \]

**Example 2.2.** For example, if \( X \) is a genus \( g \) surface and \( U \cong S^1 \times (-\epsilon, \epsilon), \epsilon > 0 \) a tubular neighbourhood of a non-separating embedded circle then the local cut construction removes a circle from \( X \) and replaces it with two copies of the point \( U/\mathbb{S}^1 \), to produce a surface of genus \( g - 1 \).

**Example 2.3.** Let \( X \) be a cubic surface, that is, a blow-up of \( \mathbb{P}^2 \) at six general points. Consider the degeneration of \( X \) to the toric surface \( X_0 \) obtained by blowing up \( \mathbb{P}^2 \) six times at torus fixed points, so that the original fixed points have exceptional locus \( E_i, i = 1, 2, 3 \) the union of two projective lines, each with Chern number zero.

The family of anti-canonical embeddings gives a family of closed two-forms on \( X \) that are symplectic away from the union of these exceptional spheres, and gives a degeneration of \( X \) to \( X_0 \), where \( X_0 \) is the toric surface with three \( A_2 \) singularities obtained from the anticanonical embedding of \( X_0 \). Similar to the discussion in Harada-Kaveh [36] the symplectic connection on this family an identification \( X \to X_0 \) which restricts to a diffeomorphism on the complement of a union \( E_1 \cup E_2 \cup E_3 \) of Lagrangian spheres \( E_k \cong S^2 \cup S^2, k \in \{1, 2, 3\} \). In Figure 1 each of these spheres \( E_j, j = 1, 2, 3 \subset X_0 \) maps to a vertex. Compare Sheridan [63, Appendix B] and Chan-Lau-Cho-Tseng [18].

We recast the definition of a symplectic cut in terms of a tropical Hamiltonian circle action.

**Definition 2.4.** (Tropical Hamiltonian circle actions) A tropical Hamiltonian circle action is a triple \( (X, \Phi, c) \) where \( X \) is a compact symplectic manifold, \( \Phi : X \to \mathbb{R} \) is a smooth map and \( c \in \mathbb{R} \) is a regular value such that in a neighbourhood of \( \Phi^{-1}(c) \), \( \Phi \) is a moment map for an \( \mathbb{S}^1 \)-action.

**Definition 2.5.** (Cut spaces) The cut spaces for a tropical Hamiltonian circle action \((X, \Phi, c)\) are

(a) the symplectic manifolds \( X_+, X_-, X_c \) defined as

\[ X_\pm := \{ \pm \Phi \geq c \}/\sim, \quad X_c := \Phi^{-1}(c)/\mathbb{S}^1, \]

where in all instances \( \sim \) mods out the boundary \( \Phi^{-1}(c) \) by the \( S^1 \)-action:

\[ x \sim tx, \quad \forall x \in \Phi^{-1}(c), \quad t \in S^1; \]

(b) together with embeddings \( i_{\pm} : X_c \to X_\pm \), whose images are symplectic submanifolds.
2.2. Neck-stretching. In this section, we define a family of neck-stretched almost-complex structures which are tamed by the symplectic form, so that the family of almost complex manifolds degenerates into the union of cut spaces. The neck region for a tropical Hamiltonian circle action is a neighbourhood of the separating hypersurface. One can stretch the neck to obtain a family of almost complex structures which converge, roughly speaking, to the almost complex structure on the broken manifold.

Definition 2.6. (Neck-stretched manifold) Let \((X, \Phi, c)\) be a tropical Hamiltonian circle action. Let

\[
Z := \Phi^{-1}(c) \subset X
\]

be the separating hypersurface. Let \(\epsilon > 0\) be sufficiently small so that

\[
U := \Phi^{-1}([c - \epsilon, c + \epsilon])
\]

is a tubular neighbourhood of \(Z\) with an identification \(U \simeq Z \times [-\epsilon, \epsilon]\). We call \(U\) the neck region. Let

\[
X^\bullet = X - U,
\]

be the complement and let

\[
Z_\pm = Z \times \{\pm \epsilon\} \subset \partial X^\bullet
\]

be the copies of \(Z\) on the boundary of \(X^\bullet\). For any \(\nu > 0\) let

\[
X^\nu = X^\bullet \bigcup_{\nu \leq \nu} [-\nu, \nu] \times Z
\]

\[
Z_+ = \{-\nu\} \times Z
\]

\[
Z_- = \{\nu\} \times Z
\]

obtained by gluing together the ends \(Z_-, Z_+\) of \(X^\bullet\) using a neck \(Z \times [-\nu, \nu]\) of length \(2\nu\).

Definition 2.7. (Broken manifold) Suppose the manifold \(X\) has a tropical Hamiltonian circle action. The broken manifold corresponding to \(X\) is the disjoint union

\[
\mathfrak{X} := X_+ \cup X_- \cup Z(\mathbb{P}^1)
\]

where \(X_+, X_-\) are the cut spaces from (2.1). The third space

\[
Z(\mathbb{P}^1) := Z \times_{S^1} \mathbb{P}^1
\]

is a \(\mathbb{P}^1\)-bundle on \(X_0\), which can be viewed as the thickening of the cut space \(X_0\). The space \(Z(\mathbb{P}^1)\) has two copies of \(X_0\), called \(X_0^0, X_0^\infty\), as divisors corresponding to the \(S^1\)-fixed points \(0, \infty \in \mathbb{P}^1\).

The neck regions of the manifolds \(X^\nu\) are equipped with a cylindrical almost complex structure which we describe next. A cylindrical almost complex structure is defined on the product \(Z \times \mathbb{R}\), where \(Z \subset X\) is the separating hypersurface. The separating hypersurface is a \(S^1\)-principal bundle on \(X_0 := \Phi^{-1}(0)/S^1\). The manifold \(Z \times \mathbb{R}\) has a natural \(\mathbb{C}^\times\)-action given by

\[
s \exp(it)(s_0, z) = (s_0 + s, \exp(it)z).
\]
Definition 2.8. (Cylindrical almost complex structure) An almost complex structure $J \in \mathcal{J}(\mathbb{R} \times Z)$ is of cylindrical form if

(a) there exists an almost complex structure $J_{X_0}$ on $X_0$ such that the projection $\pi_{X_0} : \mathbb{R} \times Z \to X_0$ is almost complex:

$$D\pi_{X_0} J = J_{X_0} D\pi_{X_0},$$

(b) $J$ is invariant under the $\mathbb{C}^\times$-action in (2.4),

(c) and $J$ acts on the fiber directions as

$$J\partial_a = \partial_\theta$$

where $\partial_a \in \text{Vect}(\mathbb{R} \times Z)$ is the unit vector field in the vertical direction and $\partial_\theta \in \text{Vect}(\mathbb{R} \times Z)$ generates the $S^1$-action on $\mathbb{R} \times Z$.

The space of cylindrical almost complex structures is denoted $\mathcal{J}^{cyl}(\mathbb{R} \times Z)$.

To describe cylindrical structures on broken resp. neck-stretched manifolds, we require an identification of the neck region with a subset of a cylinder. The identification is chosen to be compatible with the symplectic form.

Definition 2.9. (Cylindrical symplectic structure) Let $(X, \Phi, 0)$ be a tropical Hamiltonian circle action with separating hypersurface $Z := \Phi^{-1}(0)$. Let $\pi : Z \to X_0$ be the projection onto the $S^1$-quotient. A cylindrical symplectic structure on $(X, \Phi, 0)$ is an $S^1$-equivariant symplectomorphic embedding

$$\phi : ([-\epsilon, \epsilon] \times Z, \pi^* \omega_{X_0} + d(t\alpha)) \to (X, \omega)$$

that maps $Z \times \{0\}$ identically to $Z$, and $\alpha \in \Omega^1(Z)$ is a connection one-form on $Z \to X$. The identification $\phi$ yields maps

$$\phi_\nu : [-\nu, \nu] \times Z \to X^\nu, \quad \phi_\pm : \mathbb{R} \pm \times Z \to X_\pm.$$ 

The map $\phi_\nu$ is uniquely determined, and $\phi_\pm$ is uniquely defined up to $\mathbb{R}$-translation on the domain.

Definition 2.10. Let $(X, \Phi, 0)$ be a tropical Hamiltonian circle action with a fixed cylindrical symplectic structure.

(a) (Cylindrical almost complex structure on $X^\nu$) An almost complex structure $J^\nu$ on $X^\nu$ is cylindrical if the image of $\phi_\nu$ in (2.6) has a cylindrical almost complex structure.

(b) (Cylindrical almost complex structure on $\mathfrak{X}$) An almost complex structure $\mathfrak{J}$ on $\mathfrak{X}$ is cylindrical if there is a cylindrical almost complex structure $J_0$ on $X_0$ such that

$$\mathfrak{J}|(Z(\mathbb{P}^1) \setminus (X^0_0 \cup X^\infty_0)) = J_0, \quad \mathfrak{J}| \text{im}(\phi_\pm) \equiv (\phi_\pm)_*, J_0,$$

where $\phi_\pm$ is the cylindrical structure map from (2.6).

(c) (Tameness) A cylindrical almost complex structure $\mathfrak{J}$ on $\mathfrak{X}$ is tame resp. compatible if $\mathfrak{J}| X_\pm \in \mathcal{J}(X_\pm)$ and $J_0 \in \mathcal{J}(X_0)$ are both tame resp. compatible.
Remark 2.11. (Consistent coordinates on cylindrical ends) A choice of cylindrical structure on \((X, \Phi)\) gives embeddings

\[ i_{\pm} : U_{\pm} \setminus X_0 \to \mathbb{R}_{\pm} \times \mathbb{Z} \]

where \( U_{\pm} \subset X_{\pm} \) is a neighborhood of the divisor \( X_0 = \Phi^{-1}(0)/S^1 \) in \( X_{\pm} \). There is a consistent choice of the pair \((i_{+}, i_{-})\) so that the neck-stretched manifold \( X^\nu \) can be recovered from \( X \) (as an almost complex manifold) as

\[ X^\nu = (X^0_+ \cup X^0_-)/\sim_{\nu}, \]

where \( X^0_{\pm} \) is the complement of the divisor \( X_0 \) in \( X_{\pm} \), and the equivalence relation \( \sim_{\nu} \) is

\[ X^0_{\nu} \ni x_+ \sim_{\nu} x_- \in X^0_\nu \iff x_{\pm} \in U_{\pm}, \quad i_{\pm}(x_{\pm}) = e^{i\nu}i_{\mp}(x_{\mp}). \]

The consistent choice of the pair \((i_{+}, i_{-})\) is uniquely determined up to post-composition by \( \mathbb{C}^\times\)-action on the target cylinder \( \mathbb{R} \times \mathbb{Z} \). Consistent coordinates do not play an important role in analyzing curves in a broken manifold with a single cut, but do play an important role in the multiple cut case (2.25).

2.3. Multiple cuts in a symplectic manifold. We define multiple cuts corresponding to polyhedral decompositions of a symplectic manifold. These polyhedral decompositions appeared in, for example, Meinrenken [43]. In addition to the treatment in Ionel and Parker’s work [37], [50], more recent treatment of the symplectic analog of normal crossing singularities appears in [26].

Definition 2.12. (Simple resp. Delzant polytopes) Let \( T \) be a torus with Lie algebra \( t \). Let \( t_{\mathbb{Z}} \subset t \) denote the coweight lattice of points that map to the identity under the exponential map, so that \( T \cong t/t_{\mathbb{Z}} \). A convex polytope \( P \) in \( t^\vee \) is described by a collection of linear inequalities determined by constants \( c_F \in \mathbb{R} \) and normal vectors \( \nu_F \in t \):

\[ P = \{ \lambda \in t^\vee \mid \langle \lambda, \nu_F \rangle \geq c_F, \quad \forall F \subset P \text{ facets} \}. \]

The polytope \( P \) is a simple resp. Delzant polytope if, for each vertex point \( \lambda \in P \), the normal primitive vectors \( \nu_F \in t_{\mathbb{Z}} \) to the facets \( F \subset P \) containing \( \lambda \) form a basis resp. lattice basis:

\[ \text{span}(\nu_F, F \ni \lambda) \cap t_{\mathbb{Z}} = \text{span}_{\mathbb{Z}}(\nu_F, F \ni \lambda). \]

By Delzant [21], there is a bijection between Delzant polytopes \( P \) and smooth compact symplectic manifolds \( V_P \) equipped with a completely integrable generically free torus action and moment map and polytope

\[ \Psi : V_P \to t^\vee, \quad \Psi(V_P) = P. \]

Definition 2.13. (Tropical Hamiltonian action) A tropical Hamiltonian action of a torus \( T \) with Lie algebra \( t \) is a triple \((X, \Phi, \mathcal{P})\) consisting of a

(a) compact symplectic manifold \( X \);
(b) a decomposition

\[ t^\vee = \bigcup_{P \in \mathcal{P}} \text{int}(P), \quad \mathcal{P} = \{ P \subset t^\vee \} \]
of $t^\vee$ into the disjoint union of the interiors of Delzant (or simple, if one allows orbifold singularities) polytopes $P \in \mathcal{P}$ such that

(i) if $P_0, P_1 \in \mathcal{P}$ have non-empty intersection, then $P_0 \cap P_1 \in \mathcal{P}$ and is a face of both $P_0$ and $P_1$,

(c) a tropical moment map compatible with $\mathcal{P}$

$$\Phi : X \to t^\vee$$

in the following sense. For any $P \in \mathcal{P}$, we denote by

$$t_P := \text{ann}(TP) \subset t$$

the annihilator of the tangent space of $P$ at any point $p \in P$, and by

$$T_P = \exp(t_P)$$

the torus whose Lie algebra is $t_P$. Let $t_{P,\mathbb{Z}}$ be the coweight lattice in $t_P$ so that

$$T_P \cong t_P/t_{P,\mathbb{Z}}.$$ 

For any $P \in \mathcal{P}$, there exists an open neighbourhood $U_P$ of $\Phi^{-1}(P)$ such that the composition

$$\pi_P \circ \Phi : U_P \to t_P^\vee$$

is a moment map for a free action of $T_P$ on $U_P$.

In the single breaking case, we defined the tropical manifold as the datum $(X, \Phi, c)$. This should be interpreted as the set of polytopes

$$\mathcal{P} = \{(-\infty, c], \{c\}, [c, \infty)\}.$$

**Definition 2.14.** (Cut space for a multiple cut) Given a tropical symplectic manifold $(X, \Phi, \mathcal{P})$ for every polytope $P \in \mathcal{P}$ define a symplectic manifold

$$X_P := \Phi^{-1}(P)/\sim,$$

where the equivalence $\sim$ mods out by the following torus actions:

$$x \sim tx, \quad \forall x \in \Phi^{-1}(Q^\circ), t \in T_Q$$

for all polytopes $Q \subseteq P$ contained in $\mathcal{P}$. Thus the open set $\Phi^{-1}(P^\circ) \subset \Phi^{-1}(P)$ is modded out by the $T_P$-action and the quotient

$$X_P^\circ := \Phi^{-1}(P^\circ)/T_P$$

is an open subset of $X_P$. The cut spaces for $(X, \Phi, \mathcal{P})$ are the manifolds $X_P$, $P \in \mathcal{P}$, together with the data of inclusions

$$X_Q \hookrightarrow X_P, \quad P \subset Q.$$ 

For a face $Q \subset P$ with $\text{codim}_P(Q) = 1$, the corresponding subset $X_Q$ is called a boundary divisor of $X_P$. The boundary divisors $X_Q$ intersect normally, and the intersections correspond to a manifold $X_Q$ for some $Q \in \mathcal{P}$. 
Example 2.15. The multiple cut in Figure 5 is made up of two simultaneous single cuts along hypersurfaces that intersect $\omega$-orthogonally. The set of polytopes is

$$\mathcal{P} = \{P_i, 0 \leq i \leq 3, P_{ij}, j = (i+1) \mod 4, P_\cap\}.$$  

The manifolds $X_{P_{(i+1)}}, X_{P_{(i-1)}}$ are boundary divisors of $X_{P_i}$.

Remark 2.16. A more general construction allows targets equipped with an integral affine structures as follows. Let $B$ be a compact differentiable manifold of dimension $n$.

(Tropical manifolds) An integral affine structure on $B$ (see for example Gross [35]) is a torsion-free flat connection on the tangent bundle $TB$ with holonomy contained in $SL(n, \mathbb{Z})$. A lattice system is a system of lattices $T_\mathbb{Z}B \subset TB$ invariant under parallel transport. The quotients

$$TB/T_\mathbb{Z}B, \quad T^\vee B/T_\mathbb{Z}^\vee B$$

are the torus resp. dual torus bundles. A polyhedral structure on $B$ is a decomposition of $B$ into Delzant polytopes

$$B = \bigcup_{P \in \mathcal{P}} P$$

so that $B$ is the disjoint union of the relative interiors $P^\circ, P \in \mathcal{P}$ and each $TP^\circ$ is a flat sub-bundle of $TP$. The polyhedral structure is Delzant if at each point $b \in P$, the local system of primitive normal vectors $v_1, \ldots, v_k \in T_\mathbb{Z}^\vee B$ to faces of $P$ forms an integral basis for its span $\text{span}(v_1, \ldots, v_k)$. The pair $(B, \mathcal{P})$ is called a tropical manifold.

(Tropical moment maps) Given such a structure denote by $t_P \subset T^\vee B$ the annihilator of $TP$ at any point $\lambda \in P^\circ$, independent up to isomorphism, and $T_P = t_P/(t_P \cap T_\mathbb{Z}^\vee B)$ the corresponding torus. so that $\dim(T_P) = \text{codim}(P)$. Extend the sub-bundle $TP \subset TB$ to an open neighbourhood $U_P$ of $P$ by parallel transport. For a sufficiently small neighbourhood $U_P$ is foliated by submanifolds obtained by translation of $P$ and we denote by $\pi_P : U_P \to t_P^\vee$ the natural quotient mapping $P$ to 0. A moment map to the tropical manifold $B$ is a continuous, not necessarily smooth, map $\Phi : X \to B$ such that for any $P \in \mathcal{P}$ there exists an neighbourhood $U_P$ such that the composition

$$\pi_P \circ \Phi : U_P \to t_P^\vee$$
is smooth and is the moment map for a free $T_P$-action in a neighbourhood of $\Phi^{-1}(P)$. Given these data one may form the cut space $X_P = \Phi^{-1}(P)/\sim$ for any $P \in \mathcal{P}$ as above.

2.4. Multiple neck-stretching. Starting from a symplectic manifold with a tropical Hamiltonian action, we describe a family of almost complex structures corresponding to neck-stretching. The construction of neck-stretched manifolds requires a choice of gluing datum, for which we introduce some definitions.

**Definition 2.17.** (a) (Cone at faces of polytopes) Let $P$ be a Delzant polytope. For a face $Q \subset P$, the cone of $P$ at $Q$ is

\begin{equation}
\text{Cone}_Q(P) := \mathbb{R}_{\geq 0}(P - \lambda)
\end{equation}

where $\lambda$ is an arbitrary point in the interior of $Q$. The normal cone of $P$ at $Q$ is

\begin{equation}
\text{Ncone}_Q(P) := \text{Cone}_Q(P)/T_\lambda Q
\end{equation}

The definitions are independent of the choice of $\lambda$.

(b) (Normal fan) Let $P = \{P \subset t^\vee\}$ be a set of Delzant polytopes as in Definition 2.13 of a tropical Hamiltonian action. For a polytope $P \in \mathcal{P}$ the normal fan is

\begin{equation}
C^\perp(P) := \{\text{Ncone}_P(P_0) : P_0 \supseteq P\}.
\end{equation}

(c) (Fan of a convex polytope) Let $P \subset t^\vee$ be a convex polytope. The dual cone at a face $Q \subseteq P$ is

\begin{equation}
\text{Cone}_Q(P)^\vee := \{v \in t : \langle v, \mu \rangle \geq 0 \quad \forall \mu \in \text{Cone}_Q(P)\}
\end{equation}

The fan of $P$ is

\begin{equation}
\mathcal{C}(P) := \{\text{Cone}_Q(P)^\vee : Q \subseteq P\}.
\end{equation}

**Definition 2.18.** A gluing datum for a tropical action $(X, \Phi, \mathcal{P})$ is

(a) a collection of polytopes

\begin{equation}
P^\vee \subset t^\vee_P, \quad P \in \mathcal{P}
\end{equation}

such that

(i) for each $P$, there is an isomorphism

\begin{equation}
\text{C}(P^\vee) \xrightarrow{\phi_P} C^\perp(P),
\end{equation}

consisting of a bijection of cones and a linear isomorphism $\phi_P$ for each cone $P \in \text{C}(P^\vee)$ that restricts to $\phi_Q$ on faces $Q \subset P$.

(ii) the polytopes $\{P^\vee\}_{P \in \mathcal{P}}$ are compatible in the sense that if $Q$ is a face of $P$, then $P^\vee$ is the projection of a face of $Q^\vee$ onto $t^\vee_P$;

(b) and embeddings

\begin{equation}
i_P : P \times P^\vee \subset t^\vee, \quad P \in \mathcal{P}
\end{equation}

for which

(i) $P \times \{0\}$ is mapped to $P$, and so $P \times P^\vee$ is a tubular neighbourhood of $P$ in $t^\vee$. 

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...
(ii) the action of $T_P$ on $\Phi^{-1}(P \times P')$ is free, and
(iii) projection to the $P'$ factor is the moment map for the $T_P$-action, so that

$$\text{(2.13)} \quad \text{im}(\Phi) = \bigcup_{P \in \mathcal{P}} \Phi^{-1}(P \times P') / \sim,$$

where $P^\bullet$ is the complement of neighborhoods of faces defined as

$$\text{(2.14)} \quad P^\bullet := P \setminus \left( \bigcup_{Q \subset P} i_P(Q \times Q') \right),$$

and the equivalence relation $\sim$ identifies, for any pair $Q \subseteq P$, the boundary component $Q^\bullet \times P'$ in $Q^\bullet \times Q'$ and $P^\bullet \times P'$.

**Figure 6.** Stretching

*Example 2.19.* In Figure 6 the dual complex indeed satisfies the fan isomorphism (2.11) as follows. For any top-dimensional polytope, say $P_4 \in \mathcal{P}$, both $\mathcal{C}(P_4')$ and $\mathcal{C}(P_4^\vee)$ only consist of the trivial cone \{0\}. For a one-dimensional polytope, say $P_1$, both $\mathcal{C}(P_1)$ and $\mathcal{C}(P_1^\vee)$ are isomorphic to a complete one-dimensional fan \{R \geq 0, \{0\}, -R \geq 0\}. Finally for $P_0$, both $\mathcal{C}(P_0)$ and $\mathcal{C}(P_0^\vee)$ are isomorphic to a complete two-dimensional fan consisting of three top-dimensional cones.

Using the gluing datum we define neck-stretched manifolds. They are constructed by ‘stretching’ the region $\Phi^{-1}(P \times P')$ in the $P'$-direction, see Figure 6. The stretched pieces fit together because of the relation (2.13). In the limit neck-stretched manifolds degenerate into a broken manifold.

**Definition 2.20.** (Neck-stretched manifolds) For any $\nu \in \mathbb{R}_{>0}$, define a neck-stretched manifold as

$$\text{(2.15)} \quad X^\nu := \left( \bigcup_{P \in \mathcal{P}} \Phi^{-1}(P^\bullet) \times \nu P'^\vee \right) / \sim,$$

where the identifications are along the boundaries of the pieces as in (2.13).
Remark 2.21. In the above definition of neck-stretched manifolds, the identification of the boundaries, and hence the equivalence map \( \sim \), is well-defined only up to smooth homotopy equivalence. Indeed the boundary identifications are induced by choosing for each \( P \) a diffeomorphism \( \Phi^{-1}(P \times P^\vee) \to \Phi^{-1}(P) \times P^\vee \) that is equal to \( \text{Id} \times \{0\} \) on \( \Phi^{-1}(P \times \{0\}) \). Therefore we have defined \( X^\nu \) up to diffeomorphism equivalence. Later, in order to define almost complex structures on neck-stretched manifolds, the identifications are specified precisely via a choice of a cylindrical structure in the neck regions. See Definition 2.27.

Definition 2.22. (Broken manifold) The broken manifold corresponding to the tropical manifold \((X, \Phi, P)\) is a disjoint union

\[
X := \bigsqcup_{P \in P} X_P,
\]

where \( X_P \) is the compactification of the space

\[
X_P^\circ := X_{P \times P^\vee}^\circ := \Phi^{-1}(P^\circ \times P^\circ) \setminus \Phi^{-1}(Q \times \{0\})
\]

by adding divisors at the non-compact ends, and is defined as

\[
(2.16) \quad X_P := X_{P \times P^\vee} := \Phi^{-1}(P \times P^\vee)/\sim.
\]

Here \( \sim \) mods out boundary components

\[
\Phi^{-1}(Q \times P^\vee), Q \subset P, \quad \text{resp.} \quad \Phi^{-1}(P \times P_0^\vee), P \subset P_0
\]

by the torus action of \( T_Q/T_P \) resp. \( T_P/T_{P_0} \).

Definition 2.23. (Boundary divisors in a broken manifold) Let \( Q \) be a facet of the product polytope \( P \times P^\vee \). A boundary divisor \( D_Q \) of \( X_{P \times P^\vee} \) is the space

\[
D_Q = \Phi^{-1}(Q)/\sim,
\]

where the equivalence relation \( \sim \) is same as the one in the definition of \( X_P \) in (2.16). Thus \( X_P^\circ \) is the complement of all the boundary divisors \( X_Q, Q \subset P \) of \( X_P \). The boundary divisor is horizontal resp. vertical if the facet \( Q \subset P \times P^\vee \) is the inverse image of a facet \( Q' \) of \( P \) resp. \( Q'' \) of \( P^\vee \) under the projection \( P \times P^\vee \) to \( P \) resp. \( P^\vee \).

Remark 2.24. The components of the broken manifold are fibrations over cut spaces, and therefore can be regarded as thickenings of the cut spaces. The manifold \( X_P = X_{P \times P^\vee} \) fibers as

\[
(2.17) \quad V_{P^\vee} \to X_{P \times P^\vee} \xrightarrow{\pi_P} X_P,
\]

where \( V_{P^\vee} \) is a \( P \)-toric manifold with moment polytope \( P^\vee \).

Example 2.25. For the multiple cut in Figure 5 the dual complex is a rectangle and the broken manifold \( X \) is as in Figure 7. Relative submanifolds \( X_{P_{ij}} \) and \( X_{P_{ij}} \) are thickened into toric fibrations \( X_{P_{ij}} \) and \( X_{P_{ij}} \)

\[
\mathbb{P}^1 \to X_{P_{ij}} \to X_{P_{ij}} \quad \text{and} \quad (\mathbb{P}^1)^2 \to X_{P_{ij}} \to X_{P_{ij}}
\]

in the broken manifold.
2.5. **Cylindrical almost complex structures.** The broken manifold and the family of neck-stretched manifolds are equipped with cylindrical almost complex structures which we describe next.

**Definition 2.26.** (Cylindrical almost complex structure)

(a) (P-cylinder) For a polytope $P \in \mathcal{P}$ a P-cylinder is the space

$$Z_{P,\mathbb{C}} := Z_P \times t_P'$$

where $Z_P \to X_P$ is a $T_P$-bundle defined as

$$Z_P := \Phi^{-1}(P)/\sim$$

where $\sim$ mods out $\Phi^{-1}(Q)$, $Q \subset P$ by the action of $T_Q/T_P$ (identified to a sub-torus of $T_Q$ via the pairing (2.24)). The space

$$Z_P' := \Phi^{-1}(P') \text{ resp. } Z_{P,\mathbb{C}}' := \Phi^{-1}(P') \times t_P'$$

is a complement of a codimension two subset of $Z_P$ resp. $Z_{P,\mathbb{C}}$, and is a bundle over the open subset $X_P' \subset X_P$.

(b) (P-cylindrical symmetries) The torus $T_{P,\mathbb{C}}$ acts on $Z_{P,\mathbb{C}}$ by

$$te^{is}(z, s_0) = (tz, s_0 + s), \quad t \in T_P, s \in t_P$$

using the identification $t_P \simeq t_P'$ arising from an inner product

$$t_P \times t_P \to \mathbb{R}.$$ (2.19)

(c) An almost complex structure $J \in \mathcal{J}(Z_{P,\mathbb{C}})$ is P-cylindrical iff

(i) there exists an almost complex structure $J_{X_P}$ on $X_P$ such that the projection

$$\pi_P : Z_{P,\mathbb{C}} \to X_P$$

is almost complex:

$$D\pi_P J = J_{X_P} D\pi_P,$$

Figure 7. Dual polytope and broken manifold for the multiple cut in Figure 5.
(ii) there exists a connection one-form $\alpha_P \in \Omega^1(Z_P, t_P)$ on the $T_P$-bundle $Z_P \to X_P$ such that the horizontal sub-bundle

$$H_P := \ker(\alpha_P) \subset TZ_P \subset TZ_{P,\mathbb{C}}$$

is $J$-invariant.

(iii) and the almost complex structure $J$ is fixed on the fiber of $\pi_P$ by the equation

$$(2.20) \quad J\xi_X = \xi^\vee, \quad \forall \xi \in t,$$

where $\xi_X \in \text{Vect}(Z_{P,\mathbb{C}})$ is the vector field generated by $\xi$ and $\xi \mapsto \xi^\vee$ is the identification $t_P \to t_P^\vee$ in (2.19).

As a result, $J$ is invariant under the $T_P, C$-action on $Z_{P,\mathbb{C}}$. Denote by

$$\mathcal{J}^{\text{cyl}}(Z_{P,\mathbb{C}}) = \{ J \in \mathcal{J}(Z_{P,\mathbb{C}}) \mid (i) - (iii) \}$$

the space of $P$-cylindrical almost complex structures on $Z_{P,\mathbb{C}}$.

To define cylindrical almost complex structures on neck-stretched manifolds, the subset of the manifold $X$ with the action of the torus $T_P$ has to be identified with a $P$-cylinder. This identification is made via a symplectic cylindrical structure defined below.

**Definition 2.27.** (Symplectic cylindrical structure on tropical manifolds) Let $(X, \mathcal{P}, \Phi)$ be a symplectic manifold with a tropical Hamiltonian action. A *symplectic cylindrical structure* $\phi = (\phi_P)_{P \in \mathcal{P}}$ is a collection of maps consisting of a $T_P$-equivariant symplectomorphism

$$(2.21) \quad \Phi^{-1}(P \times P^\vee) \xrightarrow{\phi_P} (\Phi^{-1}(P) \times P^\vee, \varpi), \quad \varpi := (\omega_X|_{\Phi^{-1}(P)}) + d\langle \alpha_P, \pi_P \rangle$$

for all polytopes $P$, where

(a) $\alpha_P \in \Omega^1(\Phi^{-1}(P), t_P)$ is a $T_P$-connection one-form

(b) the second component of $\phi_P$ is the $T_P$-moment map

$$\pi_P \circ \phi_P = \Phi,$$

and

(c) the maps $\phi$ satisfy the (Patching) condition described below.

Composing the identification $\phi_P$ with a quotient map $\Phi^{-1}(P) \to \Phi^{-1}(P)/T_P$, we obtain a projection

$$\pi_P : \Phi^{-1}(P \times P^\vee) \to \Phi^{-1}(P)/T_P$$

with $T_P$-invariant fibers. Further if the fibers are equipped with the standard almost complex structure $J_P$ the $T_P$-action extends to a partial $T_{P,\mathbb{C}}$-action on the fibers.

(Patching) For any pair $Q \subset P$, in the overlap region $\overline{Q} \cap \overline{P}$ the $T_{P,\mathbb{C}}$-action induced by $\phi_P$ is the restriction of the $T_{Q,\mathbb{C}}$-action induced by $\phi_Q$.

This ends the Definition.

**Remark 2.28.** The cylindrical structure maps $\{\phi_P\}_{P \in \mathcal{P}}$ in (2.21) are fixed throughout the paper.
Remark 2.29. In the above definition the (Patching) condition automatically places the following consistency condition on connection one-forms \((\alpha_P)_{P \in \mathcal{P}}\). For a pair of polytopes \(Q \subset P\) the connection \(\alpha_P\) on \(\Phi^{-1}(P)\) is determined by \(\alpha_Q\) in a neighborhood of \(\Phi^{-1}(Q)\) in \(\Phi^{-1}(P)\) via the consistency condition.

Definition 2.30. (Consistency for connection one-forms) Let \((\phi_P)_{P \in \mathcal{P}}\) be a collection of \(T_P\)-equivariant diffeomorphisms

\[ \Phi^{-1}(P \times P^\vee) \xrightarrow{\phi_P} \Phi^{-1}(P) \times P^\vee. \]

A collection of connection one-forms \((\alpha_P)_{P \in \mathcal{P}}\) is consistent if for a pair of polytopes \(Q \subset P\), and a point \(x \in \Phi^{-1}(P)\) in a neighborhood of \(\Phi^{-1}(Q)\) in \(\Phi^{-1}(P)\) via the consistency condition.

(2.22)

\[ \ker \alpha_P(x) := J_Q(t_P x)^\omega \cap T_x \Phi^{-1}(P) \simeq (t_P x + J_Q t_P x)^\omega, \]

where \(J_Q\) is the standard almost complex structure defined on fibers of \(\pi_Q\) (see (2.20)).

For the moment, we assume the existence of such structures and use them to define neck-stretching. The proof of existence of symplectic cylindrical structures on tropical manifolds is deferred to Lemma 2.39 at the end of this section. Recall the definition of a neck-stretched manifold \(X^\nu\), \(\nu \geq 1\):

(2.23)

\[ X^\nu := \left( \bigcup_{P \in \mathcal{P}} \Phi^{-1}(P^\bullet) \times \nu P^\vee \right) / \sim, \]

where \(P^\bullet := P \setminus \cup_{Q \subset P} (Q \times Q^\vee)\). The identifications on the boundaries are induced by polytope identifications from (2.13), and the maps \(\phi_P\) which are part of the symplectic cylindrical structure from 2.27. For any \(P \in \mathcal{P}\), the subset \(\Phi^{-1}(P^\bullet) \times \nu P^\vee\) is a fibration over \(\Phi^{-1}(P^\bullet)/T_P\), and the base space has a symplectic form.

Remark 2.31. (The inner product on \(t\)) We recall that the definition of \(P\)-cylindrical almost complex structures required a choice of an inner product on \(t_P\), see (2.19). To define cylindrical almost complex structures on neck-stretched manifolds we fix an inner product

(2.24)

\[ t \times t \to \mathbb{R} \]

such that any polytope \(P \in \mathcal{P}\) is orthogonal to the dual polytope \(P^\vee\). The inner product is fixed for the proof of the homotopy equivalence of the broken and unbroken Fukaya algebras (Theorem 1.3). Once the equivalence is proved we allow the inner product to vary across components of the broken manifold.

Definition 2.32. (Cylindrical almost complex structures on neck-stretched manifolds) Let \((X, \mathcal{P})\) be a tropical Hamiltonian action. Let \(X\) be equipped with a symplectic cylindrical structure as in 2.27, and let \(X^\nu\) be a sequence of neck-stretched manifolds defined as in (2.23).

(a) An almost complex structure \(J^\nu\) on the neck-stretched manifold \(X^\nu\) is cylindrical if \(J^\nu\) is \(P\)-cylindrical in the sense of 2.26 in the subset

\[ \Phi^{-1}(P^\bullet) \times \nu P^\vee, 0 \subset X^\nu. \]
Remark 2.33. Tameness resp. compatibility of $J^\nu$ implies that on the $P$-cylindrical set $\Phi^{-1}(P^\bullet) \times \nu P^\nu$, $J^\nu$ projects to a tame resp. compatible almost complex structure on $\Phi^{-1}(P^\bullet)/T_P$. 

Remark 2.34. In the space of cylindrical tamed almost complex structures the cylindrical coordinate maps, taking values in $\Phi^{-1}(P^\bullet) \times \nu P^\nu$, are held fixed, but the connection one-forms $\alpha_{P,J}$, $P \in \mathcal{P}$ underlying a cylindrical almost complex structure $J$ are allowed to vary. The connection one-forms $\alpha_{P,J} \in \Omega^1(\Phi^{-1}(P), t_P)$ satisfy the consistency condition (2.22).

The complement of boundary divisors in a broken manifold is the degenerate limit of neck-stretched almost complex manifolds. A cylindrical almost complex structure on a broken manifold is the limit of a sequence of cylindrical almost complex structures on neck-stretched almost complex structures on neck-stretched manifolds, and can alternately be defined as follows.

Definition 2.35. (Cylindrical almost complex structures on broken manifolds) Let $\mathcal{X}_\mathcal{P}$ be a broken manifold.

(a) An almost complex structure $J = (J_{P \times P^\nu})_{P \in \mathcal{P}}$ on the broken manifold $\mathcal{X} = (X_{P \times P^\nu})_{P \in \mathcal{P}}$ is cylindrical if there is a set of consistent connection one-forms $\alpha = (\alpha_P)_{P \in \mathcal{P}}$ (as in Definition 2.30) that satisfies the following:

(i) (Fiber) Each $J_{P \times P^\nu}$ is $P$-cylindrical in the fibers of the map $X_{P \times P^\nu} \rightarrow X_P$, and the $T_P$-connection associated to $J_{P \times P^\nu}$ is equal to $\alpha_P$.

(ii) (Base) For any pair of polytopes $Q \subseteq P$, the base almost complex structure $J_P$ is $Q$-cylindrical in a neighbourhood $U_{X_Q} \subseteq X_P$ of $X_Q$, and the $T_Q$-connection one-form associated to $J_P$ on $Q$-cylindrical end is equal to $\alpha_Q$.

(iii) (Consistency) The base almost complex structures are consistent:

$$\forall P, P_0 \in \mathcal{P}, \ (P \subset P_0) \Rightarrow (J_{P_0}|_{X_P} = J_P).$$

(b) (Tameness) A cylindrical almost complex structure $J = (J_{P \times P^\nu})_{P \in \mathcal{P}}$ on $\mathcal{X}$ is tame resp. compatible if $J$ is the degenerate limit of tame resp. compatible almost complex structures on neck-stretched manifolds $X_{P^\nu}$. The space of tamed cylindrical almost complex structures on $\mathcal{X}$ is denoted by

$$J^cyl_\tau(\mathcal{X}) = \{J = (J_{P \times P^\nu})_{P \in \mathcal{P}}\}.$$
In the space \( \mathcal{J}^{\text{cyl}}(X) \), the connection one-forms associated to the almost complex structure are allowed to vary.

**Remark 2.36.** (Cylindrical ends on broken manifolds) Since the broken manifold is a ‘degenerate limit’ of a sequence of neck-stretched almost complex manifolds, the complement of the boundary divisors in \( X \) may be viewed as a manifold with cylindrical ends. There is a natural choice of cylindrical coordinates on these ends which we fix for the rest of the paper. For any pair of polytopes \( Q \subset P \), \( \text{codim}(P) = 0 \), there is an embedding of the neighborhood \( U_{X_P}X_Q \subset X_P \) of \( X_Q \) into the \( Q \)-cylinder

\[
(2.25) \quad i^P_Q : U_{X_P}X_Q \cap X^\nu_P \to Z_Q^\nu \subset Z_Q^\nu \times \{0\}.
\]

The map \( i^P_Q \) arises naturally from the identification of the \( Q \)-cylindrical subset of \( X^\nu \) with \( \Phi^{-1}(Q) \times \nu Q^\nu \) and is well-defined up to \( T_Q^\nu \)-translations. The \( T_Q^\nu \)-freedom is fixed by requiring that \( i^P_Q \) maps \( \Phi^{-1}(Q) \times \nu Q^\nu \) to \( Z_Q^\nu \times \{0\} \). (Recall that \( P^\nu \) is a point.) With this choice,

\[
(2.26) \quad \text{image}(i^P_Q) = Z_Q^\nu \times \text{Cone}_{P^\nu} Q^\nu.
\]

The domain of the map \( i^P_Q \) is called the \( Q \)-cylindrical end of the complement \( X^\nu_P \) of \( X_P \). The map \( i^P_Q \) is viewed as a coordinate on a \( Q \)-cylindrical end, taking values in \( Z_Q^\nu \subset Z_Q^\nu \times \{0\} \).

Remark 2.37. (Symplectic versus almost complex broken manifolds) We have defined cut spaces \( X^\omega_P \subset P \) both as a symplectic and an almost complex manifold. The two spaces are evidently diffeomorphic but there is no natural diffeomorphism between. For top-dimensional polytopes \( P_0, P_1 \), and \( Q := P_0 \cap P_1 \),

\[
(2.27) \quad X^\nu_{P_0} \ni x_0 \sim x_1 \in X^\nu_{P_1} \iff x_0 \in U_{P_0}Q, \; x_1 \in U_{P_1}Q, \; i^P_Q(x_0) = e^{\pi \nu P^\nu_0 - \nu P^\nu_1} i^P_Q(x_1).
\]

Here \( P^\nu_0 - P^\nu_0 \in t^\nu_Q \) and \( \pi \) is the identification \( t^\nu_Q \simeq t^\nu_Q \) from (2.24). This ends the Remark.

Remark 2.38. (Symplectic versus almost complex broken manifolds) We have defined cut spaces \( X^\omega_P, P \in \mathcal{P} \) both as a symplectic and an almost complex manifold. The two spaces are evidently diffeomorphic but there is no natural diffeomorphism between. From now on we refer to the symplectic cut space \( X^\omega_P \) to distinguish it from the almost complex space \( X_P \). In the complement of the cylindrical ends \( \cup_{Q \subset P} U_{X_P}X_Q \), there is a natural embedding

\[
i_P : X_P \setminus (\cup_{Q \subset P} U_{X_P}X_Q) \to X^\omega_P.
\]

On the cylindrical ends, there is no fixed choice of embedding. Thus \( i^\omega_P \omega_P \) is a symplectic form on \( X_P \) in the complement of \( \cup_{Q \subset P} U_{X_P}X_Q \).
The cylindrical coordinates give rise to the following projection to a cone

$$\pi_{B^\nu} : X_P^\circ \to \text{Cone}_{P^\nu} B^\nu, \ P \in \mathcal{P}$$

on components of a broken manifold $X_P^\circ$. For top-dimensional $P \in \mathcal{P}$, we define for any $Q \subseteq P$

$$\pi_{B^\nu} := \pi_{Q^\nu} \circ i_Q^P, \quad \text{on } U_P \cup \cup_{R \subseteq Q} U_P^R.$$  

The complement of cylindrical ends $X_P^\circ \cup \cup_{Q \subseteq P} U_P Q$ is mapped to the vertex $P^\nu$ of the cone. For lower dimensional $Q$, $\pi_{B^\nu} | X_Q^\circ$ is defined so that it is some $t_Q^\nu$-translate of $\pi_Q^\nu$ on the $Q$-cylindrical end on $X_P^\circ$, where $P \in \mathcal{P}$ is any top-dimensional polytope containing $Q$. Thus on $X_Q^\circ$ the projection $\pi_{B^\nu}$ is well-defined up to $t_Q^\nu$-translation.

Neck-stretched manifolds and components of a broken manifold are equipped with a cylindrical metric. Choose a non-degenerate metric $(\cdot, \cdot) : t \times t \to \mathbb{R}$ restricting to a map $t \times t \to \mathbb{Z}$.

Definition 2.38. (Cylindrical metric) A metric $g_P$ on $Z_P^\nu$ is $P$-cylindrical if $g_P$ is a product metric, that is, the product of the given linear metric $(\cdot, \cdot)$ on $t_P^\nu$ and a $T_P$-invariant metric $g_{Z_P}$ on $Z_P$ that satisfies

$$|\xi_{Z_P}|_{g_{Z_P}} = |\xi| \quad \xi \in t_P.$$

On the multiply-stretched manifolds $X^\nu$, a metric $g_\nu$ is cylindrical if for any $P \in \mathcal{P}$, $g_\nu$ is $P$-cylindrical in the region $\Phi^{-1}(P^\nu) \times \nu P^\nu$.

We finally prove that tropical Hamiltonian manifolds possess symplectic cylindrical structures.

Lemma 2.39. (Existence of symplectic cylindrical structures) There exists a symplectic cylindrical structure (see 2.27) for a tropical Hamiltonian manifold $(X, \mathcal{P}, \Phi)$.

The symplectic cylindrical structure is constructed via an analogous structure on cut spaces, which we define next.

Definition 2.40. (Symplectic structure on cut spaces) A symplectic structure on cut spaces consists is a collection of maps $(\phi^Q_P)_{Q \subseteq P}$ where $P \in \mathcal{P}$ ranges over top-dimensional polytope. For any pair $Q \subseteq P$, the map $\phi^Q_P$ is a $T_Q$-equivariant symplectomorphism on a neighborhood $U_{X_P} X_Q \subset X_P$ of $X_Q$:

$$\phi^Q_P : U_{X_P} X_Q \to (\text{Cone}_{P^\nu}(Q^\nu) \times Z_Q, \omega_Q^\nu) / \sim, \quad \omega_Q^\nu := \omega_{X_Q} + d\langle \alpha_Q, \pi_Q^\nu \rangle,$$

where

(a) the equivalence $\sim$ mods out boundaries $\partial Q^\nu$ by circle actions,
(b) $X_Q$ is identically mapped by $\phi^Q_P$ to $\{P^\nu\} \times Z_Q / T_Q$,
(c) $\alpha_Q \in \Omega^1(Z_Q, t_Q)$ is a connection one-form such that the collection $(\alpha_P)_{P \in \mathcal{P}}$ is consistent.

Since $P^\nu$ is a vertex of the polytope $Q^\nu$, $\text{Cone}_{P^\nu}(Q^\nu)$ is linearly isomorphic to a quadrant $(\mathbb{R}_{\geq 0})^{\text{codim}_P(Q)}$ in $t_Q^\nu$.
Remark 2.41. Any symplectic cylindrical structure induces projection maps on neighborhoods of boundary submanifolds

\[(2.29) \quad \pi_Q : U_{X_P X_Q} \rightarrow X_Q\]

for all pairs of polytopes $Q \subset P$.

Proof of Lemma 2.39. We first construct a symplectic cylindrical structure as in 2.27 for a broken manifold. This data includes a consistent collection of connection one-forms $(\alpha_P)_{P \in \mathcal{P}}$ and symplectomorphisms $(\phi^Q_P)_{Q \subset P}$ in (2.28). For a fixed top-dimensional polytope $P$, the maps $\phi^Q_P$ are constructed by induction on the dimension of $Q$. At every step of the induction the connection one-form $\alpha_{Q_0}$ on $Z_{Q_0} \rightarrow X_{Q_0}$ is pre-determined in a neighborhood $U_{X_P X_{Q_0} X_{Q_0}}$ of $X_{Q_0}$ for strict subsets $Q \subsetneq Q_0$. We may choose any extension of $\alpha_{Q_0}|(\cup Q U_{X_P X_{Q_0}} X_{Q_0})$ to all of $X_{Q_0}$. The symplectomorphism $\phi^Q_{Q_0}$ is also pre-determined in the neighborhoods $U_{X_P X_{Q_0}}$, and is extended to all of $U_{X_{Q_0}}$ by the relative symplectic neighborhood theorem (Lemma 2.43).

A symplectic cylindrical structure on $X$ is obtained by gluing the structure maps on cut spaces. Indeed, for a polytope $Q \in \mathcal{P}$, codim$(Q) > 0$, we can invert the multiple cut operation on the domain and target spaces of the maps

\[(2.30) \quad \phi^Q_P : U_{X_P X_Q} \rightarrow (\text{Cone}_{P^\lor}(Q^\lor) \times Z_{Q_0}, \omega_{Q_0})/\sim, \quad P \supset Q, \quad \text{codim}(P) = 0,
\]

to yield the symplectomorphism $\phi_Q$ of (2.21), because the connection one-form $\alpha_Q$ underlying the symplectic form is the same in each of the target spaces in (2.30). \(\Box\)

Remark 2.42. (On cylindrical structures on broken manifolds) We first summarize the constructions carried out in this section:

Step 1: First we constructed a symplectic cylindrical structure on cut spaces in the proof of Lemma 2.39.

Step 2: The symplectic cylindrical structure on the cut spaces yielded a symplectic cylindrical structure on a tropical Hamiltonian manifold (see proof of Lemma 2.39).

Step 3: Cylindrical almost complex structures on neck-stretched manifolds were defined using the symplectic cylindrical structure, see Definition 2.32.

Step 4: Finally, the degeneration of neck-stretched manifolds endowed the limit broken almost complex manifold with a cylindrical structure, see Remark 2.36.

However, there is no natural diffeomorphism between the almost complex broken manifold in Step 4 and the cut symplectic manifolds in Step 1 that is taming and preserves the cylindrical structure.

Lemma 2.43. (Relative symplectic neighbourhood theorem) Let $Y \subset X$ be a compact symplectic submanifold of a symplectic manifold $(X, \omega)$. Let $\mathcal{N}$ be a neighborhood of the zero section in the normal bundle $N_X Y$ that is equipped with a symplectic form $\omega_{\mathcal{N}}$. Let $\psi : N \rightarrow U$ be a diffeomorphism onto a neighborhood $U \subset X$ of $Y$ that is identity on $Y$. Further, let $Y \subset S \subset \mathcal{N}$ be a subset satisfying

\[s \in S \implies ts \in S \quad \forall t \in [0, 1], \quad \text{and} \quad (\omega_{\mathcal{N}} - \psi^* \omega)|_{T_s \mathcal{N}} = 0.\]
Then, there is a smaller neighborhood $N' \subset N$ of the zero section that contains $S$ and so that $\psi$ can be homotoped to a symplectomorphism $\phi : N' \to U$ satisfying $\phi|_S = \psi|_S$.

The proof of the ordinary symplectic neighborhood ([45, Lemma 3.14]) can be used to prove the slightly stronger statement of Lemma 2.43.

2.6. Dual complex. The combinatorial type of our broken maps will be a graph equipped with a map to the dual complex associated to the degeneration, defined as follows.

Definition 2.44. The dual complex $B^\vee$ for the gluing datum $(P^\vee, P \in P)$ is the topological space

$$B^\vee := \left( \bigcup_{P \in P} P^\vee / \sim \right) \subset t^\vee$$

where the equivalence $\sim$ is given by

$$\lambda_P \sim \lambda_Q \text{ whenever }\lambda_P \in P^\vee, \lambda_Q \in Q^\vee \text{ is on the face corresponding to } P^\vee, \text{ and } \lambda_Q \text{ maps to } \lambda_P \text{ under the projection map } t_Q^\vee \to t_P^\vee.$$

Example 2.45. In the case of a simple cut $P = \{(-\infty, 0], \{0\}, [0, \infty)\}$, the polytopes $P^\vee$ consist $\{0\}, [-1, 1], \{0\}$ with identifications given by inclusion of the endpoints. Hence the identification produces the interval $B^\vee \cong [-1, 1]$.

Components of the broken manifold may be identified with regions in neck-stretched manifolds. In particular, for any $P \in P$, the broken manifold $X^\circ_{P \times P^\vee}$ is identified with the following subset of the neck-stretched manifold:

Definition 2.46. ($P$-cylindrical region on neck-stretched manifolds) The $P$-cylindrical region in $X^\nu$ is

$$(2.31) \quad X^\nu_P := \left( \bigcup_{Q \in P, Q \subset P} \Phi^{-1}(Q^\bullet) \times \nu Q^\vee \right) / \sim.$$

where $\sim$ is the relation in Definition 2.44.

Remark 2.47. To examine the convergence behavior of maps in neck-stretched manifolds to a limit map in the broken manifold, we need to embed $P$-cylindrical regions of the neck-stretched manifold into the $P$-cylindrical component of the broken manifold. The embedding is not unique, since any embedding can be post-composed with the torus action on the fibers of $X_P \to X_P$. The following is a parametrized family of embeddings. For any $\nu$, there is a natural projection map

$$(2.32) \quad \pi^\nu_P : X^\nu_P \to X_P,$$

and the images of $\pi^\nu_P$ exhaust $X^\circ_P$ as $\nu \to \infty$. The fibers are subsets of $T_{P, C}$. Choose lifts of $\pi^\nu_P$ to maps between bundles $X^\nu_P \to X^\circ_P$ as follows. For any $t \in \nu P^\vee$, denote by

$$(2.33) \quad e^{-t} : X^\nu_P \to X^\circ_{P \times P^\vee}.$$
a lift of $\pi_p^\nu$ that maps a level set $\{\Phi_p^\nu = c\} \subset X_p^\nu$ to $Z_p \times \{c - t\} \subset X_p^\nu \times \nu P^\nu$. For any $t \in \nu P^\nu$, the inverse of the map $e^{-t}$ is well-defined on a sequence of exhausting subsets $X_{p,x,p^\nu,\nu}^\nu \subset X_{p,x,p^\nu,\nu}^\nu$:

\begin{equation}
(2.34) \quad e^t := (e^{-t})^{-1} : X_{p,x,p^\nu}^\nu \supset X_{p,x,p^\nu,\nu}^\nu \to X_p^\nu.
\end{equation}

Remark 2.48. The family of embeddings above is parametrized by the dual complex as follows. Consider any $\nu > 0$. Suppose $Q \subset P$ is a facet and $i : \nu P^\nu \to \nu Q^\nu$ is the inclusion map whose image is a facet of $\nu Q^\nu$. Then, for any $t \in \nu P^\nu$, the map $e^{-t}$ restricts to the map $e^{-i(t)}$ on $X_Q^\nu$. Since $e^{-t}$ and $e^{-i(t)}$ may be viewed as the same map, we view the translation $t$ as an element in the scaled dual complex defined as

\begin{equation}
\nu B^\nu := (\cup_{p \in \mathcal{P}} \nu P^\nu) / \sim.
\end{equation}

Here $\sim$ is as in the definition of $B^\nu$. Given sequences $x_{\nu}, t_{\nu} \in \nu B^\nu$, the statement

\begin{equation}
"e^{-t_{\nu}} x_{\nu} \text{ converges to } x \in X".
\end{equation}

means the following: There is a polytope $P \in \mathcal{P}$ such that $x \in X_{p,x,p^\nu}^\nu$ and $t_{\nu} \in \nu P^\nu$, and the sequence $e^{-t_{\nu}} x_{\nu}$ converges to $x$ in $X_{p,x,p^\nu}^\nu$.

Example 2.49. We illustrate the concepts introduced for multiple cuts using an example with two non-intersecting single cuts. Consider the tropical manifold $(X, \Phi, \mathcal{P})$ where the torus is $T = S^1$, and the polytopes in $\mathcal{P}$ are $P_0, P_1, P_2, P_{01}, P_{12} \subset \mathbb{R}$:

```
\begin{center}
\begin{tikzpicture}
\fill (0,0) circle (2pt) node[below] {$c_0$};
\fill (2,0) circle (2pt) node[below] {$c_1$};
\draw[->] (0,0) -- (2,0);
\draw (0,0) -- (0,1);
\draw (2,0) -- (2,1);
\draw (0,0) node[above] {$P_0^\nu$} -- (0,1) node[below] {$P_{01}^\nu$} -- (2,0) node[below] {$P_1^\nu$} -- (2,1) node[below] {$P_{12}^\nu$} -- (0,0);\end{tikzpicture}
\end{center}
```

The dual complex

```
\begin{center}
\begin{tikzpicture}
\fill (0,0) circle (2pt) node[below] {$c_0$};
\fill (2,0) circle (2pt) node[below] {$c_1$};
\draw[->] (0,0) -- (2,0);
\draw (0,0) -- (0,1);
\draw (2,0) -- (2,1);
\draw (0,0) node[above] {$P_0^\nu$} -- (0,1) node[below] {$P_{01}^\nu$} -- (2,0) node[below] {$P_1^\nu$} -- (2,1) node[below] {$P_{12}^\nu$} -- (0,0);\end{tikzpicture}
\end{center}
```

is a subset of $\mathbb{R}$. Let the point $P_i^\nu$ be $g_i$ for $i = 0, 1, 2$. For $\nu > 0$, the neck-stretched manifold is $X^\nu := X_0^\nu \cup ([\nu g_0, \nu g_1] \times Z_0) \cup X_1^\nu \cup ([\nu g_1, \nu g_2] \times Z_1) \cup X_2^\nu / \sim$, where $Z_i := \Phi^{-1}(c_i)$, $X_{P_i}^\nu$ is $X_{P_i}$ minus a tubular neighbourhood of boundary divisors, and $\sim$ identifies the copies of $Z_0$ and $Z_1$ on the boundaries. Let $t \in \nu B^\nu$ be a translation. The following possibilities arise:

- $t \in P_i^\nu$ for $i = 0, 1$ or $2$. We consider the case $i = 1$, since the others are similar. Then $t = \nu g_1$ and $e^{-t}$ is the embedding

\begin{equation}
([\nu g_0, \nu g_1] \times Z_0) \cup Z_0 \cup Z_1 \cup ([\nu g_1, \nu g_2] \times Z_1)
\rightarrow X_{P_1}^\nu \simeq ((-\infty, 0] \times Z_0) \cup Z_0 \cup Z_1 \cup ([0, \infty) \times Z_1).
\end{equation}

- or $t \in P_{i(i+1)}^\nu$. Then $t \in [\nu g_i, \nu g_{i+1}]$ and $e^{-t} : [\nu g_i, \nu g_{i+1}] \times Z_i \to \mathbb{R} \times Z_i$ maps $\{c\} \times Z_i$ to $\{c - t\} \times Z_i$. 

3. Broken disks

The goal of this section is to define broken treed holomorphic disks. These are an analog of what Parker [50] calls exploded holomorphic maps. These structures combine the features of treed holomorphic disks and tropical maps. Treed holomorphic disks consist of surface components that are holomorphic disks or spheres in a symplectic manifold whose boundary lies in a Lagrangian submanifold; and treed segments attached to the boundary of disks which map to line segments in the dual complex associated to the degeneration.

3.1. Treed disks. The domains of our pseudoholomorphic maps are treed disks, which are analogues of pearly trajectories of Biran-Cornea [7], Cornea-Lalonde [20] and Seidel [60]. A treed disk is a combination of trees, nodal disks, and nodal spheres.

Definition 3.1. (a) (Nodal disks) A nodal disk $S$ is a union

$$S = \left( \bigcup_{\alpha=1,\ldots,d(\circ)} S_{\alpha,\circ} \right) \cup \left( \bigcup_{\beta=1,\ldots,d(\bullet)} S_{\beta,\bullet} \right) / \sim$$

of a collection of disk components $S_{\alpha,\circ}$ each biholomorphic to a unit disk $D^2 \subset \mathbb{C}$, and sphere components $S_{\beta,\bullet}$ each biholomorphic to the projective line $\mathbb{P}^1$, glued together by an equivalence relation $\sim$. The equivalence relation $\sim$ is generated by pairs of interior or boundary nodal points

$$w_e = (w_+(e), w_-(e)) \in (\cup_{\alpha} S_{\alpha,\circ})^2 \cup (\cup_{\beta} S_{\beta,\bullet})^2$$

with the property that the boundary $\partial S$ is connected and there are no cycles of components. A marking of a nodal disk is a collection of boundary and interior points

$$z_{\circ} = (z_{\circ,i} \in \partial S_i, i = 1, \ldots, d(\circ)), \quad z_{\bullet} = (z_{\bullet,i} \in S \setminus \partial S_i, i = 1, \ldots, d(\bullet))$$

distinct from the nodes. A marked nodal disk is stable if it admits no automorphisms, or equivalently, if for each disk component $S_{\alpha,i}$ the sum of the number of special (nodal or marked) boundary points and twice the number of interior special points is at least three, and each sphere component $S_{\bullet,i}$ has at least three special points.

(b) (Combinatorial type) The combinatorial type of a nodal disk $\text{Vert}(\Gamma)$ is the tree $\Gamma$ whose vertices

$$\text{Vert}(\Gamma) = \text{Vert}_{\bullet}(\Gamma) \cup \text{Vert}_{\circ}(\Gamma)$$

correspond to disk or sphere components, and whose edges $\text{Edge}(\Gamma)$ correspond to markings or nodes. The tree $\Gamma$ is equipped with a ribbon structure, which is a cyclic ordering on the set of boundary markings and boundary nodes on each disk component. The edges corresponding to markings are called leaves. Thus $\text{Edge}(\Gamma)$ admits two partitions

$$\text{Edge}(\Gamma) = \text{Edge}_{\circ}(\Gamma) \cup \text{Edge}_{\bullet}(\Gamma)$$
corresponding to whether the edge is of boundary or interior type, and
\[ \text{Edge}(\Gamma) = \text{Edge}_-(\Gamma) \cup \text{Edge}_-(\Gamma) \]
corresponding to whether the edge represents a marking or node. The edge \( e_0 \) corresponding to the boundary marking \( z_0 \) is an outgoing edge and is called the root edge. All the other boundary markings are incoming edges. All edges corresponding to nodes are oriented to point towards the root. Given a boundary vertex \( v \in \text{Vert}_-(\Gamma) \) the set of edges \( e \in \text{Edge}_-(\Gamma) \) incident to \( v \) is equipped with a cyclic ordering, corresponding to the ordering of the nodes and markings along the boundary.

(c) (Treed disk) A treed segment is obtained from a collection of (possibly infinite) intervals \( I_1, \ldots, I_k \) by gluing along infinite endpoints, e.g.
\[ [0, \infty) \cup \infty((-\infty, 0]) \]
is a treed segment with three components. Each treed segment \( T \) has a length \( l(T) \in [0, \infty] \) and a number of breakings \( b(T) \in \mathbb{Z}_{\geq 0} \), with \( l(T) = \infty \) if \( b(T) > 0 \). A treed nodal disk \( C = S \cup T \) is
(i) either obtained from a nodal disk \( S_0 \) by replacing each boundary node \( w_e, e \in \text{Edge}_{\infty, -}(\Gamma) \) with a treed segment \( T_e \cong [0, \ell(e)] \) with finite endpoints, and each boundary marking \( w_e, e \in \text{Edge}_{\infty, -}(\Gamma) \) one of whose endpoints is infinite;
(ii) or is a treed segment \( T_e \cong (-\infty, \infty) \), both whose endpoints are infinite.

(d) (Combinatorial type of a treed disk) The combinatorial type of a treed disk consists of the combinatorial type \( \Gamma \) of the underlying nodal disk and a partition
\[ \text{Edge}_{\infty, -}(\Gamma) = \text{Edge}_{\infty, -}^0(\Gamma) \cup \text{Edge}_{\infty, -}^{(0, \infty)} \cup \text{Edge}_{\infty, -}^\infty(\Gamma) \]
of edges corresponding to boundary nodes with zero, finite, and infinite length edges.

(e) (Stable treed disk) A treed nodal disk \( C \) is stable if the underlying disk \( S \) is stable, the treed segment at any node \( T_e, e \in \text{Edge}_{\infty, -}(\Gamma) \) has at most one breaking, and treed segments at markings \( T_e, e \in \text{Edge}_{\infty, -}(\Gamma) \) are unbroken. A treed disk with no surface component is not stable.

For integers \( d(\bullet), d(\circ) \geq 0 \), denote by \( \overline{\mathcal{M}}_{d(\bullet), d(\circ)} \) the moduli space of isomorphism classes of stable treed disks with \( d(\circ) \) incoming boundary markings, one outgoing boundary marking and \( d(\bullet) \) interior markings. For each combinatorial type \( \Gamma \), denote by \( \mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{d(\bullet), d(\circ)} \) the set of isomorphism classes of stable treed disks of type \( \Gamma \) so that the moduli spaces decomposes into strata of fixed type
\[ \overline{\mathcal{M}}_{d(\bullet), d(\circ)} = \bigcup_\Gamma \mathcal{M}_\Gamma, \]
whose dimension is \( d(\circ) + 2d(\bullet) - 2 \).

The moduli spaces admit universal curves, which admit partitions into one and two-dimensional parts. For any combinatorial type \( \Gamma \) let \( \mathcal{U}_\Gamma \) denote the universal treed disk consisting of isomorphism classes of pairs \( (C, z) \) where \( C \) is a treed disk
of type $\Gamma$ and $z$ is a point in $C$, possibly on a disk component, sphere component, or one of the edges of the tree. The map

$$U_\Gamma \to M_\Gamma, \quad [C, z] \mapsto [C]$$

is the universal projection, whose fiber over $[C]$ is a copy of $C$. The union over types $\Gamma'$ with $M_{\Gamma'} \subset M_\Gamma$ is denoted $\overline{U}_\Gamma$. Denote by

$$\mathcal{S}_\Gamma, \quad \text{resp.} \quad \mathcal{T}_\Gamma,$$

the locus of points $[C, z] \in \overline{U}_\Gamma$ where $z$ lies on a disk or sphere component resp. an edge of $C$. Hence $U_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma$, and $\mathcal{S}_\Gamma \cap \mathcal{T}_\Gamma$ is the set of points on the boundary of the disks meeting the edges of the tree.

### 3.2. Treed pseudoholomorphic disks

Treed pseudoholomorphic disks are maps from treed disks to a symplectic manifold equipped with a tamed base almost complex structure. On the two-dimensional part of the treed disk the map is pseudoholomorphic and the boundaries of the disks map to the Lagrangian submanifold. On the one-dimensional part of the domain, the map is a gradient flow line of a Morse function on the Lagrangian submanifold, whose length is same as the length of the tree edge. Later the almost complex structure, the Morse function, and the metric on the Lagrangian will be given domain-dependent perturbations in order to regularize the moduli spaces of treed holomorphic disks.

We introduce the necessary notation for defining treed holomorphic disks. Let $(X, \omega)$ be a symplectic manifold and $L \subset X$ be a Lagrangian submanifold. Let $J$ be an $\omega$-tame almost complex structure. Let $G_L$ be a Riemannian metric on $L$ and let $F \in C^\infty (L, \mathbb{R})$ be a Morse function such that the pair $(F, G_L)$ is Morse-Smale. The **gradient vector field** is defined by the condition

$$\text{grad}_F \in \text{Vect}(L), \quad df(\cdot) = G_L(\text{grad}_F, \cdot).$$

**Definition 3.2.** A **treed holomorphic disk** with boundary in $L \subset X$ consists of a treed disk $C = S \cup T$ and a continuous map

$$u : C \to X$$

satisfying the following conditions:

(a) The map $u|S$ is a pseudoholomorphic map on the surface part: $Jd(u|S) = d(u|S) \circ j$.

(b) $u|T$ is a union of gradient trajectories: $\frac{d}{ds} u|T = -\text{grad}_F(u|T)$ where $s$ is a unit velocity coordinate on $T$; and

(c) the tree components $T$ and the boundary $\partial S$ of the surface components are mapped to the Lagrangian submanifold $L : u(T \cup \partial S) \subset L$.

A holomorphic treed disk $u : C = S \cup T \to X$ is **stable** if it has no non-trivial automorphisms, that is, $\# \text{Aut}(u) = 1$, or equivalently

(a) each surface component $S_v \subset S$ on which the map $u$ is constant is stable as a component of a nodal disk $S$ (see Definition 3.1);

(b) each treed segment $T_v$ on which the map $u$ is constant has at most one infinite end, that is, one of the ends of $T_v$ is an attaching point to a sphere or disk $S_v \subset S$. 
Note that the case $C \cong \mathbb{R}$ equipped with a non-constant Morse trajectory $u : C \to L$ is allowed under this stability condition. The *energy* $E(u)$ of a treed disk $u : C \to X$ is the sum of the energies of the surface components,

$$E(u) = \int_S \frac{1}{2} |du|^2 \text{Vol}_S.$$ 

For a holomorphic disk $u$ the energy is equal to the symplectic area

$$\text{Area}(u) = \int_S (u|S)^* \omega.$$ 

### 3.3. Multiply-broken disks.

A broken map is a map from a nodal curve to a multiply cut manifold that is discontinuous at nodes. Different components of the nodal curve map to different pieces of the multiply cut manifold, and the lifts of the nodal points satisfy an edge matching condition. The nodal points carry an additional data of intersection multiplicity with boundary divisors. This data is packaged into a *tropical structure*, which is part of the combinatorial type of the broken map.

**Definition 3.3.** (Tropical graph) Let $B^\vee \subset t^\vee$ be the dual complex for a set of polytopes $P = \{P \subset t^\vee\}$ as in Definition 2.44. A *tropical structure* $\mathcal{T}$ on a graph $\Gamma$ relative to $B$ consists of

(a) a *polytope map*

$$\text{Vert}(\Gamma) \to P, \quad v \mapsto P(v)$$

such that if if $v_-, v_+$ are contained in an edge $e$ then

$$P(e) := P(v_+) \cap P(v_-)$$

is non-empty and a face of both $P(v_-)$ and $P(v_+)$;

(b) a *slope* for each edge

$$\mathcal{T}(e) \in t_{P(e), \mathbb{Z}}.$$ 

so that there exists a *tropical weight*

(3.2) $$\text{Vert}(\Gamma) \to (P(v))^\vee \subset B^\vee, \quad v \mapsto \mathcal{T}(v)$$

satisfying the following.

(a) The difference of weights $\mathcal{T}(v_\pm)$ between any two vertices $v_-, v_+$ connected by an oriented edge $e$ from $v_-$ to $v_+$ (that is $e = (v_+, v_-)$) is a non-negative multiple of $\mathcal{T}(e)$:

(Slope condition) $$\mathcal{T}(v_+) - \mathcal{T}(v_-) \in \mathbb{R}_{\geq 0} \mathcal{T}(e).$$

Here we view $\mathcal{T}(e)$ as an element in $t_{P(e)}^\vee$ via the identification $t \simeq t^\vee$ in (2.24). If an edge has zero slope then the polytopes and weights are equal:

$$\mathcal{T}(e) = 0 \implies (P(v_+) = P(v_-)) \text{ and } \mathcal{T}(v_+) = \mathcal{T}(v_-).$$ 

(b) (Slopes for boundary edges) Suppose $\Gamma$ is the type of a treed disk. Then, in a tropical structure on $\Gamma$, we require that all boundary edges have zero slope:

$$\mathcal{T}(e) = 0 \quad \forall e \in \text{Edge}_\partial(\Gamma).$$
The treed graph $\Gamma$ together with a tropical structure is called a *tropical graph*. The set of tropical weights on a tropical graph $\Gamma$ is denoted

\[
W(\Gamma) = \{(T(v), v \in \text{Vert}(\Gamma))\}.
\]

A *tropical isomorphism* between tropical graphs $(\Gamma_1, T_1)$ and $(\Gamma_2, T_2)$ is a graph isomorphism $\Gamma_1 \cong \Gamma_2$ that preserves the polytopes $P(v)$ and edge slopes $T(e)$. This ends the definition.

**Definition 3.4.** (Broken map) Let $(X, P)$ be a broken manifold. Let $P_0 \in P$ be a polytope for which the torus $T_{P_0}$ is trivial, and so, $X_{P_0} = X_{P_0} \times P_0\vee$. Let $L \subset X_{P_0}$ be a Lagrangian submanifold that does not intersect boundary divisors $X_P, P \subset P_0$.

A *broken map* $u$ to $X$ is a datum consisting of

1. (Type) a *type* $\Gamma$, which is the combinatorial type of a treed disk with a tropical structure for which disk vertices map to $P_0$:
   \[ v \in \text{Vert}_c(\Gamma) \implies P(v) = P_0; \]

2. (Domain curve) a treed nodal disk $C$ of type $\Gamma$; we denote for any $v \in \text{Vert}(\Gamma)$ the corresponding component of $C$ by $C_v$ and the complement of the interior nodes by $C_0 \subset C_v$;

3. (Map) a collection of treed holomorphic maps $u_v : C_v \to X_{P(v)}$, $X_{P(v)} := X_{P(v) \times P(v)\vee}$, $v \in \text{Vert}(\Gamma)$ denoted $u : C \to X$ for short for which
   \[ u_v(C_0) \subset X_0; \]

4. (Framing) and a linear isomorphism
   \[ \text{fr}_e : T_{w_+(e)}C_{P(v_+)} \oplus T_{w_-(e)}C_{P(v_-)} \to \mathbb{C}, \]
   for all interior nodes $w_e$ corresponding to edges $e = (v_+, v_-) \in \text{Edge}_e(\Gamma)$ with non-zero slope $T(e) \neq 0$,

such that the collection $(u_v)_v$ satisfies the continuity and edge matching condition as follows.

- (Continuity) The restriction of the map $u$ to disk components and treed parts
  \[ u|((\cup_{v \in \text{Vert}_c(\Gamma)} C_0_v) \cup (\cup_{e \in \text{Edge}_e(\Gamma)} T_e)) \]
  is continuous.

- (Matching at edges with zero slope) If an interior edge $e \in \text{Edge}_e(\Gamma)$ has zero slope $T(e) = 0$, the corresponding nodal points $w_\pm(e) \in C_{v \pm}$ map to $X_{P(v_\pm)}$ and the map $u$ is continuous at the node:
  \[ u(w_+(e)) = u(w_-(e)) \in X_{P(v_\pm)}; \]

- (Matching at edges with non-zero slope) For an edge $e = (v_+, v_-) \in \text{Edge}_e(\Gamma)$ with $T(e) \neq 0$, the maps
  \[ u_- = u|_{C_{v_-}}, u_+ = u|_{C_{v_+}} \]
  with nodes $w_-(e), w_+(e) \in C_{v \pm}$ satisfy the following.
(a) The evaluations of $u_{±}$ at the nodal points $w_{±}(e)$ lie on the space $X_{P(e)}$ corresponding to the face $P(e)$ in $X_{P_{(v±)}}$. That is,
\[
\left(\pi_{P(v±)} \circ u_{±}\right)(w_{±}(e)) \in X_{P(e)} \subset X_{P_{(v±)}},
\]
where $\pi_{P(v±)} : X_{P_{(v±)}} \to X_{P(e)}$ is the fibration map from (2.17). Further,

\[
(\text{3.4) (Horizontal matching)} \quad \pi_{P_{(v−)}}(u_{−}(w_{−}(e)))) = \pi_{P_{(v+)}}(u_{+}(w_{+}(e))) = x(e).
\]

(b) (Vertical matching) The leading order terms in the vertical direction agree on both sides of the node in the following sense. There exist local coordinates
\[
z_{±} : (U_{w_{±}(e)}, w_{±}(e)) \to (\mathbb{C}, 0),
\]
in the neighbourhood $U_{w(e)} = U_{w_{+}(e)} \cup U_{w_{−}(e)}$ of the node $w$ that respect the framing
\[
dz_{+}(w_{+}(e)) \otimes dz_{−}(w_{−}(e)) = \text{fr}_e
\]
and such that

\[
(\text{3.5) (Vertical matching)} \quad \lim_{z_{−} \to 0} z_{−}{^T}(e)u_{−}(z_{−}) = \lim_{z_{+} \to 0} z_{+}{^T}(e)u_{+}(z_{+})
\]

Here, $u_{−}(z_{−})$, $u_{±}(z_{±})$ are viewed as points in the $P(e)$-cylinder $Z_{P(e),\mathbb{C}}$ via the embedding
\[
X_{P_{(v±)}}^o \to Z_{P(e),\mathbb{C}}
\]
from (2.25) which is defined in a neighborhood of $\pi_{−1}P_{(v)}(X_{P(e)})$. The coordinates $z_{+}$, $z_{−}$ are called matching coordinates at the node $w_{e}$.

This ends the Definition.

Remark 3.5. The vertical matching condition (3.5) subsumes the horizontal matching condition (3.4).

As usual one obtains a Hausdorff moduli space after imposing a stability condition:

Definition 3.6. A broken map $u : C \to \mathfrak{X}$ is stable if either of the following equivalent conditions hold:

(a) For each $v \in \text{Vert}(\Gamma)$, such that the horizontal projection $\pi_{P(v)} \circ u_{v} : C_{v} \to X_{P(v)}$ is constant, $C_{v}$ is stable as a marked curve, and any tree component $T_{e}$ on which $u\mid T_{e}$ is constant does not contain an infinite segment;

(b) the reduced automorphism group $\text{Aut}_r(C, u)$ (see Definition 3.29) is trivial.

Remark 3.7. The definition of stability differs from the standard definition of stability for holomorphic maps in that the latter rules out trivial cylinders: A trivial cylinder is a map of the form
\[
\mathbb{P}^1 \setminus \{0, \infty\} \to X_{\mathfrak{F}}, \quad z \mapsto z^\mu x
\]
for some $\mu \in T_{P,e}$ and $x \in X_{\mathfrak{F}}$, and $0, \infty$ are the only special points on the domain.
Figure 8. Broken maps and their dual graphs in the broken manifold of Figure 7.

Remark 3.8. (Intersection multiplicity at nodes) At a node of a broken map, the intersection multiplicity with boundary divisors is specified by the slope of the corresponding edge in the tropical graph. Let $e = (v_+, v_-) \in \text{Edge}_{*, -}(\Gamma)$ be an edge. Suppose the map $u_{v_+} : C_{v_+} \to X_{\mathcal{P}(v_+)}$ intersects the boundary divisors $D_1, \ldots, D_k \subset X_{\mathcal{P}(v_+)}$ at the nodal point $w_+$. Let $\nu_i \in t$ be the primitive outward normal for the facet in $P(v_+) \times P(v_+) \vee$ corresponding to the boundary divisor $D_i$. The intersection multiplicities of $u_{v_+}$ at $w_+(e)$ with boundary divisors are same as that of the vertical cylinder $u_{v, \text{vert}} = z^{T(e)} x$ at $z = 0$. Consequently $T(e)$ lies in the $\mathbb{Z}_+$-span of the normal vectors $\nu_1, \ldots, \nu_k$, and

$$T(e) = \sum_i m_{w(e)}(u_{v_+}, D_i) \nu_i \quad \text{where} \quad m_{w(e)}(u_{v_+}, D_i) \in \mathbb{Z}_{>0} \quad \text{(3.6)}$$

is the intersection multiplicity of $u_{v_+}$ with $D_i$ at $w(e)$. There is a similar formula for the other end of the node $v_-$, in which all occurrences of $T(e)$ are replaced by $-T(e)$.

Remark 3.9. (Intersection multiplicity in the orbifold case) In the orbifold case intersection multiplicities may be fractions. Indeed, in (3.6) the slope of the edge $T(e) \in t_{P(e), \mathbb{Z}}$ lies in the $\mathbb{Q}_+$-span of the normal vectors $\nu_1, \ldots, \nu_k$.

Example 3.10. In Figure 8 suppose the slope of the edge $e$ is $T(e) = (\mu_1, \mu_2) \in t_{P(e), \mathbb{Z}} \simeq \mathbb{Z}^2$. If the basis of $t_{P(e)}$ is chosen to consist of the outward pointing normal vectors of $P_{d_1}$ and $P_{d_2}$ in $P_1$, then $\mu_1, \mu_2 < 0$. At the node $w_e$ the intersection multiplicity of $u_1$ with $X_{P_{d_1}}$ resp. $X_{P_{d_2}}$ is $-\mu_1$ resp. $-\mu_2$.

Remark 3.11. (Number of framings) For a broken map without framing, there is a finite number of possible framings. Let $u$ be a broken map without framing of type $\Gamma$. Consider an edge $e \in \text{Edge}_{*, -}(\Gamma)$ with non-zero slope $T(e) \neq 0$. Let $n_e \in \mathbb{Z}_{>0}$
be the largest integer such that \( \frac{1}{n_e} T_e \in t_{P(e)} \mathbb{Z} \). Then, the trivial cylinder \( z \mapsto z^T(e) \) is an \( n_e \)-cover. Consequently, if \( f_r \) is a framing for the node \( w_e \), then \( e^{2\pi i k/n_e} f_r \) is also a framing for \( k = 0, 1, \ldots, n_e - 1 \). Thus, the total number of possible framings for \( u \) is \( \prod_{e \in \text{Edge}_{\alpha, \beta}(\Gamma)} n_e \).

In the following sequence of remarks, we give various alternate statements (Remark 3.13, Remark 3.15, Remark 3.16 and Remark 3.18) for the edge matching condition for broken maps.

**Remark 3.12.** *(Vertical component of a map near a node)* In the neighborhood of a nodal point \( w(e) \) corresponding to an edge \( e \in \text{Edge}(\Gamma) \), a map \( u : C \rightarrow X \) of type \( \Gamma \) splits into horizontal and vertical components relative to the base \( X_{P(e)} \). The vertical component is useful in describing the edge matching condition and in stating an exponential convergence result.

More generally, consider an almost complex manifold \( (X, J) \) for which \( Y_1, \ldots, Y_n \) are a set of transversely intersecting almost complex codimension two submanifolds, and the almost complex structure \( J \) is cylindrical in a neighborhood of each \( Y_i \).

Let \( Y := Y_1 \cap \ldots \cap Y_n \subset X \).

The cylindricity of \( J \) implies that there is a tubular neighbourhood map from the normal bundle \( N_Y \) onto a neighbourhood \( U_Y \) of \( Y \)

\[
(3.7) \quad N_Y \stackrel{i}{\rightarrow} U_Y \subset (X, J),
\]

such that the fibers of \( N_Y \rightarrow Y \) are \( i^*J \)-holomorphic, and in the decomposition \( N_Y = \bigoplus_i N_Y_i \) for each of the summands the fibers of \( N_Y_i|Y \rightarrow Y \) are \( i^*J \)-holomorphic. Thus a neighbourhood of the zero section of the normal bundle can be identified with a neighbourhood of \( Y \) without making any new choices.

This leads to the following description of a map in terms of horizontal and vertical parts. Suppose \( X \) and \( Y \) are as in the previous paragraph. Let \( D \subset \mathbb{C} \) be the unit disk and \( u : D \rightarrow X \) be a holomorphic map that intersects \( Y \) at \( 0 \in D \) with multiplicity \( \mu \in \mathbb{Z}^n \). That is, the intersection of \( u \) with the divisor \( Y_i \) is of order \( \mu_i \), where \( \mu = (\mu_1, \ldots, \mu_n) \).

(a) *(Horizontal and vertical parts)* Viewing \( u \) as a map to \( N_Y \) in a neighbourhood of \( 0 \), the horizontal part of the map \( u \) is the projection \( u_H : D \rightarrow Y \). The vertical part

\[
u_v : D \rightarrow (N_Y)_{u_Y(0)} \simeq \bigoplus_i (N_Y_i)_{u_Y(0)} \simeq \mathbb{C}^n\]

is the projection of \( u \) to the vertical direction for a choice of holomorphic trivialization \( u_H^* N_Y \cong C \times \mathbb{C} \) of the pullback bundle \( u_H^* N_Y \rightarrow C \).

(b) *(Vertical cylinder)* The vertical cylinder asymptotically close to the map \( u \) at \( 0 \) is the leading order term of the Taylor expansion of each of the summands of \( u_v \):

\[
(3.8) \quad u_{\text{vert}} : D \rightarrow (N_Y)_{u_Y(0)}, \quad z \mapsto (z^{\mu_i} x_i)_i = z^\mu x,
\]

where \( x \in N_Y \setminus \cup_i Y_i \), which is a \( (\mathbb{C}^\times)^n \)-bundle over \( Y \). Via the identifications \( (3.7) \) and \( (2.25) \), the vertical cylinder may alternately be viewed as a map to
a cylinder

\[(3.9) \quad u_{\text{vert}} : C \setminus \{0\} \to Z \times \mathbb{R}^n, \quad z \mapsto z^\mu x, \quad x \in Z \times \mathbb{R}^n\]

where \(Z \to Y\) is an \((S^1)^n\)-bundle which is the product of the unit circle bundles of \(NY_i|Y\).

The vertical cylinder \(u_{\text{vert}} : D \to (NY)_{u_Y(0)}\) does not depend on the choice of holomorphic trivialization \(u_i^* NY \cong D \times \mathbb{C}\) of \(NY\) because \(x_i\) is the \(\mu_i\)-th derivative of the vertical part and the lower order derivatives vanish. A change in trivialization amounts to multiplying \((u_i)_i\) by a holomorphic function that is identity at the origin, and this transformation does not alter \(x_i\). This ends the Remark.

**Remark 3.13.** (Edge matching via vertical cylinders) We state the vertical matching condition for broken maps using the vertical component of a map near a node. The vertical matching condition at a node \(w_e\) means that there exist holomorphic local coordinates in the neighborhood of the node

\[z_\pm : (U_\pm, w_\pm(e)) \to (\mathbb{C}, 0) \text{ satisfying } dz_+(w_+(e)) \otimes dz_-(w_-(e)) = \text{fr}_e\]

so that the vertical parts of the maps \(u_\pm\) coincide, that is, there exists \(x \in Z_{P(e), \mathbb{C}}\) such that

\[(3.10) \quad u_{\pm, \text{vert}}(z_\pm) = z_\pm^{\text{fr}_e} x.\]

The right-hand-side in (3.10) lies in \(X^o_{P(v_\pm)}\), because the \(P(e)\)-cylindrical end in \(X^o_{P(v_\pm)}\) has a fixed identification to the cylinder \(Z_{P(e), \mathbb{C}}\) via (2.25).

As usual, holomorphic curves with finite energy satisfy exponential convergence conditions.

**Lemma 3.14.** (Exponential convergence to the vertical cylinder) Let \(u : (D, 0) \to (X, Y)\) be a holomorphic map as in Remark 3.12, and let \(u_{\text{vert}}\) be its vertical component at \(0 \in D\). There is a constant \(c > 0\) such that

\[d(u(s, t), u_{\text{vert}}(s, t)) \leq ce^{-s}, \quad s \geq s_0,\]

using the cylindrical metric in \(Z \times \mathbb{R}^n\) and exponential coordinates in the domain \(z = e^{-s-it}\) in the domain.

**Proof.** In the horizontal projection to \(Y\), we have

\[d(\pi_Y(u(z)), \pi_Y(u_{\text{vert}}(z))) \leq c|z|, \quad z \in D,\]

because \(\pi_Y(u_{\text{vert}}(z))\) is the constant map \(\pi_Y(u(0))\). Switching to an exponential coordinate in the domain gives an exponential decay. In the vertical projection to the fiber \((NY)_{u_Y(0)}\), we consider each summand in \(NY = \oplus_{i=1}^n NY_i\) separately, and obtain

\[d(u_{v,i}(z), \pi_Y(u_{\text{vert}, i}(z))) \leq c|z|^\mu_i+1, \quad z \in D.\]

Transforming both domain and target coordinates we get the required estimate of exponential decay. \(\square\)
Remark 3.15. (Edge matching via higher order evaluation maps) The vertical matching condition at a node can be stated as follows: There exist framing coordinates $z_\pm$ in the neighborhood of a node $w_\pm(e)$ such that the leading order derivatives $\text{ev}^T(e) u_{\pm}(0)$ on both adjacent components $u_{\pm}(e)$ are the same at the nodal point $w_\pm(e)$.

To establish this, we first set up the notation for leading order derivatives for a collection of orthogonal divisors. In the setting of Remark 3.12, at the point $0 \in D$ the derivative normal to the divisor $Y_i$ vanishes up to order $\mu_i - 1$. Therefore the $\mu_i$-th normal derivative, or the $\mu_i$-jet normal to $Y_i$, denoted by

$$\text{ev}^{\mu_i} u(0) \in NY_i \backslash Y_i$$

is well-defined (see [16, Section 6]). Denote the tuple of $\mu_i$ derivatives by

$$\text{ev}^\mu u(0) = \oplus_i \text{ev}^{\mu_i} u(0) \in NY \cup_i Y_i.$$  \hfill (3.11)

We refer to $\text{ev}^\mu u(0)$ as the $\mu$-th derivative normal to $Y_i \cap Y_j$. The derivative $\text{ev}^\mu u(0)$ is equal to $x$ in (3.8) in the description of the vertical cylinder. The vertical matching condition at an edge $e = (v_+, v_-) \in \text{Edge}_*(\Gamma)$ is equivalent to the following condition: There exist framing coordinates $z_\pm$ in the neighborhood of $w_\pm(e)$ such that

$$\text{ev}^T(e) u_{v_+}(w_+(e)) = \text{ev}^T(e) u_{v_-}(w_-(e)).$$  \hfill (3.12)

Both sides of (3.12) lie in $Z_{P(e), C}$ via the identification (2.25) of a neighborhood of $X_{\mathcal{P}(e)} \subset X_{\mathcal{P}(e), C}$ with the $P(e)$-cylinder.

Remark 3.16. (Vertical matching for unframed broken maps) For unframed broken maps the vertical matching condition can be stated without using domain coordinates as:

$$\pi_T^T(e)(\text{ev}^T(e) u_{v_+}(w_+(e))) = \pi_T^T(e)(\text{ev}^T(e) u_{v_-}(w_-(e))) \in Z_{P(e), C}/T_{\mathcal{T}(e), C},$$  \hfill (3.13)

where $\pi_T^T(e) : Z_{P(e), C} \to Z_{P(e), C}/T_{\mathcal{T}(e), C}$ is the projection map, and $T_{\mathcal{T}(e), C} \subset T_{P(e), C}$ is the torus generated by $\mathcal{T}(e) \in t_{P(e), C}$. Indeed, if for a choice of domain coordinate $z_+$, $\text{ev}^T(e) u_{v_+}(w_+(e)) = x$, then changing the domain coordinate to $az_+$ for some $a \in \mathbb{C}^\times$ changes the evaluation to

$$\text{ev}^T(e) u_{v_+}(w_+(e)) = x + \langle \mu, a \rangle \in Z_{P(e), C}.$$  

Remark 3.17. (Comparison to Ionel’s refined matching) In [37, p14], Ionel states the edge matching condition using a ‘refined evaluation map’. In the terminology of (3.8), the refined evaluation map $\text{ev}^r(u, 0)$ for the map $u$ and the point $0 \in C$ is a point in the weighted projective space

$$\text{ev}^r(u, 0) = (x_1, \ldots, x_n) \in \mathbb{P}_{(\mu_1, \ldots, \mu_n)}(\oplus_i NY_i).$$

The point $\text{ev}^r(u, 0)$ is well-defined since replacing the domain coordinate $z$ by $az$ for some $a \in \mathbb{C}^\times$ has the effect of changing $(x_1, \ldots, x_n)$ to $(a^{\mu_1} x_1, \ldots, a^{\mu_n} x_n)$. The evaluation we use in stating the matching condition in (3.13)

$$\pi_\mu^\perp(\text{ev}^\mu(0)) \in (Z \times \mathbb{R}^n)/T_{\mu, C}$$
Remark 3.18. (Edge matching via evaluations of projected maps) We give an alternate characterization of the edge matching condition for unframed broken maps. Suppose $z_\pm$ is a node with edge multiplicity $T(e) \in t_{P(e)}$. The punctured curves $U_{z_+ \setminus \{z_\pm\}}$ are mapped to $Z_{P(e),\mathbb{C}}$, which is a $T_{P(e),\mathbb{C}}$-bundle over $X_{P(e)}$. Let

$$T_{T(e),\mathbb{C}} := \exp(CT(e)) \subset T_{P(e),\mathbb{C}}$$

be the complex one-parameter subgroup generated by $T(e) \in t_{P(e)}$. The map

$$\pi_{T(e)}^+(u) : U_{z_+ \setminus \{z_\pm\}} \to Z_{P(e),\mathbb{C}}/T_{T(e),\mathbb{C}}$$

has a pre-compact image. Therefore the evaluation

$$(\pi_{T(e)}^+ \circ u)(z_+) \in Z_{P(e),\mathbb{C}}/T_{T(e),\mathbb{C}}$$

is well-defined. The matching condition is

$$(3.14) \quad (\text{Edge Matching}) \quad (\pi_{T(e)}^+ \circ u)(z_+) = (\pi_{T(e)}^+ \circ u)(z_-).$$

Remark 3.19. The edge slopes $T(e)$ for the edges $e \in \text{Edge}(\Gamma)$ coming out of any vertex $v \in \text{Vert}(\Gamma)$ satisfy a balancing property in $t_{P(v)}$

$$\sum_{e \ni v} T(e)_{\text{vert}} = c_1((\pi_{P(v)} \circ u_v)^*Z_{P(v)} \to X_{P(v)});$$

Here $(\pi_{P(v)} \circ u_v) : C_v \to X_{P(v)}$ is the projection of the component $u_v$ corresponding to $v \in \text{Vert}(\mathcal{T})$, and $T(e)_{\text{vert}} \in t_{P(v)}$ is the vector sum of intersection multiplicities (see (3.6)) with vertical boundary divisors $X_{P(e)}$ at the node $w_e$. Indeed, the map $u_v$ gives a section of the $T_{P(v)}$-principal bundle $(\pi_{P(v)} \circ u_v)^*Z_{P(v)} \to X_{P(v)}$ on the complement of the intersections with the boundary divisors $X_{P(e)}$ corresponding to the edges $e \ni v$, and the monodromy of the section around each such intersection is determined by $T(e)_{\text{vert}}$. In particular, if the horizontal projection $\pi_{P(v)}(u_v)$ is constant (for example, if $X_{P(v)}$ is a point) then one obtains the standard balancing condition

$$\sum_{e \ni v} T(e)_{\text{vert}} = 0.$$

Remark 3.20. If the components $X_P$ of the broken manifold $X$ have toric orbifold singularities (that is, the polytopes $P \in \mathcal{P}$, $P^\vee$ may have non-smooth vertices), broken maps to $X$ may be viewed as collections of maps $u_v : C_v \to X_{P(v)}^0$ defined on punctured nodal curves $C^0 := C \setminus \{\text{nodes}\}$ whose target space $X$ is the complement of boundary divisors $X_Q$ in a broken manifold. With this viewpoint, all the exposition about broken maps in manifolds extends to the orbifold case. We make explicit remarks where there are differences.
Remark 3.21. (Broken maps for a single cut) In case of a single cut, the target space of a broken map is the broken manifold $\mathcal{X} = X_+ \cup Z(\mathbb{P}^1) \cup X_-$ defined in Section 2.2. The dual complex $B$ is a line segment $B \cong [-1, 1]$ whose endpoints $\{\pm 1\}$ correspond to the cut pieces $X_\pm$. The tropical type of a broken map $u : C \to \mathcal{X}$ modelled on a graph $\Gamma$ consists of the following information: For any vertex $v \in \text{Vert}(\Gamma)$ whether the curve component $C_v$ maps to $X_+$, $X_-$ or the neck piece $Z(\mathbb{P}^1)$, and for any edge $e = (v_+, v_-) \in \text{Edge}_\bullet(\Gamma)$, a slope $T(e) \in \mathbb{Z}$. The sign of $T(e)$ indicates the orientation of $e$ in the dual complex $B$, and $|T(e)|$ is equal to the intersection multiplicity of the map with the divisor $Y$ at the nodal point $w_+$, $w_-$. In particular, the intersection multiplicities are equal for both lifts of the node:

$$m_{w_+}(e)(u_{v_+}, Y) = m_{w_-}(e)(u_{v_-}, Y).$$

The matching condition for an edge $e$ only consists of a horizontal matching condition:

$$\pi_Y(u_{v_+}(w_+(e))) = \pi_Y(u_{v_-}(w_-(e))).$$

The vertical matching condition is vacuous: the fiber $N_Y \to Y$ is one-dimensional (in $\mathbb{C}$), and therefore the coefficient of the leading order term can be matched by adjusting the domain coordinates. This ends the Remark.

Remark 3.22. (A comparison with symplectic field theory) Holomorphic buildings in symplectic field theory [9] are comparable to broken maps on manifolds with a single cut. The target space of a holomorphic building consists of $X^+$, $X^-$ and $k-1$ copies of the neck piece $Z(\mathbb{P}^1)$ for some $k \geq 1$:

$$\mathcal{X}[k] := X_+ \cup_Y Z(\mathbb{P}^1) \cup_Y \cdots \cup_Y Z(\mathbb{P}^1) \cup_Y X_-,$$

and any pair of consecutive pieces are identified along a divisor $Y$. A holomorphic building $u : C \to \mathcal{X}[k]$ is a continuous map, where nodes map to the divisor $Y$, and intersection multiplicities $m_{w_+}(e)(u_{v_+}, Y)$ are equal on both sides.

A holomorphic building differs from a broken map in two ways: it is a continuous map, and it remembers an ordering for the neck piece components. In the broken map view, this ordering is not important: With suitable regularity assumptions a broken $u$ map with $m$ components in neck pieces can be glued to give a $2m$-dimensional family of unbroken maps in $X^\nu$ for any $\nu$. Any sequence of maps $u_\nu : C_\nu \to X^\nu$ lying in the glued family converges to a broken map $u' : C' \to \mathcal{X}$ that is related to $u : C \to \mathcal{X}$ by a tropical symmetry (as in Definition 3.23). The choice of the sequence $\{u_\nu\}_\nu$ determines to which of the neck pieces $X[k]$ a $S_\nu \subset C$ component maps. For a broken map, the ordering of the pieces is not a combinatorial invariant of the moduli space. We do not fully prove the equivalence in this paper, and instead give a gluing proof only for rigid types, see Theorem 8.1 below. One effect of the differing definitions is the following: Unlike holomorphic buildings, broken maps do not have trivial cylinders. In holomorphic buildings trivial cylinders have to be inserted whenever there is a node between components that are not in adjacent levels in order to achieve continuity. This ends the Remark.
3.4. **Symmetries and isomorphisms of broken maps.** The definition of equivalence for broken maps involves not only automorphisms of the domains, but the torus actions on the “neck pieces” in the degeneration. These equivalences are encoded in the action of a *tropical symmetry group* defined as follows.

**Definition 3.23.** (Tropical symmetry) A *tropical symmetry* for a tropical graph $\Gamma$ is a tuple

$$
g = ((g_v)_{v \in \text{Vert}(\Gamma)}, (z_e)_{e \in \text{Edge}(\Gamma)}), \quad g_v \in T_{P(v), \mathbb{C}}, \quad z_e \in \mathbb{C}^\times
$$

consisting of a translation $g_v$ for each vertex and a change of local coordinate $z_e$ for each edge that satisfies

$$
g_{v_+} g_{v_-}^{-1} = z_e^{T(e)} \quad \forall e = (v_+, v_-) \in \text{Edge}(\Gamma).
$$

A tropical symmetry $(g, z)$ acts on a broken map $(u,f)$ as

$$
u_v \mapsto g_v u_v, \quad f_e \mapsto z_e f_e.
$$

The group of tropical symmetries is denoted

$$
T_{\text{trop}}(\Gamma) := \{(g_v)_{v \in \text{Vert}(\Gamma)}, (z_e)_{e \in \text{Edge}(\Gamma)} | (3.15)\}.
$$

We often drop the term $(z_e)_{e}$ from the notation of a tropical symmetry element.

The condition (3.15) is a necessary and sufficient condition for the translations $(g_v)_v$ to preserve the matching condition at nodes.

**Remark 3.24.** (Framing symmetry) There are finitely many tropical symmetries $(g, v) \in T_{\text{trop}}(\Gamma)$, called *framing symmetries*, for which the action on the unframed map $u$ of type $\Gamma$ is trivial, that is $g_v = \text{Id}$ for all vertices $v \in \text{Vert}(\Gamma)$. The group of framing symmetries is the product $\oplus_{e \in \text{Edge}(\Gamma)} \mathbb{Z}_{n_e}$, where $n_e$ is the order of ramification of the broken map at the nodal point $w_e$, see Remark 3.11.

The next result shows that the identity component of the tropical symmetry group is generated by tropical weights of vertices of the tropical graph.

**Lemma 3.25.** (Tropical weights generate tropical symmetries)

(a) For a tropical graph $\Gamma$, the set of tropical weights $W(\Gamma)$ (defined in (3.3)) is convex.

(b) (From tropical weights to tropical symmetries) If a tropical graph $\Gamma$ has two distinct tropical weights

$$
(T_0(v), v \in \text{Vert}(\Gamma)), \quad (T_1(v), v \in \text{Vert}(\Gamma))
$$

then the difference $T_1 - T_0$ generates a real-two-dimensional subgroup $\exp((T_1 - T_0)(\cdot))$ of the tropical symmetry group $T_{\text{trop}}(\Gamma)$ (defined in (3.16)).

(c) The subgroup

$$
T_{\text{trop}, W}(\Gamma) := \langle T_1 - T_0 | T_0, T_1 \in W(\Gamma) \rangle
$$

generated by tropical weights is the identity component of $T_{\text{trop}}(\Gamma)$. 

Proof. For the first statement, if $T_0, T_1 \in \mathcal{W}(\Gamma)$ are weights for a tropical graph $\Gamma$, then for any $t \in [0, 1]$

$$(1 - t)T_0 + tT_1 \in \mathcal{W}(\Gamma)$$

is a tropical weight. Assuming that the weights $T_0, T_1$ are distinct, and that the numbers $l_{e,i} \in \mathbb{R}$ satisfy

$$T_i(v_+) - T_i(v_-) = l_{i,e}T_e, \quad i = 0, 1, e \in \text{Edge}_e(\Gamma),$$

then

$$g : \mathbb{C} \to T_{t}\text{rop}(\Gamma), \quad z \mapsto \begin{cases} e^{(T_1(v) - T_0(v))z}, & v \in \text{Vert}(\Gamma) \\ e^{(l_{1,e} - l_{0,e})z}, & e \in \text{Edge}_e(\Gamma) \end{cases}$$

is a non-trivial group of symmetries for broken maps modelled on $\Gamma$. In a similar way, we see that the Lie algebra $T_{t}\text{rop}(\Gamma)$ is equal to the vector space generated by differences of tropical weights $T_1 - T_0, T_0, T_1 \in \mathcal{W}(\Gamma)$, from where (c) follows. □

Definition 3.26. A tropical graph is rigid if its tropical symmetry group $T_{t}\text{rop}(\Gamma)$ is finite.

![Figure 9](image-url) The dual complex $B^\lor$ for the multiple cut $B = \bigcup_{P \in \mathcal{P}} P$ in Figure 5 is a rectangle. The tropical graph $\Gamma_1$ (left figure) is rigid, but $\Gamma_2$ (right figure) is not rigid since the vertices inside the square can be moved to the dotted positions.

Lemma 3.27. For any tropical graph, the quotient $T_{t}\text{rop}(\Gamma)/T_{t}\text{rop,\mathcal{W}}(\Gamma)$ is finite.

Proof. The quotient is discrete because by Lemma 3.25 $T_{t}\text{rop,\mathcal{W}}(\Gamma)$ is the identity component of $T_{t}\text{rop}(\Gamma)$. Furthermore, every connected component of $T_{t}\text{rop}(\Gamma)$ deformation retracts to a component of the maximal compact subtorus

$$T_{t}\text{rop}(\Gamma)^{\text{im}} := \{(g, z) \in T_{t}\text{rop}(\Gamma) : g_v \in T_P(v), |z_e| = 1\}.$$

Indeed, an element $(g, z) \in T_{t}\text{rop}(\Gamma)$ can be written as

$$g_v = k_v e^{i\gamma_v}, \quad k_v \in T_P(v), s_v \in t_P(v), \quad z_e = \theta_e e^{\alpha_e}, \quad \theta_e \in S^1, \alpha_e \in \mathbb{R},$$

and $(k, \theta) := ((k_v)_{v}, (\theta_e)_e)$ is a tropical symmetry element in $T_{t}\text{rop}(\Gamma)^{\text{im}}$ which is connected to $(g, z)$ via the path

$$[0, 1] \ni \tau \mapsto ((k_v e^{i\gamma_v})_{v}, (\theta_e e^{\alpha_e})_e) \in T_{t}\text{rop}(\Gamma).$$

The quotient $T_{t}\text{rop}(\Gamma)/T_{t}\text{rop,\mathcal{W}}(\Gamma)$ is finite because it is in bijection with the connected components of the compact subgroup $T_{t}\text{rop}(\Gamma)^{\text{im}}$. □
Example 3.28. The tropical graphs in Figure 10 are rigid, but have non-trivial tropical symmetry groups. Suppose the tropical graph $\Gamma_1$ has edge slopes

$$T_{e_1} = (-1, -1), \quad T_{e_2} = (-1, 2), \quad T_{e_3} = (2, -1).$$

A tropical symmetry $(g, z)$ on $\Gamma_1$ satisfies the equations

$$g_1 = g_2 = g_3 = \text{Id}, \quad g_1g_0^{-1} = z_{e_1}^{(-1, -1)}, \quad g_2g_0^{-1} = z_{e_2}^{(-1, 2)}, \quad g_3g_0^{-1} = z_{e_3}^{(2, -1)},$$

and is therefore given by

$$g_0 = \omega, \quad z_{e_1} = z_{e_2} = z_{e_3} = (\omega, \omega),$$

where $\omega \in e^{2\pi i k/3}$ is a cube root of unity. Thus $|T_{\text{trop}}(\Gamma_1)| = 3$. This tropical graph is similar to the example studied by Abramovich-Chen-Gross-Siebert [2, p51] and Tehrani [24, Section 6]. The tropical graph $\Gamma_2$ has edge slopes

$$T_{e_1} = (-1, 1), \quad T_{e_2} = (-1, -1).$$

By a similar calculation $|T_{\text{trop}}(\Gamma_2)| = 2$.

Definition 3.29. (Isomorphisms)

(a) An isomorphism between two broken maps $u : C \to \mathfrak{X}, u' : C' \to \mathfrak{X}$ is a biholomorphism $\phi : C \to C'$ such that $u = u' \circ \phi$. The group of automorphism of a map $u : C \to \mathfrak{X}$ is denoted $\text{Aut}(C, u)$.

(b) A reduced isomorphism between two broken maps $u : C \to \mathfrak{X}, u' : C' \to \mathfrak{X}$ is an isomorphism $\phi : C \to C'$ and a tropical symmetry

$$(g_v \in T_{P(v), C}), v \in \text{Vert}(\Gamma))$$

intertwining the maps $u, u'$ in the sense that

$$u'_v = g_v u_v \circ (\phi|_{C_v}), \quad \forall v \in \text{Vert}(\Gamma).$$

The group of reduced automorphism of a map $u : C \to \mathfrak{X}$ is denoted $\text{Aut}_r(C, u)$.
4. Stabilizing divisors

We use domain-dependent perturbations of the almost complex structure to regularize the moduli space of pseudoholomorphic disks/spheres. As in Cieliebak-Mohnke [16], we use Donaldson divisors to stabilize the domains so that domain-dependent perturbations are possible. We will require the interior markings on the domains of disks/spheres to coincide with the intersections points of the pseudoholomorphic map with the Donaldson divisor. If the divisor has high enough degree, all domain components are stable as curves. Consequently the almost complex structure can be perturbed to attain regularity on all domain components. Of course, one could envision using any of the current perturbation schemes to achieve virtual fundamental chains; the stabilizing divisor approach is merely a convenience.

4.1. Stabilizing divisors in smooth symplectic manifolds. We recall results about stabilizing divisors for unbroken manifolds. To define and construct Donaldson divisors we assume that all the symplectic forms are rational.

Definition 4.1. The symplectic manifold \((X,\omega)\) is rational if \([\omega] \in H^2(X,\mathbb{Q})\). A prequantum line bundle is a line-bundle-with-connection \(\tilde{X} \to X\) whose curvature is \((2\pi/i)\omega\). A Lagrangian \(L \subset X\) is (strongly) rational if there exists a prequantum bundle \(\tilde{X}\) an integer \(k\) and a flat section of the restriction \(\tilde{X} \otimes k|L\).

Definition 4.2. (a) A divisor in \(X\) is symplectic submanifold \(D \subset X\) of real codimension \(\text{codim}(D) = 2\).

(b) A tamed almost complex structure \(J \in J_\tau(X)\) is adapted to \(D\) if \(J(TD) = TD\). The space of tamed almost complex structures that are adapted to \(D\) is denoted by
\[
J_\tau(X, D) = \{J \in J_\tau(X) | J(TD) = TD\}.
\]

(c) Let \(D \subset X - L\) be a divisor disjoint from the Lagrangian \(L\). For \(E > 0\), an adapted almost complex structure \(J_D \in J_\tau(X, D)\) is \(E\)-stabilized by a divisor \(D\) iff
(i) (Non-constant spheres) \(D\) does not contain any \(J_D\)-holomorphic sphere \(u: \mathbb{P}^1 \to X\) with energy < \(E\),
(ii) (Sufficient intersections) any \(J_D\)-holomorphic sphere in \(X\) with energy < \(E\) has at least 3 distinct points of intersection with \(D\), and any \(J_D\)-holomorphic disk with energy < \(E\) has at least one intersection with \(D\).

A pair \((J, D)\) consisting of a divisor \(D\) and \(J \in J_\tau(X, D)\) is stabilizing if \(J\) is \(E\)-stabilizing for all \(E > 0\).

Proposition 4.3. (Existence of a stabilizing pair, [12, Section 4]) Suppose \(\omega \in \Omega^2(X)\) is a rational symplectic form and \(L \subset X\) is a compact (strongly) rational Lagrangian submanifold. Then, there exists a stabilizing pair \((J_0, D)\), and for every \(E > 0\) an open neighbourhood \(J(X, D; J_0, E)\) of \(J_0\) consisting of \(E\)-stabilizing almost complex structures adapted to \(D\). Additionally, \(L\) is exact in \(X - D\).
Remark 4.4. We recall some details in the proof of Proposition 4.3 which are used in constructing stabilizing pairs for broken manifolds. For a tame almost complex structure $J \in J_{\text{r}}(X)$, an integer $k_0 \in \mathbb{N}$ is a degree bound for stabilization if for any homology class $\alpha \in H^2(X, \mathbb{Z})$ that is represented by a non-constant $J$-holomorphic sphere $u : \mathbb{P}^1 \to X$,

\begin{equation}
  k_0 \omega(\alpha) \geq 2(c_1(X), \alpha) + \dim(X) + 1.
\end{equation}

If a divisor $D$ is Poincaré dual to $k_0[\omega]$ satisfying (4.1), and if $J$ is generic, then $(J, D)$ is stabilizing by [16], [12, Section 4].

For the proof of Proposition 4.3 one starts with a preliminary almost complex structure $J_{\text{pre}}$ that is $\omega$-compatible. For any $\epsilon > 0$ there exists a constant $k(\epsilon, J_{\text{pre}})$ that is a degree bound for stabilizing for any $J \in J_{\text{r}}(X)$ satisfying $\|J - J_{\text{pre}}\|_{C^0} \leq \epsilon$. This fact gives enough wiggle room to find a stabilizing pair. Indeed, by the adaptation of Donaldson’s construction in Auroux-Gayet-Mohsen [4], one can find an approximately $J_{\text{pre}}$-holomorphic divisor $X - L$. There is an open subset of $J(X, D)$ contained in $B_\epsilon(J_{\text{pre}})$. Therefore for a generic $J_0$ in $J(X, D) \cap B_\epsilon(J_{\text{pre}})$, the pair $(J_0, D)$ is stabilizing. See [16, Section 8] for details.

We recall from [16, Lemma 8.13] how the uniform degree bound $k(\epsilon, J_{\text{pre}})$ for stabilization is determined. Let $\alpha \in \Omega^2(X)$ be a representative of $c_1(TX) \in H^2(X)$. For any $0 < \epsilon < 1$, we have

\begin{equation}
  \alpha(v, K v) \leq k_* \omega(v, K v), \quad \forall v \in TX, K \in J^r(X) \cap B_\epsilon(J_0),
\end{equation}

where

\[
  k_* = \frac{(1 + \epsilon)}{(1 - \epsilon)} \|\alpha\|.
\]

The estimate (4.2) implies that

\[
  c_1(\alpha) \leq k_* \omega(\alpha)
\]

for any homology class $\alpha \in H_2(X, \mathbb{Z})$ that is represented by a $K$-holomorphic sphere for $K \in J^r(X) \cap B_\epsilon(J_0)$. Then, any constant

\[
  k \geq 2k_* + 2n + 1
\]

is a degree bound for stabilizing for almost complex structures in a $\epsilon$-neighbourhood of $J_0$.

By [12, Theorem 3.6], if the Lagrangian $L$ is rational, then for any divisor $D$ in $X - L$ produced by Auroux-Gayet-Mohsen [4], $L$ is exact in $X - D$. Further, if $X$ is a broken manifold, $D \subset X - L$ is a broken submanifold as in definition 4.8 below, $L$ is contained in a cut space $X_0 \subset X$, and $L$ is exact in $(X_0, D \cap X_0)$, then, $L$ is exact in the glued family $X^\nu - D^\nu$. This ends the Remark.

Remark 4.5. Exactness allows the following relation of area to intersection numbers. If $L$ is exact in $X - D$, then, the intersection number of a disk $u : (C, \partial C) \to (X, L)$ with the divisor $D$ is proportional to the area of the disk, see [12, Lemma 3.4]. In particular, if $[D]^\nu = k[\omega]$ for some $k \in \mathbb{Z}$, then,

\begin{equation}
  \# u^{-1}(D) = k \int_C u^* \omega.
\end{equation}
4.2. Stabilizing divisors in broken manifolds. In this section, we construct stabilizing divisors in a broken manifold by a modification of Donaldson’s construction.

We recall from (2.29) that the symplectic cylindrical structure on cut spaces induces projection maps on neighborhoods of boundary submanifolds

\[ \pi_Q : U_{X_P} X_Q \rightarrow X_Q \]

for all pairs of polytopes \( Q \subset P \).

**Definition 4.6.** (Strong compatibility on broken manifolds) Let \( X \) be a broken manifold with a symplectic cylindrical structure, see 2.27. An almost complex structure \( J = (J_P)_{P \in \mathcal{P}} \) is strongly compatible if

(a) \( J \) satisfies the properties of a cylindrical almost complex structure in Definition 2.35 except that the inner product on \( t \) from (2.24) is not fixed.

(b) For all \( P \in \mathcal{P} \) the projection \( d\pi_P(J_P) \) is \( \omega_P \)-compatible on \( X_P \).

**Remark 4.7.** We defined a notion of strong compatibility for neck-stretched manifolds (see Definition 2.32), which is different from strong compatibility for neck-stretched manifolds. The \( t \)-inner product is fixed by (2.24) for strongly compatible almost complex structures on neck-stretched manifolds. Therefore, a family \( J^\nu \) of strongly compatible almost complex structures on neck-stretched manifolds \( X^\nu \) does not converge to a strongly compatible almost complex structure \( J \) on \( X \).

**Definition 4.8.** (Divisor in a broken manifold) Suppose \( X \) is the broken manifold associated to a collection of polytopes \( \mathcal{P} \). A broken divisor \( \mathcal{D} \) in \( X \) is a collection of divisors

\[ \mathcal{D} = \{ D_P \subset X_P, \ P \in \mathcal{P} \} \]

such that

(a) \( D_P \) is a lift of a symplectic divisor \( D_P \subset X_P \) by the projection map \( \pi_P : X_P \rightarrow X_P \), and

(b) for any pair of polytopes \( Q \subset P \), \( D_Q \) is the intersection \( D_P \cap X_Q \).

The broken divisor \( \mathcal{D} \) is cylindrical, if for any pair of polytopes \( Q \subset P \), \( \text{codim}(P) = 0 \), the divisor \( D_P \subset X_P \) is \( Q \)-cylindrical in the neighbourhood of \( X_Q \). That is, in the neighbourhood \( U_P^Q \subset X_P \) of \( X_Q \) from (4.4), \( D_P \cap U_Q \) is equal to \( \pi_Q^{-1}(D_Q) \).

The following is the main result of the section.

**Proposition 4.9.** Let \( X \) be a broken manifold with a strongly compatible almost complex structure \( J \). For any \( \theta > 0 \), there is a cylindrical \( \theta \)-approximately holomorphic divisor \( \mathcal{D} \subset X \).

We refer the reader to [22] for the definition of \( \theta \)-approximate holomorphicity. Before constructing the divisor, we show that a broken divisor \( \mathcal{D} \) can be isotoped to a cylindrical divisor if \( \mathcal{D} \) intersects boundary divisors in \( X \) \( \omega \)-orthogonally.

**Lemma 4.10.** (Making a broken divisor cylindrical) Suppose \( X \) is a broken manifold with a strongly compatible cylindrical almost complex structure \( J \). For any \( 0 < \theta_0 < 1 \), there exists \( \theta_1 \) such that the following is satisfied. Let \( \mathcal{D} \subset X \) be a broken
divisor that is $\theta_1$-approximately $J$-holomorphic and intersects boundary divisors $\omega$-orthogonally. Then there is a cylindrical broken divisor $D_1$ that is homotopic to $D$ and which is $\theta_0$-approximately holomorphic.

Proof of Lemma 4.10. The proof is an induction on the face structure. For any pair $Q \subset P$, $\text{codim}(P) = 0$ let

$$\psi_Q^P : U_{X,P}X_Q \to (\text{Cone}_{P'}(Q') \times Z_Q, \omega_Q) / \sim$$

be the symplectic cylindrical structure on a neighborhood $U_{X,P}X_Q \subset X_P$ of boundary submanifold $X_Q$, where the symplectic form $\omega_Q$ is as in (2.28) and the equivalence relation $\sim$ mods out boundaries by circle actions as described in 2.28. Since the divisor $D$ intersects boundary submanifolds $\omega$-orthogonally, by the relative symplectic neighbourhood theorem 2.43, there is a different symplectic cylindrical structure $\phi_Q^P : U_{X,P}X_Q \to (\text{Cone}_{P'}(Q') \times Z_Q, \omega_Q)$ that is adapted to $D$.

The map $(\psi_Q^P)^{-1} \circ \phi_Q^P$ is homotopic to the identity, and is equal to identity on $X_Q \setminus \cup_{R \subset Q} U_{X,R}X_R$. Therefore, there exist truncations of the cylindrical ends $U'_{X,P}X_Q \subset U_{X,P}X_Q$ for all $Q \subset P$ and diffeomorphisms $\phi_p : X_P \to X_P$ for all top-dimensional $P$,

- that are equal to $(\psi_Q^P)^{-1} \circ \phi_Q^P$ on $U'_{X,P}X_Q \subset U_{X,P}X_Q$;
- that are equal to the identity in the complement of all the cylindrical neighborhoods $U_{X,P}X_Q$; and
- for any face $Q \subset P$, $\phi_P(X_Q) = X_Q$ and for any pair of top-dimensional polytopes $P_0, P_1$ containing $Q$, $\phi_{P_0}|X_Q = \phi_{P_1}|X_Q$.

The map $\phi_P$ is constructed inductively. It is first defined on $X_Q$ where $Q \in P$ is the least dimensional face of $P$ and then extended to higher dimensional faces. The broken divisor

$$D_1 := (D_{1,P} \subset X_P)_{P \in P, \text{codim}(P) = 0}, \quad D_{1,P} := \phi_P(D)$$

is cylindrical. If the truncated neighborhoods $U_{X_P}X_Q$ are small enough, $\phi_P$ is small in the $C^1$-norm, and $D_1$ is $\theta_0$-approximately holomorphic. Therefore the broken divisor $D_1$ satisfies the properties required by the Proposition. \qed

In the rest of this section, we construct a stabilizing divisor in the cut spaces of a broken manifold that intersects boundary divisors $\omega$-orthogonally. The construction of the divisors is via a slight modification of Donaldson’s technique [22]. Let $\tilde{X} \to X$ be a line-bundle with connection $\alpha$ over $X$ whose curvature two-form $\text{curv}(\alpha)$ satisfies $\text{curv}(\alpha) = (2\pi/i)\omega$; since our symplectic manifolds are rational we may always assume this to be the case after taking a suitable integer multiple of the symplectic form.

Definition 4.11. (Asymptotically holomorphic sequences of sections) Let $(s_k)_{k \geq 0}$ be a sequence of sections of $\tilde{X}^k \to X$. 

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In both definitions the norms of the derivatives $s_k$ on $X_k$ as functions of $k$, the sequence $\{s_k\}_{k \geq 0}$ is asymptotically holomorphic if there exists a constant $C$ and integer $k_0$ such that for $k \geq k_0$,

$$|s_k| + |\nabla s_k| + |\nabla^2 s_k| \leq C k^{-1/2}.$$  

(b) The sequence $\{s_k\}_{k \geq 0}$ is uniformly transverse to 0 if there exists a constant $\eta$ independent of $k$, and $k_0 \in \mathbb{Z}$ such that for any $x \in X$ and $k \geq k_0$ with $|s_k(x)| < \eta$, the derivative $\nabla s_k$ of $s_k$ is surjective at any point and satisfies $|\nabla s_k(x)| \geq \eta$.

In both definitions the norms of the derivatives $\nabla s_k$ are evaluated using the metric $g_k = k\omega(\cdot, J\cdot)$.

**Proposition 4.12.** (Construction of a broken divisor) Suppose $X$ is a broken manifold, for which the symplectic form $\omega_{X_P}$ on each of the cut spaces $X_P$, $P \in \mathcal{P}$ is rational. Given a strongly compatible cylindrical almost complex structure $J$ on $X$, there is a sequence of asymptotically holomorphic and uniformly transverse sections $\{s_{k,P} : X_P \to \tilde{X}_P\}$ for all $P \in \mathcal{P}$ such that for any pair of polytopes $Q \subset P$, $s_{k,P}|_{X_Q} = s_{k,Q}$ and the zero set $s_{k,P}^{-1}(0)$ intersects $X_Q$ $\omega$-orthogonally.

**Proof.** The sections are constructed by running Donaldson’s procedure simultaneously for all the manifolds in the set $\{X_P\}_{P \in \mathcal{P}}$. To study sections on the line bundle $\tilde{X}_P \to X_P$, we use the metric

$$g_k := k\omega(\cdot, J\cdot)$$

on $X_P$. Under this metric, the effects of the non-integrability of $J$ become negligible as $k$ increases.

We first describe a set of center points for the Gaussian section for each tensor power of the given line bundle. Given $k \gg 0$, a set of center points $\Lambda_{k,P} \subset X_P$ is defined so that it contains $C_k^{\dim(X_P)}$ number of points, and $X_P$ is covered by the following neighbourhoods :

$$X_P = (\cup_{x \in \Lambda_{k,P}} B_{g_k}(x, 1)) \cup (\cup_{Q \subset P} B_{g_k}(X_Q, 1)).$$

The set of center points also satisfies the condition that for any pair of polytopes $Q \subset P$

$$\Lambda_{k,Q} \subset \Lambda_{k,P}, \quad \text{and} \quad x \in \Lambda_{k,P} \setminus \Lambda_{k,Q} \implies d_{g_k}(x, X_Q) > 1.$$  

Let $\Lambda_k$ be the union $\cup_P \Lambda_{k,P}$. The set $\Lambda_k$ is partitioned into subsets $I_1, \ldots, I_N$ where $N$ is independent of $k$ while satisfying the following: For any pair $x \in \Lambda_{k,P} \setminus \cup_{Q \subset P} \Lambda_{k,Q}$ and $y \in \cup_{Q \subset P} \Lambda_{k,Q}$

$$x \in I_\alpha, y \in I_\beta \implies \beta < \alpha.$$  

For any center point $p$, we now define an asymptotically holomorphic Gaussian section centered at $p$ for every manifold $X_P$ that contains $p$. Let $P_0 \in \mathcal{P}$ be the smallest polytope so that $X_{P_0}$ contains $p$. Let $(z_1, \ldots, z_m)$ be Darboux coordinates on $X_{P_0}$ centered at $p$ which are complex linear at $p$. Fix a local trivialization $\tilde{X}^k_{P_0} |_{B_{g_k}(x, 1)} \cong B_{g_k}(x, 1) \times \mathbb{C}$ of the line bundle $\tilde{X}^k_{P_0} \to X_{P_0}$ so that the connection is given by the one-form $\sum_i (z_i d\bar{z}_i - \bar{z}_i dz_i)$. Define a Gaussian section $\sigma_{k,p,P_0}(z)$ as
\[ e^{-k|z|^2} \text{ multiplied by a cutoff function } \rho_{k,p} \text{ vanishing at a } g_k\text{-distance of } k^{1/6} \text{ and extended by zero.} \]

Extensions of Gaussian sections to higher dimensional manifolds are defined using identifications of normal bundles to neighbourhoods of submanifolds. Since \( \omega \) and \( J \) have the same cylindrical structure, there is a set of consistent symplectomorphisms \( \{ \phi_Q^p \}_{Q \subset P} \)

\[
\phi_Q^p : N_{X_p} X_Q \to U_{X_p} X_Q,
\]
each of which maps the neighborhood of the zero section of the normal bundle \( N_{X_p} X_Q \) to a neighborhood \( U_{X_p} X_Q \) of \( X_Q \) in \( X_P \), and the fibers are biholomorphic to a neighborhood of the origin in \( \mathbb{C}^{\text{codim}(Q)} \). For \( p \in \Lambda_{k,Q} \), Darboux coordinates \((z_1, \ldots, z_n)\) in \( X_P \) are defined so that a subset of components, say \((z_1, \ldots, z_{\dim Q})\), is the lift of Darboux coordinates in \( X_Q \). The other components \((z_{\dim Q+1}, \ldots, z_n)\) are coordinates in the fiber of the normal bundle \( N_{X_p} X_Q \), and for any \( z_i, i > \dim Q \), the zero set \( \{ z_i = 0 \} \) is a boundary divisor \( X_P^{\prime} \) of \( X_P \) that contains \( X_Q \). Next, choose a trivialization of \( \hat{X}_P \) that extends the trivialization of \( \hat{X}_Q \) for any \( Q \subset P \), \( p \in \Lambda_{k,Q} \), and so that the connection is \( \sum i \kappa(z_i d\bar{z}_i - \bar{z}_i dz_i) \) plus terms higher order in \( z_i, \bar{z}_i \). Finally, define Gaussian sections \( \sigma_{k,p, p}(z) \) as the product of \( e^{-k|z|^2} \) with a cutoff function \( \rho_{k,p, p} \) vanishing at a \( g_k \)-distance of \( k^{1/6} \) and extended by zero. The Gaussian sections are approximately holomorphic.

The Gaussian sections are approximately holomorphic and have the following properties for polytopes \( Q \subset P \):

(a) (Restriction) if \( p \in \Lambda_{k,Q} \subset \Lambda_{k,P} \), then \( \sigma_{k,p, p}|X_Q = \sigma_{k,p, Q} \),

(b) (Transversality) if for some constants \( \{ w_p \}_{p \in \Lambda_{k,Q}} \), the section

\[
\sigma_{w,Q} := \sum_{p \in \Lambda_{k,Q}} w_p \sigma_{k,p, Q}
\]

intersects the zero section \( \eta \)-transversely, then in a neighbourhood \( B_{g_k}(X_Q, 1) \subset X_P \), the section

\[
\sigma_{w,P} := \sum_{p \in \Lambda_{k,Q}} w_p \sigma_{k,p, P}
\]

is \( \eta/e \)-transversal.

(c) (\( \omega \)-orthogonality) If the transversality in the previous item holds, then the zero set \( \sigma_{w, p}^{-1}(0) \subset X_P \) is a \( Q \)-cylindrical divisor in a neighbourhood of \( X_Q \), and projects to \( \sigma_{w, Q}^{-1}(0) \) on \( X_Q \). Consequently for any polytope \( P_0 : Q \subset P_0 \subset P \), the zero set \( \sigma_{w, P}^{-1}(0) \subset X_P \) intersects the submanifold \( X_{P_0} \) \( \omega \)-orthogonally.

Gaussian sections whose centers are close to boundary divisors are multiplied by a factor that vanishes up to order two at the boundary divisor. Consider a lattice point \( p \) and let \( P_0 \) be the smallest polytope that contains \( p \). We know that \( d_{g_k}(p, Q) \geq 1 \) for all facets \( Q \subset P_0 \). Let \( Q_1, \ldots, Q_\alpha \subset P_0 \) be facets for which \( 1 \leq d_{g_k}(p, Q_i) \leq k^{-1/6} \). Since the identifications of normal bundles in (4.6) are holomorphic on fibers, and the fiber coordinates can be chosen to be holomorphic on fibers. The fiber coordinate \( z_i \),
on the normal bundle $N_{P_i}Q_i$ vanishes on $Q_i$ and $\overline{\partial}z_i(p) = 0$. The modified Gaussian section defined as
\[
\sigma'_{k,p,P} = \sigma_{k,p,P} \cdot \prod_{i=1}^n z_i^2 / z_i(p)^2
\]
is asymptotically holomorphic and has a uniform lower bound on $B_{gk}(p,1)$. (The second power ensures that not only does the factor $z_i^2 / z_i(p)^2$ vanish along the relative divisor, but the derivative of $z_i^2 / z_i(p)^2$ also vanishes, so that the resulting stabilizing divisor will be $\omega$-orthogonal to the relative divisor.) If $p$ is not close to any boundary divisors then $\alpha_i = 0$, and the Gaussian section $\sigma_{k,p,P}$ at $p$ is left unchanged. Such a section $\sigma_{k,p,P}$ vanishes on boundary divisors $X_{Q_1}, \ldots, X_{Q_\alpha}$ because of the cutoff function $\rho_{k,p,P}$.

The globalization process consists of finding coefficients $\{w_p \in \mathbb{C}\}_{p \in \Lambda_k}$ such that
\[
\sum_{p \in \Lambda_k, P} w_p \sigma'_{k,p,P}
\]
is a uniformly transverse sequence of sections for each $P$. The proof of globalization carries over from [22]. The only new feature is to determine each coefficient $w_p$ in a $P$-independent way. This can be done by Lemma 4.13 which is a modification of [22, Theorem 20].

Finally we show that the zero set of the transverse section intersects boundary divisors orthogonally. Suppose the globalization process produces constants $w_p \in \mathbb{C}$, $p \in \Lambda_k$ such that the section $s_{w,P} := \sum_{p \in \Lambda_k, P} w_p \sigma'_{k,p,P}$ is transverse to the zero section for all $P$. Consider a manifold $X_P$ and a boundary divisor $X_Q \subset X_P$. The section $s_{w,Q}$ vanishes near $X_Q$ if $d_k(p,X_Q) > k^{-1/6}$. If for a lattice point $1 < d_k(p,X_Q) < k^{-1/6}$, then both the section $\sigma'_{k,p,P}$ and its derivative normal to $X_Q$ vanish along $X_Q$. Therefore, on $X_Q$, the section $s_{w,P}$ is equal to $s_{w,Q}$, and $\omega$-orthogonality follows from that of the section $\sum_{p \in \Lambda_k, Q} w_p \sigma'_{k,p,P}$, which is equal to $s_{w,Q} e^{\lvert z \rvert^2}$.

**Lemma 4.13.** (Quantitative Sard’s theorem) Given a tuple of positive integers $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ there is an integer $p$ for which the following is satisfied. Suppose $0 < \delta < \frac{1}{4}$, and $f_i : B^n_+ \to \mathbb{C}$ is a set of $k$ functions on balls $B^n_+ \subset \mathbb{C}^{n_i}$ of radius $\frac{11}{10}$ that satisfy $\lVert f_i \rVert_{C^1} \leq \eta$, where $\eta := \delta \log(\delta^{-1})^{-p}$. Then, there exists $w \in \mathbb{C}$, $\lvert w \rvert \leq \delta$ such that $f_i w$ is $\eta$-transverse to 0 over the interior ball $B^n_+$ of radius 1.

The case $k = 1$ is Theorem 20 of [22], and the proof is by bounding the size of the image $f(B_+)$ in the range. For a finite $k$, the volume is multiplied by a constant, and the proof in [22] can be replicated by altering the constants.

**Proof of Proposition 4.9.** The proof is a consequence of Lemma 4.10 and Proposition 4.12. \qed
4.3. Stabilizing pairs in neck-stretched manifolds.

**Definition 4.14.** (Adapted broken almost complex structure) Given a broken divisor $D \subset X$, we denote by

$$\mathcal{J}^{\text{cyl}}(X, D) = \{ J = (J_P) | J_P(TD_P) = TD_P \forall P \in \mathcal{P} \}$$

the space of cylindrical almost complex structures that are adapted to $D$. The subset of tamed adapted almost complex structures is denoted by $\mathcal{J}^{\text{cyl}}(X, D)$. 

**Definition 4.15.** (Stabilizing pair in a broken manifold) Let $D \subset X$ be a broken cylindrical divisor which is disjoint from the Lagrangian $L \subset X$. For $E > 0$, an adapted almost complex structure $J \in \mathcal{J}^{\text{cyl}}(X, D)$, is $E$-stabilized if for every polytope $P \in \mathcal{P}$, the almost complex structure $J|X_P$ is $E$-stabilized in $(X_P, \omega_{X_P}, D_P)$. 

A pair $(J, D)$ is stabilizing if $J$ is $E$-stabilizing for all $E > 0$.

The existence theorem for stabilizing pairs extends to the broken case.

**Proposition 4.16.** (Stabilizing pair in a broken manifold) Suppose $X_\nu$ is a broken manifold, such that the symplectic form $\omega_{X_\nu}$ on all of the cut space $X_\nu$, $\nu \in \mathcal{P}$, is rational. Suppose $J^{\text{pre}} \in \mathcal{J}(X)$ is a strongly compatible almost complex structure. Then, there exists a stabilizing pair $(J_0, D)$ in $X$ such that

(a) the family $(J_\nu, D_\nu)$ obtained by gluing consists of stabilizing pairs for all $\nu \in [1, \infty)$,

(b) and for every $E > 0$ a neighbourhood $\mathcal{J}^{\text{cyl}}(X, D; J_0, E)$ of $J_0$ consisting of $E$-stabilizing cylindrical almost complex structures adapted to $D$.

The proof is based on the following Lemma which shows that a degree bound for stabilization can be chosen uniformly for all glued manifolds $X_\nu$.

**Proof of Proposition 4.16.** First we choose a uniform degree bound for stabilization that works for $X$ and neck-stretched manifolds $X_\nu$. To derive the bound we need, on each $X_\nu$, a choice of representative of the class $[\omega]$ of the symplectic form and the first Chern class $[c_1(TX)]$. We choose the classes to be basic forms $\omega^{\text{bas}}_\nu$, $\alpha_\nu \in \Omega^2(X_\nu)$: For any $Q$ these forms are pull-backs of forms on $X_\nu$ by the projection map $\pi_Q$

$$X_\nu \supset \Phi^{-1}(Q^*) \times Q^* \xrightarrow{\pi_Q} X_\nu^0.$$ 

Such forms indeed exist and converge to basic forms on the broken manifold by Proposition 6.27. An advantage of basic forms is that for a family of cylindrical neck-stretched almost complex structures $(J_\nu)_\nu$, if for some $\nu$ there exists $k$ such that

$$\alpha_\nu(v, J_\nu v) \leq kc_1(v, J_\nu v) \forall v \in TX_\nu$$

then the same estimate holds for all $\nu$.

We now derive the degree bound for stabilization. On the space of almost complex structures we define a metric by the summing the horizontal projections on cylindrical parts. For any $J_0, J_1$ that are cylindrical or strongly compatible, define

$$d^\bullet(J_0, J_1) := \sum_{P \in \mathcal{P}} \| d\pi_P(J_0) - d\pi_P(J_1) \|_{C^0(X_P^0)}.$$
There is a tamed cylindrical almost complex structure \( J_0 \) such that \( d^*(J_{\text{pre}}, J_0) = 0 \) where \( J_{\text{pre}} \) is the strongly compatible almost complex structure on \( X \) in the hypothesis of the Proposition. Indeed, we obtain \( J_0 \) from \( J_{\text{pre}} \) by changing the \( t \)-inner product to that fixed in (2.24), and this operation does not affect the horizontal projection of the almost complex structure. By the argument in Remark 4.4 for a small \( \epsilon \), and any \( J \in \mathcal{J}_{\text{cyl}}(X) \)

\[
\alpha(v, Jv) \leq k_* \omega(v, Jv) \quad \forall v \in TX^\nu, \ J \in B_\epsilon(J_0),
\]

where

\[
k_* = \frac{(1 + \epsilon)}{(1 - \epsilon)} \max_{P \in \mathcal{P}} ||\alpha||_{X^\bullet_P}
\]

where the norm on \( \alpha \) is with respect to the metric \( \omega_P(\cdot, J_0 \cdot) \) on \( X^\bullet_P \). Thus \( k_* \) is a degree bound for stabilization for \( J \in B_\epsilon(J_0) \cap \mathcal{J}_{\text{cyl}}(X) \).

The rest of the proof is as in the unbroken case in Cieliebak-Mohnke [16]. For a small enough \( \theta > 0 \), if \( D \) is a cylindrical divisor that is \( \theta \)-approximately \( J_{\text{pre}} \)-holomorphic, the set of adapted almost complex structures

\[
(4.7) \quad \mathcal{J}_{\text{cyl}}(X, D, J_0, \epsilon) := \mathcal{J}_{\text{cyl}}(X, D) \cap B_\epsilon(J_0)
\]

is non-empty. We remark that the almost complex structures in the above space are only \( d^* \)-close to \( J_{\text{pre}} \). This is not a problem because since \( D \) is cylindrical, so it is possible to change the underlying inner product on \( t \) by a large amount without affecting adaptedness. For all \( J \in \mathcal{J}_{\text{cyl}}(X, D) \cap B_\epsilon(J_0) \) and all \( J \)-holomorphic spheres and disks \( u \)

\[
c_1(u) \leq \omega(u).
\]

Therefore by calculations in Cieliebak-Mohnke [16], generic almost complex structures in the set \( \mathcal{J}_{\text{cyl}}(X, D, J_0, \epsilon) \) are stabilizing.

Next, we claim that for any \( E > 0 \), an open and dense subset of \( \mathcal{J}_{\text{cyl}}(X, D, J_0, \epsilon) \) is \( E \)-stabilizing for all neck lengths. For this purpose, it is enough to show that the set

\[
\mathcal{J}^* := \{(J, \nu) \in \mathcal{J}_{\text{cyl}}(X, D, J_0, \epsilon) \times [1, \infty] : J^\nu \text{ is } E \text{-stabilizing for } X^\nu\}
\]

is open and dense in \( \mathcal{J}_{\text{cyl}}(X, D, J_0, \epsilon) \times [1, \infty] \). The openness is a consequence of Gromov’s compactness, see the proof of [16, Corollary 8.16]. For multiply breaking manifolds, openness at the infinite neck length parameter is proved by Lemma 4.17.

The stabilizing condition is generic leading to density. Finally, let \( E_k \to \infty \) be any sequence of real numbers with limit infinity. The set of almost complex structures

\[
\mathcal{J}^\text{reg} = \bigcap_{k=1}^{\infty} \mathcal{J}^\text{reg,E_k}
\]

that is stabilizing for all \( \nu \in [1, \infty] \) is the intersection of the set of \( E_k \)-stabilizing almost complex structures for all \( k \). The intersection \( \mathcal{J}^\text{reg} \) is non-empty because of each of the sets in the intersection is open and dense.

The following Lemma, used above in the proof of Proposition 4.12, is an openness statement for stabilizing almost complex structures at \( \nu = \infty \).
Lemma 4.17. Suppose $D \subset X$ is a cylindrical broken divisor, and $\mathcal{J} \in \mathcal{J}^{\text{cyl}}(X, D)$ is a tamed adapted almost complex structure that is $E$-stabilizing. Suppose the divisors $D^\nu \subset X^\nu$ are obtained by gluing, and the sequence $J^\nu \in \mathcal{J}(X^\nu, D^\nu)$ converges to $\mathcal{J}$. Then, there exists $\nu_0$ such that $J^\nu$ is $E$-stabilizing for $\nu \geq \nu_0$.

Proof. Suppose $u_\nu : \mathbb{P}^1 \to X^\nu$ is a sequence of non-constant $J^\nu$-holomorphic spheres with area $\leq E$ that are not stabilized. That is, either the images are contained in the divisor $D^\nu$ or they have $\leq 2$ distinct points of intersection with the divisor. We will show that there is a unstabilized sphere in $X_P$ for some polytope $P \in \mathcal{P}$.

First consider the situation that the derivatives of $u_\nu$ are uniformly bounded. Then, the sequence $u_\nu$ converges horizontally in some polytope $P$. We then choose a translation sequence $t_\nu \in \nu P^\nu$ such that $e^{t_\nu}u_\nu$ is contained in a compact subset of $Z_{P,\mathbb{C}}$. Therefore, the sequence $e^{t_\nu}u_\nu$ converges uniformly to a $J_P$-holomorphic map $u : \mathbb{P}^1 \to X_P$, that is unstabilized. The basic area forms $u_\nu^*\omega^\nu$ converge to $u^*\omega_P$, and therefore the area of $\pi_P(u)$ is positive and $\leq E$.

If the derivatives on $u_\nu$ are not uniformly bounded, we produce a sphere by hard rescaling. By following the procedure in Step 2 of the proof of Theorem 7.1, we obtain a rescaled sequence of $J^\nu$-holomorphic maps $v_\nu : B_{r_\nu} \to X^\nu$ on balls $B_{r_\nu}$ that exhaust $\mathbb{C}$, a polytope $P$, and a sequence of translations $t_\nu \in \nu P^\nu$ such that $e^{t_\nu}u_\nu$ converges in $C^\infty_{\log}$ to a non-constant map $v : \mathbb{C} \to Z_{P,\mathbb{C}}$. As in the proof of Theorem 7.1, the map $v$ extends to $v : \mathbb{P}^1 \to X_P$. The projection $\pi_P \circ v : \mathbb{P}^1 \to X_P$ is non-constant: otherwise the image of $v$ is in a fiber $V_P$, which is a toric variety, and there is only one point $\infty \in \mathbb{P}^1$ that maps to toric divisors $V_Q, Q \subset P^\nu$ of $V_P$, and therefore, $v$ is constant. Since the basic area forms $v_\nu^*\phi_{R_{\nu,t}}^*\omega$ for $v_\nu$ converge to the basic area form $v^*\phi_{R_{\nu,t}}^*\omega$ for $v$, so Area($v$) $\leq E$. The sphere $v$ is not stabilized and the Lemma follows. $\blacksquare$
5. Coherent perturbations and regularity

In order to obtain the necessary transversality our Morse functions and almost complex structures must be allowed to depend on a point in the domain. A domain-dependent perturbation is defined as a map from a universal curve to the space of tamed almost complex structures. Any holomorphic map has as its domain a fiber of the universal moduli space of domain curves.

5.1. Domain-dependent perturbations. We first fix subsets of the universal treed disk on which to perturb. Let $\Gamma$ be a combinatorial type of treed disk and $\overline{U}_\Gamma = \overline{S}_\Gamma \cup \overline{T}_\Gamma$ its universal curve from (3.1). Fix a compact subset $\overline{T}_\Gamma^{cp} \subset \overline{T}_\Gamma$ containing, in its interior, at least one point $z \in T_e$ on the treed segment corresponding to every boundary edge $e \in \text{Edge}_e(\Gamma)$. Thus the complement $\overline{T}_\Gamma - \overline{T}_\Gamma^{cp} \subset \overline{T}_\Gamma$ is a neighbourhood of infinity on each edge. Also fix a compact subset $\overline{S}_\Gamma^{cp} \subset \overline{S}_\Gamma - \{w_e \in \overline{S}_\Gamma, e \in \text{Edge}_-(\Gamma)\}$ disjoint from the boundary and spherical nodes $w(e) \in C, e \in \text{Edge}(\Gamma)$, containing in its interior at least one point $z \in S_v$ on each sphere and disk component $S_v \subset C$ for each fiber $C \subset \overline{U}_\Gamma$. Furthermore, the complement $\overline{S}_\Gamma - \overline{S}_\Gamma^{cp} \subset \overline{S}_\Gamma$ is a neighbourhood of the boundary and nodes; these neighborhoods must be chosen compatibly with those already chosen on the boundary for the inductive construction later.

Following Floer [27], we use a $C^\varepsilon$-topology on the space of almost complex structures. For a section $\xi$ of a vector bundle $E \to X$, the $C^\varepsilon$-norm is

$$\|\xi\|_{C^\varepsilon} := \sum_{k=0}^\infty \varepsilon_k \|\xi\|_{C^k(X,E)}.$$  

Here $\varepsilon = (\varepsilon_i)_{i \in \mathbb{N}}$ is a fixed sequence of positive numbers that converges fast enough to 0 as $i \to \infty$. If the convergence is sufficiently rapid, then the space of sections with a bounded norm is a Banach space [27, Lemma 5.1] and contains sections supported in arbitrarily small neighbourhoods of $X$.

**Definition 5.1.** (a) (Domain-dependent Morse functions) Suppose that $\Gamma$ is a type of stable treed disk, and $\overline{U}_\Gamma \subset \overline{T}_\Gamma$ is the tree part of the universal treed disk. Let

$$(F : L \to \mathbb{R}, G : T^\otimes 2 L \to \mathbb{R})$$

be a Morse-Smale pair. For an integer $l \geq 0$ a domain-dependent perturbation of $F$ of class $C^l$ is a $C^l$ map

$$F_\Gamma : \overline{T}_\Gamma \times L \to \mathbb{R}$$

equal to the given function $F$ away from the compact part:

$$F_\Gamma |_{(\overline{T}_\Gamma - \overline{T}_\Gamma^{cp})} = \pi_2^* F$$

where $\pi_2$ is the projection on the second factor in (5.1).
(b) (Domain-dependent almost complex structure) Let $J_0 \in J_{cy}(\mathcal{X})$ be a strongly tamed cylindrical almost complex structure. A domain-dependent almost complex structure of class $C^c$ for treed disks of type $\Gamma$ and base $J_0$ is a map from the two-dimensional part $\mathfrak{S}_\Gamma$ of the universal curve $\mathcal{U}_\Gamma$ to $J_{cy}(\mathcal{X})$ given by

$$J_\Gamma : \mathfrak{S}_\Gamma \to J_{cy}(\mathcal{X})$$

equal to the given $J_0$ away from the compact part:

$$J_\Gamma|_{(\mathfrak{S}_\Gamma - \mathfrak{S}_\Gamma^p)} = J_0,$$

and for any fiber $S_\mathfrak{F} \subset \mathfrak{S}_\Gamma$, $J_\Gamma - J_0$ has finite norm in $C^c(S_\mathfrak{F} \times \mathfrak{X}, \text{End}(T\mathfrak{X}))$.

**Definition 5.2.** (Perturbation data) A perturbation datum for a type $\Gamma$ of stable treed disks is a pair $p_\Gamma = (F_\Gamma, J_\Gamma)$ consisting of a domain-dependent Morse function $F_\Gamma$ and a domain-dependent almost complex structure $J_\Gamma$.

The following are operations on perturbation data given morphisms on types of stable treed disks.

**Definition 5.3.**

(a) (Cutting edges) Suppose that $\Gamma$, $\Gamma'$ are combinatorial types of stable treed disks such that $\Gamma$ is obtained by cutting an edge $e \in \text{Edge}_{\infty,-}(\Gamma')$ that contains a breaking, see Figure 11. (The edge $e$ is necessarily a boundary edge.) A perturbation datum for $\Gamma$ gives rise to a perturbation datum for $\Gamma'$ by pushing forward $p_\Gamma$ under the map $\pi_{\Gamma'} : \mathcal{U}_{\Gamma'} \to \mathcal{U}_{\Gamma'}$. That is, define

$$J_{\Gamma'}(z', x) = J_\Gamma(z, x), \quad \forall z \in (\pi_{\Gamma'}^*)^{-1}(z).$$

The definition for $F_{\Gamma'}$ is similar.

(b) (Collapsing edges/making an edge length finite or non-zero) Suppose that $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making an edge length finite/non-zero (in case of a boundary edge). Any perturbation datum $p_\Gamma$ for $\Gamma$ induces a datum for $\Gamma'$ by pullback of $p_\Gamma$ under $\ell_{\Gamma'} : \mathcal{U}_{\Gamma'} \to \mathcal{U}_{\Gamma}$.

![Figure 11. Cutting an edge $e$ relabels the boundary and interior markings.](image)

We are now ready to define coherent collections of perturbation data. These are data that behave well with each type of operation in Definition 5.3.

**Definition 5.4.** (Coherent families of perturbation data) A collection of perturbation data $p = (p_\Gamma)_{\Gamma}$ is coherent if it is compatible with the morphisms of moduli spaces of different types in the sense that

(a) (Cutting edges) if $\Gamma$ is obtained from $\Gamma'$ by cutting a boundary edge $e \in \text{Edge}_{\infty,-}(\Gamma')$ of infinite length, then $p_{\Gamma'}$ is the push-forward of $p_\Gamma$;

...
(b) (Collapsing edges/making an edge length finite/non-zero) if $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making an edge finite/non-zero, then $p_{\Gamma'}$ is the pullback of $p_{\Gamma}$;

(c) (Products) if $\Gamma$ is the union of types $\Gamma_1, \Gamma_2$ obtained by cutting an edge of $\Gamma'$, then $p_{\Gamma}$ is obtained from $p_{\Gamma_1}$ and $p_{\Gamma_2}$ as follows: Let

\[ \pi_k : \mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2} \to \mathcal{M}_{\Gamma_k} \]

denote the projection on the $k$-th factor. Then $\mathcal{U}_\Gamma$ is the union of $\pi_1^* \mathcal{U}_{\Gamma_1}$ and $\pi_2^* \mathcal{U}_{\Gamma_2}$. Then we require that $p_{\Gamma}$ is equal to the pullback of $p_{\Gamma_k}$ on $\pi_k^* \mathcal{U}_{\Gamma_k}$:

\[ p_{\Gamma} \vert_{\mathcal{U}_{\Gamma_k}} = \pi_k^* p_{\Gamma_k}. \]

(5.4)

We also require the perturbation data to satisfy the following locality axiom which ensures that the perturbations on any component only depend on special points on that component, and the length of the treed segments on the boundary of the disk.

We first set up some notation: For a type $\Gamma$ underlying treed disks, and a vertex $v \in \text{Vert}(\Gamma)$, let $\Gamma(v)$ be the sub-tree consisting of the vertex $v$, and edges meeting a vertex $v$. Let $U_{\Gamma,v} \subset U_{\Gamma}$ be a fibration over $\mathcal{M}_{\Gamma}$ whose fiber over $m \in \mathcal{M}_{\Gamma}$ consists of the curve component represented by $v$. Define a map

\[ \pi_v : U_{\Gamma,v} \to U_{\Gamma(v)} \times ([0, \infty])^{\text{Edge}_{0,-}(\Gamma)}, \]

whose first component $U_{\Gamma,v} \to U_{\Gamma(v)}$ is the natural projection map, and the second component is the length function on boundary edges $e \in \text{Edge}_{0,-}(\Gamma)$.

(Locality Axiom) The restriction of the perturbation datum $p_{\Gamma}$ to $U_{\Gamma,v}$ is the pullback via $\pi_v$ of some datum on $U_{\Gamma(v)} \times ([0, \infty])^{\text{Edge}_{0,-}(\Gamma)}$.

This ends the Definition.

Let $C$ be a possibly unstable treed disk of type $\Gamma$. The stabilization of $C$ is the stable treed disk $\text{st}(C)$ of some type $\text{st}(\Gamma)$ obtained by collapsing unstable surface and tree components. Thus the stabilization $\text{st}(C)$ of any treed disk $C$ is the fiber of a universal treed disk $U_{\text{st}(\Gamma)}$. Given perturbation datum for the type $\text{st}(\Gamma)$, we obtain a domain-dependent almost complex structure and Morse function for $C$, still denoted $J_{\Gamma}, F_{\Gamma}$, by pull-back under the map $C \to U_{\text{st}(\Gamma)}$. If $\Gamma$ does not contain vertices, i.e. if $C$ is a single infinite segment $T_e, e \in \text{Edge}(\Gamma)$, then the perturbation $p_{\Gamma}$ vanishes on $C$.

**Remark 5.5**. (On the locality axiom) The locality axiom ensures that forgetting a marking $z_e$ on a treed curve affects the perturbation datum only on the component containing $z_e$. This feature is used in Proposition 7.38. The dependence on the boundary edge lengths is useful for the following reason. Suppose $\Gamma$ is a combinatorial type of treed disk depicted in Figure 11. By cutting an edge $e$ in $\Gamma$, we obtain two identical types. The cutting edge axiom requires that a coherent perturbation datum $P_{\Gamma}$ for $\Gamma$ is equal after restriction to the universal curve for the type $\Gamma'$ on both sides of the edge $e$. If in the locality axiom, the perturbation is $P_{\Gamma}$ defined by pulling back by the map $\pi_v : U_{\Gamma,v} \to U_{\Gamma(v)}$, then the perturbation datum on both surface components will be required to be equal even when the edge length of $e$ is finite. This creates a problem, because in order to obtain transversality in the
case $\ell(e) = 0$, we need the perturbation datum on both surface components to be independent of each other.

### 5.2. Perturbed maps

Given domain-dependent perturbations as in the previous section, the equations defining the moduli space of pseudoholomorphic maps are perturbed as follows.

**Definition 5.6.** (Perturbed pseudoholomorphic treed disks) Given a coherent perturbation datum $p = \{p_\Gamma\}_\Gamma$, a pseudoholomorphic treed disk in $X$ with boundary in $L$ consists of

- a treed disk $C = S \cup T$ with stabilized type $\Gamma := \text{st}(C)$,
- and continuous maps $u : C \to X$, such that the following hold:
  
  (a) **(Boundary condition)** The tree components and the boundary of the surface components are mapped to the Lagrangian submanifold: $u(\partial S \cup T) \subset L$.
  
  (b) **(Surface equation)** On the surface part $S$ of $C$ the map $u$ is $J$-pseudoholomorphic for the given domain-dependent almost complex structure: if $j$ denotes the complex structure on $S$ then
    $$J_\Gamma(z, u(z)) \, du_S = du_S \, j.$$  
  
  (c) **(Boundary tree equation)** On the boundary tree part $T \subset C$ the map $u$ is a collection of gradient trajectories:
    $$\frac{d}{ds} u_T = - \text{grad}_{F_\Gamma(s, u(s))} u_T$$
    where $s$ is a coordinate on the segment so that the segment has the given length. Thus for each treed edge $e \in \text{Edge}_{\text{int}}(\Gamma)$ the length of the trajectory $u|_{T_e}$ is equal to $\ell(e)$.

**Definition 5.7.** (Perturbed pseudoholomorphic broken treed disks) Given a coherent perturbation datum $p = \{p_\Gamma\}_\Gamma$, a holomorphic broken treed disk in a broken manifold $X_P$ with boundary in $L \subset X_{P_0}$, $P_0 \in P$, consists of

- a treed disk $C = S \cup T$ with type $\Gamma'$ whose stabilization $\text{st}(C)$ is $\Gamma$;
- a tropical structure $\mathcal{T}$ on $\Gamma$;
- and a collection of maps $u_v : C_v \to X_v, v \in \text{Vert}(\Gamma), \quad v \in \text{Vert}_v(\Gamma)$
    that is $p_\Gamma$-holomorphic, and satisfies the edge matching condition (3.14) at all interior edges $e \in \text{Edge}_v(\mathcal{T})$ and the Lagrangian boundary condition $u_v((\partial C)_v) \subset L$.

The stability condition for treed pseudoholomorphic disks is the following.

**Definition 5.8.** A treed pseudoholomorphic disk $u : C \to X$ is **stable** if
(a) If $S_v \subset S, v \in \text{Vert}(\Gamma)$ is a surface component and $u|S_v$ has zero area (and so is horizontally constant) then $S_v$ with its special points is a stable curve.

(b) If $T_e \subset T, e \in \text{Edge}(\Gamma)$ is an infinite segment and $u|T_e$ is constant, then $T_e$ has a finite end, that is, $T_e$ is infinite in both directions.

We will not introduce notation for the moduli space of broken maps, but instead work directly with the regularized moduli spaces using Donaldson hypersurfaces and domain-dependent almost complex structures.

**Definition 5.9.** (Perturbations adapted to a stabilizing divisor) Let $k \gg 0$, and $(\mathcal{D}, \mathcal{O})$ be a stabilizing pair on the broken manifold $X$, such that $D_P \subset X_P$ is dual to $k[\omega_{X_P}]$ for all polytopes $P \in \mathcal{P}$. Further $\mathcal{O}$ is disjoint from the Lagrangian submanifold $L$. Suppose $\Gamma$ is a type for treed disks. A perturbation datum $p_\Gamma(X, L) = (J_\Gamma, F_\Gamma)$ is adapted to the pair $(\mathcal{D}, \mathcal{O})$ if $J_\Gamma$ is the base almost complex structure for $J_\Gamma$, and for any treed curve $C = S \cup T$, and a connected component $S' \subset S$ with $d_*(S')$ interior markings,

$$J_\Gamma(S') \subset J^{\text{cyl}}(X, D; J_0, \frac{1}{k} d_*(S')),$$

where $J^{\text{cyl}}(X, D; J_0, \frac{1}{k} d_*(S')) \subset J^{\text{cyl}}(X, D)$ is the neighbourhood of $J_0$ consisting of $\frac{1}{k} d_*(S')$-stabilizing almost complex structures (see Proposition 4.16). The set of perturbation data adapted to $(\mathcal{D}, D)$ is denoted by $\mathcal{F}_\Gamma(X, J_0, D)$. The motivation to define perturbations of the form (5.6) will be explained in Remark 5.13.

A cylindrical divisor $\mathcal{D}$ in a broken manifold $X$ can be glued to give a family of divisors $D^\nu$ in neck-stretched manifolds $X^\nu$. There is a natural correspondence of tamed cylindrical divisor-adapted almost complex structures

$$\rho_\nu : J^{\text{cyl}}(X, \mathcal{D}) \to J^{\text{cyl}}(X^\nu, D^\nu).$$

**Definition 5.10.** (Breaking perturbation datum) Let $p$ be a perturbation datum for the broken manifold $X$ adapted to a cylindrical divisor $\mathcal{D}$. A breaking perturbation datum is a family of perturbation data $\{p_\nu\}_{\nu \in [\rho_0, \infty]}$ for the neck-stretched manifolds $(X^\nu, D^\nu)$ defined as $p_{\nu, \Gamma} := \rho_\nu(p_\Gamma)$.

**Definition 5.11.** (Convergence of breaking perturbations) A sequence of perturbation data $p_\nu = (p_{\nu, \Gamma})_\Gamma$ on $X^\nu$ converges to a perturbation datum $p_\infty$ on the broken manifold $X$ if the sequence $\rho^{-1}_\nu(p_\nu)$ converges in $C^\infty_{\text{loc}}$ in $U_\Gamma \times X$ for all $\Gamma$.

5.3. **Adapted maps.** We consider adapted holomorphic maps whose domains are equipped with markings are required to map to a Donaldson divisor.

**Definition 5.12.** (Adapted stable treed disks) A stable broken treed holomorphic disk $u : (C, \partial C) \to (X, L)$ is adapted to a divisor $\mathcal{D}$ iff each interior marking $z_e, e \in \text{Edge}(\Gamma)$ maps to $D$ under $u$ and each connected component $C'$ of $u^{-1}(\mathcal{D}) \subset C$ contains an interior marking.

**Remark 5.13.** We will show later that for our choice of divisors, disks occurring in the compactification of moduli spaces (of expected dimension 0 or 1) of adapted stable trees are also adapted. The condition (5.6) ensures that a sequence of adapted
perturbed holomorphic maps converges to an adapted map. That is, the limit map does not have components lying in the stabilizing divisor, and each spherical resp. disk domain component in the limit has at least three resp. one distinct intersections with the stabilizing divisor.

The moduli space of treed holomorphic disks is stratified by combinatorial type.

**Definition 5.14.** The combinatorial type of an adapted holomorphic broken treed disk \( u : C \to X \) adapted to a divisor \( D \subset X - L \) consists of

(a) the combinatorial type \( \Gamma \) of its domain \( C \),
(b) the tropical structure on \( \Gamma \), which consists of an assignment of polytopes for vertices, and slopes for edges:

\[
\text{Vert}(\Gamma) \ni v \mapsto P(v) \in \mathcal{P}, \quad \text{Edge}(\Gamma) \ni e \mapsto T(e) \in \mathcal{T},
\]

(c) a labelling

\[
d : \text{Vert}(\Gamma) \to (\bigcup_{P \in \mathcal{P}} H_2(X_P)) \cup H_2(X_{P_0}, L)
\]

that maps each vertex \( v \) of \( \Gamma \) to the hology class \( d(v) \) of the disk/sphere \( \Pi_{P(v)} \circ u(C_v) \),

(d) a labelling

\[
\mu_D : \text{Edge}(\Gamma) \to \mathbb{Z}_{>0}
\]

that records the order of tangency of the map \( u \) to the divisor \( D \) at markings that do not lie on horizontally constant components (with the convention that a transverse intersection has order 1).

The type is denoted simply as \( \Gamma \), suppressing \( T, d, \mu_D \) in the notation, or by \( \Gamma_X \) if we wish to distinguish a type of map \( \Gamma_X \) from a type of treed disk \( \Gamma \).

We introduce the following notations for moduli spaces. Let \( \widehat{M}_{\text{brok}}(L, D) \) be the moduli space of isomorphism classes of stable treed broken holomorphic disks in \( \mathcal{X} \) with boundary in \( L \) adapted to \( D \), where isomorphism is modulo reparametrizations of domains and is defined in Definition 3.29. Let

\[
\widehat{M}_{\Gamma, \text{brok}}(L, D) \subset \widehat{M}(L, D)
\]

be the locus of combinatorial type \( \Gamma \). The group of tropical symmetries \( T_{\text{trop}}(\Gamma) \) acts naturally on \( \widehat{M}_{\Gamma, \text{brok}}(L, D) \). The quotient

\[
\mathcal{M}_{\Gamma, \text{brok}}(L, D) := \widehat{M}_{\Gamma, \text{brok}}(L, D)/T_{\text{trop}}(\Gamma)
\]

is the moduli space of reduced isomorphism classes of (adapted, stable, treed, broken) holomorphic disks in \( \mathcal{X} \) with boundary in \( L \). For \( x \in I(L)^{d_{(0)} + 1} \), let

\[
\mathcal{M}_{\Gamma, \text{brok}}(L, D, x) \subset \mathcal{M}_{\Gamma, \text{brok}}(L, D)
\]

denote the adapted subset made of holomorphic treed disks of type \( \Gamma \) adapted to \( D \) with limits \( x = (x_0, \ldots, x_{d_{(0)}}) \in I(L) \) along the root and leaves. The union over all types with \( d_{(0)} \) incoming leaves is denoted

\[
\mathcal{M}_{d_{(0)}}(L, D) = \bigcup_{\Gamma, x} \mathcal{M}_{\Gamma, \text{brok}}(L, D, x).
\]
The space $\mathcal{M}_{d(c)}^{\text{brok}}(L, D)$ has a natural topology for which convergence is a version of Gromov convergence defined in Section 7. Given any energy bound $E > 0$, the locus of treed pseudoholomorphic disks satisfying that bound is $\mathcal{M}_{d(c)}^{E, \text{brok}}(L, D)$. Assuming the perturbations satisfy the coherence conditions above, $\mathcal{M}_{d(c)}^{E, \text{brok}}(L, D)$ is compact for any energy bound $E$ and any number of edges $d(c)$.

In this paper we show that the compactifications of zero and one-dimensional components of the moduli space $\tilde{\mathcal{M}}_{d(c)}^{\text{brok}}(L, D)_{< E}$ are manifolds. For one-dimensional components, the boundary consists of configurations with disk bubbling. For higher dimensional moduli spaces, we cannot expect $\tilde{\mathcal{M}}_{d(c)}^{\text{brok}}(L, D)_{< E}$ to be compact because of the possibility of a positive dimensional tropical symmetry group. However we expect the following conjecture to hold, assuming suitable regularity for maps.

**Conjecture 5.15.** For generic perturbations the space $\mathcal{M}_{d(c)}^{\text{brok}}(L, D)_{< E}$ is the coarse moduli space of a smooth Deligne-Mumford stack. In any connected component, generic points have a finite tropical symmetry group. For any tropical graph $\Gamma$, the moduli space $\mathcal{M}_{d(c)}^{\text{brok}}(L, D)_{< E}$ is a stratum of codimension $\dim(T_{\text{trop}, \Xi}(\Gamma))$ in $\mathcal{M}_{d(c)}^{\text{brok}}(L, D)_{< E}$.

### 5.4. Fredholm theory for broken maps.

We introduce a weighted Sobolev space needed for the transversality result. The norm is defined by viewing the domain as having punctures that map to cylindrical ends in the target. With this Sobolev completion, we can enforce higher order tangencies with boundary divisors. The Sobolev norm is defined component-wise for a broken map. In this section, we focus on surface components of a broken map. The transversality result uses a standard norm on tree components.

**Definition 5.16.** (Relative map) A relative type is a graph $\Gamma$ with a single vertex $v$, and edges and

$$P(v) \in \mathcal{P}, \quad T(e) \in t_{P(v), \mathbb{Z}}, \quad e \in \text{Edge}(\Gamma)$$

homology data $d(v) \in H_2(X_{\mathcal{T}}(v))$, and tangency data for ordinary markings with the stabilizing divisor $D_{\mathcal{T}}(v)$. A relative map is a map $u : C \to X_{\mathcal{P}}(v)$ with the given homology and tangency data and intersections with the boundary.

We introduce the space of smooth maps whose Sobolev completion will be given later. Let $(C, j)$ be a connected Riemann surface with a set $\{z_e : e \in \text{Edge}_\bullet(\Gamma)\}$ of interior special points. Let

$$C^0 := C \setminus \{z_e : e \in \text{Edge}_\bullet(\Gamma)\}$$

be the curve with the interior special points removed. For a map $u : C \to X_{\mathcal{T}}$ of type $\Gamma$, the restriction $u|C^0$ maps to

$$X^0_{\mathcal{T}} := X_{\mathcal{T}} - \text{boundary divisors}$$

and $X^0_{\mathcal{T}}$ has cylindrical metrics in the neighbourhoods of all its boundary divisors. Let

$$\text{Map}_\Gamma(C^0, X^0_{\mathcal{T}})$$
be the set of smooth maps, which can be extended smoothly over $C$ and the orders of tangencies with boundary divisors at the points in $C \setminus C^o$ are as prescribed by $\Gamma$. The tangent space at $u$ consists of sections
\[ \xi \in \Gamma(C^o, u^*TX^o) \]
for which the limit
\[ \xi_e := \lim_{z \to z_e} \xi \]
exists for any $e \in \text{Edge}_\bullet(\Gamma)$ in the space $TX_{P(e)} \oplus t_{P(e), C}$. (We recall from (2.25) that a neighborhood of $z_e$ in $C$ maps to a neighborhood in $X_P$ that has an identification to a $P(e)$-cylinder.) Further, if the vertical component of $u|C$ at $z_e$ (see (3.9)) is $z \mapsto zT^{(e)}x_e$ for some $x_e \in Z_{P(e), C}$, then the vertical component of $\exp_u \xi$ at $z_e$ is $z \mapsto zT^{(e)}(\exp_{z_e}(\xi_e))$.

We now define the Sobolev norm. We assume that the neighbourhood of each puncture in $C^o$ has holomorphic coordinates $(s_e, t_e) \in \mathbb{R}_{\geq 0} \times S^1$. Define a cutoff function
\[ \beta \in C^\infty(\mathbb{R}, [0, 1]), \quad \begin{cases} \beta(s) = 0 & s \leq 0 \\ \beta(s) = 1 & s \geq 1 \end{cases} \]
Let
\[ \kappa : C^o \to \mathbb{R} \]
be equal to $\beta(s_e)s_e$ near the puncture corresponding to $z_e$, for any edge $e$, and 0 outside all the edge neighbourhoods. For a section $\xi : C^o \to E$ of a vector bundle $E$ with connection $\nabla$, integers $k \geq 0$, $p > 1$ and constant $\lambda \in (0, 1)$ define the $W^{k,p,\lambda}$-norm of $\xi$ as
\[ \|\xi\|_{W^{k,p,\lambda}}^p := \sum_{0 \leq i \leq k} \int_{C^o} |\nabla^i \xi|^p \exp(\lambda \kappa) \, dv_{C^o}. \]
The norm on a section $\xi$ is
\[ \|\xi\|_{\Gamma} := \sum_{e} \|\xi_e\| + \|\xi - \sum_{e} \beta T_u \xi_e\|_{W^{1,p,\lambda}}, \]
where the notations involved are as follows:
\[ \xi_e := \lim_{z \to z_e} \xi \in TX_{P(e)} \oplus t_{P(e), C}, \]
$u_{\text{vert}}$ is the vertical component of $u$ in the sense of (3.9), and
\[ T_u : u_{\text{vert}}^*TX_P \to u^*TX_P \]
is the parallel transport map, and in the second appearance of $\xi_e$ it is a constant section
\[ \xi_e : U_{z_e} \setminus \{z_e\} \to u_{\text{vert}}^*TX^o_P. \]

There are similar Sobolev spaces of maps and one-forms. Let $\text{Map}^{1,p,\lambda}_\Gamma(C^o, X^o_P)$ be the Banach completion of the space of maps $\text{Map}_\Gamma(C^o, X^o_P)$ under the norm (5.10). That is, for a smooth map $u \in \text{Map}_\Gamma(C^o, X^o_P)$ and a section $\xi \in \Gamma(C, u^*TX_P)$
satisfying $\|\xi\|_{\Gamma}^\circ < \infty$, the map $\exp_u \xi$ belongs to the completion $\Map^1_{\Gamma}(C^\circ, X^\circ_P)$. Let
\[ \mathcal{E}^{p,\lambda} \to \Map^1_{\Gamma}(C^\circ, X^\circ_P) \]
be the vector bundle whose fiber
\[ \mathcal{E}_u^{p,\lambda} = \Omega^{0,1}(C^\circ, u^*TX^\circ_P)_{L^p,\lambda} \]
is the space of $(0,1)$-forms with respect to $(j, J)$, where $J$ is a fixed cylindrical almost complex structure. The vertical projection of the linearization of the Cauchy-Riemann operator $\overline{\partial}_{j,J}$ at $u$
\[ D_u^0 : T_u \Map^1_{\Gamma}(C^\circ, X^\circ_P) \to \mathcal{E}_u^{p,\lambda} \]
is a Fredholm operator by results of Lockhart-McOwen \cite{42}.

There is a related Fredholm operator on sections with compactified domain which is described as follows, and will be is used only to compute indices. Choose integers
\[ k \in \mathbb{Z}_{>0}, \quad p > 1, \quad k > \mu + 2/p \]
where $\mu$ is the highest order of tangency $m_{z_e}(u, D_Q)$ to any boundary divisor $D_Q \subset X_P$ over all the edges $e \in \text{Edge}_e(\Gamma)$. Let $\Map_{\Gamma}^{k,p}(C, X_P)$ be the set of $W^{k,p}$ maps from $C$ to $X_P$ that intersect the boundary divisors at marked points with the prescribed order of tangency. Consider the Banach bundle
\[ \mathcal{E}^{k-1,p} \to \Map_{\Gamma}^{k,p}(C, X_P) \]
whose fiber over any map $u$ is
\[ \mathcal{E}_u^{k-1,p} = W^{k-1,p}_{\Gamma-1}(\Omega^{0,1}_{j,J}(C, u^*TX_P)), \]
where the space $W^{k-1,p}_{\Gamma-1}$ consists of sections, which have zeros of order one less than that prescribed by $\Gamma$ at marked points. More rigorously, a $u^*TX_P$-valued $(0,1)$-form $\eta$ is in $W^{k-1,p}_{\Gamma-1}$ if for any marked point $z_e$ and a boundary divisor $Y$ with prescribed intersection multiplicity $m_{z_e}(u, Y) \geq 2$, the projection of $\eta$ to the normal bundle $NY$ has a zero of order $m_{z_e}(u, Y) - 1$. The linearization of the Cauchy-Riemann operator at $u$
\[ D_u : T_u \Map_{\Gamma}^{k,p}(C, X_P) \to \mathcal{E}_u^{k-1,p}. \]
is a Fredholm operator.

**Proposition 5.17.** Let $u \in \Map_{\Gamma}(C, X_P)$ be a holomorphic map. There are isomorphisms
\[ \ker(D_u) \simeq \ker(D_u^0), \quad \coker(D_u) \simeq \coker(D_u^0). \]

**Proof.** It is enough to prove the result for a map whose domain is a single curve component with a single marked point, as all other cases are analogous. We assume that the marked point is $0 \in C$. We first consider the case that the marked point is mapped to a single boundary divisor $Y \subset X_P$, and the intersection multiplicity is $\mu$. In this proof, we denote $X := X_P$ and $X^\circ := X^\circ_P$.

The proof is based on the fact that in a neighbourhood of the divisor the tangent space splits into a vertical and a horizontal subspace. Indeed, because of the cylindrical almost complex structure, there is a neighbourhood $U_Y \subset X$ of $Y$ for which
there is a projection \( \pi_Y : U_Y \to Y \) with holomorphic fibers. The tangent space splits into a \( J \)-holomorphic horizontal and vertical part:

\[
(5.11) \quad TX|U_Y \cong V \oplus H, \quad H := \pi_Y^* TY, \quad V := \ker(d\pi_Y).
\]

The operator \( D_u \) is defined by the Levi-Civita connection of the cylindrical metric (Definition 2.38). Therefore under the splitting (5.11),

\[
(5.12) \quad D_u = \left( \begin{array}{cc} \overline{\partial} & A^0 \\ 0 & D_u^0 \end{array} \right)
\]

in a neighbourhood \( U_0 \subset C \) of the marked point. Here \( \overline{\partial} : \Gamma(U_0, u^* V) \to \Omega^{0,1}(U_0, u^* V) \) is the standard Cauchy-Riemann operator, \( A : \Gamma(U_0, u^* H) \to \Omega^{0,1}(U_0, u^* V) \) is multiplication with a tensor and \( \ker(D_u^0) \) is the lift of the linearized operator \( D_{uY} \), where \( u_Y := \pi_Y \circ u \).

The correspondence for the kernels follows by decay estimates. For an element \( \xi \in W^{k,p}(C, u^* TX) \), we denote the horizontal part and vertical part by

\[
\xi_h \in W^{k,p}(U_0, u^* H), \quad \xi_v \in W^{k,p}(U_0, u^* V).
\]

in the neighbourhood \( U_0 \subset C \) of 0. Denote by \( \cdot \) resp. \( \cdot\) the ordinary metric on \( TX \) resp. the cylindrical metric on \( X \setminus Y \). Then, we have

\[
(5.13) \quad |\xi_h(z)|^\circ \sim |\xi_h(z)|, \quad |\xi_v(z)|^\circ \sim |\xi_v(z)|/|u(z)|, \quad z \neq 0,
\]

where the norm on \( u(z) \) is a norm on the fiber of the projection \( U_Y \to Y \). Since

\[
|\xi_h| \leq c, \quad |\xi_v(z)| \leq c|z|^\mu, \quad |u(z)| \sim c|z|^\mu,
\]

we conclude that \( |\xi_h|^\circ \) and \( |\xi_v(z)|^\circ \) are uniformly bounded and have a limit as \( z \to 0 \). Since \( \xi \) is smooth, the convergence is exponential:

\[
|\xi(z) - \xi(0)|^\circ, |\nabla \xi_h(z)|^\circ, |\nabla \xi_v(z)|^\circ \sim c|z| \sim ce^{-s}
\]

where \((s, t)\) are cylindrical coordinates on \( \mathbb{P}^1 \setminus \{0\} \) near the puncture. We conclude

\[
\xi \in W^{k,p}_1(C, X) \implies \xi - \xi(0) \in W^{1,\lambda,\circ}_{k,p}(C^\circ, u^* TX^\circ)
\]

for any \( 0 < \lambda < 1 \), which implies \( \ker(D_u) \subset \ker(D_u^0) \). The converse \( \ker(D_u^0) \subset \ker(D_u) \) is proved using removal of singularity, elliptic regularity, and the estimate (5.13).

A similar argument holds for the cokernels. We first prove the inclusion \( \coker(D_u) \subset \coker(D_u^0) \). Consider \( \eta \in \coker(D_u) \). We view \( \coker(D_u) \) as a subspace in the \( (W^{k,1-p})^\bigvee \)-completion of \( \Omega^{1,0}(C, u^* (TX^\circ)) \), which is a space of distributions. (We use \( \Omega^{1,0}(u^* (TX)) \) instead of \( \Omega^{0,1}(u^* TX) \) to avoid making choices of metrics on \( C \) and \( X \).) By elliptic regularity, the distribution \( \eta \) is represented by a smooth section in the complement of marked points. So we focus attention in a neighbourhood \( U_0 \subset C \) of 0. For any \( W^{k,p} \)-section \( \xi : C \to u^* TX \) that is supported in \( U_0 \), and is vertical, we have \( z^\mu \xi \in W^{k,p}_1(C, u^* TX) \). So for any such \( \xi \),

\[
0 = \int_C \langle \overline{\partial}(z^\mu \xi), \eta \rangle.
\]
Therefore, \( \overline{\nabla}(z^\mu \eta_v) = 0 \) weakly in \( U_0 \) and so, \( z^\mu \eta_v \) can be represented by a smooth function. Next, using the split form (5.12), we observe that any section \( \xi \) that is supported in \( U_0 \) and is horizontal satisfies

\[
(A\xi, \eta_v) + (D_u^Y \xi, \eta_h) = 0,
\]

and so, \( A^*\eta_v + (D_u^Y)^*\xi_h = 0 \) weakly in \( U_0 \). The tensor \( A \) vanishes to order \( \mu \) at \( 0 \in C \) in the \( |\cdot| \) -norm, as a consequence of the transformation relation (5.13) and the fact that \( A \) is bounded in the \( |\cdot|^0 \) -norm. So, \( A^*\eta_v \) is smooth in \( U_0 \). By elliptic regularity \( \xi_h \) is smooth in \( U_0 \). Finally, \( \eta \) is in \( \text{coker}(D_u^0) \) because of the following transformations valid in \( U_0 \):

\[
|\eta_h(z)|^0 \sim |\eta_v(z)|, \quad |\eta_v(z)|^0 \sim |\eta_v(z)| \cdot |u(z)|, \quad z \neq 0.
\]

Indeed, \( |\eta(z)|^0 \) is bounded, and therefore is in \( L^{p,\lambda} \) the dual space of \( L^{p,\lambda} \). The reverse inclusion \( \text{coker}(D_u^0) \subset \text{coker}(D_u) \) follows formally from the inclusion relation

\[
W_{\mu-1, p}^0(\Omega^{0,1}(L, u^*TX)) \to L^{p,\lambda}(\Omega^{0,1}(C^0, u^*TX^0))
\]

between the target spaces of \( D_u \) and \( D_u^0 \).

Finally, the proof carries over to the general case where a marked point is mapped to an intersection of boundary divisors, say \( \bar{Y} = \cap_i Y_i \), where \( Y_i \subset X \) is a boundary divisor. The new feature is that the vertical component of the tangent space splits into a sum of normal bundles \( NY_i \). In each of the summands, the same arguments can be used. \( \square \)

The following Proposition implies that collapsing edges with non-zero slope in a tropical graph does not change the expected dimension of the moduli space of broken maps.

**Proposition 5.18.** (Expected dimension) Suppose \( \Gamma \) is the combinatorial type of a broken map with limits \( \bar{x} \in \mathcal{I}(L)^{d(\cdot) + 1} \) along the root and leaves. Suppose \( \Gamma_{\text{glue}} \) is the unbroken type obtained by collapsing all edges \( e \in \text{Edge}_{\bullet\bullet}(\Gamma) \) with non-zero tropical slope \( T(e) \neq 0 \). The expected dimension of \( \overline{\mathcal{M}}_{\mu, k}^{\text{brok}}(L, \mathcal{D}, \bar{x}) \) is given by

\[
i(\Gamma, \bar{x}) := i(x_0) - \sum_{i=1}^{d(\cdot)} i(x_i) + d(\cdot) - 2 + \sum_{v \in \text{Vert}(\Gamma_{\text{glue}})} I(\Gamma_v)
- 2|\text{Edge}_{\bullet\bullet}(\Gamma_{\text{glue}}) - |\text{Edge}_{\bullet\bullet}^0 - |\text{Edge}_{\bullet\bullet}^\infty | + 2\sum_{e \in \text{Edge}_{\bullet\bullet\cdot\cdot}(\Gamma)} (\mu_D(e) - 1),
\]

where \( I(\Gamma_v) \) is the Maslov index of the sphere/disk with homology type prescribed for \( v \in \text{Vert}(\Gamma_{\text{glue}}) \). Thus,

\[
i(\Gamma, \bar{x}) = i(\Gamma_{\text{glue}}, \bar{x}).
\]

**Proof of Proposition 5.18.** It is enough to carry out the proof in case of a tropical graph with two vertices \( v_\pm \) with an edge \( e \) between them and the curve components of the broken map are

\[
u_{\pm e} : C_\pm \to X_\pm,
\]
where $X_{\pm} := X_{P(v_{\pm})}$ are pieces of the broken manifold $X$. All other cases can be worked out similarly. The node corresponding to $e$ lies on $X_{P(e)}$, which is a transverse intersection of boundary divisors, denoted by $X_{Q_1}, \ldots, X_{Q_k}$. The intersection multiplicity is then a tuple

\[(5.16) \quad T(e) = (\mu_1, \ldots, \mu_k) \in (\mathbb{Z}_{>0})^k.\]

By Riemann-Roch, the dimension of the moduli space of broken maps is

\[
\sum_{* \in \pm} \left( \frac{1}{2} \dim(X)(\chi(C_*)) + I(u_*) + 2 - 2\sum_{i=1}^k \mu_i - 6 \right) = (\dim(X) - 2),
\]

where $\chi(C_{\pm})$ is the Euler characteristic of the domain curve $C_{\pm}$, and $I(\cdot)$ is the Maslov index. In the summation, $+2$ comes from the choice of nodal point, the term $2\sum_{i=1}^k \mu_i$ comes from the constraint on intersection multiplicity at the node, and $6$ is the dimension of the automorphism of the domain. The last term $(\dim(X) - 2)$ is from the edge matching condition. The indices for spheres are $I(u_{\pm}) = 2c_1(u_{\pm})$.

The conclusion follows from the fact that the Euler characteristics are related as

\[
\chi(C_{\text{glue}}) = \chi(C_+) + \chi(C_-) - 2,
\]

and the first Chern classes are related as

\[(5.17) \quad c_1(u_{\text{glue}}) = c_1(u_+) + c_1(u_-) - 2\sum_{i=1}^k \mu_i.\]

The formula (5.17) is justified as follows. Because of the cylindrical complex structure on the ends, a neighbourhood of $X_{P(e)}$ in $X_{\pm}$ can be viewed as a neighbourhood of the zero section of a direct sum of complex line bundles

\[\pi_{\pm} : \bigoplus_{i=1}^k L_i \to X_{P(e)}.\]

In this identification, the divisor $D_i$ is the total space $\bigoplus_{j \neq i} L_j$. In this region, the tangent space splits as the sum of horizontal and vertical sub-bundles

\[TX = H \oplus \sum_i L_i, \quad H = \pi_{\pm}^* TX_{P(e)}.\]

The maps $u_+, u_-$ are obtained by cutting $u_{\text{glue}}$ to yield $u^+_e, u^-_e$ and adding a capping disk to either side. The horizontal bundle $(u^\pm_e)^* H$ extends trivially over the capping disk, but for $(u_\pm^e)^* L_i$, the bundle over the capping disk is glued in with a $\mu_i$ twist leading to (5.17).

Finally, a marking $w_e$ whose intersection with the stabilizing divisor with the stabilizing divisor is $\mu_D(e)$ cuts down the dimension of the moduli space by $2(\mu_D(e) - 1)$. \qed

**Remark** 5.19. Index results for the orbifold case follow in an analogous way. The index for the operator $D_u$ is given by the Kawasaki-Riemann-Roch formula, see [14, Lemma 3.2.4].

\[1\] The proof of Proposition 5.17 can be applied in the orbifold case after passing to ramified covers in the neighborhoods of lifts of nodal points, to obtain $\text{ind}(D_u) = \text{ind}(D^\circ_p)$. Proposition 5.18 is also valid in the orbifold case. The only difference in the proof is that the tuple of intersection multiplicities consists of fractions, that is in (5.16), $(\mu_1, \ldots, \mu_k) \in (\mathbb{Q}_{>0})^k$.

\[1\] See [3, p3] and [15, p9] for exposition on the first Chern number for orbi-bundles and the index formula.
5.5. **Transversality.** The perturbation scheme we use here only achieves transversality for certain combinatorial types, as in Cieliebak-Mohnke [16].

**Definition 5.20.** A combinatorial type $\Gamma$ of broken map $u : C \to X$ is *crowded* if there exists a connected subgraph $\Gamma' \subset \Gamma$ such that $u|_{S_v}$ is horizontally constant on all vertices $v \in \text{Vert}(\Gamma')$ and $\Gamma'$ contains more than one interior leaf.

**Definition 5.21.** For a type $\Gamma$ of treed disks, a perturbation datum $p^{\Gamma}$ is *regular* if for any uncrowded type $\Gamma$ of tropical treed disks, adapted maps of type $\Gamma$ are regular, and so the moduli space $M^{\Gamma}(X, L, D)$ is a manifold of expected dimension.

**Theorem 5.22.** (Transversality) Let $X$ be a broken manifold with a symplectic cylindrical structure as in 2.27. Let $D \subset X$ be a cylindrical broken divisor and $J_0 \in J^{\text{cyl}}(X, D)$ be a strongly tamed cylindrical almost complex structure adapted $D$ such that $(D, J_0)$ is a stabilizing pair. Suppose $\Gamma$ is a type of stable treed disks, and regular perturbation data for types $\Gamma'$ of stable treed disks with $\Gamma' < \Gamma$ are given. Then there exists a comeager subset $p^{\text{reg}}_{\Gamma}(D) \subset p_{\Gamma}(D)$ of regular perturbation data for type $\Gamma$ coherent with the previously chosen perturbation data. For any regular perturbation $p \in p_{\Gamma}$, and an uncrowded type $\Gamma_X$ of broken maps whose domain type is $\Gamma$, the moduli space $M_{\Gamma_X}(X, L, D)$ is a smooth manifold of expected dimension.

In the statement of the above Theorem, the ordering on types of stable treed disks is defined as follows: $\Gamma' < \Gamma$ if $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making the length of a boundary edge finite or non-zero.

**Proof of Theorem 5.22.** Transversality, as in the unbroken case, is an application of Sard-Smale as in Cieliebak-Mohnke [16], Charest-Woodward [13] on the universal space of maps. The new feature is that in neck pieces of the broken manifold the almost complex structure is fixed in the fiber direction. This does not pose any issues for maps whose horizontal projection is non-constant. Components of the map whose horizontal projection is constant will shown to be automatically transversal.

The moduli space is cut out as a zero set of a section of a Banach bundle which we now describe. We restrict our attention to types of maps for which all intersections with the stabilizing divisor have multiplicity one. Other types are discussed later. We construct the moduli space of broken maps without framing, since the framed version is a finite cover of the unframed one. We cover the moduli space of treed disks $M_{\Gamma}$ by charts $\cup_i M^{\Gamma}_i$, so that on a trivialization of the universal curve $U_{\Gamma}$, each of the fibers is a fixed treed curve $C = S \cup T$ with fixed special points. The complex structure on $S$ varies smoothly in the sense that it is given by a map $M^{\Gamma}_i \to J(S_{\Gamma}), \ m \mapsto j(m)$.

In order to apply Sard-Smale, we pass to maps from the normalized curve of a fixed Sobolev class. Let

$$\tilde{C} := \bigsqcup_{v \in \text{Vert}(\Gamma')} S_v \sqcup \bigsqcup_{e \in \text{Vert}(\Gamma')} T_e$$
denote the normalized curve in the sense that spherical nodes in \(C\) are lifted to double points in \(\tilde{C}\) and the tree components in \(C\) are detached from the surface components. Choose \(p > 2\) and \(\lambda \in (0, 1)\). Let \(\text{Map}_{\Gamma}^{1, \mu, \lambda}(\tilde{C}, \mathcal{X}, L, \mathcal{D})\) denote the completion of maps under the weighted Sobolev norm \(\|\cdot\|_{\Gamma}\) in (5.10) for surface components, and the ordinary \(W^{1,p}\)-norm for the tree components. That is, an element of this space consists of

(a) a collection of \(W^{1,p,\lambda}\)-maps

\[ u_v : S_v \to X_{\mathcal{P}(v)}, \quad (\partial u)_v : (\partial S)_v \to L, \quad v \in \text{Vert}(\Gamma) \]

for each vertex \(v\) with tangency of order \(T(e)\) with the boundary divisors at the node corresponding to any interior edge \(e \in \text{Edge}(\Gamma)\);

(b) and \(W^{1,p}\) maps

\[ u_e : T_e \to L, \quad e \in \text{Edge}(\Gamma) \]

from tree components to \(L\) asymptotic to the critical points \(x\) on the leaves \(T_e, e \in \text{Edge}(\Gamma)\).

The metric on \(X^2_{\mathcal{P}(v)}\) is chosen to be cylindrical in the neighborhood of boundary divisors, and so that \(L\) is totally geodesic. Let

\[ P_\Gamma = \{ p_\Gamma = (J_\Gamma, F_\Gamma) \} \]

be the space of perturbation data defined on \(U_\Gamma\) whose distance from the base data \((J_0, F_0)\) is bounded in the \(C^\varepsilon\)-norm, and that agree with the given perturbation datum on a fixed open neighbourhood of \(U_\Gamma\) for the strata \(\Gamma' < \Gamma\); this neighbourhood should be chosen sufficiently small so that any such perturbation is already regular for maps \(u\) lying over it. We use the \(C^\varepsilon\)-norm for \(J_\Gamma\) and a \(C^{l}\)-norm for \(F_\Gamma\) where \(l > 1\) is a fixed number. For any broken curve \(C\) of type \(\Gamma\) we obtain perturbation data on \(C\) by identifying it conformally with a fiber of the universal tree disk \(U_\Gamma\).

Let

\[ \mathcal{B}^i_{p,\lambda,l,\Gamma} : = \mathcal{M}_\Gamma^i \times \text{Map}^{1,p,\lambda}_\Gamma(\tilde{C}, \mathcal{X}, L, \mathcal{D}) \times P_\Gamma(\mathcal{X}, \mathcal{D}). \]

Let \(\mathcal{E}^i = \mathcal{E}^i_{p,\lambda,\Gamma}\) be the Banach bundle over \(\mathcal{B}^i_{p,\lambda,l,\Gamma}\) given by

\[ (\mathcal{E}^i_{p,\lambda,\Gamma})_{j,u,J} \subset L^{p,\lambda}(\Omega^{0,1}_{j,u,J}(S, (u|S)^*T\mathcal{X})) \oplus L^p(\Omega^{1}(T, (u|T)^*TL)). \]

Here the first summand is the space of 0,1-forms with respect to \((j(m), J)\). The Cauchy-Riemann and shifted gradient operators applied to the restrictions \(u|S\) resp. \(u|T\) of \(u\) to the two resp. one dimensional parts of \(C = S \cup T\) define a \(C^{l-1}\) section (5.18)

\[ \partial_\Gamma : \mathcal{B}^i_{p,\lambda,l,\Gamma} \to \mathcal{E}^i_{p,\lambda,\Gamma}, \quad (C, u, (J_\Gamma, F_\Gamma)) \mapsto \left( \partial_{j(m),J_\Gamma} u|S, \left( \frac{d}{ds} + \text{grad}_{F_\Gamma} \right) u|T \right) \]

where \(s\) is a local coordinate on the tree components with unit speed. The evaluation maps at lifts of nodal points, markings, and lifts of \(S \cap T\) give a smooth map

(5.19) \[ \text{ev}_\Gamma : \mathcal{B}^i_{p,\lambda,l,\Gamma} \to \mathcal{X}(\Gamma) \]
where

\[(5.20)\]

\[
\mathcal{X}(\Gamma) = \left( \prod_{e \in \text{Edge}_{\Gamma}, \gamma \in \Gamma, \tau(e) \neq 0} (Z_{P(e),\mathcal{L}}/T_{T(e),\mathcal{L}})^2 \right) \times \left( \prod_{e \in \text{Edge}_{\Gamma}, \gamma \in \Gamma, \tau(e) = 0} (X_{P(v(e))})^2 \right) \times \left( \prod_{x \in S \cap T} L^2 \right) \times \left( \prod_{e \in \text{Edge}_{\Gamma}, \gamma \in \Gamma} X_{P(v(e))}^2 \right).
\]

The first two factors in (11.4) correspond to lifts of interior nodes, the third term is a lift of \(S \cap T\), the fourth is a lift of a boundary node with a zero length edge, and the last term corresponds to evaluation at an interior marking. All the factors of \(ev_{\Gamma}\) are standard evaluation maps, except for the first factor which is the evaluation of a leading order derivative as in (3.13). Let

\[\Delta(\Gamma) \subset X(\Gamma)\]

be the submanifold that is the product of diagonals in the first four factors of \(X(\Gamma)\) in (11.4), and the stabilizing divisor \(D_{P(v(e))} \subset X_{P(v(e))}\) in the last factor. The local universal moduli space is

\[\mathcal{M}_{\text{univ}}^{i,j}(L, \mathcal{D}) = (\bar{\mathcal{D}}, ev_{\Gamma})^{-1}(B_{p,\lambda,\mu}^{i,j}, \Delta(\Gamma)),\]

where \(B_{p,\lambda,\mu}^{i,j}\) is embedded as the zero section in \(E_{p,\lambda,\mu}\).

We will next show that this subspace is cut out transversely. We first consider two-dimensional components of \(\tilde{C}\) on which the map is not horizontally constant, and show that the linearization of \((\bar{\mathcal{D}}, ev_{\Gamma})\) is surjective. For components whose target space is not a neck piece, the surjectivity of the differential \(d(\bar{\mathcal{D}}, ev_{\Gamma})\) follows from [46, Proposition 3.4.2]. Recall from 2.17 that neck pieces \(X_{P}\) admit fibrations

\[V_{Pv} \to X_{P} \to X_P\]

where \(V_{Pv}\) is a \(T_P\)-toric manifold, and the almost complex structure on \(X_{P}\) is \(P\)-cylindrical. For non-neck pieces \(T_P\) is trivial. Consider a component \(S_v \simeq \mathbb{P}^1\) that maps to a neck piece \(X_{P}\), and the horizontal projection \(\Pi_P \circ u : \mathbb{P}^1 \to X_P\) is non-constant. The linearized operator

\[D_{u,j}(\xi, K) = D_u \xi + \frac{1}{2} K D u_j.\]

is surjective as follows. Let

\[\eta \in \text{coker}(D_{u,j}) \subset \Omega^{0,1}(u^*TX_{P})\]

be a one-form in the cokernel of \(D_{u,j}\). Variations of tamed almost complex structure of cylindrical type are \(J\)-antilinear maps

\[K : TX_{P} \to TX_{P}\]

that vanish on the vertical sub-bundle and are \(T_{P,\mathcal{L}}\)-invariant. Since the horizontal part of \(D_2 u\) is non-zero at some \(z \in C\), we may find an infinitesimal variation \(K\) of almost complex structure of cylindrical type by choosing \(K(z)\) so that \(K(z)D_2 u_j(z)\) is an arbitrary \((j(z), J(z))\)-antilinear map from \(T_z C\) to \(T_{u(z)} X_{P}\). Choose \(K(z)\) so
that $K(z)D_z u j(z)$ pairs non-trivially with $\eta(u(z))$ and extend $K(z)$ to an infinitesimal almost complex structure $K$ by a cutoff function on the domain curve. For two-dimensional components that are horizontally constant, by Corollary 5.25 the linearization $d\bar{\partial}$ is surjective, and additionally the evaluation map at a single marked point is surjective. Note that $d\text{ev}_\Gamma$ may not be surjective on these components. Finally, for tree components, the linearization of the shifted gradient operator and the evaluation map at the finite end is surjective, since we can perturb the Morse function on the Lagrangian.

From the discussion so far, we conclude that matching conditions are cut out transversely except if both components incident at the node are horizontally constant. Now we handle this exceptional case. We consider a maximal connected subgraph $\Gamma' \subset \Gamma$ so that the map is horizontally constant on the vertices of $\Gamma'$, and $\Gamma'$ does not have tree components. By uncrowdedness, there is at most one marked point in $\Gamma'$. So, it is possible to choose at most one special point (marked point or a lift of a nodal point) on each component of $\tilde{C}_{\Gamma'}$, so that for every nodal point one of its ends is chosen. By Corollary 5.25, for a horizontally constant component with a single marked point $z$, the linearized map $D(\partial, \text{ev}_z)$ is surjective. Since the evaluation map is surjective at each of the chosen lifts, an inverse of the linearized map $D(\partial, \text{ev}_\Gamma)$ can be constructed inductively, see [13, p63].

For types where the map has higher order intersections with the stabilizing divisor, the universal moduli space is cut out inductively as in [16, Lemma 6.5]. Each step of the induction cuts out a moduli space where the tangencies at one of the markings is increased by one. We start out with a moduli space cut out of $W^{k,p,\lambda}$ where $k - \frac{2}{p} > \mu$ and $\mu$ is the largest order of tangency with the stabilizing divisor $D$ that occur in the type $\Gamma$.

By the implicit function theorem, $\mathcal{M}^{\text{univ},i}_\Gamma(L, \mathcal{D})$ is a smooth Banach manifold, and the forgetful morphism

$$\varphi_i : \mathcal{M}^{\text{univ},i}_\Gamma(L, \mathcal{D})_{k,p,l} \to \mathcal{P}_\Gamma(L, \mathcal{D})_{l}$$

is a smooth Fredholm map. By the Sard-Smale theorem, the set of regular values $\mathcal{P}^{\text{reg},i}_{\Gamma}(L, \mathcal{D})$ of $\varphi_i$ on $\mathcal{M}^{\text{univ},i}_\Gamma(L, \mathcal{D})$ in $\mathcal{P}_\Gamma(L, \mathcal{D})$ is comeager. Let

$$\mathcal{P}^{\text{reg},i}_{\Gamma}(L, \mathcal{D}) = \bigcap_i \mathcal{P}^{\text{reg},i}_{\Gamma}(L, \mathcal{D}).$$

A standard argument shows that the set of smooth domain-dependent $\mathcal{P}^{\text{reg},i}_{\Gamma}(L, \mathcal{D})$ is also comeager. Fix $(J_\Gamma, F_\Gamma) \in \mathcal{P}^{\text{reg},i}_{\Gamma}(L, \mathcal{D})$. By elliptic regularity, every element of $\mathcal{M}^{\text{reg},i}_\Gamma(L, \mathcal{D})$ is smooth. The transition maps for the local trivializations of the universal bundle define smooth maps

$$\mathcal{M}^{\text{reg},i}_{\Gamma}(L, \mathcal{D})|_{\mathcal{M}^{\ell}_\Gamma \cap \mathcal{M}^{\text{reg},i}_{\Gamma}} \to \mathcal{M}^{\ell}_{\Gamma}(L, \mathcal{D})_{\mathcal{M}^{\ell}_\Gamma \cap \mathcal{M}^{\text{reg},i}_{\Gamma}}.$$ 

This construction equips the space

$$\mathcal{M}_{\Gamma}(L, \mathcal{D}) = \cup_i \mathcal{M}^{\ell}_{\Gamma}(L, \mathcal{D})$$

with a smooth atlas. Since $\mathcal{M}_{\Gamma}$ is Hausdorff and second-countable, so is $\mathcal{M}_{\Gamma}(L, \mathcal{D})$ and it follows that $\mathcal{M}_{\Gamma}(L, \mathcal{D})$ has the structure of a smooth manifold.
For regular perturbations, the tangent space to the moduli space can be identified with the kernel of an operator called the linearized operator. Using notations from the proof of transversality, we write down the linearized operator for future reference. For a regular perturbation $p \Gamma$, consider the map $(\partial, ev_\Gamma) : B^i_{k,p,\Gamma} \to E^i_{k,p,\Gamma} \times X(\Gamma)$ where $ev_\Gamma$ is defined in (5.19). Its linearization is denoted
\begin{equation}
D_u : T[C],u B^i_{k,p,\Gamma} \to (E^i_{k,p,\Gamma}|C],u \oplus ev_\Gamma^* T X(\Gamma)/T \Delta(\Gamma).
\end{equation}
For a regular broken map $u$ the operator $D_u$ is surjective.

5.6. The toric case. In this section, we consider the question of regularity for maps in a toric variety with the standard complex structure. The results are also useful for maps in toric fibrations that project to a constant in the base space.

Assumption 5.6.1. Suppose $P \in \mathcal{P}$ is a top-dimensional polytope and the tropical moment map $\Phi$ generates a Hamiltonian $T$-action which makes $X_P$ a non-singular toric variety, and the Lagrangian $L$ is a torus orbit in $X_P$.

Suppose that all torus-invariant divisors of $X_P$ are boundary divisors of the broken manifold $X$ (see Definition 2.23) in the following sense. Let $\mathcal{F}(P)$ be the set of facets of $P$. For any $Q \in \mathcal{F}(P)$, $\Phi^{-1}(Q)$ is a torus-invariant divisor of $X_P$. That is, $\mathcal{F}(P) \subset \mathcal{P}$.

Recall from Definition 5.16 that the domain of a relative map has a single surface component and no tree components. Thus the type of the map prescribes the order of intersection of the map with boundary divisors at each of the marked points.

Proposition 5.23. Let $P \in \mathcal{P}$ be a top-dimensional polytope. Suppose $X := X_P$ is a toric piece of a broken manifold $X$ as in the previous paragraph, all whose torus-invariant divisors are boundary divisors of $X$. Let $L \subset X$ be a toric Lagrangian. Let $\Gamma$ be a type of relative map (see definition 5.16). Any map $u$ of type $\Gamma$ is regular, and therefore, the moduli space of maps $M_{\Gamma}(X)$ is a manifold of expected dimension.

Let $z_e$ be a marked point corresponding to an edge $e \in \text{Edge}(\Gamma)$ that maps to a torus-invariant submanifold $Y \subset X$. Then, the leading order evaluation map (as in (3.11))
\[ ev^{T(e)}_{z_e} : M_{\Gamma}(X) \to NY \]
is submersive.

Remark 5.24. The maps considered in Proposition 5.23 can not map to a torus-invariant divisor of $X$. Indeed since all such divisors are boundary divisors of the broken manifold, a relative map can intersect those divisors only at relative marked points.

\[ \text{Which is to say that every divisor in } X_P \text{ represents a stretching direction.} \]
Proof of Proposition 5.23. Suppose $X$ is the toric variety corresponding to a moment polytope $\Delta \subset t^\vee$. Each facet $Q_1, \ldots, Q_N$ defines a prime $T_C$-invariant divisor $X_{Q_1}, \ldots, X_{Q_N}$ in $X$, whose union is a representative of the first Chern class. Holomorphic disks to $X$ with boundary on $L$ are regular by Cho-Oh [19]. Holomorphic spheres meeting the interior of $X$ are also regular by the following argument. As in Delzant [21], $X$ can be viewed as a geometric invariant theory quotient $C^N//G$ where $N$ is the number of prime torus-invariant boundary divisors of $X$, and $G \subset (\mathbb{C}^\times)^N$ is a complex torus whose quotient $(\mathbb{C}^\times)^N/G$ is $T_C$. Each of the boundary divisors $X_{Q_1}, \ldots, X_{Q_N}$ in $X$ lifts to a coordinate hyperplane $\{z_1 = 0\}, \ldots, \{z_N = 0\}$ in $\mathbb{C}^N$. Consider a holomorphic sphere $u : \mathbb{P}^1 \to X$, that is not contained in any toric divisor. The vector bundle $u^*TX \oplus \mathfrak{g}$ on $\mathbb{P}^1$ is a sum of line bundles $u^*TX \oplus \mathfrak{g} = \bigoplus_{i=1}^N u^*O(X_{Q_i})$ where $\mathfrak{g} := \mathfrak{g} \times \mathbb{P}^1$ is the trivial bundle. The degree $\deg(u^*O(X_{Q_i}))$ of the line bundle $u^*O(X_{Q_i})$ is given by the intersection of $u$ with $X_{Q_i}$. Hence each of the degrees $\deg(u^*O(X_{Q_i}))$ is non-negative. As a result, the operator $\partial : \Gamma(\mathbb{P}^1, \bigoplus_i u^*O(X_{Q_i})) \to \Omega^{0,1}(\mathbb{P}^1, \bigoplus_i u^*O(X_{Q_i}))$ is onto. Consequently the cohomology group $H^{0,1}(\mathbb{P}^1, \bigoplus_i u^*O(X_{Q_i})) \simeq H^1(\mathbb{P}^1, \bigoplus_i u^*O(X_{Q_i}))$ vanishes. Consider the long exact sequence in Čech cohomology, corresponding to the short exact sequence of sheaves $0 \to \mathfrak{g} \to \bigoplus_i u^*O(X_{Q_i}) \to u^*TX \to 0.$ Vanishing of the zeroth resp. first cohomology of the first resp. second terms implies that the first cohomology of the third term $H^{0,1}(\mathbb{P}^1, u^*TX) \simeq H^1(\mathbb{P}^1, u^*TX)$ also vanishes. Therefore the sphere $u$ is regular in $X$. Next, we will show that the moduli spaces of spheres or disks with prescribed tangencies at relative marked points are cut out transversely. The Maslov index $I(u)$ of a holomorphic disk or sphere $u : C \to X$ (not contained in a boundary divisor) is twice the sum of its intersection multiplicities $(u.X_{Q_i})$ with all boundary divisors $X_{Q_i}$. In fact, the moduli space containing $u$ is parametrized by the set of intersection points $z \in C$ with the toric divisors. For maps with a higher order tangency with a toric divisor $X_{Q_i}$, some subsets of intersection points of $u$ with $X_{Q_i}$ coincide. Since the zeros of sections of positive line bundles may be chosen arbitrarily in genus zero, the locus of such maps is cut out transversely from the space of all maps. Therefore the moduli space of relative maps of any type $\Gamma$ is regular. The linearization of the evaluation map at relative marked points is surjective because of the torus-equivariance of the evaluation map. Recall that if the marked point $z_e$ maps to an intersection of boundary divisors $Y = \cap_i Y_i$, the target space for
a higher order evaluation map is the normal bundle $NY$. The normal bundle can be identified with a neighbourhood of $Y$ in a standard way. With this identification,

$$\text{ev}_{z_e}^{T(e)}(tu) = t \text{ev}_{z_e}^{T(e)}(u), \quad \forall t \in T_C.$$  

A $T_C$-orbit is an open set in $X$, and therefore $\text{ev}_{z_e}^{T(e)}(u)$ is a regular value of the map $\text{ev}_{z_e}^{T(e)}$.

Proposition 5.23 implies an analogous result for toric fibrations, for which all torus-invariant divisors are boundary divisors. This result was used to prove transversality for horizontally constant components $u : \mathbb{P}^1 \to X_{\mathcal{T}}$ in a broken map. The target space $X_{\mathcal{T}}$ is a fibration $X_{\mathcal{T}} \to X_P$, whose fibers are smooth symplectic toric varieties with the action of a compact torus $T_P$. The $T$-action on the fibers is the restriction of a $T_P$-action on the total space $X_{\mathcal{T}}$. The torus-invariant divisors of $X_{\mathcal{T}}$ are precisely the vertical divisors of $X_{\mathcal{T}}$.

**Corollary 5.25.** Suppose $P \in \mathcal{P}$ is a polytope and the toric fibration $X_{\mathcal{T}} \to X_P$ has an almost complex structure that is standard on the fibers. Let $\Gamma$ be a type of relative map that has a single sphere component $S_v, v \in \text{Vert}(\Gamma)$ mapping to $X_{\mathcal{T}}$, which projects to a constant $\pi_P \circ u_v : S_v \to X_P$, and has no tree components. Any map $u$ of type $\Gamma$ is regular, and therefore, the moduli space of maps $\mathcal{M}_\Gamma(X_{\mathcal{T}})$ is a manifold of expected dimension. For a marking $z_e$ corresponding to any edge $e \in \text{Edge}_e(\Gamma)$ that maps to a torus-invariant submanifold $Y \subset X_{\mathcal{T}}$, the leading order evaluation map (as in (3.11))

$$\text{ev}_{z_e}^{T(e)} : \mathcal{M}_\Gamma(X) \to NY$$

is submersive.
6. Hofer energy and exponential decay

Compactness for sequences of broken pseudoholomorphic maps requires bounds on area. In this section we define various notions of area in the context of multiple cutting.

Neck-stretched manifolds and cut spaces have a natural second cohomology class that contains symplectic forms. Pairing curves with this cohomology class gives a notion of area. We recall that neck-stretched manifolds $X^\nu$ were constructed from a symplectic manifold $(X, \omega)$ with a tropical structure $(\Phi, P)$. The cut spaces corresponding to $(X, \Phi, P)$ are symplectic manifolds $(X_P, \omega_P, P \in P)$.

**Definition 6.1.**

(a) (Area on a neck-stretched manifold) Let $C$ be a complex curve with boundary. The area of a map $u : (C, \partial C) \to (X^\nu, L)$ is

$$\text{Area}(u) := \langle u_*[C], [\omega] \rangle.$$

(b) (Area of a broken curve) Let $C$ be a treed nodal curve and let $u : C \to X$ be a broken map. The area of a surface component $(u|S_v) : S_v \to X_{P(v)}$ is

$$\text{Area}(u|S_v) = \int_C (\pi_{P(v)} \circ u)^* \omega_{X_{P(v)}}$$

where $\omega_{X_{P(v)}}$ is the symplectic form on the base space of the projection $\pi_{P(v)} : X_{P(v)} \to X_P$. The area of a broken map $u$ is the sum of the areas of the surface components

$$\text{Area}(u) = \sum_{v \in \text{Vert}(\Gamma)} \text{Area}(u|S_v).$$

One needs an explicit two-form representing the second cohomology class in order to define area for regions in a curve in neck-stretched manifolds. Neck-stretched manifolds do not have a standard symplectic form, rather there is a family of cohomologous symplectic forms that we now describe.

6.1. Symplectic forms on neck-stretched manifolds. Neck-stretched almost complex manifolds are embedded into a compact symplectic manifold via increasing maps of dual complexes, which are defined next.

**Definition 6.2.** (Increasing maps on dual complexes) For any $\nu \geq 1$, a diffeomorphism $\aleph : \nu B^\vee \to B^\vee$ is said to be increasing if

(a) for any polytope $P \in \mathcal{P}$, the restriction $\aleph|_{\nu P} : \nu P \to P$ is a diffeomorphism onto $P$,

(b) and

$$\langle (d\aleph|_P)_x(\xi), \xi \rangle > 0, \quad \forall x \in \nu P, \xi \in t_P.$$

**Definition 6.3.** An increasing map $\aleph$ induces a sequence of diffeomorphisms $\psi^\nu$ between multiply-stretched manifolds $\{X^\nu\}_\nu$ and the compact symplectic manifold $(X, \omega)$ as follows. Recall from (2.21) that a symplectic cylindrical structure is a collection of symplectomorphisms $\psi_P : P \to P$ defined on neighborhoods $U_P \subset X$ of $\Phi^{-1}(P), P \in \mathcal{P}$:

$$U_P \xrightarrow{\phi_P} (\Phi^{-1}(P) \times P^\vee, \omega_{X_P} + d\langle \alpha_P, \pi_{P^\vee} \rangle), \quad \forall P \in \mathcal{P}.$$
Any increasing map \( \mathcal{N} : \nu P^\vee \to P^\vee \) induces a diffeomorphism \( \psi_{\mathcal{N}}^P \) on the \( P \)-neck region:

\[
X^\nu = \Phi^{-1}(P) \times \nu P^\vee \xrightarrow{\psi_{\mathcal{N}}^P} U_P.
\]

The maps \( \{ \psi_{\mathcal{N}}^P \}_{P \in \mathcal{P}} \) patch to produce a diffeomorphism

\[
\psi^\nu : X^\nu \to (X, \omega).
\]

This ends the Definition.

**Lemma 6.4.** For any strongly tamed cylindrical almost complex structure \( J^\nu \) on \( X^\nu \) and an increasing map \( \mathcal{N} : \nu B^\vee \to B^\vee \), the symplectic form \((\psi_{\mathcal{N}}^P)^* \omega \) tames \( J^\nu \).

**Proof.** Strong tamedness in Definition 2.35 implies that there is a projection map \( \pi_P : \Phi^{-1}(P) \times \nu P^\vee \to X^\nu \) whose fibers are \( J^\nu \)-holomorphic, and the horizontal tangent sub-bundle \( \ker(\alpha_P) \subset T \Phi^{-1}(P) \) is \( J^\nu \)-invariant. The symplectic form \((\psi_{\mathcal{N}}^P)^* \omega \) is equal to \( \omega_X + d(\mathcal{N} \alpha_P) \). On the vertical sub-bundle \( \ker(d\pi_P) \), the form is taming because of the increasing condition (6.1):

\[
(d\mathcal{N} \alpha_P)(v, J^\nu v) > 0, \quad v \in \ker(d\pi_P).
\]

On the horizontal sub-bundle \( \ker(\alpha_P) \subset T \Phi^{-1}(P) \), the form is \( \omega_X + \mathcal{N} d\alpha_P \) which is taming. \( \square \)

### 6.2. Hofer energy for neck-stretched manifolds.

Hofer energy is the supremum of areas of a perturbed holomorphic map, where the symplectic form is given by a class of maps into a fixed symplectic manifold. The allowable class of maps is given by squashing maps between polytopes.

**Definition 6.5.** (Squashing maps)

(a) **(Squashing datum)** Let \( P, Q \subset V \) be top-dimensional polytopes in a vector space \( V \) with an inner product. (The domain polytope \( Q \) is typically much ‘larger’ than the target \( P \).) An squashing datum consists of

(i) a partition \( Q = \bigcup_{i=1}^n Q_i \) of the target space into polytopes, and

(ii) elements \( \tau_1, \ldots, \tau_n \in V^\vee \) satisfying the following:

(i) For any \( i \), \( P_i := Q_i - \tau_i \subset P \) and the polytopes \( P_1, \ldots, P_n \) are disjoint.

(ii) (Orthogonality) For an intersecting pair \( Q_i \cap Q_j \neq \emptyset \), \( \tau_i - \tau_j \) is perpendicular to \( P_i \cap P_j \). If \( \text{codim}(P_i \cap P_j) = 1 \) then the vector \( \tau_i - \tau_j \) is directed from \( P_i \) to \( P_j \).

(b) **(Squashing map between polytopes)** Given squashing datum as above, the corresponding undilated squashing map \( \mathcal{N} : P \to Q \) is defined by

(i) \( x \mapsto x + \tau_i \) for all \( x \in P_i \),

(ii) and for any \( x \in P \setminus \bigcup_i P_i \), \( \mathcal{N}(x) := \mathcal{N}(y) \), where \( y \) is the point in \( \bigcup_i P_i \) that is closest to \( x \).

(c) **(Squashing maps for neck-stretched manifolds)** A squashing map for a neck-stretched manifold \( X^\nu \) is a map \( \mathcal{N} : \nu B^\vee \to B^\vee \) that satisfies \( \mathcal{N}(\nu P^\vee) \subseteq P^\vee \) for all polytopes \( P \in \mathcal{P} \) and \( \mathcal{N} : \nu P^\vee \to P^\vee \) is a squashing map of polytopes.
Remark 6.6. A squashing map \( \aleph \) of polytopes is surjective, continuous, and piecewise smooth. The above method defines a map \( \aleph \) uniquely. Indeed if a point \( x \in P \setminus \bigcup_i P_i \) has more than one closest point in \( \bigcup_i P_i \), then all the closest points have the same value of \( \aleph \) as a consequence of the (Orthogonality) condition.

Lemma 6.7. Let \( \aleph : P \to Q \) be a squashing map of polytopes. There is a partition \( P = \bigcup_{i=1}^m P_i \) by polytopes such that

(a) each \( P_i \) is a product \( P_i = R_i \times R_i' \) where \( R_i, R_i' \subset V \) are polytopes that are orthogonal to each other and have complementary dimension,

(b) and \( \aleph|_{P_i} \) is a projection map to \( R_i \) composed with a translation in \( V \).

The proof of the lemma is left to the reader.

Definition 6.8. (Hofer energy for neck-stretched manifolds) Let \( \{ X^\nu \}_\nu \) be a family of neck-stretched manifolds equipped with a cylindrical structure as in 2.27. The Hofer energy of a map \( u : C \to X^\nu \) is

\[
E_{\text{Hof}}(u) = \sup_{\aleph \nu} B^\nu \to B^\nu \int_C (\psi^\nu \circ \aleph)^* \omega,
\]

where the supremum is over all squashing maps \( \aleph \), and \( \psi^\nu \) is as in (6.3).

The next result shows that the quantity \( (\psi \circ \aleph)^* \omega \) is pointwise non-negative if the map \( u \) is pseudoholomorphic with respect to an almost complex structure that is close to a strongly tamed almost complex structure.

Definition 6.9. An almost complex structure is weakly taming if \( \omega(v, Jv) \geq 0 \) for all tangent vectors \( v \).

Lemma 6.10. (Monotonicity of Hofer energy) Suppose \( J_0 \in J^\text{cyl}(X) \) is a strongly tamed cylindrical almost complex structure. There is a \( C^0 \)-small neighbourhood \( U_{J_1} \subset J^\text{cyl}(X) \) of \( J_0 \) such that the following is satisfied for any \( J_1 \in U_{J} \) and any \( \nu : X^\nu \to C \):

(a) (Weakly taming) For any squashing map \( \aleph \),

\[
\forall \nu \in TX^\nu \quad \phi^\nu \omega(v, J_1^\nu v) \geq 0,
\]

where \( J_1^\nu \in J^\text{cyl}(X^\nu) \) is obtained by gluing \( J_1 \) at cylindrical ends.

(b) (Monotonic) Let \( u : C \to X^\nu \) be a perturbed \( J_1^\nu \)-holomorphic map for a domain dependent almost complex structure \( J^\nu : C \to U_{J_1}(J_0) \). For any \( J_1 \in U_{J}, \) and open subset \( \Omega \subset C \),

\[
E_{\text{Hof}}(u, \Omega) \leq E_{\text{Hof}}(u, C).
\]

Proof. We consider a polytope \( P \in \mathcal{P} \), and prove the (Weakly taming) property in the \( P \)-cylindrical region

\[
X_P^{\nu \bullet} := X_P^\nu \setminus \bigcup_{Q \subset P} X_Q^\nu \subset X^\nu.
\]

We recall that an almost complex structure on \( X_P^{\nu \bullet} \) is the pullback of a \( P \)-cylindrical almost complex structure on the \( P \)-cylinder via the embedding \( X_P^{\nu \bullet} \to Z_{P,C} \) in (2.25). The \( P \)-cylinder is a \( T_{P,C} \)-bundle \( \pi_P : Z_{P,C} \to X_P \). An almost complex
structure $J$ on $Z_{P,C}$ is determined by its projection to $X_P$, namely $d\pi_P(J)$, and the associated connection $\alpha_{P,J} \in \Omega^1(Z_P, t_P)$.

We first consider perturbations to the horizontal projection of the almost complex structure $\mathfrak{J}_0$. The horizontal part of the two-form $\psi^*\omega$ is on the piece corresponding to $P$ given by $\omega_{X_P} + \mathfrak{R}d\alpha_{J_{P,0}}$, which is a symplectic form by (2.21) that tames $d\pi_P(J_{P,0})$ by Definition 2.32. Since $\mathfrak{R}$ takes values in a compact set $P^\nu$, and tameness on a symplectic manifold is a $C^0$-open condition, we conclude the following: If $J_{P,1}$ is $C^0$-close enough to $J_{P,0}$, $d\pi_P(J_{P,1})$ is tamed by $\omega_{X_P} + \mathfrak{R}d\alpha_{J_{P,0}}$.

Next, we study the effect of changing the connection associated to the almost complex structure. Suppose $\mathfrak{J}_1 \in \mathcal{F}^{cyl}$ is a cylindrical almost complex structures whose horizontal projection $J_{P,1}$ on the $P$-piece is $C^0$-close to $d\pi_P(J_{P,0})$ as in the previous paragraph. Let $J_{P,10} \in \mathcal{F}^{cyl}(Z_{P,C})$ be such that $d\pi_P(J_{P,10}) = J_P$, and whose $P$-connection one-form is same as that of $J_{P,0}$. Denote the $P$-connection forms by

$$\alpha_0 := \alpha_{P,J_{P,10}} = \alpha_{P,J_{P,0}}, \quad \alpha_1 := \alpha_{P,J_{P,1}}.$$

The difference

$$A := \alpha_1 - \alpha_0$$

descends to a $t_P$-valued one-form on $X_P$. For a vector $(v,t) \in TZ_{P,C}$, where $v \in TZ_P$, $t \in t_P^\nu$, the difference between $J_{P,1}$ and $J_{P,10}$ is

$$(J_{P,1} - J_{P,10})(v,t) = -(A(J_Pv_P))_Z + J_F(Av_P)_Z, \quad v_P := d\pi_P(v),$$

where $J_F$ is the complex structure on the fibers of $\pi_P$, and for any $\xi \in t$, $\xi_Z \in \text{Vect}(Z)$ is the vector field generated by $\xi$. Then,

$$\psi^*\omega((v,t), \mathfrak{J}_1(v,t)) = (\psi^*\omega)((v,t), \mathfrak{J}_{10}(v,t)) + (\psi^*\omega)((v,t), (\mathfrak{J}_1 - \mathfrak{J}_{10})(v,t)).$$

By the previous paragraph, there is a constant $c > 0$ such that the first term in the right-hand-side of (6.4) is bounded on the $P$-region as

$$\psi^*\omega((v,t), \mathfrak{J}_{10}(v,t)) \geq c|v_P|^2 + \langle d\mathcal{R}_x(\alpha_0(v)), \alpha_0(v) \rangle + \langle d\mathcal{R}_x(t), t \rangle.$$

In the second term in the right-hand-side of (6.4), the difference $(\mathfrak{J}_1 - \mathfrak{J}_{10})(v,t)$ is in the fiber direction. The form $\psi^*\omega$ in the fiber is $d\mathcal{R} \wedge \alpha_0$ and therefore,

$$\psi^*\omega((v,t), (\mathfrak{J}_1 - \mathfrak{J}_{10})(v,t)) = \langle \alpha_0(v), d\mathcal{R}_x(A(v_P)) \rangle + \langle d\mathcal{R}_x(t), A(J_Pv_P) \rangle.$$

By the definition of squashing maps and Lemma 6.7, the derivative $d\mathcal{R}_x$ is diagonalizable with orthogonal eigenvectors and eigenvalues $n_1, \ldots, n_k \in [0,1]$. For any element $\xi \in t_P$ resp. $t_P^\nu$, we denote by $\xi_i \in t_P$ resp. $t_P^\nu$ the projection of $\xi$ to the $i$-th eigenspace. We write

$$d\mathcal{R}(\xi) = \sum_{i=1}^k n_i \xi_i.$$

Using this eigen-decomposition, and the equations (6.4), (6.5) and (6.6), we get

$$\psi^*\omega((v,t), \mathfrak{J}_1(v,t)) \geq c|v_P|^2 + \sum_{i=1}^k n_i(|\alpha_0(v)i|^2 + |t_i|^2 + \alpha_0(v)A(v_P)i + t_iA(J_Pv_P)i)$$

$$\geq \sum_{i=1}^k n_i(|t_i|^2 + |\alpha_0(v)i|^2 + |t_i|^2 + |\alpha_0(v)A(v_P)i + t_iA(J_Pv_P)i|).$$
The last two terms are bounded as
\[ \alpha_0(v)A(vP) \geq -\frac{|A|}{2}(|v|^2 + |\alpha_0(v)|^2), \quad t_i A(JPvP) \geq -\frac{|A|}{2}(|v|^2 + |t_i|^2) \]
where \( |A| := \|A\|_{C^0} \). Therefore, \((\psi^*\omega)((v,t),\mathcal{J}_1(v,t))\) is non-negative if \(|A| \leq \frac{c}{k}\), leading to the proof of the (Weakly Taming) property. The (Monotonic) property follows from (Weakly Taming).

Figure 12. Examples of squashing maps. In both examples, solidly shaded regions are mapped to points, ruled regions are mapped to lines by contracting each ruling to a point, and blank regions are mapped isometrically.

Remark 6.11. (Translational squashing maps) The set of allowed maps \( \mathcal{N} \) in the definition of Hofer energy may appear restrictive. However, note that there exist squashing maps that map into any \( B^\gamma \)-shaped region in \( \nu B^\gamma \). That is, for any map \( \Delta : P^\gamma \rightarrow \nu P^\gamma \) that is the restriction of a translation in \( t^\gamma \), there exists a squashing \( \mathcal{N} : \nu P^\gamma \rightarrow P^\gamma \) for which \( \mathcal{N} \circ \Delta = \text{Id}_{P^\gamma} \). Such a map \( \mathcal{N} \) is constructed as in the example in the left half of Figure 12.

Remark 6.12. (Comparison to Bourgeois et al. [9]) The above definition of Hofer energy is inspired by the definition of energy in [9]. There are two differences between our definition and the one in [9].

(a) The first difference is that we take supremum over the set of squashing maps \( \mathcal{N} \), which is a subset of the set of non-decreasing maps. We need the broader definition for dealing with perturbed holomorphic maps in the case of multiple cuts. In case of a single cut, the squashing condition is same as the non-decreasing condition.

(b) The two-form \( \psi^*\omega \) in our definition appears different from the two-form in [9]. The different versions give equivalent values of energy as we explain below. Consider a cylindrical almost complex structure \( J \) on \( X^\nu \). Recall the \( P \)-cylindrical region in \( X^\nu \) is denoted by \( X^\nu_P \). The region \( X^\nu_P \) can be viewed as a subset of \( Z_{P,C} \) which is a \( T_{P,C} \)-bundle over \( X_P \). Suppose \( J \) is associated to the \( T_P \)-connection form \( \alpha_P \). For an increasing diffeomorphism \( \mathcal{N} : \nu B^\gamma \rightarrow B^\gamma \) the symplectic form \( \psi^\mathcal{N}_x \omega \) on \( X^\nu_P \) is
\[ \pi^\nu_P \omega_{X_P} + d\langle \mathcal{N}, \alpha_P \rangle. \]
The expression for energy in [9] is given by \( \pi^\nu_P \omega_{X_P} + \langle d\mathcal{N}, \alpha_P \rangle \). The difference is a horizontal term \( \langle \mathcal{N}, d\alpha_P \rangle \), which we claim is uniformly bounded by
cπ^*ω_X_P for some c < 1. Indeed for any value of \( \aleph \in P^\nu \),

\[
(\omega_X P + \langle \aleph, d\alpha_P \rangle)(v, Jv) > 0 \quad v \in TX_P^\nu
\]

since it is the horizontal component of the pullback symplectic form \((\phi_{\aleph})^*\omega\) in (6.2), and \(X_P^\nu\) is compact.

This ends the Remark.

6.3. **Hofer energy on a broken manifold.** Hofer energy on a broken manifold is defined in a similar way as that of neck-stretched manifolds. We recall that cylindrical ends of \(X\) have a consistent choice of cylindrical coordinates: for a pair of polytopes \(Q \subset P\), the \(Q\)-cylindrical end \(U_{X_P}X_Q\) has cylindrical coordinates taking values in \(\text{Cone}_{P^\nu}Q^\nu \times Z_Q\) where \(\text{Cone}_{P^\nu}Q^\nu \subset v_Q^\nu\) and is defined in (2.8). To define Hofer energy the \(Q\)-cylindrical region in the component \(X_P\) of \(X\) is embedded into the \(Q\)-cylindrical region in \((X, \omega)\) via a map

\[
\aleph_Q : \text{Cone}_{P^\nu}Q^\nu \rightarrow Q^\nu
\]
satisfying the squashing condition. For a component \(X_P \subset X\), the domain of squashing maps is the union of cones

\[
\text{Cone}_{P^\nu}B^\nu := \left( \bigcup_{Q \subseteq P} \text{Cone}_{P^\nu}Q^\nu \right) / \sim,
\]

where, for any pair \(Q_0 \subset Q_1\), the equivalence relation \(\sim\) identifies \(\text{Cone}_{P^\nu}Q_1^\nu\) to a face of \(\text{Cone}_{P^\nu}Q_0^\nu\) as in the definition of the dual complex. For \(X_P\) the target space for squashing maps is

\[
B^\nu_P := \left( \bigcup_{Q \subseteq P} Q^\nu \right) / \sim \subset B^\nu;
\]

and \(\sim\) is the equivalence relation in the definition of \(B^\nu\).

**Definition 6.13.** (Squashing map for a broken manifold) Let \(X_P\) be a broken manifold and let \(P \in \mathcal{P}\) be a polytope. A **squashing map** for the component \(X_P \subset X_P\) is a map

\[
\aleph : \text{Cone}_{P^\nu}B^\nu \rightarrow B^\nu_P
\]

for which \(\aleph(\text{Cone}_{P^\nu}Q^\nu) \subset Q^\nu\), and \(\aleph : \text{Cone}_{P^\nu}Q^\nu \rightarrow Q^\nu\) is a squashing map of polytopes. See Figure 13 for examples.

We define squashing maps with some special properties which are used in the next Section.

**Definition 6.14.** (Unpartitioned squashing map) We say that a squashing function \(\aleph : P \rightarrow Q\) between polytopes \(P, Q\) is **unpartitioned** if the underlying partition of the target space \(Q\) has a single element, which is \(Q\) itself.

**Definition 6.15.** (Zero translation squashing map) An unpartitioned squashing map \(\aleph : \text{Cone}_{P^\nu}B^\nu \rightarrow B^\nu_P\) has **zero translation** if

\[
\{\aleph^{-1}(P^\nu) = \{P^\nu\}, \quad \dim(P^\nu) > 0, \aleph^{-1}(P^\nu, 0) \subset \text{Cone}_{P^\nu}P^\nu, \quad \dim(P^\nu) > 0.
\]
In Figure 13 both examples are unpartitioned and have zero translation. Given a cylindrical structure on $X_P$ a squashing map $\mathcal{N}$ induces a continuous surjection

$$\phi_\mathcal{N} : (X_P, J_0) \to U_P \subset (X, \omega),$$

that is piecewise a smooth submersion.

**Definition 6.16.** (Hofer energy for a broken manifold) Let $X_P$ be a broken manifold with a symplectic cylindrical structure as in 2.27 and let $P \in \mathcal{P}$ be a polytope. The **Hofer energy** of a map $u : C \to \overset{\circ}{X_P}$ is

$$E^{\text{Hof}}(u) = \sup_{\mathcal{N}} \int_C (\phi_\mathcal{N} \circ u)^* \omega,$$

where the supremum is over all squashing maps $\mathcal{N}$ for $X_P^\circ$.

The results involving Hofer energy heavily use unpartitioned squashing maps. Therefore we define another version of Hofer energy that is a supremum over unpartitioned squashing maps.

**Definition 6.17.** For a polytope $P \in \mathcal{P}$ and map $u : C \to \overset{\circ}{X_P}$, the **translational Hofer energy** is

$$(6.8)\quad E^{\text{tHof}}(u) = \sup_{\mathcal{N}} \int_C (\phi_\mathcal{N} \circ u)^* \omega,$$

where the supremum is over all unpartitioned squashing maps $\mathcal{N}$ for $X_P^\circ$.

**Remark 6.18.** The definition of the squashing area form $\phi_N^* \omega$ for a component $X_P^\circ$ of a broken manifold $X$ does not require the full data of the tropical Hamiltonian manifold, but rather the datum of neighborhoods of $\Phi^{-1}(R) \subset X$ for polytopes $R \in \mathcal{P}$, $Q \in R \subset P$. More precisely the definition of squashing area uses the symplectic manifolds

$$(6.9)\quad \Phi^{-1}(\mathcal{i}_R(R^\bullet \times R^\nu)) \subset X, \quad Q \in R \subset P$$

and the Hamiltonian $T_R$-action. Here $R^\bullet \subset R$ from (2.14) is the complement of a neighborhood of facets.
Given the datum in (6.9), and assuming that \( Q \in \mathcal{P} \) is the only zero-dimensional polytope, one may define another version of a squashing area form by replacing \( Q^\vee \) by a sub-polytope \( Q^\vee_+ \subset t^\vee \). Let \( H \subset t^\vee \) be a hyperplane that divides \( Q^\vee \) into two sub-polytopes \( Q^\vee_+, Q^\vee_- \), such that \( Q^\vee_- \) is a simplex that contains a part or the whole of the face \( P^\vee \). Thus for any \( P \subset R \subset Q \), \( R^\vee := R^\vee \cap Q^\vee_- \) is a face of \( Q^\vee_- \). We define squashing area forms via squashing maps \( \text{Cone}_{P^\vee} Q^\vee_- \rightarrow Q^\vee_- \). Indeed the datum required for the definition can be read off from (6.9).

**Definition 6.19.** For any map \( u : C \rightarrow X^\circ_P \) define

\[
E_{\text{Hof},Q^\vee_-}(u) = \sup_{\mathcal{N}} \int_C (\phi_{\mathcal{N}} \circ u)^* \omega,
\]

where the supremum is over squashing maps \( \mathcal{N} : \text{Cone}_{P^\vee} Q^\vee_- \rightarrow Q^\vee_- \). The translational Hofer energy

\[
E_{\text{Hof},Q^\vee_-}^*(u)
\]

can be defined analogously as in (6.8).

**Lemma 6.20.** For any map \( u : C \rightarrow X^\circ_P \) with \( E_{\text{Hof}}(u) < \infty \),

\[
E_{\text{Hof},Q^\vee_-}^*(u) \leq E_{\text{Hof}}(u).
\]

**Proof.** For any unpartitioned squashing map \( \mathcal{N}_- : \text{Cone}_{P^\vee} Q^\vee_- \rightarrow Q^\vee_- \), and a map \( u : C \rightarrow X^\circ_P \) we will show that

\[
(6.10) \quad \int_C (\phi_{\mathcal{N}_-} \circ u)^* \omega \leq E_{\text{Hof}}(u).
\]

The bound is proved by defining a family of squashing maps \( \mathcal{N}_t : \text{Cone}_{P^\vee} Q^\vee_- \rightarrow Q^\vee_- \) that extend \( \mathcal{N}_- \). Suppose the unpartitioned map \( \mathcal{N} \) is determined by a translation \( \tau_- \in t^\vee \). Suppose \( \xi \in t^\vee \) is normal to the separating hyperplane \( H \) and points towards \( Q^\vee_- \). Define a squashing map \( \mathcal{N}_t : \text{Cone}_{P^\vee} Q^\vee_- \rightarrow Q^\vee_- \) whose target space partition consists of \( Q^\vee_- \), \( Q^\vee_+ \) and translations are \( \tau_- \) and \( \tau_- + t\xi \) respectively. For any \( t \), let

\[
U_t := \{ \mathcal{N}_t = \mathcal{N}_- \} \subset \text{Cone}_{P^\vee} Q^\vee_-.
\]

Then

\[
\int_{u^{-1}(U_t)} (\phi_{\mathcal{N}_-} \circ u)^* \omega = \int_{u^{-1}(U_t)} (\phi_{\mathcal{N}_t} \circ u)^* \omega \leq E_{\text{Hof}}(u).
\]

Since the family of subsets \( U_t \) exhausts \( \text{Cone}_{P^\vee} Q^\vee_- \) as \( t \rightarrow \infty \), the estimate (6.10) follows. The Lemma is a consequence of (6.10). \( \square \)

6.4. **Hofer energy on quotients.** We define a formula for Hofer energy on quotients that occur as cut spaces. That is, we consider maps to \( X^\circ_P = X^\circ_{P^\vee}/T_{P,\mathbb{C}} \) for \( P \in \mathcal{P} \). Hofer energy on these quotients is defined as a supremum over squashing area forms that pullbacks of the symplectic form on a symplectic quotient.

We describe the symplectic quotient. Recall that the subset \( \cup_{Q \subseteq P} \Phi^{-1}(Q \times Q^\vee) \) of \( (X, P, \Phi) \) has a Hamiltonian action of the torus \( T_P \). Denote the moment map by \( \Phi_{P^\vee} : \cup_{Q \subseteq P} \Phi^{-1}(Q \times Q^\vee) \rightarrow P^\vee \).
We assume $0 \in P^\vee$ is such that $\Phi^{-1}(P) \subset \Phi^{-1}_{P^\vee}(0)$.

**Definition 6.21.** The symplectic quotient of $X$ at $P$ is the quotient $\Phi^{-1}_{P^\vee}(0)/T_P$ equipped with symplectic form $\omega_{/P}$. The *squashed area forms* are the pullbacks $\phi_\mathbb{N}^*\omega_{/P}$ where

$$\phi_\mathbb{N} : X_P^\circ \to \Phi^{-1}_{P^\vee}(0)/T_P$$

is a surjective continuous piecewise smooth map induced by the squashing function

$$\mathbb{N} : \text{Cone}_{P^\vee} B^\vee \cap H_P \to B^\vee_P \cap H_P.$$  

Here

$$H_P := \{\pi_{\mathbb{N}}^P = 0\} \subset t^\vee_P$$

is orthogonal to $t^\vee_P$, and $\mathbb{N}$ maps $\text{Cone}_{P^\vee} Q^\vee \cap H_P$ to $Q^\vee \cap H_P$ for any $Q \subset P$.

**Remark 6.22.** The quotient map $X_P^\circ \to X^\circ_P$ descends via $\pi_{P^\vee}$ to the orthogonal projection $\text{Cone}_{P^\vee} B^\vee \to \text{Cone}_{P^\vee} B^\vee \cap H_P$. Indeed the projection $\pi_{B^\vee}$ to $\text{Cone}_{B^\vee} B^\vee$ maps a $T_{P,C}$-orbit to an affine subspace parallel to $t^\vee_B$.

**Definition 6.23.** (Hofer energy for quotients) Let $P \in \mathcal{P}$ be a polytope with $\text{codim}(P) > 0$. For a map $u : C \to X^\circ_P$, the *Hofer energy* is

$$E_{\text{Hof},/P}(u) := \sup_{\mathbb{N}} \int_C (\phi_\mathbb{N} \circ u)^*\omega_{/P}$$

where the supremum is over squashing maps $\mathbb{N}$ between the spaces in (6.12) and $\phi_\mathbb{N}$ is as in (6.11).

Hofer energy is monotonic for perturbed holomorphic maps if the almost complex structure lies in a neighborhood $U_J$ as in Lemma 6.6.

**Proposition 6.24.** Let $Q \in \mathcal{P}$ be a polytope with $\text{codim}(Q) > 0$. Let $u : C \to X^\circ_Q$ be a perturbed holomorphic map with respect to a domain-dependent almost complex structure $J : C \to U_J$. Let $\pi_Q : X^\circ_Q \to X^\circ_Q$ be the quotient under the action of $T_{Q,C}$. Then,

$$E_{\text{Hof},/Q}(\pi_Q \circ u) \leq E_{\text{Hof}}(u).$$

**Proof.** We will prove the result assuming $\text{codim}(Q) = 1$. The general case follows by repeated application of the result for codimension one polytopes. For any quotient squashing function

$$\mathbb{N} : \text{Cone}_{Q^\vee} B^\vee \cap H_Q \to B^\vee_Q \cap H_Q,$$

and a map $u : C \to X^\circ_Q$ we will show that

$$\int_C (\phi_\mathbb{N} \circ \pi_Q \circ u)^*\omega_{/Q} \leq E_{\text{Hof}}(u).$$

Suppose the squashing map $\mathbb{N}$ is determined by a domain partition $B^\vee_Q \cap H_Q = \cup_{i=1}^n B_i$ and translations $\tau_i \in t^\vee \cap T_H$. We define a family of squashing maps

$$\mathbb{N}_t : \text{Cone}_{Q^\vee} B^\vee \to B^\vee_Q, \quad t \geq 0.$$
as follows. Let $\xi$ be a generator of $t_Q$. Consider a domain partition of $B_Q^\vee$ consisting of top-dimensional polytopes $B_1^+, B_1^-, \ldots, B_n^+, B_n^-$ that satisfy

(a) $B_1^+ \cap B_1^- = B_1$, and

(b) $\pm \langle \xi, v \rangle \geq 0$ for all $v \in \cup_i B_i^\pm$, or in other words, $B_i^+$ resp. $B_i^-$ lies in the $+\xi$ resp. $-\xi$-side of the hyperplane $B_Q^\vee \cap H_Q$ in $B_Q^\vee$.

For any $t \geq 0$, the squashing map $\mathcal{N}_t$ is defined by the domain partition $\{B_i^+, B_i^-\}_i$ and translations $\tau_i \pm t\xi$ for $B_i^\pm$. The map $\mathcal{N}_t$ is well-defined because for any $t$ the translate $B_i^+ + t\xi$ lies in Cone $Q^\vee B^\vee$ since the facets of Cone $Q^\vee B^\vee$ are parallel to $\xi$. On the set $U_t := \mathcal{N}_t^{-1}(B^\vee)$

$$\mathcal{N}_t = \mathcal{N} \circ \pi_{H_Q},$$

where $\pi_{H_Q} : t^\vee \to H_Q$ maps any point to its closest point in $H_Q$. Then,

$$\int_{\pi_{H_Q}^{-1}(U_t)} (\phi_{\mathcal{N}} \circ \pi_Q \circ u)^* \omega_Q = \int_{\pi_{H_Q}^{-1}(U_t)} (\phi_{\mathcal{N}} \circ u)^* \omega \leq E_{Hof}(u).$$

The sets $U_t$ exhaust Cone $Q^\vee B^\vee$ as $t \to \infty$ because the inverse image $\mathcal{N}_t^{-1}(\partial B^\vee)$ moves away from $H_Q$ as $t$ grows. Therefore the inequality (6.13) holds.

\[ \square \]

**Figure 14.** The squashing map $\mathcal{N}_t$ extends $\mathcal{N}$ as in the proof of Proposition 6.24.

### 6.5. Basic area forms.

We sometimes need a fixed family of area forms on the family of neck-stretching manifolds. For example, such a family is useful for the proof of convergence of areas in Gromov convergence in Section 7. We define a sequence of two-forms called ‘basic area forms’ on neck-stretched manifolds that resemble symplectic forms. They converge to the pull back of the base symplectic form in the components of the broken manifold.

The basic area form is given by a fixed squashing map which we now describe, beginning with the following remark.

**Remark 6.25.** Any undilated squashing map $\mathcal{N} : \nu B^\vee \to B^\vee$ as in the right side of Figure 12 is determined by a set of points

$$x(Q) \in Q^{\vee,0}, \quad Q \in \mathcal{P}, \dim(Q) = 0,$$

chosen to satisfy the following. If a polytope $q \in \mathcal{P}$ contains two distinct point polytopes $Q_0, Q_1 \in \mathcal{P}$, and $x(q, Q_i)$ is the point in $q^\vee$ that is closest to $x(Q_i)$, then
\[ x(q, Q_0) = x(q, Q_1). \] Thus for any \( q \) there is a point \( x(q) \) such that for all point polytopes \( Q \in q, x(q) = x(q, Q) \). The domain partition corresponding to \( \mathfrak{N}|Q_\nu \) is given by the partition

\[
Q^\nu = \bigcup_{P \in P, P^\nu \subseteq Q^\nu, \dim(P^\nu) = 0} \mathfrak{p}(P)
\]

where \( \mathfrak{p}(P) \) is the convex hull of the set of points \( \{x(q): q \in P, P^\nu \subseteq q^\nu \subseteq Q^\nu\} \). Finally \( \mathfrak{N} \) is an undilated squashing map whose underlying translation of \( \mathfrak{p}(P) \) is such that that \( \mathfrak{N}^{-1}(P^\nu) \) is a single point \( \nu P^\nu \). The resulting squashed area form on \( X^\nu \) is

\[
\omega^\nu_{\text{bas}} := \phi^*_{\mathfrak{N}, \nu} \omega \in \Omega^2(X^\nu)
\]

**Definition 6.26.** Given \( \mathfrak{N}_\nu \) as above define a sequence of dilated squashing maps \( \mathfrak{N}_{\nu,t} \) for all \( t \geq 1 \) as

\[
\mathfrak{N}_{\nu,t} := \delta_t \circ \mathfrak{N}_{\nu/t} : \nu B^\nu \to B^\nu, \quad \text{where} \quad \delta_t : (\nu/t)B^\nu \to B^\nu,
\]

which is dilation by a factor of \( t \). The maps define *dilated basic area forms* as

\[
\omega^\nu_{\nu,t} := \phi^*_{\mathfrak{N}_{\nu,t}, \nu} \omega \in \Omega^2(X^\nu).
\]

**Proposition 6.27.** (Basic area forms on neck stretched manifolds) The basic area forms \( \omega^\nu_{\nu} \) and \( \omega^\nu_{\nu,t} \) satisfy the following properties:

(a) The form \( \omega^\nu_{\nu} \) is cohomologous to \( \omega \in \Omega^2(X) \) and is weakly taming for \( J \in U_J \subset J^{\text{cyl}}(X) \) from Lemma 6.6.

(b) (Convergence to horizontal forms) There exist two-forms \( \omega^\nu_{\nu,t} \in \Omega^2(X_P) \), \( P \in \mathcal{P} \) satisfying the following. Suppose for a polytope \( P \in \mathcal{P}, t^\nu \in \nu P^\nu \) is a sequence of translations \( t^\nu \in \nu P^\nu \) satisfying \( d(t^\nu, \nu Q^\nu) \to \infty \) for any \( Q \supseteq P \). Then \( (\varepsilon^\nu)^* \omega^\nu_{\nu} \) converges to \( \pi^*_{P^\nu} \omega^\nu_{\nu,t} \), where the embedding \( \varepsilon^\nu : X^\nu_P \to X^\nu_{P^{\nu,P}} \) is defined in (2.34).

(c) (Non-degeneracy on compact sets) For any \( t \geq 1 \) there is an open set \( U_{P,t} \subset X^\nu_P \) on which \( \omega^\nu_{\nu,t} \) is non-degenerate, and the family \( \{U_{P,t}\}_t \) exhausts \( X^\nu_P \) as \( t \to \infty \).

The proof is left to the reader.

**Definition 6.28.** (Area of a region of a map) Let \( \Omega \) be a complex curve possibly with boundary, and let \( u : \Omega \to X^\nu \) be a perturbed holomorphic map. The basic area of the map is

\[
\text{Area}^{\text{bas}}(u, \Omega) := \int_{\Omega} u^* \omega^\nu_{\nu},
\]

where \( \omega^\nu_{\nu} \in \Omega^2(X^\nu) \) is the basic area form on \( X^\nu \). If a map \( u \) has closed domain \( \Omega \), or it maps the boundary of the domain \( \partial \Omega \) to the Lagrangian \( L \), then \( \text{Area}(u) = \text{Area}^{\text{bas}}(u) \). For a perturbed holomorphic map \( u : \Omega \to X^\nu \) as in Definition 6.28, the *dilated basic area* is

\[
\text{Area}^{\text{bas}}_{t}(u, \Omega) := \int_{\Omega} u^* \omega^\nu_{\nu,t}.
\]
6.6. Removal of singularities. In this section, we prove a removal of singularities result for holomorphic maps in a piece of a broken manifold. Of course, each of the pieces in a broken manifold has a compactification into a compact symplectic manifold, so the main question is boundedness of symplectic area. For the following, see for example [66, Proposition 3.12].

Proposition 6.29. (Monotonicity) Let \((X, \omega)\) be a compact symplectic manifold, and let \(J\) be a tame almost complex structure. There exist constants \(c, r_0 > 0\) such that for any \(x \in X, 0 < r \leq r_0\), a Riemann surface \(C\) with boundary \(\partial C\) and a \(J\)-holomorphic map \(u : C \to X\) whose image contains \(x\) and \(u(\partial C) \subset \partial B(x, r)\),

\[
\int_C u^* \omega \geq cr^2.
\]

For the removal of singularities result we consider maps on a punctured disk \(B_1 \setminus \{0\}\) which is holomorphically identified with \(\text{Cyl} := \mathbb{R}_{\geq 0} \times S^1\).

For any \(l \geq 0\), we refer to a truncated semi-infinite cylinder by \(\text{Cyl}(l) := [l, \infty) \times S^1\).

Proposition 6.30. (Removal of singularities) Suppose \(u : [0, \infty) \times S^1 \to X_P^o\) is a perturbed \(J\)-holomorphic curve with respect to the domain-dependent almost complex structure \(J : B_1 \to U_J\) (holomorphically identifying \(\text{Cyl} \simeq B_1 \setminus \{0\}\)) with

\[
E_{\text{Hof}}(u) < \infty, \quad \|du\|_{L^\infty(\text{Cyl})} < \infty.
\]

Then, \(u\) extends to a holomorphic map \(u : B_1 \to X_P\).

We prove the Proposition in the case of a single cut. The proof in the single cut case provides motivation for some of the definitions needed for the proof in the multiple cut case.

Proof of Proposition 6.30 in the case of a single cut. We set up some notation first. For a single cut the set of polytopes is \(\{P_0, P_\gamma, P_1\}\), where \(P_0, P_1\) are one-dimensional and \(P_\gamma\) is a point. The dual polytope \(B^\vee\) is a segment \([0, 1]\) whose end-points are \(P_0^\vee, P_1^\vee\). Choose any \(t \in \mathbb{R}\) such that \(S := [t, t+1] \times Z \subset X_P^o\).

Claim 6.31. There is a finite number of connected components \(C\) of \(u^{-1}(S)\) for whom the image \(u(C)\) intersects both boundary components of \(S\).

A connected component \(C\) as in the above Claim is called a crossing.

Proof of Claim. Let \(C\) be a crossing. The squashed area form \(\phi_{\delta S}^\vee \omega\) is a symplectic form on \(\delta S\). Choose a constant \(0 < \epsilon < \frac{1}{4}\). There exists \(z \in C\) such that \(u(z) \in \delta S \setminus B_\epsilon(\delta (\delta S))\). By the monotonicity theorem there is a constant \(c\) such that if \(\epsilon\) is small enough the squashed area, that is, the integral of \((\phi_{\delta S} \circ u)^* \omega\) over \(u^{-1}(B_\epsilon(u(z))) \cap C\), is at least \(c\epsilon^2\). Since \(\int_{\mathbb{R}_+ \times S^1} (\phi_{\delta S} \circ u)^* \omega\) is bounded by \(E_{\text{Hof}}(u)\) we conclude that there are a finite number of crossings. \(\square\)
Next we show that there is either an upper or a lower bound for the cylindrical \( R \) on the image of \( u \). Suppose \( S_0, S_1 \) are the components of the complement \( X_0^\circ \) of \( S \) so that \( \partial S_i = \{ t + i \} \times Z \). If all crossings of the map \( u \) are compact then there exists \( l \) such that \( u(\text{Cyl}(l)) \) is contained in the complement of crossings, and is therefore contained in either \( S_0 \) or \( S_1 \). Suppose a crossing \( C \subset \text{Cyl} \) is not compact. Then there exists \( l \) such that \( C \) intersects \( \{ l' \} \times S^1 \) for any \( l' \geq l \). Then \( u(\text{Cyl}(l)) \) is contained within a radius \( 2\pi|du|_{L^\infty} \) of \( S \). In this latter case the \( R \)-cylindrical coordinate on the image of \( u \) has both an upper and lower bound.

We now prove the result for the case that \( u \) maps to the neck piece \( X_0^\circ \). Let \( \pi_\cap : X_0^\circ_{\mathcal{P}_\cap} \to X_{\mathcal{P}_\cap} \) be the quotient by the \( \mathbb{C}^\times \)-action. Since \( \omega_{X_\cap}(\pi_\cap \circ u) \leq E_{\text{Hof}}(u) \), the projected map extends holomorphically to
\[
u_\cap := \pi_\cap \circ u : B_1 \to X_{\mathcal{P}_\cap},
\]
Consider a holomorphic trivialization \( u^*_\cap X_0^\circ_{\mathcal{P}_\cap} \). The projection of \( u \) to the fiber, denoted by
\[
u_\cap : B_1 \setminus \{ 0 \} \to S^1 \times \mathbb{R},
\]
is holomorphic. Since the \( \mathbb{R} \)-coordinate of \( u_\cap \) has a one-sided bound, \( u_\cap \) extends over \( 0 \) to a holomorphic map in \( \mathbb{P}^1 \). Consequently, \( u \) extends over \( 0 \) in the compactified space \( X_{\mathcal{P}} \).

Next suppose that \( u \) maps to \( X_0_{\mathcal{P}_\cap} \), the case of \( X_0^\circ_{\mathcal{P}_\cap} \) being similar. The image of \( u \) either lies in the \( \mathcal{P}_\cap \)-cylindrical end or it lies in a compact subset \( K \subset X_0_{\mathcal{P}_\cap} \). The former case is handled in the same way as when \( u \) maps to \( X_0^\circ_{\mathcal{P}_\cap} \). In the latter case, there is a basic area form \( \omega^\text{bas} \) on \( X_{\mathcal{P}_\cap} \) which is a symplectic form on the compact set \( K \). The \( \omega^\text{bas} \)-area of \( u \) is bounded by Hofer energy and so it is finite. The proof of the Proposition follows by the removal of singularity theorem for compact symplectic manifolds. This finishes the proof. \( \square \)

The proof of removal of singularities in the case of a single cut relied on partitioning the target space by level sets of the moment map, and showing that the image is entirely contained in one of the subsets. For multiple cuts, we do not have a component of the moment map whose level set disconnects the manifold. Instead we partition the target spaced via a partition of the dual polytope into ‘sectors’.

**Definition 6.32. (Sectors induced by a squashing map)** Let \( \mathcal{P} \) be the set of polytopes underlying a tropical Hamiltonian action, which contains a unique zero-dimensional polytope \( Q \in \mathcal{P} \). For \( \mathcal{P} \in \mathcal{P} \) let \( \pi : \text{Cone}^{\mathcal{P}\vee} B^\vee \to B^\vee_{\mathcal{P}} \) be an unpartitioned squashing map as in Definition 6.14.

**Definition 6.33.** Let \( p \in \mathcal{P} \) be a point. The *sector* corresponding to \( p \) is
\[ S_p := \pi^{-1}(\bigcup_{R: R \subseteq p} R^{\cap, p}) \]
where the union is over polytopes \( R \in \mathcal{P} \).

**Remark 6.34.** If \( p^\vee_0, \ldots, p^\vee_n \) are the vertices of \( Q^\vee \), then \( \text{Cone}_{\mathcal{P}\vee} B^\vee \) is the union of sectors \( \bigcup_{i=0}^n S_{p_i} \). See Figure 15.
Remark 6.35. The following is a motivation for defining sectors as above. If the dual polytope $Q^\vee$ corresponding to the polytope $Q$ is a simplex, then on a sector in $\text{Cone}_{P^\vee} Q^\vee$ either there is a bound on all cylindrical coordinates, or there is a torus $T_R$ satisfying $P \subset R \subseteq Q$ such that the sector lies in the $R$-cylindrical end. This fact is justified and used in the proof of Proposition 6.30.

The following result is the main technical ingredient in the proof of the removal of singularities. The proof is given later in the section.

Lemma 6.36. (Sector lemma) Let $P \in \mathcal{P}$ and let $Q \in P$ be a vertex. Let $u : \text{Cyl} \to X_0^P$ be a holomorphic map whose image does not intersect any $Q_1$-cylindrical end where $Q_1 \subset P$ is a face not containing $Q$. Suppose $u$ satisfies
\[ E_{\text{Hof}}^*(u) < \infty, \quad \| du \|_{L^\infty(\text{Cyl})} < \infty. \]
Then there exists $\ell_0 \geq 0$, an unpartitioned squashing function $\aleph : \text{Cone}_{P^\vee} Q^\vee \to Q^\vee$, and a vertex $p^\vee$ of $Q^\vee$ such that the image $u([\ell_0, \infty) \times S^1)$ is contained in the sector $\pi_{B^1}^{-1}(S_p^0) \subset X_0^P$.

One of the hypotheses of the sector lemma 6.36 is that the polytopal decomposition $\mathcal{P}$ consists of a unique zero-dimensional polytope. A zero-dimensional polytope in $P$ is a torus orbit in the tropical Hamiltonian manifold $(X, \mathcal{P}, \Phi)$ where many cut loci intersect. Two such intersection points are 'well-separated' in the symplectic manifold. Therefore a semi-infinite holomorphic cylinder with finite Hofer energy can accumulate in the neighborhood of at most one such intersection point. This fact is proved in the following Lemma. To state it we introduce notation for unions of cylindrical ends. For any $Q \subset P$, $Q \in P$, $\dim(Q) = 0$, let
\[ U_Q := \bigcup_{R : Q \subseteq R \subseteq P} \text{R-cylindrical end in } X_0^P \simeq \bigcup_{R : Q \subseteq R \subseteq P} \pi_{B^1}^{-1}(R^{\vee, 0}). \]
We remark that by the definition of tropical Hamiltonian actions any polytope $P \in \mathcal{P}$ has at least one vertex $Q$ in $\mathcal{P}$. The following Lemma is of interest if $U_Q \neq X_0^P$, which happens only if there is another vertex $Q_1$ of $P$ in $\mathcal{P}$.

Lemma 6.37. Let $u : \text{Cyl} \to X_0^P$ be a holomorphic map. Then there exists a constant $l \geq 0$ and a unique vertex $Q \subset P$ in $\mathcal{P}$ such that $u(\text{Cyl}(l)) \subset U_Q$.

Definition 6.38. A cone $C$ in an affine space $V$ is a proper cone if it has a non-empty interior and it is the intersection of $n$ half-spaces where $n := \dim(V)$.

Proof of Proposition 6.30. By Lemma 6.37 there is at most one vertex $Q \in \mathcal{P}$ such that the image of the cylinder $u$ accumulates in $R$-cylindrical ends for $Q \subset R \subseteq P$. So we may assume that $U_Q = X_0^P$ and that the polytopal decomposition $\mathcal{P}$ has only one zero-dimensional polytope $Q$. Automatically $P \subseteq Q$.

Claim 6.39. There is a polytope $q$ with $Q \subseteq q \subseteq P$ such that, possibly after restricting $u$ to $\text{Cyl}(l)$ for some $l$

(a) the image of $u$ lies in the $q$-cylindrical end of $X_0^P$, and therefore the projections
\[ \pi_q \circ u : \text{Cyl} \to X_0^P, \quad \pi_{q^l} \circ u : l^l \to q^l, \]
are well-defined, where \( \pi_{t^q_i} \) is the projection to \( t^q_i \) defined on the \( q \)-cylindrical end.

(b) The image of \( \pi_q \circ u \) lies in a compact subset of \( X_q^0 \).

(c) The image of \( \pi_{t^q_i} \circ u \) is contained in a proper cone of \( t^q_i \).

**Proof.** The proof of the Claim is by induction on \( \dim(Q^v) = \dim(t^v) \). The base case of the induction corresponds to \( t \) being a trivial torus, and is therefore vacuously true.

Consider a hyperplane \( H \) in \( t^v \) that divides the polytope \( Q^v \) into subpolytopes \( Q^v_+ \), \( Q^v_- \) such that \( Q^v_- \) is a simplex and it intersects the face \( P^v \). The simplex \( Q^v_- \) thus has a face \( F \) and a vertex \( P_0^v \in P^v \setminus F \), where \( P_0 \in \mathcal{P} \). By Lemma 6.20,

\[
E_{\text{Hof}}(u, Q^v_-) \leq E_{\text{Hof}}(u).
\]

We apply the sector Lemma 6.36 to the map \( u : \text{Cyl} \to \mathcal{U}_Q \) equipped with \( Q_- \)-Hofer energy. Suppose \( p_1, \ldots, p_n \) are vertices of the polytope \( Q^v_- \), where we assume that exactly \( p_1 \ldots p_k \) lie on the face \( P^v \subset Q^v_- \). By the sector Lemma 6.36 we conclude that, after truncating the domain by a finite amount, the image of \( u \) lies in the sector \( S(p_i) \) for some \( p_i \).

The cases \( i \leq k \) and \( i > k \) are treated separately. Suppose first that \( i \leq k \). Suppose that \( P^v \) is point and so the map \( u \) lies in the sector \( S_P \). Then the image of the map \( u \) lies in a compact subset of \( X_P^0 \), and the Claim follows. Next assume \( P^v \) is positive dimensional. The sector \( S(p_i) \) is a proper cone in \( Q^v_- \). The 1-edges of \( Q^v_- \) emanating from \( p_i \) that lie in \( P^v \) form a basis of \( t^v_{q_i} \). Each of these 1-edges is normal to a facet of the cone \( S(p_i) \). Therefore the orthogonal projection of \( S(p_i) \) to \( t^v_{q_i} \) is contained in a proper cone of \( t^v_{q_i} \). The induction hypothesis is applicable on the projection \( \pi_P \circ u : \text{Cyl} \to X_P \) since by Proposition 6.24 \( E_{\text{Hof}}(\pi_P \circ u) \leq E_{\text{Hof}}(u) \). The Claim is a consequence of the induction hypothesis and the fact that the image of \( \pi_{t_{q_i}} \circ u \) is contained in a proper cone of \( t_{q_i} \).

Next suppose \( i > k \). There exists \( Q_i \in \mathcal{P} \), \( \text{codim}(Q_i) = 1 \), such that \( p_i \) lies on the edge \( Q^v_i \) of \( Q^v_- \). (In fact \( Q^v_i \) is the edge connecting \( P_0^v \) and \( p_i \).) Further \( Q^v_i \nsubseteq P^v \). The sector \( S_{p_i} \) is contained in a half-plane of \( t^v \) whose boundary is normal to \( Q^v_i \) and on which the \( Q_i \)-cylindrical coordinate is unbounded. Consequently the sector \( S_{p_i} \) lies in the \( Q_i \)-cylindrical end. Let \( R_i := P \cap Q_i \). Then \( T_{R_i} \) is the minimal torus containing both \( T_P \) and \( T_{Q_i} \), and \( R^v_i \) is the smallest face of \( Q^v_i \) that contains both \( P^v \) and \( Q^v_i \). We have shown that the image of \( u \) lies in a subset of \( X_P^0 \) that has a projection \( \pi_{R^v_i} \) to \( X_{R^v_i}^0 \). The projection \( \pi_{t_{R^v_i}} \circ u \) lies in a proper cone of \( t_{R^v_i} \) by the same reasoning as in the previous paragraph. We may apply the induction hypothesis to the projection \( \pi_{R^v_i} \circ u : \text{Cyl} \to X_{R^v_i}^0 \) since by Proposition 6.24 \( E_{\text{Hof}}(\pi_{R^v_i} \circ u) \leq E_{\text{Hof}}(u) \). The Claim follows from the induction hypothesis and the fact that the image of \( \pi_{t_{R^v_i}} \circ u \) is contained in a proper cone of \( t_{R^v_i} \).

The proof of Proposition 6.30 follows from the above Claim. By Proposition 6.24, the Hofer energy \( E_{\text{Hof}}(\pi_q \circ u) \) of the projected map \( \pi_q \circ u \) is finite, where \( q \) is from the conclusion of the Claim. By Proposition 6.27 (c) there a neighborhood of the compact set \( K \subset X_q^0 \) containing the image of \( \pi_q \circ u \) that admits a taming form.
The removal of singularity theorem on compact symplectic manifolds applies, and we obtain an extension \( u_q : B_1 \to X_q \). Consider a holomorphic trivialisation of \( u_q^* X_q^\circ \to B_1 \). The projection of \( u \) to the vertical direction \( u_\parallel : B_1 \setminus \{0\} \to T_{q,C} \) is holomorphic, and extends to a smooth map \( u_\parallel : B_1 \to V_q^\circ \), where \( V_q^\circ \) is a \( T_q \)-toric variety, and therefore is a compactification of \( T_{q,C} \). Indeed \( u_\parallel \) does not have an essential singularity at \( z = 0 \) because the image of \( \pi_t \circ u_\parallel \) is contained in a proper cone of \( t_q^\circ \). □

**Figure 15.** Sectors induced by the squashing map \( R : \text{Cone}_{P^\circ} Q^\circ \to Q^\circ \). The thickened boundaries \( S_{P_i} \subset S_{P_i} \) of sectors are dotted.

We introduce some more notation for the proof of the Lemma. In the setting of Definition 6.32, the **thickened boundary** of the sector \( S_p \) is

\[
S_p := S_p \setminus (R^{-1}(p^\circ)) \simeq R^{-1}(\cup_{S_p} R^{\circ,q}).
\]

The complement of a thickened boundary \( S_p \) in \( \text{Cone}_{P^\circ} B^\circ \) has two components, namely \( R^{-1}(p^\circ) \) and \( \text{Cone}_{P^\circ} B^\circ \setminus S_p \). The shared boundaries of the complexes are denoted by

\[
\partial_+ S_p := S_p \cap (\text{Cone}_{P^\circ} B^\circ \setminus S_p), \quad \partial_- S_p := S_p \cap R^{-1}(p^\circ).
\]

**Proof of Lemma 6.36.** We claim that for any sector the annulus can only ‘cross’ its thickened boundary a finite number of times. This is achieved by proving that on each crossing there is a positive lower bound on the contribution to \( R \)-area, that is, the integral of \((\phi_R \circ u)^*\omega\).

The details are as follows. Consider a vertex \( p \) of \( Q^\circ \), and consider the crossings of the thickened boundary \( S_p \) by the map \( u \). Say that a connected component \( C \subset \text{Cyl} \) of \( u^{-1} \pi_B^{-1}(S_p) \) is a **crossing** if \( \pi_B u(C) \) intersects both \( \partial_+ S_p \) and \( \partial_- S_p \). We will show a uniform lower bound on \( \int_C (\phi_R \circ u)^*\omega \) for all crossings \( C \).

The proof is by applying the monotonicity lemma. Consider a polytope \( q \in P \) such that \( Q \subseteq q \subseteq P \). Let \( Y_q \) be a symplectic quotient of a subset of \((X,\omega)\) defined as

\[
Y_q := \Phi^{-1}(Q \cap i_q(q^\bullet \times q^\circ))/K_q.
\]
Here $i_q$ is the embedding $q \times q^\vee \to t^\vee$ from (2.12), $K_q \subset T$ is the torus whose Lie algebra $\mathfrak{t}_q \subset \mathfrak{t}$ is the orthogonal complement of $q \subset \mathfrak{t}$, and $q^\bullet \subset q$ from (2.14) is the complement of a neighborhood of the union of facets. Note that $\Phi^{-1}(Q \cap (q^\bullet \times q^\vee))$ lies in a level set of the projection of $\Phi$ to $t^\vee$. The quotient $Y_q$ is a symplectic manifold with boundary. The subspace $t_q \subset t$ is a complement of $t_q$ and so, the $t_q$-action descends to $Y_q$ with moment polytope $q^\vee$. Denote the moment map by

\begin{equation}
\Phi_q : Y_q \to q^\vee.
\end{equation}

First we give the proof for the case that $P = Q$ is a point. The space $X_Q^0$ is a $T_\mathbb{C}$-bundle $X_Q^0 \to X_Q^Q$. Denote by $\omega_{X_Q}$ the symplectic form on the base $X_Q$ (there are no cylindrical ends on $X_Q$) and write

\begin{equation}
\phi^*_q \omega = \omega_{X_Q} + d(\alpha_Q, \mathcal{N}).
\end{equation}

Choose an arbitrary unpartitioned squashing function $\mathcal{N}$ on $\text{Cone}_{Q^\vee} B^\vee$, which in this case is just $t^\vee$.

We partition the subset of the broken manifold corresponding to the thickened boundary $S_p$ into subsets, each of which is a fibration over a manifold $Y_q$ (see (6.16) above). Later we will consider the part of the map $u$ whose image lies in each of these subsets, and apply the monotonicity lemma by projecting to $Y_q$. The partition of the manifold $X$ is the $\pi_{B^\vee}$-inverse of a partition of the polytope $Q^\vee$ by subsets $\{V_q : p^\vee \in q^\vee \subset Q^\vee\}$. Here $V_q$ is a thickening of $q^\vee$ in $Q^\vee$ minus a neighborhood of faces $r^\vee \subset q^\vee$. The quantitative description is as follows. Let $\epsilon > 0$ be a small constant. For any face $q \subset p$, define

\begin{align*}
\epsilon_q &:= \text{codim}_{Q^\vee}(q^\vee) \epsilon, \\
\bar{\epsilon}_q &:= \max\{\epsilon_q - \epsilon, 0\}.
\end{align*}

Let $U(q^\vee) \subset Q^\vee$ be a tubular neighborhood of the face $q^\vee$ such that for the orthogonal projection

\begin{equation}
\pi_q : U(q^\vee) \to q^\vee
\end{equation}

the fibers $\pi_q^{-1}(x)$ are affine isomorphic to each other for $x \in q^\vee \setminus \cup_{r^\vee \subset q^\vee} B_{\bar{\epsilon}_q}(r)$. Define

\begin{equation}
V_q := \begin{cases} 
\mathbb{N}^{-1}(\pi_q^{-1}(q^\vee \setminus \cup_{r^\vee \subset q^\vee} B_{\bar{\epsilon}_q}(r))), & q^\vee \subset Q^\vee, \\
\mathbb{N}^{-1}(Q^\vee \setminus B_\epsilon(\partial Q^\vee)), & q^\vee = Q^\vee.
\end{cases}
\end{equation}

For proper faces $q^\vee \subset Q^\vee$ we also define an enlargement of $V_q$ as

\begin{equation}
\tilde{V}_q := \mathbb{N}^{-1}(\pi_q^{-1}(q^\vee \setminus \cup_{r^\vee \subset q^\vee} B_{\bar{\epsilon}_q}(r))), \quad \text{codim}(q^\vee) > 0.
\end{equation}

We assume the tubular neighborhoods are large enough that

\begin{equation}
\cup_{q^\vee \in q^\vee \subset Q^\vee} V_q = S_p \setminus B_\epsilon(\partial \pm S_p).
\end{equation}

The orthogonal projection $V_q \to q$ yields a fibration of manifolds

\begin{equation}
\pi_q : \pi_q^{-1}(V_q) \to Y_q.
\end{equation}

The fibers of $\pi_q$ are holomorphic because the almost complex structure is $Q$-cylindrical and the projection is orthogonal. On $\pi_{B^\vee}(V_q)$ the two-form $\phi_q^* \omega$ satisfies for any $J \in U_J$

\begin{equation}
\pi_q^* \omega \eta_q(v, Jv) \leq \phi_q^* \omega(v, Jv),
\end{equation}
which is an equality on \( \pi_{B^\vee}^{-1}(q^{\vee,0}) \). Indeed on \( \pi_{B^\vee}^{-1}(q^{\vee,0}) \), \( \pi_q \) is equal to \( \phi_K \) composed with the quotient map by \( K_q \).

Finally we apply the monotonicity theorem. Let \( q^\vee \subset Q^\vee \) be the highest dimensional polytope such that the image \( u(C) \) intersects \( \pi_{B^\vee}^{-1}(V_q) \). First consider the case that \( q = Q^\vee \). Then there exists a point \( x \in u(C) \cap \pi_{B^\vee}(V_{Q^\vee}) \) and \( B_q(\pi_{B^\vee}(x)) \) is an equality on \( Q^\vee \). Since \( \pi_{B^\vee} \) is uniformly continuous, there exists a constant \( \delta(\epsilon) \) such that \( B_q(x) \subset \pi_{B^\vee}^{-1}(N^{-1}(Q^\vee,0)) \). Consequently \( \phi_K \) symplectomorphically embeds \( B_q(x) \) into \( \mathcal{Y}_{Q^\vee} \), which is an open set in \( (X, \omega) \). By the monotonicity theorem applied to the ball \( B_q(x) \), we conclude that there is a constant \( c(\delta) \) such that \( \int_{C_q}(\phi_K \circ u)^* \omega \geq c \). Next, consider the case when \( q^\vee \subset Q^\vee \), that is, \( \text{codim}(q^\vee) > 0 \). The fibration \( \pi_q : \pi_{B^\vee}^{-1}(V_q) \to \mathcal{Y}_q \) has

- a vertical boundary \( \partial_v(\pi_{B^\vee}^{-1}V_q) \) which is the union of the boundaries of the fibers of \( \pi_q \),
- a horizontal boundary \( \partial_h(\pi_{B^\vee}^{-1}V_q) \) which is the \( \pi_q \)-inverse image of the boundary on the base \( \pi_q(\pi_{B^\vee}^{-1}(V_q)) \subset \mathcal{Y}_q \).

The vertical boundary \( \partial_v(\pi_{B^\vee}^{-1}V_q) \) is contained in \( \cup_{q^\vee \supset q^\vee} \pi_{B^\vee}^{-1}(V_{q^\vee}) \). By the maximality of the dimension of \( q^\vee \) the image \( u(C) \) does not intersect the vertical boundary \( \partial_v(\pi_{B^\vee}^{-1}V_q) \). Denote \( C_q := C \cap u^{-1}(\pi_{B^\vee}^{-1}(V_q)) \). The projection \( u_q := \pi_q \circ u : C_q \to \mathcal{Y}_q \) is a holomorphic map to a symplectic manifold. Recall from (6.17) that there is a moment map \( \Phi_q : Y_q \to q^\vee \). We also recall

\[
\pi_q(\pi_{B^\vee}^{-1}(V_q)) = \Phi_q^{-1}(q^\vee \setminus B_{\epsilon_q}(v^{\vee})), \quad \pi_q(\pi_{B^\vee}^{-1}(V_q)) = \Phi_q^{-1}(q^\vee \setminus B_{\epsilon_q}(v^{\vee})),
\]

and \( \epsilon_q = \epsilon_q - \epsilon \). Therefore, there exists a \( \epsilon = \delta(\epsilon) > 0 \) such that for any \( z \in \pi_q(\pi_{B^\vee}^{-1}(V_q)) \), \( B_q(x) \subset \pi_q(\pi_{B^\vee}^{-1}(V_q)) \). Choose \( x \in u_q(C_q) \cap \pi_q(\pi_{B^\vee}^{-1}(V_q)) \). Indeed by our definition of \( q \) the intersection is non-empty. Also, \( u_q(\partial C_q) \) is contained in the boundary \( \partial \pi(q_{V_q}) \). Finally we apply the monotonicity lemma 6.29 to \( u_q \) and the ball \( B_q(z) \). The conclusion is that \( \int_{C_q} u_q^* \omega \geq c \). By (6.20), \( \int_{C_q}(\phi_K \circ u)^* \omega \geq c \) as well.

Next we consider the case when \( Q \subsetneq P \). In that case we choose \( \mathcal{N} \) to have zero translation as in Definition 6.15. Consider a vertex \( p^\vee \) of \( Q^\vee \). Define a cover

\[
S_p \setminus B_{\epsilon}(\partial_+ S_p) = V_{q^\vee} \cup \bigcup_{q^\vee \subset C_q \cap Q^\vee, p^\vee \notin q^\vee} V_{q^\vee}
\]

where \( V_{q^\vee} := V_q \) from (6.19) if \( q \neq Q \), and

\[
V_{q^\vee} := \bigcup_{q : p^\vee \notin q^\vee \subset Q^\vee} V_q.
\]

As in the earlier proof, to apply the monotonicity theorem, we consider the largest dimensional polytope \( q^\vee \) for which the image of the crossing intersects \( \pi_{B^\vee}^{-1}(V_q^\vee) \). If \( q = Q \) then there is a symplectic form on \( \pi_{B^\vee}^{-1}(V_Q^\vee) \), although now \( \pi_{B^\vee}^{-1}(V_q^\vee) \) is a larger set than \( \pi_{B^\vee}^{-1}(V_q) \). The rest of the proof is the same as before.

We now finish the proof. So far we have shown that for any vertex \( p \) of \( Q \) the map \( u \) has a finite number of crossings with the thickened boundary of the sector \( S_p \). Let \( C_{cr} \subset Cyl \) be the union of all crossings for all \( p \in Q \). For any connected
component \( C \) of the complement \( \text{Cyl}\backslash \text{Cyl}_{cr} \) there exists \( p_0' \in Q^\circ \) such that \( u(C) \) is contained in the sector \( \pi_{B^\circ}^{-1}(S_{p_0}) \). If \( C_{cr} \) is compact then one of the connected components \( C \subset \text{Cyl}\backslash \text{Cyl}_{cr} \) contains \( \text{Cyl}(l) \) for a large enough \( l \) and \( u(\text{Cyl}(l)) \) lies in a sector of \( X_{T} \). Otherwise there is a non-compact crossing \( C \subset \text{Cyl} \) for a thickened boundary \( S_{p_0} \). There exists \( l_0 \) such that \( u(\{l\} \times S^1) \) intersects \( C \) for all \( l \geq l_0 \). Since \( du \) is uniformly bounded, the image \( \pi_{B^\circ}(u(\text{Cyl}(l_0))) \) lies within a bounded distance of \( S_{p_0} \). Dilating the squashing map \( R \) by a large enough factor \( t \) ensures that \( \pi_{B^\circ}(u(\text{Cyl}(l_0))) \) is contained in \( S_{p_0} \). Consequently the image \( u(\text{Cyl}(l_0)) \) lies in the sector \( S_{p_0} \), proving the Lemma.

The next result is for an annulus in a component \( X_{T} \) in the broken manifold \( \mathcal{X} \). The result says that after possibly truncating the annulus by a finite length, the image of the annulus cannot intersect disjoint cylindrical ends of \( X_{T} \).

**Proof of Lemma 6.37.** We may assume that \( P \) is top-dimensional, otherwise we carry out the proof for the projection \( \pi_P \circ u : \text{Cyl} \to X_{P}^\circ \). We use a basic area form \( \omega_{l}^{bas} \) on \( X_{P}^\circ \). Suppose the lemma is not true. Then the polytope \( P \) has at least two vertices \( Q_0, Q_1 \) in \( P \) such that for any \( l \), \( u(\text{Cyl}(l)) \) intersects both \( \mathcal{U}Q_0 \cup \mathcal{U}Q_1 \) and \( \mathcal{U}Q_0 \cup \mathcal{U}Q_1 \). The two sets are separated by

\[
S := \bigcup_{Q \leq P, Q_0, Q_1 \in Q} \pi_{B^\circ}^{-1}(Q^\circ).
\]

That is there are distinct boundary components \( \partial_0 S, \partial_1 S \) of \( S \), which are boundaries of \( \mathcal{U}Q_0 \cup \mathcal{U}Q_1 \) and \( \mathcal{U}Q_0 \cup \mathcal{U}Q_1 \) respectively. The proof of the result follows from the following Claim.

**Claim 6.40.** There are a finite number of connected components \( C \) of \( u^{-1}(S) \) whose image intersects both \( \partial_0 S, \partial_1 S \).

We will show a uniform lower bound on \( \int_C u^* \omega_{bas} \) where \( C \) is as in the Claim. The lower bound proves the Claim because the \( \omega_{bas} \)-area of \( u \) on \( R_+ \times S^1 \) is bounded by \( E_{\text{Hof}}(u) \) which is finite. The image \( u(C) \) intersects \( S' := S \backslash B_{\epsilon}(\partial_0 S \cup \partial_1 S) \) for a small \( \epsilon \). Suppose \( u(C) \) intersects \( \pi_{B^\circ}^{-1}(P) \cap S' \). For a small enough \( \epsilon > 0 \), \( B_{\epsilon}(\pi_{B^\circ}^{-1}(P)) \) is contained in \( S \) and \( \omega_{bas} \) is a symplectic form on it. The monotonicity theorem then gives a lower bound on the \( \omega_{bas} \)-area of \( u^{-1}(B_{\epsilon}(\pi_{B^\circ}^{-1}(P))) \). Otherwise let \( Q \subset P \) containing \( Q_0, Q_1 \) be a maximal dimensional polytope such that \( u(C) \) intersects \( \pi_{B^\circ}^{-1}(Q) \). The proof is by applying the monotonicity lemma in the \( T_{Q,C} \)-quotient of a \( Q \)-cylindrical subset of \( X_{P}^\circ \) in a manner similar to the proof of Lemma 6.36. The details are left to the reader. \( \square \)

### 6.7. Hofer energy for Gromov compactness of broken maps

So far we have given a definition of Hofer energy for the proof of Gromov compactness of maps in neck-stretched manifolds. For the proof of convergence of a sequence of broken maps (in the upcoming Section 7.5), we choose a different version of Hofer energy so that Hofer energy is bounded by area. We describe such a choice in this section.
Definition 6.41. Consider a polytope $P \in \mathcal{P}$. The space

$$ (X^\omega_P, \omega_P) := \Phi^{-1}(P \times P^\vee)/\sim $$

is a toric fibration over $X^\omega_P := \Phi^{-1}(P)/\sim$. The fiber of $X^\omega_P \to X^\omega_P$ is the symplectic $T_P$-toric manifold $V_{P^\vee}$, see (2.17). A fibered squashing map is a pair $\mathcal{R} = (\mathcal{R}_b, \mathcal{R}_f)$ consisting of a squashing map $\mathcal{R}_b$ resp. $\mathcal{R}_f$ for the base resp. the fiber of $X^\omega_P$.

Remark 6.42. The base squashing map is same as the map for quotients in Section 6.4. In the fiber direction, the map is $\mathcal{R}_f : t^\vee_P \to P^\vee$. The pair $\mathcal{R} = (\mathcal{R}_b, \mathcal{R}_f)$ induces a map

$$ \phi_\mathcal{R} : X^\circ_P \to X^\omega_P $$

which is weakly taming in a neighborhood $U_\mathcal{R} \subset J^{\text{cy}}_P(\mathcal{X})$.

Definition 6.43. For a perturbed holomorphic map $u : C \to X^\circ_P$ define the fibered Hofer energy as

$$ E_{P,\text{Hof}} = \sup_{\mathcal{R}} \int_C (\phi_\mathcal{R} \circ u)^* \omega_P, $$

where $\omega_P$ is the symplectic form on $X^\circ_P$ and $\mathcal{R}$ ranges over fibered squashing maps.

The following result says that in a cylindrical manifold, Hofer energy is the sum of horizontal area, and a weighted sum of the intersection multiplicities.

Proposition 6.44. For any polytope $P \in \mathcal{P}$, there is a constant $c$ such that for a perturbed holomorphic map $u : \mathbb{P}^1 \to X_P$ that intersects boundary divisors at the points $z_1, \ldots, z_d(\bullet)$ with multiplicities $T(z_i) \in t_P$,

$$ E_{P,\text{Hof}}(u) \leq \int_C (\pi_P \circ u)^* \omega_{X_P} + c \sum_{i=1}^d |T(z_i)|. $$

If $u(z_i)$ maps to an intersection of boundary divisors, then $|T(z_i)|$ is the sum of the intersection multiplicities of the map with each of the divisors at $z_i$.

Proof. For a map $u$ from a closed domain to $X^\circ_P$, Hofer energy $E_{\text{Hof}}(u)$ is equal to the pairing of $[u]$ with the cohomology class of the symplectic form $\omega_P$ in $X^\circ_P$. We bound the topological quantity $\langle [u], \omega_P \rangle$ as follows. Suppose the moment polytope for the fiber of $X^\circ_P \to X_P$ is

$$ P^\vee = \{ x \in t^\vee_P : \langle \mu_i, x \rangle \leq c_i, i = 1, \ldots, N \} $$

where $c_i \in \mathbb{R}$ and $\mu_i \in t_{P;\mathbb{Z}}, i = 1, \ldots, N$ are primitive outward pointing normals to the facets of $P^\vee$. Suppose the base symplectic form $\omega_P$ on $X_P$ is given by reduction at the origin $0 \in t^\vee_P$. Then,

$$ (6.21) \quad \omega_P(u) = \omega_P(\pi_P \circ u) + 2\pi \sum_{i=1}^d \sum_{j} c_j \cdot m(z_i, D_j), $$

where $m(z_i, D_j)$ is the intersection multiplicity at $z_i$ with the toric divisor $D_j = \{ \langle \mu_j, x \rangle = c_j \}$. Since the origin $0 \in t^\vee_P$ is in the interior of $P^\vee$, the constant $c_j$ is positive for all $j$. The Proposition follows from (6.21).
7. Gromov compactness

We prove the two versions of Gromov compactness with breaking along multiple divisors. The first theorem concerns the limit of holomorphic maps to the neck-stretched manifolds $X^\nu$ in the limit $\nu \to \infty$ and requires only an area bound:

**Theorem 7.1.** (Gromov convergence for breaking maps) Suppose $(J_0, D)$ is a stabilizing pair for the broken manifold $X$, and $p^\infty$ is a perturbation datum on $X$ adapted to $(J_0, D)$. Suppose $\{p^\nu\}_\nu$ is a sequence of perturbation data on neck-stretched manifolds $\{X^\nu\}_\nu$ that converges to $p^\infty$. Let $u_\nu : C^\nu \to X^\nu$ be a sequence of treed $p^\nu$-adapted disks with uniformly bounded area $\text{Area}(u_\nu)$. There is a subsequence of $\{u_\nu\}_\nu$ that Gromov converges to a $p^\infty$-adapted stable broken disk $u : C \to X_P$ modelled on a tropical graph $\Gamma$. The limit $u$ is unique up to domain reparametrizations and the action of the identity component of the tropical symmetry group $T_{\text{trop}, W}(\Gamma)$ (see (3.17)).

Gromov convergence is defined in the following section. Adapted perturbations are defined in 5.9, and convergence of perturbation data on neck-stretched manifolds is defined in 5.11.

The second theorem is a compactness result for sequences of broken maps.

**Theorem 7.2.** (Gromov convergence for broken maps) Suppose $(J_0, D)$ is a stabilizing pair for the broken manifold $X_P$, and $p$ is a perturbation datum on $X_P$ adapted to $(J_0, D)$. Suppose $u_\nu : C_\nu \to X_P$ is a sequence of adapted broken $p$-holomorphic disks with uniformly bounded area.

After passing to a subsequence the type of the broken disks $u_\nu$ is $\nu$-independent, and is equal to, say, $\Gamma$. The sequence $u_\nu$ Gromov converges to a broken adapted $p$-disk $u : C \to X_P$ of type $\Gamma'$ for which there is a tropical edge-collapse morphism $\Gamma' \to \Gamma$. The limit $u$ is unique up to domain reparametrizations and the action of the identity component of the tropical symmetry group $T_{\text{trop}, W}(\Gamma)$ (see (3.17)).

The edge-collapse $\Gamma' \to \Gamma$ is trivial (in the sense of Definition 7.25) if and only if there exist disks $u'_\nu$ in the $T_{\text{trop}, W}(\Gamma)$-orbit of $u_\nu$ which converge to the limit disk $u$ with a trivial translation sequence $t'_\nu$ (that is $t'_\nu(v) = 0$ for all vertices $v$).

**Remark 7.3.** In theorems 7.1 and 7.2, the limit broken map is uniquely determined if it is rigid. Indeed for a rigid tropical graph $\Gamma$ the tropical symmetry group is finite.

7.1. Gromov convergence. In this section, we define the notion of convergence in the compactness results, Theorem 7.1 and Theorem 7.2.

We first fix identifications between domain curves that are close to each other in the compactified moduli space of stable curves. This is done via a choice of an exponential map in the neighborhood of nodes. Different choices lead to the same notion of convergence. We start by recalling a construction of deformation of a stable nodal curve. Let $\Gamma$ be a curve type with only interior nodes. The moduli space of curves $M_\Gamma$ is a submanifold of the compactified moduli space $\overline{M}_d$ whose tubular neighborhood can be described as follows. For a curve $C$ in $\overline{M}_\Gamma$, let
\( \tilde{C} \) be the normalization of \( C \) at the nodal points. For any edge \( e \in \text{Edge}_{\bullet,-}(\Gamma) \), we fix a map

\[
\exp_{w_{\pm}(e)}^{C} : (U(T_{w_{\pm}(e)}\tilde{C}), 0) \to (U_{w_{\pm}(e)}(\tilde{C}), w_{\pm}(e)),
\]

that biholomorphically maps a neighborhood \( U(T_{w_{\pm}(e)}\tilde{C}) \) of the origin in the tangent space onto a neighborhood \( U_{w_{\pm}(e)}(\tilde{C}) \) of the lift of the node \( w_{e} \), satisfies \( d\exp_{w_{\pm}(e)}^{C}(0) = \text{Id} \), and varies smoothly with \( C \). The family of maps \( \{\exp_{w_{\pm}(e)}^{C} : e \in \text{Edge}_{\bullet,-}(\Gamma)\} \) is fixed for the rest of this paper. Whenever we choose complex coordinates

\[
z_{\pm} : (U_{w_{\pm}(e)}(\tilde{C}), w_{\pm}(e)) \to (\mathbb{C}, 0)
\]

in neighborhoods of the node we assume that they are compatible with \( \exp_{w_{\pm}(e)}^{C} \). That is, \( z_{\pm} \) is the composition of \( \exp_{w_{\pm}} \) with a linear map on the tangent space.

On a neighborhood

\[
U_{M_{\Gamma}} \subset M_{d(\circ),d(\bullet)}
\]

of \( M_{\Gamma} \), there is a projection map

\[
\pi_{\Gamma} : U_{M_{\Gamma}} \to M_{\Gamma},
\]

such that curves in a fiber \( \pi_{\Gamma}^{-1}(C) \) are obtained by gluing the interior nodes of \( C \) as follows.

**Definition 7.4.** A *collection of gluing parameters* is a tuple

\[
\delta = (\delta_{e})_{e} : \delta_{e} \in T_{w_{\pm}(e)}\tilde{C} \otimes T_{w_{-}(e)}\tilde{C}.
\]

The *curve corresponding to a gluing parameter* \( \delta \) is defined by

\[
C^{\delta} := (C \setminus \cup_{e} U'_{w_{\pm}(e)}(\tilde{C})) / \sim,
\]

\[
z_{+} \sim z_{-} \iff (\exp_{w_{\pm}(e)}^{C})^{-1}(z_{+}) \otimes (\exp_{w_{-}(e)}^{C})^{-1}(z_{-}) = \delta_{e}.
\]

Here \( U'_{w_{\pm}(e)}(\tilde{C}) \subset \tilde{C} \) is a neighborhood of \( w_{\pm}(e) \) such that the boundary \( \partial U'_{w_{\pm}(e)}(\tilde{C}) \) is identified to \( \partial U_{w_{\pm}}(\tilde{C}) \) (with reversed orientation) by the equivalence relation \( \sim \).

Thus, for any node \( w_{e} \), the relation \( \sim \) identifies the pair of annuli

\[
U_{w_{+}(e)}(\tilde{C}) \setminus U'_{w_{+}(e)}(\tilde{C}) \xrightarrow{\sim} U_{w_{-}(e)}(\tilde{C}) \setminus U'_{w_{-}(e)}(\tilde{C}).
\]

The resulting annulus in \( C^{\nu} \) is called \( \text{Neck}_{e}(C^{\nu}) \), and

\[
\text{Neck}(C^{\nu}) := \cup_{e} \text{Neck}_{e}(C^{\nu}) \subset C^{\nu}.
\]

The gluing construction maps a neighborhood of zero in the space of gluing parameters to a neighborhood of \( C \) in the fiber \( \pi_{\Gamma}^{-1}(C) \). Curves in the neighborhood \( U_{M_{\Gamma}} \) of \( M_{\Gamma} \) possess a neck length function for every smoothened node

\[
nl_{e} : U_{M_{\Gamma}} \to \prod_{e \in \text{Edge}_{\bullet,-}(\Gamma)} T_{w_{\pm}(e)}\tilde{C} \otimes T_{w_{-}(e)}\tilde{C}, \quad C^{\delta} \mapsto \delta.
\]
Remark 7.5. At a node \( w_e \) if we choose a framing \( fr : T_{w_+(e)} \tilde{C} \otimes T_{w_-(e)} \tilde{C} \to \mathbb{C} \) or if we choose complex coordinates

\[
z_+ : (\tilde{C}, w_+(e)) \to (\mathbb{C}, 0), \quad z_- : (\tilde{C}, w_-(e)) \to (\mathbb{C}, 0),
\]

then the neck length \( n_{l e} \) is a complex number.

Remark 7.6. (Identifications between nearby curves) Given a nodal curve \( C \in \mathcal{M}_\Gamma \) and a curve \( C' \in \mathcal{M}_{d(\circ), d(\bullet)} \) in the neighborhood of \( C \), the complement of nodes \( C^o \) can be identified to subsets of the curve \( C' \) as follows. First suppose that \( C' \) is obtained by gluing \( C \). That is, \( C' \in \pi^{-1}_\Gamma(C) \). For a vertex \( v \in \text{Vert}(\Gamma) \), let \( C'(v) \subset C' \) be the subset corresponding to the component \( C(v) \subset C \) including the necks \( \text{Neck}_e(C') \), \( v \in e \). The subsets \( C'(v) \) cover \( C' \). By the gluing construction there are natural inclusions \( i_{C,C',v} : C'(v) \to C_v \subset C \).

Next suppose \( C' \) is not in \( \pi^{-1}_\Gamma(C) \). There is a neighborhood \( U_{\Gamma,C} \subset \mathcal{M}_{\Gamma} \) of \( C \) on which the universal curve \( U_{\Gamma} \) can be trivialized, so that the variation of the complex structure is given by a map

\[
(7.5) \quad U_{\Gamma,C} \to J(S_{\Gamma}), \quad m \mapsto j(m)
\]

for which \( j(m) \) is \( m \)-independent in the neighborhood of special points. The trivialization gives a diffeomorphism \( \phi : \pi_{\Gamma}(C') \to C \). The maps \( i_{C,C',v} \) are defined as compositions

\[
i_{C,C',v} := \phi \circ i_{\pi_{\Gamma}(C'), C', v}.
\]

If a sequence \( C_\nu \) of curves converges to \( C \), then the images \( i_{C,C_\nu,v}(C_\nu(v)) \) exhaust the complement of nodes \( C^o_v \) as \( \nu \to \infty \).

Remark 7.7. (Uniqueness of identifications) The identifications between regions of nearby curves is unique in the following sense. Suppose a sequence \( [C_\nu] \in \overline{\mathcal{M}}_{d(\circ), d(\bullet)} \) converges to a curve \([C]\). For any node \( e = (v_+, v_-) \) in \( C \), the identification in the neck region

\[
(7.6) \quad i_{C,C_\nu,v_\pm} : \text{Neck}_e(C'_\nu) \to C_{v_\pm}
\]

is uniquely determined by the choice of the exponential map \((7.1)\). On the complement of the neck region let

\[
\phi_\nu, \phi'_\nu : C'_\nu \setminus \text{Neck}(C'_\nu) \to C
\]

be two possible identifications given by trivializations of the universal curve. Then \( \phi_\nu, \phi'_\nu \) have the same image, and the maps \( \phi'_\nu \circ \phi_\nu^{-1} \) converge to the identity uniformly in all derivatives.

Definition 7.8. (Annuli converging to a node) Let \( C_\nu \) be a sequence of curves converging to a limit curve \( C \) for which the arguments \( \frac{n_{l e}(C_\nu)}{n_{l e}(C_\nu)} \) of the neck length parameters converge.
(a) A sequence of annuli $A_\nu \subset C_\nu$ converge to a node $w$ in $C$ if there are $U_{w_\pm} \subset C_{v_\pm}$ of the lifts $w_+, w_-$ of $w$ such that

$$A_\nu = i^{-1}_{C,C_\nu,v_\pm}(U_{w_+}) \cap i^{-1}_{C,C_\nu,v_-}(U_{w_-}).$$

We say that the sequence of annuli $A_\nu$ is obtained by gluing the node $w$.

(b) (Centered annuli converging to a node) Suppose the node $w \in C$ is equipped with complex coordinates $z_\pm : (U_{w_\pm}, w_\pm) \to \mathbb{C}$ on neighborhoods $U_{w_\pm} \subset C_{v_\pm}$ of its lifts $w_+, w_-$. A sequence of centered annuli converging to the node $w$ is a sequence of parametrized annuli

$$A_\nu := \left[ -\frac{l_\nu}{2}, \frac{l_\nu}{2}\right] \times \mathbb{R}/2\pi \mathbb{Z} \hookrightarrow C_\nu$$

for which there exists $\epsilon \in \mathbb{R}$ such that

$$A_\nu = i^{-1}_{C,C_\nu,v_\pm}(\{|z_+| \leq \epsilon\}) \cap i^{-1}_{C,C_\nu,v_-}(\{|z_-| \leq \epsilon\}),$$

and the map

$$A_\nu \xleftarrow{i_{C,C_\nu,v_\pm}} C_{v_\pm} \xrightarrow{z_\pm} \mathbb{C}$$

is equal to $(s, t) \mapsto \exp(\mp (s + it) - (l_\nu + i\theta_\nu)/2)$.

Here $e^{-(l_\nu + i\theta_\nu)} \in \mathbb{C}^\times$ is the neck length parameter for the curve $C_\nu$ resulting from the choice of coordinates $z_\pm$.

We remark that in the above definition the annuli $A_\nu \subset \text{Neck}_e(C_\nu)$ may not be at an equal distance from both the boundaries of $\text{Neck}_e(C_\nu)$. However the difference of the lengths of $A_\nu$ and $\text{Neck}_e(C_\nu)$ is constant : $l_\nu - l_\nu' = \epsilon$.

The following definition describes the “renormalization” of neck coordinates needed to obtain bubbles in the neck components in the neck-stretching limit.

**Definition 7.9.** (Translation sequence for a tropical graph) Suppose $\Gamma$ is a tropical graph. A $\Gamma$-translation sequence consists of a collection of sequences

$$\{t_\nu(v) \in \nu B^\vee\}_{\nu, v} \subset \text{Vert}(\Gamma)$$

such that the following hold:

(a) (Polytope) For each vertex $v$

$$t_\nu(v) \in \nu P(v)^\vee \subset \nu B^\vee.$$

(b) (Slope) For any node $e$ between vertices $v_+, v_-$, there is a sequence $l_\nu(e) \to \infty$ such that

$$t_\nu(v_+) - t_\nu(v_+) = T(e)l_\nu.$$

**Definition 7.10.** (Gromov convergence, multiple cuts) A sequence of holomorphic maps $u_\nu : (C_\nu, z_\nu) \to X_\nu$ converges to a broken map $u : C \to X_P$ with tropical graph $\Gamma$ and framing $\text{fr}$ if the following are satisfied.
(a) (Convergence of domains) The sequence of curves \( C_\nu \) converges to \( C \) and for any interior node \( w \) the arguments \( \frac{\text{nl}(C_\nu)}{\text{nl}(C_\nu)} \) of the neck length parameters converge to a limit. Using the fixed holomorphic exponential map (7.1), let \( C_\nu(v) \subset C_\nu \) be the subset corresponding to a vertex \( v \in \text{Vert}(\Gamma) \), and let
\[
i_{v,\nu} : i_{C,C_\nu,v} : C_\nu(v) \to C_v, \quad C_\nu(v) \subset C_v,
\]
be embeddings from (7.6) whose images \( i_{v,\nu}(C_\nu(v)) \) exhaust the complement of nodes \( C_v^o \) as \( \nu \to \infty \).

(b) (Convergence of maps) There is a \( \Gamma \)-translation sequence \( \{ t_\nu(v) \}_{v,\nu} \) such that for any irreducible component \( C_v \subset C \), the sequence of maps
\[
e^{-t_\nu(v)}(u_\nu \circ i_{v,\nu}) : C_\nu(v) \to X_{P(v)}
\]
converges in \( C^\infty_{\text{loc}}(C_v^o) \) to \( u : C_v \to X_{P(v)} \). The map \( e^{-t_\nu(v)} : X_{P(v)} \to X_{P(v)} \) is defined in (2.33).

(c) (Thin cylinder convergence) For a node \( w \) in \( C \) corresponding to an edge \( e = (v_+,v_-) \in \text{Edge}_\bullet(\Gamma) \), let
\[
z_\pm : (U_{w_\pm},w_\pm) \to (\mathbb{C},0)
\]
be matching coordinates (see (3.5) in Definition 3.4) on neighborhoods \( U_{w_\pm} \subset C_{v_\pm} \) of the nodal point which respect the framing \( \text{fr}_e \). Let
\[
A(l_\nu) := [-l_\nu/2,l_\nu/2] \times S^1 \subset C_v
\]
be a sequence of centered annuli converging to the node \( w \), see Definition 7.8. Then the sequence
\[
x_\nu := e^{-\frac{1}{2}(t_\nu(v_+)+t_\nu(v_-))}u_\nu(0,0) \in X_{P(e)}^o
\]
converges to a limit \( x_0 \), and the components \( u_{v_\pm} \) of the broken map are asymptotically close to
\[
z_\pm \mapsto z_\pm^{T(e)} x_0.
\]

(d) (Area convergence) \( \lim_{\nu \to \infty} \text{Area}(u_\nu) = \text{Area}(u) \).

The (Thin cylinder convergence) condition in the above Definition roughly says that for a sequence of annuli that converges to a node, the evaluation of \( u_\nu \) at the mid-points of the annuli converge to the evaluation of the node, after adjusting by translation sequences \( t_\nu \).

7.2. Horizontal convergence. We discuss a notion of compactness for sequence of points in neck-stretched manifolds, that is useful in the proof of convergence for breaking maps.

Definition 7.11. (Horizontal convergence) A sequence of points \( x_\nu \in X^\nu \) horizontally converges to a point \( x \in X_P \) for a polytope \( P \in \mathcal{P} \) if
- \( x_\nu \in X^\nu_P \) for all \( \nu \),
- the sequence \( \pi_P^\nu(x_\nu) \) converges to \( x \), where \( \pi_P^\nu : X^\nu_P \to X_P \) is the projection map from (2.32),
• $x$ lies in $X^\circ_P = X_P \setminus \bigcup_{Q \subseteq P} X_Q$,
• and for any subsequence of $\{x_\nu\}_\nu$, the above conditions are not satisfied for any polytope $P_0 \supset P$.

**Lemma 7.12.** For any sequence $x_\nu \in X^\nu$, there exists a subsequence of $\{x_{\nu_k}\}_k$ that converges horizontally. There is a unique polytope $P \in \mathcal{P}$ for which the subsequence $\{x_{\nu_k}\}_k$ converges horizontally in $X_P$.

**Proof.** The first statement is a consequence of the definition of horizontal convergence. Suppose a sequence $x_\nu \in X^\nu$ converges horizontally in both $X_{P_0}$ and $X_{P_1}$, $P_0 \neq P_1$. Let $Q = P_0 \cap P_1$. There exist translations $t_\nu \in \nu Q^\vee$ such that $e^{-t_\nu} x_\nu$ converges in $X^\nu_{Q^\vee}$. Since $x_\nu \in X^\nu_{P_1}$, $d(t_\nu, P_1^\nu)$ is uniformly bounded. Then $d(t_\nu, P^\nu)$ is also uniformly bounded where $P \in \mathcal{P}$ is such that $P^\nu = P_0^\nu \cap P_1^\nu$. In fact $P$ is the smallest polytope for which $P_0$ and $P_1$ are faces. We conclude that $x_\nu$ horizontally converges in $X_P$, contradicting our initial assumption that $P_0 \neq P_1$. \hfill \Box

The following Lemma gives criteria to determine the polytope in which a sequence of points converges horizontally.

**Lemma 7.13.** A sequence of points $x_\nu \in X^\nu$ horizontally converges in $X_P$ if and only if the following holds: There exists a sequence $t_\nu \in \nu P^\nu$ such that $e^{-t_\nu} x_\nu$ converges in $X^\nu_{P^\nu}$, and $d(t_\nu, \nu P_0^\nu) \to \infty$ for all polytopes $P_0 \supset P$.

**Proof.** We prove the ‘if’, and leave the ‘only if’ to the reader. The convergence of the translated sequence $e^{-t_\nu} x_\nu$ implies that the projected sequence $\pi_P(x_\nu) \in X^\circ_P$ converges in $X^\circ_P$. There is no subsequence of $x_\nu$ that horizontally converges in $X_{P_0}$ for $P_0 \supset P$, because of the condition $d(t_\nu, \nu P_0^\nu) \to \infty$. Indeed, the limit $\lim_\nu \pi_{P_0}(x_\nu)$ lies in $X_P \subset X_{P_0}$, and not in $X^\circ_{P_0}$. \hfill \Box

To state the next result, we recall some notation. The subset $X^\circ_P \subset X^\nu$ is a manifold with corners, and using notation from (2.15), its codimension one boundary is the union

$$
\partial X^\nu_P = \bigcup_{(P_0, Q): P_0 \cap P = Q, \text{codim}_{P_0}(Q) = 1} \Phi^{-1}(Q^\bullet) \times P_0^\nu.
$$

Indeed, $\Phi^{-1}(Q^\bullet) \times P_0^\nu$ is a subset of the boundary of $\Phi^{-1}(Q^\bullet) \times Q^\nu \subset X^\nu_0 \subset X^\nu_P$. Let $d_{X^\nu}$ denote distance with respect to the cylindrical metric on $X^\nu$.

**Lemma 7.14.** Let $x_\nu \in X^\nu$ be a sequence that converges horizontally in $X_P$. Then, $d_{X^\nu}(x_\nu, \partial X^\nu_P) \to \infty$ as $\nu \to \infty$.

**Proof.** Suppose after passing to a subsequence, there are polytopes $P_0, Q$ such that $Q \subseteq P$, $P_0 \supset Q$, $\text{codim}_{P_0}(Q) = 1$, and $d_{X^\nu}(x_\nu, \Phi^{-1}(Q^\bullet) \times P_0^\nu)$ is uniformly bounded. Then, we arrive at a contradiction because $x_\nu$ horizontally converges in a polytope $P_0^\nu \supset P_0$. \hfill \Box

The next result may be seen as the horizontal convergence version of the Arzela-Ascoli theorem.
Lemma 7.15. Suppose $C$ is a connected curve and $u_\nu : C \to X^\nu$ is a sequence of differentiable maps satisfying $\sup_\nu \|du_\nu\|_{L^\infty} < \infty$. There exists a subsequence of maps $\{u_{\nu_k}\}_k$ and a polytope $P \in \mathcal{P}$ such that for all $z \in C$ the sequence $u_{\nu_k}(z)$ converges horizontally in $X_P$. For the subsequence $\{u_{\nu_k}\}_k$, the polytope $P$ is unique.

Proof. First assume that $C$ is compact. Choose a point $z_0$ on the curve $C$. By Lemma 7.12, after passing to a subsequence, there is a unique polytope $P \in \mathcal{P}$ such that $u_\nu(z_0)$ converges horizontally in $X_P$. This implies $u_\nu(z_0) \in X_P^\nu$. By Lemma 7.14

$$d_{X_\nu}(u_\nu(z_0), \partial X_P^\nu) \to \infty \text{ as } \nu \to \infty.$$ 

By the uniform boundedness of the derivatives $du_\nu$ and the compactness of $C$, the image $u_\nu(C)$ is contained in $X_P^\nu$ for large $\nu$. Further, for all $z \in C$,

$$(7.9) \quad d_{X_\nu}(u_\nu(z_0), \partial X_P^\nu) \to \infty \text{ as } \nu \to \infty.$$ 

By Arzela-Ascoli, a subsequence of $\pi_P \circ u_\nu : C \to X_P$ converges uniformly to a limit map $u : C \to X_P$. Together with (7.9), we conclude that for any $z_1 \in C$ the sequence $u_\nu(z_1)$ horizontally converges in $X_Q$ for some $Q \subseteq P$. The preceding argument then implies that $u_\nu(z_0)$ horizontally converges in $X_Q$, which leads to the conclusion that $Q = P$. The result also holds for non-compact curves since they are exhausted by a sequence of compact curves.

7.3. Breaking annuli. The next proposition governs the behavior of annuli with small base area and bounded Hofer energy. For any $L > 0$, we denote by

$$A(L) := [-\frac{L}{2}, \frac{L}{2}] \times S^1 \cong \{ z \in \mathbb{C} \mid |z| \in [e^{-L/2}, e^{L/2}] \}$$

the annulus with length $L$.

Proposition 7.16. (Breaking annulus lemma) Let $J_0 \in U_J(\mathbb{R})$ be a cylindrical almost complex structure. There are constants $0 < \rho < 1$, $c > 0$ such that the following hold. For a sequence $l_\nu \to \infty$, let

$$u_\nu : A(l_\nu) \to X^\nu$$

be a sequence of $J_0$-holomorphic maps satisfying the following.

(a) $\sup_\nu E_{\text{Hofer}}(u_\nu) < \infty$, $\sup_{z \in \mathbb{R} \times S^1} |du_\nu(z)| < \infty$.

(b) For all $t \geq 1$,

$$(7.10) \quad \lim_{L \to \infty} \lim_{\nu \to \infty} \text{Area}_{t}^{\text{bas}}(u_\nu, A(l_\nu - L)) = 0.$$ 

(c) There exist polytopes $P_+, P_-$ such that $u_\nu(\cdot \pm \frac{L}{2})$ horizontally converges in $X_{P_\pm}$.

(d) There exists a sequence of translations $t_\nu^\pm \in \nu P_\nu^\pm$ such that the sequence $e^{-t_\nu} u_\nu(\cdot \mp \frac{L}{2})$ converges in $C^\infty_{\text{loc}}$ to

$$u_\pm : \mathbb{R}_\pm \times S^1 \to X_{P_\pm},$$

and the map extends holomorphically over $\mp \infty$.

Then, there exists $\mu \in t_{P \cap \Sigma}$, $P_\cap := P_+ \cap P_-$ for which the following hold.

(a) The sequence $t_\nu^+ - t_\nu^- - \mu l_\nu \in \nu P_\cap$ is uniformly bounded.
(b) (Horizontal matching) The points $\pi_{\tilde{P}_s}(u_+(\infty))$, $\pi_{\tilde{P}_s}(u_-(\infty))$ lie in $X_{\tilde{P}_s} \subset X_{\tilde{P}_p}$ and $\pi_{\tilde{P}_s}(u_+(\infty)) = \pi_{\tilde{P}_s}(u_-(\infty))$.

(c) (Asymptotic decay) Let $\xi_\nu$ be a section defined by the relation

$$u_\nu = \exp_{u_\nu, \text{triv}} \xi_\nu, \quad u_{\nu, \text{triv}}(s, t) := e^{l(s + t)}u_\nu(0, 0).$$

There exists $l \geq 0$ and a subsequence such that for $k = 0, 1,$

$$|D^k \xi_\nu(s, t)| \leq c(e^{l(s - \frac{l^2}{2})} + e^{l(s - \frac{-l^2}{2})}), \quad \forall s \in \left[\frac{-l_\nu}{2} + l, \frac{l_\nu}{2} - l\right].$$

Proof of Proposition 7.16. The Proposition is proved using the annulus lemma on compact symplectic manifolds.

Step 1: Proof of horizontal matching.

Suppose $x_0 := \pi_{\tilde{P}_s}(u_-(\infty)) \in X^0_{\tilde{P}_s}$. Then $P \subseteq P_-$.

First we will show that, after truncating the domain cylinders by a $\nu$-independent amount, the $\tilde{P}$-projections of the images are contained in a small neighborhood of $x_0$. Choose $t$ such that the dilated basic area form $\omega_{\tilde{P}_t}^{\text{bas}}$ is a symplectic form in a neighborhood $U_t \subset X_{\tilde{P}_t}$ of $x_0$. Since $P \subseteq P_-$ the convergence of $e^{i\bar{\nu}}u_\nu$ to $u_-$ implies that the projection $\pi_{\tilde{P}_t}(u_\nu)$ converges in $C^\infty_{\text{loc}}$ to $\pi_{\tilde{P}_t}(u_-)$. Consider a constant $\rho > 0$ such that $\overline{B}_t(x_0) \subset U_t$. Choose a sequence of points $z_\nu = s_\nu + it_\nu \in A(l_\nu)$ such that

$$x_\nu := \pi_{\tilde{P}_t}(u_\nu(z_\nu)) \in B_{\epsilon/2}(x_0), \quad |\pm \frac{l_\nu}{2} - s_\nu| \to \infty.$$ We apply the monotonicity theorem to $\pi_{\tilde{P}_t} u_\nu$ on the ball $B(x_\nu, \frac{\epsilon}{2})$ and obtain a constant $\rho_0 > 0$ such that

$$\omega_{\tilde{P}_t}(u_\nu^{-1}(B_{\epsilon/2}(x_\nu))) > \rho.$$ Away from the ends of the cylinder the basic area goes to zero as $\nu \to \infty$ by (7.10), and so, there exists $L_0$ and $\nu_0$ such that

$$\text{Area}^{\epsilon}_{\text{bas}}(u_\nu A(l_\nu - L_0)) \leq \rho.$$ Consequently

$$\pi_{\tilde{P}_s} u_\nu(A(l_\nu - L_0)) \subset B_{\epsilon/2}(x_\nu) \subset B_\epsilon(x_\nu)$$

for $\nu \geq \nu_0$. We claim $P \subseteq P_+$. Indeed by (7.12) a subsequence of $u_\nu(\cdot + \frac{l_\nu}{2})$ horizontally converges in a polytope $P' \supseteq P$. By the hypothesis of the Proposition and the uniqueness of $P'$ (by Lemma 7.12) we conclude that $P' = P_+$.

Next we prove (Horizontal Matching) using the fact that any neighborhood of $x_0$ contains the images of the cylinders truncated by a $\nu$-independent amount. Since $P \subseteq P_+$ the convergence of the translated maps $e^{-it\bar{\nu}}u_\nu(\cdot + \frac{l_\nu}{2})$ implies that the sequence $\pi_{\tilde{P}_t}(u_\nu(\cdot + \frac{l_\nu}{2}))$ converges to $\pi_{\tilde{P}_t}(u_+)$.

By (7.12) we conclude

$$\pi_{\tilde{P}_t}(u_+(\infty)) \in \overline{B}_{\epsilon/2}(z_\nu) \subset B_\epsilon(x_0).$$

Since $\epsilon$ can be chosen to be arbitrarily small,

$$\pi_{\tilde{P}_t}(u_+(\infty)) = \pi_{\tilde{P}_t}(u_-(\infty)).$$
This finishes the proof of horizontal matching modulo the proof of the fact that \( P = P_+ \cap P_- \). We have shown that \( P \subseteq P_+ \cap P_- \) and the proof of equality is postponed to the last step of the Proof.

**Step 2:** Determining the edge slope \( \mu \).

In this step we read off the slope \( \mu \) of the edge from the topology of the cylinders \( u_\nu \) and show that it is equal to the intersection multiplicities of the limit maps with boundary divisors. We have shown that there exists \( L_0 \) such that the images \( \pi_P(u_\nu(A(l_\nu - L_0))) \) lie in a neighborhood \( B_\epsilon(x_0) \subseteq X_\nu^0 \). Consider a trivialization of \( X_\nu^0 \rightarrow X_\nu^1 \)

\[
B_\epsilon(x_0) \times T_{P,C} \simeq \pi_P^{-1}(B_\epsilon(x_0)) \subseteq X_\nu^0.
\]

Viewing the target space of \( u_\nu \) as the product \( B_\epsilon \times T_{P,C} \), the homotopy class \( (u_\nu)_* [A(l_\nu)] \in \pi_1(T_P) \) corresponds to an element \( \mu_\nu \in t_{P,Z} \). Let \( \mu \in t_{P,Z} \) be the intersection multiplicity of \( u_- \) at \( \infty \) with the vertical divisors of \( X_\nu \). The intersection multiplicity with a divisor \( D \subset X_\nu \) may be viewed as the winding number of the map \( u_- : \mathbb{R}_+ \times S^1 \rightarrow X_\nu \setminus D \), and therefore \( \mu_\nu = \mu \) for large \( \nu \). Similarly \( \mu \) is also the intersection multiplicity of \( u_+ \) at \( -\infty \).

**Step 3:** The sequence of twisted maps

\[
\overline{u}_\nu(s, t) := e^{-\mu(s+|l_\nu|t)+it}(e^{-it}u_\nu)
\]

converges to a pair of disks connected at an interior point.

First we show that one of the components in the Gromov limit of \( \overline{u}_\nu \) is a disk \( \overline{u}_- \) which is a twisted version of \( u_- \). The convergence of \( u_\nu \) to \( u_- \) implies that the sequence \( \overline{u}_\nu(\cdot - \frac{l_\nu}{2}) \) converges to \( \overline{u}_- := e^{-\mu(s+it)}u_- \). The image of \( \overline{u}_- \) is compact in \( X_\nu^0 \) since \( u_- \) is asymptotically close to the \( \mu \)-cylinder \( (s, t) \mapsto e^{\mu(s+it)} \). Since \( \pi_P \circ u_- \) extends over \( \infty \), the same is also true for \( \overline{u}_- \). We denote \( \overline{u}_0 := \overline{u}_-(\infty) \).

Next, we describe the component of the limit map attached to the disk \( \overline{u}_- \) at the nodal point \( \infty \). Choose a taming symplectic form \( \omega_\nu \) defined in a neighborhood \( U_{x_0} \subseteq X_\nu^0 \) of \( x_0 \). The form \( \omega_\nu \) can, for example, be defined using a squashing map \( \mathfrak{K} \) as in Remark 6.25 but the squashing property is not relevant for us here. Let \( U_{x_0} \subseteq U_{x_0}^\nu \) and let the constants \( c, \hbar, \mu \) be from the annulus lemma on compact manifolds (Proposition 7.17) applied to \((U_{x_0}, \omega_\nu, J_0)\). Since \( \overline{u}_\nu(\cdot - \frac{l_\nu}{2}) \) converges to \( \overline{u}_- \), there exists a constant \( r_0 \) such that for any \( l \geq 0 \) there exists \( \nu_0(l) \) such that

\[
\overline{u}_\nu(\cdot - \frac{l_\nu}{2} + r_0, \frac{l_\nu}{2} + r_0 + l) \times S^1) \subset U_{x_0} \quad \text{for} \quad \nu \geq \nu_0(l).
\]

Let

\[
e_0 := \omega_\nu(\overline{u}_-, [r_0, \infty) \times S^1) + \frac{\hbar}{2}.
\]

Define

\[
r_\nu := \sup \{ r : \overline{u}_\nu(\cdot - \frac{l_\nu}{2} + r_0, r) \times S^1) \subset U_{x_0}, \quad \omega_\nu(u_\nu, [\frac{l_\nu}{2} + r_0, r] \times S^1) \leq e_0 \}.
\]

The definition implies \( \frac{l_\nu}{2} + r_\nu \rightarrow \infty \). Consider the rescaled maps

\[
\overline{u}_\nu^+(s, t) := \overline{u}_\nu(\cdot + r_\nu) : \left[ -\frac{l_\nu}{2} - r_\nu, \frac{l_\nu}{2} - r_\nu \right] \times S^1 \rightarrow X_\nu^0.
\]
Since $|d\pi^+_{\nu}|$ is uniformly bounded and $(\pi^+_\nu(\{0\} \times S^1)$ lies in a compact set of $X^+_\nu$, we conclude that a subsequence of $(\pi^+_\nu)_\nu$ converges in $C^\infty_{\text{loc}}$ to a limit $\bar{\pi}^+ : (-\infty, L_1) \times S^1 \to X^+_\nu$. Here $L_1 := \lim\nu(\nu \to \infty)$. By removal of singularities on compact symplectic manifolds, the map $\bar{\pi}^+$ extends over $-\infty$. The images of the components $\bar{\pi}^-$ and $\bar{\pi}^+$ connect at the nodal point: By applying Proposition 7.18 to the sequence of maps $\bar{\pi}^+ : -\frac{(r_0+r_\nu)}{2}$ on the cylinders $A(r_\nu - r_0)$ we conclude $\bar{\pi}^+(-\infty) = \bar{\pi}^+(\infty)$, and

$$\omega_\nu(\bar{\pi}^+) = \frac{\hbar}{2}. \leqno{(7.13)}$$

We claim that the component $\bar{\pi}^+$ is a punctured disk and not a punctured sphere. For the sake of contradiction let us assume that the domain of $\bar{\pi}^+$ is the cylinder $\mathbb{R} \times S^1$ and $L_1$ is infinite. The $\pi_\nu$-projections of $u_\nu$ and $\bar{\nu}_\nu$ are the same, so

$$\omega_\nu^{\text{bas}}(\pi_\nu \circ \bar{\pi}^+) \leq \limsup\omega_\nu^{\text{bas}}(u_\nu) \leq E_{\text{Hof}}(u_\nu) < \infty.$$ 

By $(7.12)$ the image of $\pi_\nu \circ \bar{\pi}^+$ lies in a neighborhood of $X^+_\nu$ where $\omega_\nu^{\text{bas}}$ is a symplectic form. Therefore $\pi_\nu \circ \bar{\pi}^+$ extends over $+\infty$ and is a holomorphic sphere. The projected map $\pi_\nu(\bar{\pi}^+)$ is constant because by $(7.12)$ $\pi_\nu(\bar{\pi}^+) \subset B_1(x_0)$. The map $\bar{\pi}^+$ is also constant because by Step 2 and the definition of $\bar{\pi}_\nu$, $(\bar{\pi}^+)_{\nu}[\mathbb{R} \times S^1] \cap \pi^{-1}(T_{\nu}x_0)$ is trivial. This conclusion contradicts the positive lower bound on the $\omega_\nu$-area of $\bar{\pi}^+$ in $(7.13)$. Therefore we conclude that $L_1$ is finite.

We have shown that the limit of the twisted maps is a pair of disks $(\bar{\pi}^- \cup \bar{\pi}^+)$. Since $L_1 = \lim\nu(\nu \to \infty)$ is finite, we may replace $r_\nu$ by $\frac{L_1}{2}$. The limit $\bar{\pi}^+$ will be altered by a domain reparametrization and $\bar{\pi}^+(-\infty) = \bar{\pi}^+(\infty)$ continues to hold.

Part (a) of the Proposition is now proved as follows. We have shown that the maps

$$\bar{\pi}^+_{\nu}(s,t) := e^{-\mu(s+it)}(e^{-t^\nu_{\nu} - \mu_{\nu} t} u_\nu(s + \frac{L_1}{2}, t))$$

and $e^{-t^\nu_{\nu} u_\nu(s + \frac{L_1}{2})}$ converge on $\mathbb{R}_{\geq 0} \times S^1$. At the point $(s,t) = (0,0)$ the sequences $\bar{\pi}^+_{\nu}(s,t)$ and $e^{-t^\nu_{\nu} u_\nu(s + \frac{L_1}{2})}$ differ by a translation by $e^{t^\nu_{\nu} - \mu_{\nu} t}$. Since both sequences of points converge we conclude that the limit

$$\delta := \lim\nu(-\mu_{\nu} t - t^\nu_{\nu} + t^\nu_{\nu})$$

exists (which proves (a)) and that

$$\bar{\pi}^+_{\nu}(s,t) = e^\delta e^{-\mu(s+it)} u_\nu(s,t).$$

**Step 4 : Proof of the decay estimate.**

Since the sequence of the twisted maps $\bar{\pi}_\nu$ converge to a pair of disks as in Step 3, the images of the maps lie in a compact set $\overline{U}_{\nu}$ with a taming symplectic form $\omega_\nu$, and the $\omega_\nu$-area is $< h$. We apply the annulus lemma for compact manifolds (Proposition 7.17) on the maps $\bar{\pi}_\nu$. The decay estimate for the twisted maps $\bar{\pi}_\nu$ implies the asymptotic decay estimate $(7.11)$ for $u_\nu$ required by the Proposition.

**Step 5 :** $P = P_+ \cap P_-$. The polytope $P \in \mathcal{P}$ was chosen so that the nodal point on $u_-$ is on $X^+_P : (\pi P_+ u_-)(\infty) \in \mathcal{P}$. 

Therefore at the nodal point \( \infty \), \( u_- \) has positive intersection multiplicity with all divisors \( X_Q \) of \( X_P \) which contain \( X_P \), that is, \( P \subset Q \subset P_- \). Consequently the intersection multiplicity \( \mu \) lies in \( t^\nu_P \setminus \cup_{Q \supset P} t^\nu_Q \). On the other hand since \( t^\nu_P - t^\nu_P - \nu \) is uniformly bounded, and \( t^\nu_P \in \nu P^\nu_P \subset t^\nu_P \), we conclude \( \mu \in t^\nu_P \), where \( P_\cap := P_+ \cap P_- \). Therefore \( P_\cap = P \). □

The proof of the breaking annulus lemma was based on the following result on compact symplectic manifolds.

**Proposition 7.17.** (Annulus lemma on compact manifolds) Suppose \((X, \omega)\) is a compact symplectic manifold with a tamed almost complex structure \( J \). There exist constants \( 0 < \rho < 1, \ h > 0, \ c > 0 \) such that the following holds for any \( J \)-holomorphic map \( u : A(L) \to X \) with \( E(u) \leq h \). For \( x = u(0,0) \), there is a map

\[ \xi : A(L - 1) \to T_xX \] such that \( u = \exp_x \xi \)

on \( A(L - 1) \) and for \( k = 0,1 \),

\[
|D^k \xi(s,t)| \leq c(e^{\rho(s-L)} + e^{\rho(-s-L)}), \quad \forall s \in [-L + 1, L - 1].
\]

The constants \( \rho, \ h, \ c \) depend continuously on \( J \) with respect to the \( C^2 \)-topology.

*Proof.* The proposition is a consequence of the annulus lemma, see [46, Lemma 4.7.3] and elliptic regularity for holomorphic maps. □

**Proposition 7.18.** (Convergence of long cylinders) Suppose \((X, \omega)\) is a compact symplectic manifold with a tamed almost complex structure \( J \). Let \( u_\nu : A(l_\nu) \to X \) be a sequence of holomorphic cylinders satisfying \( \omega(u_\nu) \leq h \), where \( h \) is as in Proposition 7.17. After passing to a subsequence, \( u_\nu(\cdot \pm \frac{l_\nu}{2}) \) converges in \( C^\infty_{\text{loc}} \)

\[
u_- \] converges to \( X \),

the map \( u_\pm \) extends holomorphically over \( \mp \infty \), and \( u_-(\infty) = u_+(\mp \infty) \). Further,

\[
\lim_{\nu} \text{Area}_\omega(u_\nu) = \text{Area}_\omega(u_+) + \text{Area}_\omega(u_-).
\]

*Proof.* This Proposition is proved as part of the ‘bubbles connect’ result in [46, Proposition 4.7.1]. □

7.4. **Proof of convergence for breaking maps.**

*Proof of Theorem 7.1. Step 1 : Domain components for the limit map.*

The sequence of domains \((C_\nu, z_\nu)\) converges to a stable treed nodal curve \((C, z)\) modelled on a tree \( \Gamma \). The perturbation maps \( p_\nu \) converge to a perturbation datum \( p_\infty = (J_\infty, F_\infty) \) defined on \( C \). In the next few steps we will show that there are no additional domain components in the limit map.

*Step 2 : Boundedness of derivatives.*

In this step we show a bound on the derivatives of the sequence of maps by ruling out bubble trees with unstable domains in the limit. In particular, we will show that for any \( \nu \in \text{Vert}(\Gamma) \), the derivative of

\[
u_v := u_\nu \circ i_v : C_\nu(v) \to X_\nu
\]
is uniformly bounded for all $\nu$. The norm on the derivative is with respect to the cylindrical metric on $X^\nu$. On the domain we use a metric on $C^\circ$ that is cylindrical (or strip-like) in the punctured neighborhood of nodal points and on the neck regions in $C^\nu$.

Assuming, for the sake of contradiction, that the derivatives are not uniformly bounded, we will construct a sequence of rescaled maps. After passing to a subsequence, there exists a sequence of points $z_\nu \in C^\nu(\nu) \subseteq C^\nu_0$ and a point $z_\infty \in C^\nu$ for some $\nu \in \text{Vert}(\Gamma)$ such that

$$z_\nu \to z_\infty, \quad |d u_{\nu,\nu}(z_\nu)| \to \infty.$$  

We first carry out the proof assuming that $z_\infty$ is not a nodal point, and thus $z_\infty \in C^\nu_0$. We apply Hofer’s lemma 7.19 to the function $|d u_{\nu,\nu}|$, $x = z_\nu$ and the constant $\delta = |d u_{\nu}(z_\nu)|^{-1/2}$.

**Lemma 7.19.** (Hofer’s Lemma, [46, Lemma 4.6.4]) Suppose $(X, d)$ is a metric space, $f : X \to \mathbb{R}_{\geq 0}$ is a continuous function, and $x \in X$, $\delta > 0$ are such that the ball $B_{2\delta}(x)$ is complete. Then there exists a positive constant $\epsilon \leq \delta$ and a point $\zeta \in B_{2\delta}(x)$ such that

$$\sup_{z \in B_{\epsilon}} f(z) \leq 2f(\zeta), \quad \epsilon f(\zeta) \leq \delta f(x).$$

We obtain another sequence $\zeta_\nu \in C^\nu_0$ converging to $z_\infty$, and a sequence of constants $\epsilon_\nu \to 0$ such that

$$c_\nu := |d u_{\nu,\nu}(\zeta_\nu)| \to \infty, \quad \sup_{z \in B_{c_\nu}} |d u_{\nu}(z)| \leq 2c_\nu, \quad c_\nu \epsilon_\nu \to \infty.$$  

The rescaled maps are

$$\tilde{u}_\nu := u_{\nu,\nu}(\cdot - \zeta_\nu)/c_\nu : B_{c_\nu c_\nu} \to X^\nu,$$

and $|d \tilde{u}_\nu| \leq 2$, $|d \tilde{u}_\nu(0)| = 1$.

Each of the rescaled maps converges to a limiting map with domain the affine line. By Lemma 7.15 there is a polytope $P \in \mathcal{P}$ such that, a subsequence of $\tilde{u}_\nu$ horizontally converges in $X_P$. We may view $\tilde{u}_\nu$ as mapping to the $P$-cylinder, since for any translation $t_\nu \in \nu P^\nu \subset t_P^\nu$ there is an embedding

$$\varepsilon^{-t_\nu} : X^\nu_P \to Z_{P, \mathbb{C}},$$

see (2.33). We choose the translation $t_\nu \in t_P^\nu$ so that $\pi_{\nu} \circ (\varepsilon^{t_\nu} \tilde{u}_\nu)(0) = 0$. The uniform bound on the derivatives of $\tilde{u}_\nu$ implies that for any compact set $K \subset \mathbb{C}$, the images $\varepsilon^{t_\nu} \tilde{u}_\nu(K)$ are contained in a compact set of $Z_{P, \mathbb{C}}$. By the Arzela-Ascoli theorem, the sequence $\varepsilon^{t_\nu} \tilde{u}_\nu$ converges in $C^\infty_{\text{loc}}$ to a non-constant $J_{z_\infty}$-holomorphic limit $\tilde{u} : \mathbb{C} \to Z_{P, \mathbb{C}}$.

The limit map in the previous paragraph extends to a holomorphic map from the projective line by removal of singularities. Hofer energy $E_{\text{Hof}}(u_{\nu}, C^\nu)$ is equal to area and therefore is uniformly bounded for all $\nu$. By monotonicity of Hofer energy (Lemma 6.6) $E_{\text{Hof}}(\tilde{u}_\nu, B_{\epsilon_\nu c_\nu})$ is uniformly bounded. The quantity $E_{\text{Hof}}(\tilde{u}, \mathbb{C})$ is finite because $E_{\text{Hof}}$ is preserved in the neck-stretching limit. Therefore by Proposition 6.30 the singularity at $\infty$ can be removed to produce an extension $\tilde{u} : \mathbb{P}^1 \to X_P$.  

Finally, we arrive at a contradiction by showing that the limit map is constant. The domain reparametrizations $\phi_{v,\nu}$ were derived from the stable map compactification, and so, there is at most a single marked point $z_{i,\nu}$ that is contained in each of the regions $B_{i,\nu}(\zeta_{\nu})$. Therefore, the projection $\pi_P \circ \tilde{u} : \mathbb{C} \to Z P_{v,\nu}$ either lies in the divisor $\mathcal{D}$, or it has at most one intersection with the divisor $\mathcal{D}$. Since $(\mathcal{J}_0, \mathcal{D})$ is a stabilizing pair for $\mathcal{X}$, and the perturbation $p$ is adapted to $(\mathcal{J}_0, \mathcal{D})$ neither possibilities can happen if $\pi_P \circ \tilde{u}$ is non-constant. Therefore, $\pi_P \circ \tilde{u}$ is constant, and so the image of $\tilde{u}$ lies in a single toric fiber $V_{P,\nu}$. The image $\tilde{u}(\mathbb{C})$ does not intersect boundary divisors of $V_{P,\nu}$, and therefore $\tilde{u}$ is a constant map.

We now consider the case that the sequence of points with increasing derivatives converges to a nodal point $w_v$. The sequence $z_v$ lies on the neck region

$$A_v := \left[ \frac{-l_v}{2}, \frac{l_v}{2} \right] \times S^1 \subset C_v$$

obtained by gluing the node $w_v$, and $l_v \to \infty$ as $\nu \to \infty$. Since $z_v$ converges to the node $w_v$, there is a constant $c > 0$ such that the ball $B_c(z_v)$ is contained in $A_v$ for all $\nu$. The earlier proof using Hofer’s Lemma and a rescaling can now be applied to the maps $u_{v,\nu}|B_c(z_v)$, because there are no marked points in $A_v$. We conclude that $|du_{v,\nu}|$ is uniformly bounded for all $\nu$.

**STEP 3: Determining stable components of the limit map.**

In Step 2 we showed that the derivatives of the maps $u_{v,\nu}$ are uniformly bounded, where $v$ is any vertex of $\Gamma$. Therefore, by Lemma 7.15 the maps $u_{v,\nu}$ converge horizontally in some polytope $P(v) \in \mathcal{P}$, and the image of $u_{v,\nu}$ is contained in $X_{P(v)}^{\nu}$. For any choice of translation $t_{v,\nu}(v) \in \nu P_{v} \subset t_P^\nu$, there is an embedding $e^{-t_{v,\nu}(v)} : X_{P(v)}^{v} \to Z_{P(v),\nu}$ into the $P(v)$-cylinder, see (2.33). Choose a smooth point $z_v \in C_v$ in each curve component. Fix the translation $t_{v,\nu}(v) \in \nu P(v)^{v}$ so that $\pi_{t_{P(v)}}(e^{-t_{v,\nu}(u_{v,\nu})(z_v)}) = 0$. Then, for any compact set in the complement of nodal points $K \subset C_v^0$, the images $(e^{-t_{v,\nu}(u_{v,\nu})}(K))$ are contained in a uniformly bounded region of $Z_{P(v),\nu}$. Since the derivatives of $e^{-t_{v,\nu}(u_{v,\nu})}$ are locally uniformly bounded, the sequence $e^{-t_{v,\nu}(u_{v,\nu})}$ converges in $C_v^\infty$ to a limit $u_v : C_v^\infty \to Z_{P(v),\nu}$. For any node $w$ in $C_v$ there is a neighbourhood $\Omega_{v,w} \subset C_v$ of $w$ where the almost complex structure is $J_0$:

$$J_\infty|_{\Omega_{v,w}} \equiv J_0.$$  

By arguments as in Step 2, $E_{\text{Hof}, P(v)}(u_v, \Omega_{v,w})$ is bounded. By (2.25), there is an embedding of $Z_{P(v),\nu}$ into a compact symplectic manifold $(P_{P(v)}, \omega_{P(v)})$ so that

$$\int_{\Omega_{v,w}} u_v^* \omega_{P(v)} \leq cE_{\text{Hof}, P(v)}(v, \Omega_{v,w}) < \infty.$$  

Therefore, by removal of singularity, $u_v$ extends to a holomorphic map over the node $w$. We obtain a $J_\infty$-holomorphic map $u_v : C_v \to X_{P(v)}$ that is adapted to $\mathcal{D}$.

For later use, we remark that the convergence continues to hold if the sequence $t_{v,\nu}(v)$ is replaced by a sequence $t_{v,\nu}(v)'$ for which $\sup_v |t_{v,\nu}(v)' - t_{v,\nu}(v)| < \infty$. In that case, after passing to a subsequence, the limit $u_v$ would be replaced by $e^{-t}u_v$, where $t := \lim_{\nu} (t_{v,\nu}(v)' - t_{v,\nu}(v))$. 

Step 4: There are no unstable limit components.

We have already ruled out unstable components with a single special point (rational tails) in the limit map in Step 2. In this step, we rule out limit components with two nodal points and no marked points. To do this, we will show that for any node \( w \) in \( C \), the sequence of annuli in \( C_\nu \) does not have sufficient basic area to form a bubble that is horizontally non-constant. Suppose the annuli

\[
A(l_\nu) = \left[ -\frac{l_\nu}{2}, \frac{l_\nu}{2} \right] \times S^1 \subset C_\nu
\]

are obtained by gluing the node \( w \) in \( C \). First, we pass to a subsequence so that the limit

\[
(7.15) \quad \text{Area}^{\text{bas}}(w) := \lim_{l \to \infty} \lim_{\nu \to \infty} \text{Area}^{\text{bas}}(u_\nu, \left[ -\frac{l_\nu}{2} + l, \frac{l_\nu}{2} - l \right] \times S^1)
\]

exists. We need to prove that \( \text{Area}^{\text{bas}}(w) \) vanishes. Suppose for the sake of contradiction that \( \text{Area}^{\text{bas}}(w) > 0 \). We recall that by Step 3, the sequence of maps \( e^{\nu(v_\nu)} u_\nu \cdot (\cdot + l_\nu/2) \) converges in \( C^\infty_{\text{loc}} \) to \( u_{\nu,-} : \mathbb{R}_+ \times S^1 \to X_{P_{\nu}} \). We now find a limit unstable bubble attached to \( C_{\nu,-} \) at \( \infty \). The reparametrization sequence \( r_\nu \in [0, l_\nu] \) and a subsequence of \( \{u_\nu\}_\nu \) are chosen so that

\[
\text{Area}^{\text{bas}}(u_\nu, \left[ -\frac{l_\nu}{2}, -\frac{l_\nu}{2} + r_\nu \right] \times S^1) = \text{Area}^{\text{bas}}(u_{\nu,-}, \mathbb{R}_+ \times S^1) + m_w,
\]

where

\[
m_w := \frac{1}{2} \min\{\text{Area}^{\text{bas}}(w), h\}.
\]

If \( \text{Area}^{\text{bas}}(w) > 0 \), then \( l_\nu - r_\nu \to \infty \). A sequence of rescaled maps is defined as

\[
\tilde{u}_\nu := u_\nu \cdot (-\frac{l_\nu}{2} + r_\nu).
\]

We will arrive at a contradiction by showing that the rescaled sequence of maps has a limit that is horizontally non-constant. Since the derivatives of the maps \( u_\nu \) are uniformly bounded, by Lemma 7.15 a subsequence of \( \tilde{u}_\nu \) horizontally converges in \( X_{P_{w}} \), for some polytope \( P_w \in \mathcal{P} \). The horizontal convergence in \( P_w \) implies that for any compact set \( K \subset \mathbb{R} \times S^1 \), the images \( \tilde{u}_\nu(K) \) lie in \( X_{P_{w}} \) for large enough \( \nu \). So, we may view \( \tilde{u}_\nu(K) \) as lying in \( Z_{P_{w,c}} \). Choose a sequence of translations \( t_{w,\nu} \in \nu P_{w} \) such that \( \pi_{1_{p_{w,c}}} (e^{\nu(v_\nu)} \tilde{u}_\nu(0)) = 0 \). Again using the boundedness of derivatives of \( \tilde{u}_\nu \), we conclude that the sequence \( \tilde{u}_\nu \) converges in \( C^\infty_{\text{loc}} \) to a limit \( u_w : \mathbb{R} \times S^1 \to X_{P_w} \).

By removal of singularities Proposition 6.30 the limit extends holomorphically to \( u_w : \mathbb{P}^1 \to X_{P_w} \), because of the finiteness of Hofer energy as in Step 2. By Lemma 7.13, the translations \( t_{w,\nu} \) diverge from faces of \( \nu P_{w} \), that is,

\[
d(\nu P_{0}, t_{w,\nu}^\nu) \to \infty \quad \forall P_0 \supset P_w.
\]

Therefore, by Proposition 6.27 (b), the maps \( \tilde{u}_\nu \) map to a region where the basic area form \( \omega_{\nu}^{\text{bas}} \) is \( t_{w,\nu}^\nu \)-translation invariant. Therefore, for any compact set \( K \),

\[
\lim_{\nu} \text{Area}^{\text{bas}}(\tilde{u}_\nu, K) = \text{Area}^{\text{bas}}(u_w, K).
\]

We conclude that

\[
\text{Area}^{\text{bas}}(u_w, \mathbb{R}_- \times S^1) = m_w > 0.
\]
Since there are no marked points in the neck region $A(l_{\nu})$, the sphere $\pi_{P_w} \circ u_w$ either maps to the stabilizing divisor $D_P$ or intersects the divisor at a maximum of two points denoted $+\infty, -\infty \in C_w$. Since $(J_0, D_{P_w})$ is a stabilizing pair, both possibilities are ruled out. We conclude that $\text{Area}^{\text{bas}}(w) = 0$ for all nodes.

The same arguments apply for any dilated area form. Thus we have proved for any $t \geq 1$,

$$\text{Area}^{\text{bas}}_t(w) := \lim_{l \to \infty} \lim_{\nu \to \infty} \text{Area}^{\text{bas}}_t(u_{\nu}, [-\frac{l}{2} + t, \frac{l}{2} - t] \times S^1) = 0.$$  

**Remark 7.20.** (Unstable domain components in the compactness result) In the compactness theorem, unstable domain components are ruled out because the limit is adapted with respect to a stabilizing divisor. In contexts where there is no stabilizing divisor, the proof can be modified using the technique of [9], wherein additional sequences of marked points are added to the domain curves to ensure that the limit domain curve is stable. The additional marked points are added at sequences of points where the derivative of the map blows up with respect to a hyperbolic metric on the domain. Our proof then entirely carries over.

**Step 5: Constructing translation sequences.**

We adjust the component-wise translation sequences $t_{\nu}$ in Step 2 by a uniformly bounded amount so that they satisfy the (Slope) condition across all edges. Let $(t_{\nu}(v))_{v,\nu}$ be component-wise translation sequences from Step 2. In this step we prove the following claim.

**Claim 7.21.** (a) There is a $\Gamma$-translation sequence $(t'_{\nu}(v))_{v,\nu}$ such that

$$\sup_{\nu,v} |t'_{\nu}(v) - t_{\nu}(v)| < \infty.$$  

Consequently there exists a sequence $l_{\nu} \to \infty$ such that

$$t'_{\nu}(v_+) - t'_{\nu}(v_-) = T(e)l_{\nu}.$$  

(b) For any interior node $w_e$ corresponding to an edge $e = (v_+, v_-)$, and a sequence of annuli $A(l'_{\nu})$ obtained by gluing at the node $w_e$,

$$|l_{\nu} - l'_{\nu}|$$  

is uniformly bounded.

The set of component-wise translation sequences $(t_{\nu}(v))_{v,\nu}$ is an approximate translation sequence in the sense of Definition 7.22 as we now explain. Let $A(l_{\nu}) \subset C_v$ be a sequence of annuli that are obtained by gluing a node $w_e$ corresponding to an edge $e = (v_+, v_-)$. Since the dilated basic area decays to zero on this sequence of annuli (see (7.16)), the breaking annulus lemma is applicable on the maps $u_{\nu} \mid A(l_{\nu})$. We conclude that

$$\sup_{\nu} |t_{\nu}(v_+) - t_{\nu}(v_-) - T(e)l_{\nu}| < \infty.$$  

Consequently $(t_{\nu}(v))_{v,\nu}$ satisfies the (Approximate slope) condition, and is an approximate translation sequence. Finally by Lemma 7.24, there is a $\Gamma$-translation sequence $(t'_{\nu}(v))_{v,\nu}$ such that $\sup_{v,\nu} |t'_{\nu}(v) - t_{\nu}(v)|$ is uniformly bounded. The uniform bound (7.18) is a consequence of (7.17) and (7.19).
**Step 6: Finishing the proof of convergence.**

We determine matching coordinates for nodes of the broken map and finish the proof using the breaking annulus Lemma. Consider an edge $e \in \text{Edge}(\Gamma)$, and choose a complex coordinate $z_\pm : (U_\pm, w_e) \to (\mathbb{C}, 0)$ in a neighborhood $U_\pm \subset C_{v_\pm}$ of the lift $w_\pm(e)$ of the node $w_e$. Let $A(l_\nu) \subset C_{v_\nu}$ be a sequence of centered annuli that converges to the pair of disks $\{ |z_\pm| = 1 \}$ as in Definition 7.8. By (7.18), the sequence of differences

$$t_\nu'(v_\pm) - t_\nu'(v_+) - T(e)l_\nu$$

is uniformly bounded. Multiplying a constant to the sequence of neck length parameters $l_\nu + i\theta_\nu$. Therefore, we can adjust the coordinates $z_\pm$ by scalar multiplication so that the sequence of neck length parameters $e^{-l_\nu + i\theta_\nu}$ satisfies

$$(7.20) \quad \lim_{\nu}(t_\nu'(v_\pm) - t_\nu'(v_+) - T(e)l_\nu) = 0, \quad \lim_{\nu} \theta_\nu = 0.$$ 

We will show that $(z_+, z_-)$ are matching coordinates for the broken map $u$. To simplify calculations, we use logarithmic coordinates in the neighborhood of the node

$$C_\pm \supset U_\pm \setminus \{ w_\pm(e) \} \xrightarrow{\pm \ln z_\pm} \mathbb{R}_+ \times S^1.$$ 

The annulus $A(l_\nu)$ is then identified to the limit curve by translations

$$A(l_\nu) \to U_\pm, \quad (s, t) \mapsto s + it \mp \frac{1}{2}(l_\nu + i\theta_\nu).$$

By Step 2,

$$(7.21) \quad e^{-t_\nu(v_\pm)}u_\nu(s + it \mp \frac{1}{2}(l_\nu + i\theta_\nu)) \to u_{v_\pm} \quad \text{in} \quad C_{v_\nu}^\infty(U_\pm \setminus \{ w_\pm(e) \}).$$

We apply the breaking annulus lemma on the maps $u_\nu$ on the annuli $A(l_\nu)$. The resulting decay estimate together with (7.21), (7.20) implies the convergence of the sequence

$$e^{-\frac{1}{2}(l_\nu(v_+) + l_\nu(v_-))}u_\nu(0, 0) \to x_0 \quad \text{in} \quad X_{\nu(e)}^\infty.$$ 

It follows that $u_\pm$ are asymptotically close to the cylinder $(s, t) \mapsto e^{T(e)(s+it)}x_0$. We conclude that $z_+, z_-$ are matching coordinates at $w_e$ and (Thin cylinder convergence) is satisfied. Convergence of area is a consequence of Step 4, where we prove that no basic area is lost at nodes.

**Step 7: Uniqueness of the limit.**

The limit of the domain curves is unique up to reparametrization, because the limit is a stable curve. The identifications between subsets of $C^\nu$ to the limit curve are unique in the following sense (see Remark 7.7) : The neck regions in $C^\nu$ are parametrized in a unique way, and the difference between any two choices of identifications of the complement of the neck in $C^\nu$ to $C$ converge uniformly to identity as $\nu \to \infty$. For every vertex $v \in \text{Vert}(\Gamma)$, the polytope $P(v)$ in the limit map is uniquely determined by Lemma 7.13.

Translation sequences are well-determined up to uniformly bounded perturbations as follows : Suppose $t_\nu$, $t'_\nu$ are two distinct translation sequences, such that the sequence $e^{t_\nu}u_\nu$ resp. $e^{t'_\nu}u_\nu$ converges to a broken map $u$ resp. $u'$. Then for all...
vertices $v \in \text{Vert}(\Gamma)$, the sequence $|t_\nu(v) - t_\nu'(v)|$ is uniformly bounded, because both the sequences $e^t_{\nu}(v) u_\nu$, $e^{t_\nu'}(v) u_\nu$ converge pointwise in $C_\nu^o$. After passing to a subsequence, we may assume that there exists a limit

$$t(v) := \lim_\nu t_\nu(v) - t_\nu'(v).$$

Then, for each vertex $v$, $u_\nu = e^{t(v)} u_\nu'$. Since $u_\nu$ and $u_\nu'$ satisfy matching conditions at nodes we conclude that $t$ is an element of $T_{\text{trop}, W}(\Gamma)$. \hfill \square

**Definition 7.22.** (Approximate translation sequence) Suppose $\Gamma$ is a tropical graph. An approximate $\Gamma$-translation sequence consists of sequences $\{t_\nu(v) \in \nu P(v)^\nu\}$ for each $v \in \text{Vert}(\Gamma)$ such that

- (Approximate Slope) For any edge $e = (v_+, v_-)$ with non-zero slope $\mathcal{T}(e) \neq 0$, there exists a sequence $l_\nu(e) \to 0$ such that

$$\sup_\nu (t_\nu(v_+) - t_\nu(v_-) - \mathcal{T}(e) l_\nu) < \infty.$$ 

The differences appearing in the (Approximate Slope) condition will be referred to later using the following notation:

**Definition 7.23.** On a tropical graph $\Gamma$ define the discrepancy function on any edge $e = (v_+, v_-) \in \text{Edge}_{\nu, -}(\Gamma)$ as

$$\text{Diff}_e : \oplus_{\text{Vert}(\Gamma)} t^E_\nu(v) \to t^V/\langle \mathcal{T}(e) \rangle, \quad (t_\nu)_{v \in \text{Vert}(\Gamma)} \mapsto (t_{v_+} - t_{v_-}) \mod \mathcal{T}(e).$$

**Lemma 7.24.** (From approximate to exact translation sequences) Suppose $\Gamma$ is a graph with tropical structure $\mathcal{T}$ and $t_\nu$ is an approximate $\Gamma$-translation sequence. Then, after passing to a subsequence, there is a $\Gamma$-translation sequence $\tilde{t}_\nu$ such that $|\tilde{t}_\nu(v) - t_\nu(v)|$ is uniformly bounded for all $v, v \in \text{Vert}(\Gamma)$.

**Proof.** The (Approximate slope) condition says that the sequences of discrepancies $(\text{Diff}_e(t_\nu))_\nu$ are uniformly bounded. Via uniformly bounded adjustments to $t_\nu$, we aim to make this quantity vanish for all edges.

We give an algorithm that transforms $t_\nu$ into a bounded sequence $t^k_\nu \in \oplus_{\text{Vert}(\Gamma)} t^V_\nu$, and will prove later that $t_\nu - t^k_\nu$ is an exact translation sequence. The algorithm is as follows:

**Step 1:** Relativisation.

In this step, we replace $t_\nu$ by

$$t^0_\nu(v) := t_\nu(v) - \nu \lim_\nu (t_\nu(v)/\nu) \in t^V_\nu.$$ 

The limit in the right-hand side exists after passing to a subsequence because the original translation sequences $t_\nu$ lie in $\nu B$ and $B$ is compact. For any $v \in \text{Vert}(\Gamma)$, the discrepancies across edges are preserved:

$$(7.22) \quad \text{Diff}_e(t_\nu) = \text{Diff}_e(t^0_\nu).$$

**Step 2:** Subtracting fastest growing sequences.

By a sequence of further transformations, we will change $t^0_\nu$ to a bounded sequence $t^k_\nu \in t^V$. At each step, the sequence $t^i_\nu$ is replaced by $t^{i+1}_\nu$ defined as follows. Choose
a vertex \( v_0 \in \text{Vert}(\Gamma) \) for which the rate of increase of the sequence \( |t^i_\nu(v_0)| \) is the maximum. That is, for all \( v \in \text{Vert}(\Gamma) \), \( \lim_\nu |t^i_\nu(v)|/|t^i_\nu(v_0)| \) is finite. Such a vertex can indeed be chosen, because after passing to a subsequence, the limit \( \lim_\nu |t^i_\nu(v_i)|/|t^i_\nu(v_0)| \) exists in \([0, \infty)\) for any pair of vertices. Now, define

\[
(7.23) \quad t^{i+1}_\nu(v) := t^i_\nu(v) - |t^i_\nu(v_0)| \lim_\nu \frac{t^i_\nu(v)}{|t^i_\nu(v_0)|} \in t^T_P(v).
\]

We stop the iteration when the sequence \( t^i_\nu(v) \) corresponding to every vertex is bounded, and suppose the final sequence is \( t^i_\nu \).

The process terminates in a finite number of steps. Indeed, notice that \( t^{i+1}_\nu(v_0) = 0 \) for all \( \nu \). The number of vertices \( v \in \text{Vert}(\Gamma) \) for which \( t^{i+1}_\nu(v) \) vanishes is at least one more than the number of vertices \( v \) for which \( t^i_\nu(v) \) vanishes.

The iterations of the algorithm preserve the discrepancies across edges: For all \( e \in \text{Edge}^-_{\nu}(\Gamma) \) with \( T(e) \neq 0 \)

\[
(7.24) \quad \text{Diff}_e(t^{i+1}_\nu) = \text{Diff}_e(t^i_\nu).
\]

Indeed, (7.23) implies

\[ \text{Diff}_e(t^{i+1}_\nu) = \text{Diff}_e(t^i_\nu) - |t^i_\nu(v_0)| \lim_\nu \frac{\text{Diff}_e(t^i_\nu)}{|t^i_\nu(v_0)|}, \]

and the second term in the right-hand-side vanishes because \( \text{Diff}_e(t^i_\nu) \) is uniformly bounded and \( t^i_\nu(v_0) \to \infty \) as \( \nu \to \infty \).

We claim that \( t_\nu - t^i_\nu \) is an exact translation sequence. For all vertices \( v \) the (Polytope) condition (7.7) \( (t_\nu - t^i_\nu)(v) \in P(v)^\nu \) is satisfied because \( t_\nu(v) \in P(v)^\nu \) and \( t^i_\nu(v) \in t^T_P(v) \simeq TP(v)^\nu \). The (Slope) condition is satisfied because \( \text{Diff}_e(t_\nu) - \text{Diff}_e(t^i_\nu) = 0 \) by (7.22) and (7.24). \( \square \)

### 7.5. Convergence for broken maps.
In this section we prove a Gromov compactness result for broken maps. In the limit, one may have additional bubbling into the tropical divisors. We show that such bubbling happens only in families whose dimension is at least two, and so does not occur in the zero-dimensional moduli spaces we use to define the Fukaya algebra.

**Definition 7.25.** (Collapsing edges tropically) A tropical edge collapse is a morphism of tropical graphs \( \Gamma' \xrightarrow{\kappa} \Gamma \) that collapses a subset of edges \( \text{Edge}(\Gamma') \setminus \text{Edge}(\Gamma) \) in \( \Gamma' \) inducing a surjective map on the vertex sets

\[ \kappa : \text{Vert}(\Gamma') \to \text{Vert}(\Gamma), \]

and satisfies the following conditions:

(a) the edge slope is unchanged for uncollapsed edges, i.e. if \( T', T \) are the edge slope functions for \( \Gamma, \Gamma' \), then \( T(\kappa(e)) = T'(e) \) for any uncollapsed edge \( e \in \text{Edge}(\Gamma') \) and

(b) for any vertex \( v' \in \kappa^{-1}(v), P(v') \subseteq P(v) \).

Since the edge slope function \( T' \) extends \( T \), we often use the same notation for both. A tropical edge collapse \( \Gamma' \xrightarrow{\kappa} \Gamma \) is trivial if all the collapsed edges \( e \in \)
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\[ \text{Edge}(\Gamma') \setminus \text{Edge}(\Gamma) \text{ have zero slope } T(e) = 0, \text{ and } P_{T'}(v) = P_{T}(\kappa(v)) \text{ for all } v. \] This finishes the definition.

**Example 7.26.** In Figure 9, collapsing the middle edge in \( \Gamma_2 \) gives a tropical edge collapse morphism \( \kappa : \Gamma_2 \to \Gamma_1 \).

We next define relative weights for a tropical edge collapse morphism, and show that a relative weight can be viewed as a difference between absolute weights.

**Definition 7.27.** (Relative weight) Suppose \( \kappa : \Gamma' \to \Gamma \) is a tropical edge-collapse morphism. A relative weight \( T_{\Gamma'} \) for the pair \( (\Gamma', \Gamma) \) is a set of tropical weights \( T_{\Gamma'}(v) \in \text{Cone}(\kappa, v) \subset t^\vee \), \( v \in \text{Vert}(\Gamma') \), where

\[
(7.25) \quad \text{Cone}(\kappa, v) := \text{Cone}_{P(\kappa(v))}(P(v)^\vee) := \{ t - t_0 \in t^\vee : t \in P(v)^\vee, t_0 \in P(\kappa(v))^\vee \}
\]

is the cone in the polytope \( P(v)^\vee \) based at points in \( P(\kappa(v))^\vee \); and the weights \( T_{\Gamma'}(v) \) satisfy

\[
(7.26) \quad T_{\Gamma'}(v^+) - T_{\Gamma'}(v^-) \in \begin{cases} \mathbb{R}_{\geq 0} T(e), & e \notin \text{Edge}(\Gamma), \\ \mathbb{R}T(e), & e \in \text{Edge}(\Gamma). \end{cases}
\]

This ends the Definition.

**Remark 7.28.** (Relative weight is a difference between absolute weights) Suppose \( \kappa : \Gamma' \to \Gamma \) is a tropical edge collapse morphism, and \( T, T' \) are weights on \( \Gamma, \Gamma' \). Then the difference

\[
\text{Vert}(\Gamma') \ni v \mapsto T'(v) - T(\kappa(v))
\]

is a relative weight. Further, the relative weight is non-trivial if and only if the tropical edge collapse morphism \( \kappa \) is non-trivial.

Conversely, given a relative weight \( T_{(\Gamma', \Gamma)} \) and a weight \( T \) on \( \Gamma \), a weight on \( \Gamma' \) is given by

\[
\text{Vert}(\Gamma') \ni v \mapsto T(\kappa(v)) + \alpha T_{(\Gamma, \Gamma)}(v).
\]

Here \( \alpha > 0 \) is a constant that is small enough that for any uncollapsed edge \( e = (v^+, v^-) \) in \( \Gamma' \),

\[
T'(v^+) - T'(v^-) \in \mathbb{R}_{\geq 0} T(e).
\]

This ends the remark.

**Remark 7.29.** The set of relative weights for a pair \( \Gamma' \to \Gamma \) is a cone: If \( T_{(\Gamma', \Gamma)} \) and \( T'_{(\Gamma', \Gamma)} \) are relative weights, then the sum is a relative weight, and for any positive scalar \( \alpha > 0 \), \( \alpha T_{(\Gamma', \Gamma)} \) is a relative weight.

**Lemma 7.30.** Suppose \( \Gamma' \to \Gamma \) is a non-trivial edge-collapse map between tropical graphs \( \Gamma', \Gamma \). Then, a broken map modelled on \( \Gamma' \) has non-trivial tropical symmetry group \( T_{\text{trop}}(\Gamma') \) (Definition 3.23).
Proof. We recall that the difference between two distinct weights gives a two-dimensional family of tropical symmetries. In case of a non-trivial edge collapse, there is a non-trivial relative weight $\mathcal{T}$, see Remark 7.28. It generates a one-parameter subgroup in the tropical symmetry group:

$$(7.27) \quad \mathbb{C} \mapsto T_{\text{trop}}(\Gamma'), \quad z \mapsto e^{z\mathcal{T}(v)},$$

where we view $\mathcal{T}(v)$ as lying in $\mathfrak{t}$. Since the set of relative weights is at least one-dimensional, the group of tropical symmetries in $\Gamma'$ is at least two-dimensional. $\square$

Relative translation sequences can be defined analogous to relative weights.

Definition 7.31. (Relative translation sequence) Suppose $\kappa : \Gamma' \to \Gamma$ is a tropical edge collapse. Then a relative translation sequence, also called a $(\Gamma', \Gamma)$-translation sequence, consists of a sequence $t_\nu(v) \in \text{Cone}(\kappa, v) = \text{Cone}_{P(\kappa v)} \lor (P(v) \lor)$ for every $v \in \text{Vert}(\Gamma')$, where Cone$(\kappa, v)$ is as defined in (7.25) and the sequences $t_\nu(v)$ satisfy the following conditions.

(a) (Polytope) For any vertex $v \in \text{Vert}(\Gamma')$ and a polytope $Q \in \mathcal{P}$ such that $P(v) \subset Q \subset P(\kappa(v))$,

$$d(t_\nu(v), \text{Cone}_{P(\kappa v)} \lor (Q \lor)) \to \infty.$$

(b) (Slope for collapsed edge) For any edge $e \in \text{Edge}(\Gamma_{\kappa^{-1}(v)})$ connecting vertices $v_+, v_-$, there is a sequence $l_\nu \to \infty$ such that

$$t_\nu(v_+) - t_\nu(v_-) = \mathcal{T}(e)l_\nu.$$

(c) (Slope for uncollapsed edge) For an edge $e$ of $\Gamma'$ that is not collapsed in $\Gamma$,

$$t_\nu(v_+) - t_\nu(v_-) \in \mathbb{R}\mathcal{T}(e).$$

This ends the Definition.

Definition 7.32. (Gromov convergence for broken maps) Suppose $\Gamma' \to \Gamma$ is an edge collapse morphism for tropical graphs which induces a vertex map $\kappa : \text{Vert}(\Gamma') \to \text{Vert}(\Gamma)$. A sequence of broken maps $u_\nu : C \to X_\mathcal{P}$ of type $\Gamma'$ converges to a limit broken map $u : C \to X_\mathcal{P}$ of type $\Gamma$ if there is a $(\Gamma', \Gamma)$-translation sequence $t_\nu$ such that for any vertex $v \in \text{Vert}(\Gamma)$, the sequence $u_\nu|_{C \setminus C_{\kappa^{-1}(v)}}$ Gromov converges to

$$u|_{C_{\kappa^{-1}(v)}} := u|_{\cup_{v' \in \kappa^{-1}(v)} C_{v'}}$$

with translation sequence $t_\nu|_{\kappa^{-1}(v)}$.

Proposition 7.33. (Finite number of types of broken maps) For any $E > 0$, $d(\zeta) \geq 1$ there are a finite number of tropical graphs $\Gamma$ that are types of broken maps $u : C \to X$ of area at most $E$ and $d(\zeta)$ boundary leaves.

Proof. Consider a broken map $u : C \to X$ of type $\Gamma$ and area $\leq E_0$, whose edge slopes are given by $\mathcal{T} : \text{Edge}(\Gamma) \to \mathfrak{t}_\mathfrak{z}$.

Step 1: Uniform bound on the number of vertices.

The number of interior markings in $\Gamma$ is bounded by $kE_0$, where $k$ is the degree of
the stabilizing divisor. Since \( u \) is an adapted map, all the components in its domain are stable. The number of irreducible components in a stable curve is bounded by the number of interior and boundary markings.

**Step 2:** Uniform bound on the sum of vertical component of edge slopes.

Let \( u_v : C \to X_P \) be a component of the broken map corresponding to a vertex \( v \in \text{Vert}(\Gamma) \), and \( P := P(v) \). By the balancing property (Remark 3.19), the sum of the edge slopes projected to \( t'_P \) is

\[
\sum_{e \ni v} \pi_{t'_P}^v(\mathcal{T}(e)) = c_1((\pi_P \circ u_v)^*Z_{P(v)} \to X_P))
\]

The right-hand side, which is the pairing of \((\pi_P \circ u_v)^*[C]\) and the Chern class \(c_1(Z_P \to X_P)\), is bounded since the perturbed almost complex structure \(J_\Gamma\) takes values in a small neighbourhood of a base almost complex structure \(J_XP\). Indeed, for any \(\epsilon \in (0, 1)\), there is a constant \(C\) such that for any almost complex structure \(J \in B_\epsilon(J_XP)_{C^0}\) and a \(J\)-holomorphic sphere \(u : \mathbb{P}^1 \to X_P\),

\[
\int_{\mathbb{P}^1} u^*c_1(Z_P \to X_P) \leq C \int_{\mathbb{P}^1} u^*\omega_{X_P}.
\]

This estimate is similar to the one in (4.2) and the proof is the same – by choosing a two-form on \(X_P\) representing the Chern class and bounding it pointwise by \(\omega_{X_P}\).

**Step 3:** Uniform bound on the horizontal component of edge slopes.

For a vertex \(v\) and an incident edge \(e\), the horizontal component of the slope \(\mathcal{T}(e)\) is the sum of intersection multiplicities at the node \(w_e\) with horizontal boundary divisors of \(X_P\):

\[
\pi_{t'_P}^v(\mathcal{T}(e)) = \sum_{X_Q \subset X_P} m_{w_e}(u_{v,P}, X_Q)\nu_Q, \quad u_{v,P} := \pi_P \circ u_v : C \to X_P,
\]

where \(\nu_Q\) is the normal vector of the facet \(Q \subset P\). For any boundary divisor \(X_Q \subset X_P\), the sum \(\sum_{e \ni v} m_{w_e}(u_v, D_Q)\) is bounded by \(c\omega_{X_P}(u_{v,P})\) for a uniform constant \(c(X_P, D_Q)\). The proof is similar to the vertical case, by expressing the intersection number with any divisor as an integral of a two-form.

**Step 4:** Finishing the proof.

We will show that the tropical edge slopes of \(\Gamma\) are uniformly bounded in \(t'_P\). Combining Step 2 and 3, we conclude that for a vertex \(v\) and an edge \(e_0\) incident on \(v\):

\[
\exists c(E) : |\mathcal{T}(e_0)| \leq \sum_{e \ni v, e \neq e_0} |\mathcal{T}(e)| + c(E)
\]

Recall that \(\Gamma\) is a tree, any edge \(e \in \text{Edge}_{\text{out}}(\Gamma)\) is oriented so that it points away from the root vertex. The slope of any incoming edge can be bounded by the slope of outgoing edges by (7.28). Applying (7.28) iteratively, we conclude that for any edge \(e\) in \(\Gamma\)

\[
|\mathcal{T}(e)| \leq c(E)|\text{Vert}(\Gamma)|.
\]

where the constant \(c(E)\) is same as the one in (7.28). The Proposition now follows from the bound on the number of vertices in Step 1. \(\Box\)
The following definition is used in the proof of compactness for broken maps.

**Definition 7.34.** (Approximate \((\Gamma', \Gamma)\)-translation sequence) Suppose the tropical graph \(\Gamma\) is obtained by collapsing edges in \(\Gamma'\) and the induced map on the vertex set is \(\kappa : \text{Vert}(\Gamma') \to \text{Vert}(\Gamma)\). Then, a \((\Gamma', \Gamma)\)-translation sequence consists of a sequence 

\[ t_\nu(v) \in \text{Cone}(\kappa, v), \]

for every \(v \in \text{Vert}(\Gamma')\) satisfying the following.

(a) (Slope for collapsed edges) For an edge \(e\) of \(\Gamma'\) that is not collapsed in \(\Gamma\),

\[ t_\nu(v_+) - t_\nu(v_-) \in \mathbb{R} \mathcal{T}(e). \]

(b) (Approximate slope for uncollapsed edges) For an edge \(e\) of \(\Gamma'\) that is not collapsed in \(\Gamma\),

\[ t_\nu(v_+) - t_\nu(v_-) \mod \mathcal{T}(e) \]

is a bounded sequence in \(\mathfrak{t}'/\mathbb{R} \mathcal{T}(e)\). (Recall that \(\text{Cone}(\kappa, v_\pm)\) as embedded in \(\mathfrak{t}'\) with vertex \(\mathcal{T}(v_\pm)\) mapped to the origin in \(\mathfrak{t}'\), and identify \(\mathfrak{t}\) to \(\mathfrak{t}'\) via a pairing (2.24).)

An approximate \((\Gamma', \Gamma)\)-translation sequence can be adjusted by a uniformly bounded amount to produce an actual \((\Gamma', \Gamma)\)-translation sequence.

**Lemma 7.35.** (From an approximate to an exact \((\Gamma', \Gamma)\)-translation sequence) Let \(\kappa : \Gamma' \to \Gamma\) be a tropical edge collapse, and let \(\{t_\nu\}_\nu\) be an approximate \((\Gamma', \Gamma)\)-translation sequence. There exists a \((\Gamma', \Gamma)\)-translation sequence \(\{\tilde{t}_\nu\}_\nu\) such that

\[ \sup_\nu |\tilde{t}_\nu(v) - t_\nu(v)| < \infty \]

for all \(v \in \text{Vert}(\Gamma')\).

**Proof.** The proof is by replicating the iteration in Step 2 of the proof of Lemma 7.24. At the start, we set \(\tilde{t}_0^i = t_\nu\). At the \((i+1)\)-th step, we construct \(\tilde{t}_{i+1}^\nu\) as follows. As in the proof of Lemma 7.24, there exists a vertex \(v_0 \in \text{Vert}(\Gamma')\) such that the sequence \(|t_\nu^i(v)|\) has the fastest growth rate. That is, for all \(v \in \text{Vert}(\Gamma')\),

\[ \lim_\nu |t_\nu^i(v)|/|t_\nu^i(v_0)| \]

is finite. Define

\[ \tilde{t}_{i+1}^\nu(v) := t_\nu^i(v) - |t_\nu^i(v_0)| \lim_\nu \frac{t_\nu^i(v)}{|t_\nu^i(v_0)|}. \]

For the sequences \(\{t_{i+1}^\nu(v)\}_\nu, v \in \text{Vert}(\Gamma')\), the quantity

\[ \left\{ \frac{1}{\pi_\nu^e}(t_{i+1}^\nu(v_+) - t_{i+1}^\nu(v_-)) \right\}, \quad e = (v_+, v_-) \in \text{Edge}(\Gamma') \]

vanishes for collapsed edges, and is uniformly bounded for uncollapsed edges. Further, for any vertex \(v\), \(t_{i+1}^\nu(v) \in \mathcal{T}(v_\nu)\). After, say, \(k\) steps, the sequence \(|\tilde{t}_\nu^k(v)|\) is uniformly bounded for all vertices \(v\). Then, \(\tilde{t} := \tilde{t}^k - t^k\) is an exact \((\Gamma', \Gamma)\)-translation sequence. □

The notion of horizontal convergence extends in a natural way to broken manifolds, though it is easier to state in this case.
Definition 7.36. (Horizontal convergence in a broken manifold) Let $P \in \mathcal{P}$ be a polytope. A sequence of points $x_\nu \in X_P$ horizontally converges in $Q \subseteq P$ if the sequence $\pi_P(x_\nu) \in X_P$ converges to a point $x \in X_Q \subseteq X_P$, and $x$ is not contained in a submanifold $X_{Q'}$ for any $Q' \subset Q$.

Analogs of Lemma 7.13 and 7.15 hold for horizontal convergence in broken manifolds.

Proof of Theorem 7.2. After passing to a subsequence, the tropical graph $\Gamma$ underlying the maps $u_\nu$ is $\nu$-independent. Indeed by Proposition 7.33, there is a finite number of tropical graphs that are types of broken maps with area $< E$. Since the type is fixed, the Hofer energy $E_{\text{Hof}}(u_\nu)$ of the maps $u_\nu$ is uniformly bounded. Indeed for a broken map $u$ of type $\Gamma$, the area $\text{Area}(u)$ and the intersection multiplicities $T(e)$ at nodes $w_e, e \in \text{Edge}_-(\Gamma)$ with the boundary divisors $X_P(e)$ are fixed. This implies a bound on Hofer energy $E_{\text{Hof}}(u)$ by Proposition 6.44.

We first find the limit map at each of the vertices of the tropical graph $\Gamma$. For each vertex $v \in \text{Vert}(\Gamma)$, we assume that the domain curve $C_{\nu,v}$ for $u_{\nu,v}$ has marked points corresponding to lifts of nodes $e \in \text{Edge}_-(\Gamma)$, $v \in e$, in addition to the marked points corresponding to leaves $e \in \text{Edge}_+(\Gamma)$. We apply the proof of the convergence for breaking maps to the sequence of maps $u_{\nu,v}$. The conclusion is that there is a limit map $u_v$ modelled on a tropical graph $\Gamma_v$, and the convergence is via a translation sequence $t_\nu(v'), v' \in \Gamma_v$. By connecting the graphs $\{\Gamma_v\}_{v \in \text{Vert}(\Gamma)}$ using the edges of $\Gamma$, we obtain a tropical graph $\Gamma'$. Indeed $\Gamma'$ possesses vertex tropical weights, defined by pulling back the tropical weights on $\Gamma$ by $\kappa$.

Next prove that the limit map $u$ satisfies matching conditions on the edges of $\Gamma$. The matching conditions are proved by ensuring that the relative translation sequences corresponding to each vertex of $\Gamma$, when put together, form a $(\Gamma', \Gamma)$-translation sequence. As a first step we show that the translation sequence $t_\nu$ in the previous paragraph is an approximate $(\Gamma', \Gamma)$-translation sequence. In particular, we need to prove that the (Approximate slope for uncollapsed edges) is satisfied. Consider an edge $e$ of $\Gamma$, that is incident on $v_+, v_- \in \text{Vert}(\Gamma')$, and the nodal point corresponding to $e$ is the pair $w^e_{\pm}$ on $C_{v_{\pm}}$. The edge matching condition for broken maps (see Remark 3.18) implies

\begin{equation}
\pi^+_{T(e)}(u_\nu(w^e_{\nu,+})) = \pi^+_{T(e)}(u_\nu(w^e_{\nu,-})).
\end{equation}

On a sequence of converging relative maps, the evaluations of the relative marked points converge, i.e.

\[
\pi^+_{T(e)}(e^{t_\nu(v)}u_\nu(z_{i,\nu})) \to \pi^+_{T(e)}(u(z)) \quad \text{in } Z_{P(e),\mathbb{C}}/T_{T(e),\mathbb{C}}.
\]

This convergence is a consequence of the convergence of the marked points $z_{i,\nu}$ and the convergence of maps $e^{t_\nu}u_\nu$. Therefore, for the edge $e$,

\[
d(\pi^+_{T(e)}(e^{t_\nu(v_+)}u_\nu(w^e_{\nu,+})), \pi^+_{T(e)}(e^{t_\nu(v_-)}u_\nu(w^e_{\nu,-}))) \to d(\pi^+_{T(e)}(u(w^e_{+})), \pi^+_{T(e)}(u(w^e_{-}))),
\]

where $d$ is the cylindrical metric distance $Z_{P(e),\mathbb{C}}/T_{T(e),\mathbb{C}}$. Using (7.29) the sequence

\begin{equation}
\pi^+_{T(e)}(t_\nu(v_+) - t_\nu(v_-)) \in t_{\mathbb{C}}/t_{T(e),\mathbb{C}}
\end{equation}

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is bounded, implying that \( \{ t_\nu(v) : v \in \text{Vert}(\Gamma') \} \) is an approximate \((\Gamma', \Gamma)\)-translation sequence. An approximate \((\Gamma', \Gamma)\)-translation sequence can be adjusted by a uniformly bounded amount to produce an actual \((\Gamma', \Gamma)\)-translation sequence by Lemma 7.35. After making such an adjustment the quantity in (7.30) vanishes. Then by (7.29) the limit map \( u \) satisfies the edge matching condition (3.13) for edges in \( \Gamma \), and \( u \) is therefore a broken map.

The proof of uniqueness is similar to the case of breaking maps: The domain curve is uniquely determined, and any two \((\Gamma', \Gamma)\)-translation sequences \( t_\nu, t_\nu' \) differ by a uniformly bounded amount: \( \sup_\nu |t_\nu(v) - t_\nu'(v)| < \infty \). After passing to a subsequence, the limit

\[ t_\infty(v) := \lim_\nu(t_\nu(v) - t_\nu'(v)) \]

exists, and is a \((\Gamma', \Gamma)\)-relative weight. Therefore, the limit maps \( \lim_\nu \varepsilon^t u_\nu, \lim_\nu \varepsilon^t u_\nu' \) are related by a tropical symmetry element \( e^{t_\infty} \in T_{\text{trop}}(\Gamma') \).

It remains to prove the last statement about the case where the tropical edge collapse \( \Gamma' \to \Gamma \) is trivial. First, assume that \( \Gamma' = \Gamma \). For any element \( t_\nu \) of the translation sequence, \( (e^{-t_\nu(v)})_{v \in \text{Vert}(\Gamma)} \) is in the tropical symmetry group \( T_{\text{trop}}(\Gamma) \). The sequence \( u'_\nu := e^{-t_\nu} u_\nu \) converges to the same limit \( u \) with a trivial translation sequence. In the general case, where the tropical edge collapse \( \kappa : \Gamma' \to \Gamma \) is trivial, but \( \Gamma \neq \Gamma' \), all the edges collapsed by \( \kappa \) have zero slope. Hence, for any edge \( e = (v_+, v_-) \) of \( \Gamma' \) that is collapsed by \( \kappa \), \( t_\nu(v_+) = t_\nu(v_-) \). Therefore \( t_\nu(v) \in P(v)^{\nu} \) is a sequence for each vertex \( v \) of \( \Gamma \), and \( e^{-t_\nu} \in T_{\text{trop}}(\Gamma) \). Similar to the previous case, we define \( u_\nu := e^{-t_\nu} u_\nu \), finishing the proof of the Theorem. \( \square \)

7.6. True boundary strata. We wish to show that the topological boundary of the one-dimensional component of the moduli space consists of configurations with an edge of infinite length.

**Definition 7.37.** The combinatorial type \( \Gamma \) of a broken treed holomorphic map is rigid if the tropical graph \( \mathcal{T}(\Gamma) \) is rigid, and the only edges \( e \in \text{Edge}_-(\Gamma) \) with zero slope \( \mathcal{T}(e) = 0 \) are boundary edges \( e \in \text{Edge}_-(\Gamma) \) of finite non-zero length \( \ell(e) \in (0, \infty) \), and intersection multiplicity \( m_{u(e)}(u, D_P) \) of \( u \) of type \( \Gamma \) at \( e \cap S \) with the stabilizing divisor \( \mathcal{D} = (D_P, P \in \mathcal{P}) \) is 1.

**Proposition 7.38.** Let \( \overline{p} \) be a regular perturbation datum. Let \( \Gamma \) be a type of an uncrowded broken map and let \( \mathbf{x} \in (I(L))^{d(\overline{p})} \) be a tuple of limit points of boundary leaves such that the expected dimension of the moduli space \( \mathcal{M}_\Gamma(L, \overline{p}, \mathbf{x}) \) is \( \leq 1 \). Then, for any curve \( u \) in the compactification \( \overline{\mathcal{M}}_\Gamma(L, \overline{p}, \mathbf{x}) \), there are no horizontally constant components that contain interior markings.

**Proof.** We first consider a map \( u \) in the moduli space and show that its horizontal components do not have interior markings. Suppose the map \( u \) is horizontally constant on a component \( S_v \), for a vertex \( v \) of \( \Gamma \), and suppose \( S_v \) has an interior marking \( z_v \). The restriction \( u|S_v \) cannot have more than two nodes – otherwise moving the marking \( z_v \) on the component gives a two-dimensional family of adapted regular maps. By stability there are exactly two nodes on \( S_v \). By rigidity of the type \( \Gamma \) all edges \( e \) corresponding to interior nodes \( u_e \) have non-zero slope \( \mathcal{T}(e) \neq 0 \).
Since \( u|_{C_1} \) is horizontally constant, the balancing property (see Remark 3.19) implies that the edges \( e_1, e_2 \in \text{Edge}(\Gamma) \) corresponding to both nodes have the same slope \( \mathcal{T}(e_1) = \mathcal{T}(e_2) \). This means the type \( \Gamma \) has a non-trivial tropical symmetry group \( T_{\text{trop}}(\Gamma) \), contradicting the rigidity of \( \Gamma \).

Next, we consider a map in a boundary of the moduli space. Suppose \( u_\nu \) is a sequence in \( \mathcal{M}_T(L, p, x) \) that converges to a limit \( u : C \to \mathfrak{X} \) of type \( \Gamma' \). First we prove the proposition assuming that the limit \( u \) is uncrowded. Then \( u \) is an adapted regular map, and its index is

\[
\tilde{i}(u) = \tilde{i}(u_\nu) - 2\#\{e \in \text{Edge}_{\bullet,-}(\Gamma') \setminus \text{Edge}_{\bullet,-}(\Gamma) : \mathcal{T}(e) = 0\} - |\text{Edge}^0_{\bullet,-}(\Gamma') \setminus \text{Edge}^0_{\bullet,-}(\Gamma)| - |\text{Edge}^\infty_{\bullet,-}(\Gamma') \setminus \text{Edge}^\infty_{\bullet,-}(\Gamma)|
\]

Since \( \tilde{i}(u_\nu) \leq 1 \), there are no interior edges \( e \) with zero slope \( \mathcal{T}(e) = 0 \) in \( \Gamma' \). If \( \Gamma' \to \Gamma \) is a tropical edge collapse, then \( \Gamma' \) has a tropical symmetry group \( T_{\text{trop}}(\Gamma') \) that is at least two-dimensional. This implies \( u \) is part of a family with dimension \( \geq 2 \), which is not possible. Therefore the tropical graph associated to \( \Gamma' \) is the same as that of \( \Gamma \). By the argument in the previous paragraph, there are no markings in horizontally constant components of \( u \).

Next, consider the case that the limit \( u \) has a crowded component. Forgetting all but one leaf \( e \) meeting each of the crowded components \( S_v, v \in \text{Vert}_{\text{crowded}}(\Gamma) \) yields an adapted map of \( u' \) of type \( \Gamma_s \). If a crowded component \( S_v \subset S \) becomes unstable after forgetting all but one of its leaves \( e, e \cap S_v \neq \emptyset \) it is collapsed, and in \( \Gamma_s \) the remaining leaf \( e \) is assigned a multiplicity of \( \mu(e) \) plus the number of forgotten leaves \( e' \neq e \). The limit \( u' \) is \( p_{\Gamma_s} \)-adapted because of the (Locality axiom). Indeed, forgetting markings changes the type of the limit curve, but it does not affect the perturbation datum \( p_T \) on the other curve components on which the map is horizontally non-constant. Therefore, \( u' \) is regular. If no component is collapsed in \( \Gamma' \to \Gamma_s \), then, the expected dimension of the type \( \Gamma_s \) is same as that of \( \Gamma' \). In this case \( u \) is an uncrowded map with a marking in a horizontally constant component. This possibility has been ruled out in the last paragraph. If a component is collapsed, the expected dimension of \( \Gamma_s \) is at least two lower than \( \Gamma \) and therefore, the map \( u \) does not exist.

\[\square\]

**Proposition 7.39.** (Boundary strata) Suppose \( \Gamma \) is a rigid type for a broken map and \( \underline{\mathfrak{r}} \in (\mathcal{I}(L))^{d(o)} \) is a tuple of limit points of boundary leaves such that the expected dimension \( \tilde{i}(\Gamma, \underline{\mathfrak{r}}) \) is 1. The boundary components of \( \mathcal{M}_T(\mathfrak{X}_P, L, \underline{\mathfrak{r}}) \) consist of strata corresponding to types \( \Gamma' \) with a single broken boundary trajectory \( e \in \text{Edge}_{\bullet}(\Gamma') \), \( \ell(e) = \infty \) or a single boundary edge with length zero \( \ell(e) = 0 \).

**Proof.** Suppose \( u : C \to \mathfrak{X} \) is a treed tropical map occurring as a limit of a sequence of maps \( u_\nu : C \to \mathfrak{X} \) in the moduli space \( \mathcal{M}_T(\mathfrak{X}_P, L) \). By Proposition 7.38, we conclude the limit \( u \) is regular and adapted. Interior nodes \( w_e \) corresponding to edges \( e \) of zero slope \( \mathcal{T}(e) = 0 \) are ruled out in \( u \) for dimension reasons using the index relation (7.31). We next claim that the tropical type of \( u \) is \( \Gamma \). If not it is of type \( \Gamma' \) and there is a tropical edge collapse morphism \( \Gamma' \to \Gamma \), which implies \( u \) has a non-trivial tropical symmetry \( (g, v) \in T_{\text{trop}}(\Gamma) \). The expected dimension
(Proposition 5.18) of $\mathcal{M}_\Gamma(\mathcal{X}_\mathcal{P}, L)$ is the unreduced index, which is unchanged by this degeneration, so the index of $u$ is $\leq 1$. But the positive dimensionality of the tropical symmetry group $T_{trop}(\Gamma)$ implies that the reduced index $\text{Ind}(D_u)$ is negative, which is a contradiction. The only other phenomenon $\nu$ which occurs in the limit is the formation of a boundary node $w \in C$ corresponding to an edge $e$ of length $\ell(e)$ zero, or the length of a boundary edge $\ell(e)$ going to zero or infinity. $\square$

**Figure 16.** True and fake boundary strata of a one-dimensional component of the moduli space of treed holomorphic disks. The sphere components lie in different pieces of the tropical manifold.

**Remark 7.40.** (True and fake boundary strata) There are two types of strata that occur as the codimension one boundary of a moduli space - one with a boundary edge of length zero, and the second with a boundary edge containing a breaking. The first is a fake boundary, and the second one is a true boundary. Indeed, for a configuration $u$ of type $\Gamma'$ containing an edge $e$ of zero length $\ell(e)$, one may either use disk gluing to produce a configuration $u'$ with one less disk component $C_v$, $v \in \text{Vert}_\infty(\Gamma)$, or allow the length of $\ell(e)$ the edge $e$ to become positive. This implies that the stratum $\mathcal{M}_{\Gamma'}(L, \underline{x})$ is in the boundary of two one-dimensional components $\mathcal{M}_{\Gamma_\pm}(L, \underline{x})$ and so does not represent a component in the topological boundary $\partial \mathcal{M}(\mathcal{X}_\mathcal{P}, L)$ of the moduli space $\mathcal{M}(\mathcal{X}_\mathcal{P}, L)$. This is the fake boundary in Figure 16. The only (true) boundary components of one-dimensional strata thus consist of maps with a single broken Morse trajectory, see Figure 16.
8. Gluing

In this section, we show that a rigid broken map can be glued at nodes to produce a family of unbroken maps. A fixed perturbation datum \( p \) on a broken manifold \( X \) can be glued in a natural way to produce a perturbation datum \( p^\nu \) for \( X^\nu \) which is equal to \( p \) away from the neck regions. With respect to these perturbation data, we construct a bijection between rigid maps for sufficiently large neck lengths.

**Theorem 8.1.** (Gluing) Suppose that \( u : C \to X \) is a regular broken disk of index zero. There exists \( \nu_0 > 0 \) such that if \( \nu \geq \nu_0 \) there exists a family of unbroken disks \( u^\nu : C^\nu \to X^\nu \) of index zero, with the property that \( \lim_{\nu \to \infty} [u^\nu] = [u] \). For any area bound \( E > 0 \) there exists \( \nu_0 \) such that for \( \nu \geq \nu_0 \) the correspondence \( [u] \mapsto [u^\nu] \) defines a bijection between the rigid moduli spaces \( \mathcal{M}^{<E}(X^\nu, L)_0 \) and \( \mathcal{M}^{<E}_{\text{brok}}(X, L)_0 \) for \( \nu \geq \nu_0 \).

**Remark 8.2.** The gluing operation is defined on broken maps, and not on tropical symmetry orbits of broken maps. In fact if the tropical symmetry group is non-trivial (it is necessarily finite for rigid broken maps), gluing different elements in the tropical symmetry orbit produces distinct sequences of unbroken maps. The convergence result also distinguishes between rigid maps in the same tropical symmetry orbit. Indeed, in Theorem 7.1, if the limit map is rigid then the limit is uniquely determined up to domain reparametrization.

As for other gluing theorems in pseudoholomorphic curves the proof of Theorem 8.1 is an application of a quantitative version of the implicit function theorem for Banach manifolds. The steps are: construction of an approximation solution called the pre-glued map; construction of an approximate inverse to the linearized operator; quadratic estimates; application of the contraction mapping principle, and surjectivity of the gluing construction. Through the proof of the gluing theorem, the notation \( c \) denotes a \( \nu \)-independent constant whose value is different in every occurrence.

8.1. The approximate solution. A pre-glued family for a broken map is constructed using tropical weights of the broken map. A rigid map is modelled on a rigid tropical graph \( \Gamma \). We recall that a rigid tropical graph has unique weights \( \{ T(v) : v \in \text{Vert}(\Gamma) \} \) on its vertices. These weights determine the neck lengths for the approximate solution as follows. For any edge \( e = (v_+, v_-) \) of \( \Gamma \), there exists \( l_e > 0 \) such that

\[
T(v_+) - T(v_-) = l_e T(e),
\]

where \( T(e) \in \mathbb{t}_\mathbb{Z} \) is the slope of the edge.

The domain of the glued family of maps is obtained by replacing interior nodes with necks: The curve \( C^\nu \) has a neck of length \( vl_e \) in place of the node \( w_e \) in \( C \). Denote by \( \Gamma_{\text{glue}} \) the type of the glued curve. The lift of a node \( w_e \) has matching coordinates (see (Edge matching) in Definition 3.4 of broken maps) in the neighbourhoods \( U_e^+, U_e^- \) of the lifts \( w_+(e), w_-(e) \) of the node:

\[
C_{v^\pm} \supset (U_e^+, w_+(e)) \xrightarrow{\mathbb{C}, 0}. \]
The coordinates can be chosen to be compositions of the complex exponential map (7.1) and linear functions from tangent spaces to $\mathbb{C}$. The glued curve $C'$ is obtained from $C$ by deleting a small disk in $U^e_\pm$ for every edge $e$ in $\Gamma$, and gluing the remainder of the neighbourhoods $U^e_\pm$ using the identification $z_+^e \sim e^{-i\ell_e} z_-^e$, and leaving the tree part $T$ unchanged. So the surface and tree parts of $C'$ are

$$C' = S' \cup T.$$ 

For future use in the proof we point out that the punctured neighbourhoods $U^e_- \setminus \{w_-(e)\}$ have matching logarithmic coordinates $(s_e, t_e) : U^e_- \setminus \{w_-(e)\} \rightarrow (0, \infty) \times S^1$ given by $(s_e, t_e) := \pm \ln(z_+^e)$.

We use translation sequences to map target components of the broken map to regions in the neck stretched manifolds. Again, we recall that for a rigid tropical graph, the translation sequences are unique and are given by

$$t_\nu: \text{Vert}(\Gamma) \rightarrow \nu B^\nu, \quad v \mapsto \nu T(v).$$

Translation sequences give identifications $e^{t_\nu(v)} : X^\nu_{\tilde{P}(v)} \rightarrow X^\nu$ that are well-defined away from boundary divisors of $X_{\tilde{P}(v)}$, see (2.34). The translated map $u_{\nu, v} := e^{t_\nu(v)} u_v : C^\nu \rightarrow X^\nu$ is well-defined away from the nodal points on $C_v$.

Translated maps can be glued at the nodal points to yield a sequence of approximate solutions for the holomorphic curve equation in neck-stretched manifolds. For an edge $e = (v_+, v_-)$, there is a point $x'_e \in X^\nu$ such that the map $u_{e, \nu} : C^\nu \rightarrow X^\nu$ is asymptotically close to the vertical cylinder (see (3.8))

$$u_{e, \nu}^\pm : \mathbb{R}^\pm \times S^1 \rightarrow X^\nu, \quad (s, t) \mapsto e^{t_\nu(v_\pm) + T(e)(s + it)} x'_e$$

at the nodal point corresponding to $e$. Since $t_\nu(v_+) - t_\nu(v_-) = \nu \ell_e$, the vertical cylinders agree on the glued cylinder in $C'$. In the glued curve the vertical cylinder is

$$(8.4) \quad \left[ -\nu \ell_e, \frac{\nu \ell_e}{2} \right] \times S^1 \ni (s, t) \mapsto e^{T(e)(s + it)} x'_e, \quad \text{where} \quad x_e = e^{t_\nu(v_+) + t_\nu(v_-) + T(e)(s + it)} x'_e.$$

Asymptotic decay of the maps $u_{e, \nu}$ (Lemma 3.14) implies that

$$u_{e, \nu} = \exp u_{e, \nu}^\pm c^\pm_e, \quad \| D^k c^\pm_e (s, t) \| \leq ce^{-|s|}$$

for any $k \geq 0$. Let

$$\beta \in C^\infty(\mathbb{R}, [0, 1]), \quad \begin{cases} \beta(s) = 0 & s \leq 0 \\ \beta(s) = 1 & s \geq 1. \end{cases}$$
be the cutoff function from (5.8). Define $u^\text{pre}_\nu$ to be equal to $u_\pm$ away from the neck region, while on the neck region of $C^\nu$ corresponding to an edge $e \in \text{Edge}_\bullet(\Gamma)$ with coordinates $(s,t) \in \left[\frac{v_\mu}{2}, -\frac{v_\mu}{2}\right] \times S^1$ define

$$u^\text{pre}_\nu(s,t) = \exp_{e(s+it)T(e)_{X_e}}(\zeta^\nu_e(s,t)), \quad \zeta^\nu_e(s,t) = \beta(-s)\zeta^e_-(s + \frac{v_\mu}{2}, t) + \beta(s)\zeta^\nu_+(s - \frac{v_\mu}{2}, t).$$

In other words, in the cylinder in $C_\nu$ corresponding to an edge $e$, one translates the domain on both ends by an amount $\frac{v_\mu}{2}$, and then patches the map together using the cutoff function and geodesic exponentiation.

**Remark 8.3.** We explain the construction of approximate solution in a simple special case where the broken map has a single node. Suppose $X$ is given by a set of $n$-orthogonal cuts $P = \{P_i\}$, and the dual polytope $B^\nu$ is a cube with side lengths $(\mu_1, \ldots, \mu_n)$ where $\mu_i \in \mathbb{Z}_{>0}$. Suppose that the components $u_+, u_-$ map to $X_+ := X_{P_+}$ and $X_- := X_{P_-}$ both of which are non-neck pieces, and $P := P_+ \cap P_-$ is a point. Further, we assume that the edge $e$ connecting $u_+$ and $u_-$ has slope $T(e) = \mu$. The corner of $X_+$ containing the node $w(e)$ has cylindrical ends via an isometric embedding $Z_P \times \text{Cone}(P^\nu) \subset X_\pm$. The glued target $X^\nu$ contains a region obtained by removing a part of the cylindrical end in $X_+$ and $X_-$ and identifying

$$Z_P \times P^\nu \ni (z,t) \sim (z, v\mu - t).$$

The translations $e^{t_\nu(v\pm)}$ leave the maps $u_+, u_-$ unchanged, because $t_\nu(v\pm)$ lies in $\nu P^\nu_\pm$ which is a vertex in the dual complex $\nu B^\nu$. The glued map $u$ is obtained by removing a neighbourhood of the nodes $w_\pm(e)$, and identifying the punctured neighbourhoods via the map $z_+ \sim e^{-\nu}/z_-$. The glued cylinder in the domain $C^\nu$ maps to the cylindrical region in the target space $X^\nu$, and the map $u^\nu$ is defined as in (8.7). This ends the Remark.

**Remark 8.4.** In standard gluing results the gluing parameter $\nu$ is taken to be the neck length in the domain $C^\nu$ of the glued map $u^\nu$. In our setting, the gluing parameter is the neck length $\nu$ in the target space $X^\nu$. The domain neck lengths, approximately equal to $\nu l_\nu$, are allowed to vary in the Picard iteration argument. Note that resolving nodes $w_e$ corresponding to edges $e$ with non-zero slope $T(e) \neq 0$ does not increase the index of the map. As a result gluing an index zero broken map $u$ produces an isolated map $u^\nu$ in each $X^\nu$.

### 8.2. Fredholm theory for glued maps

We define a map between suitable Banach spaces whose zeroes describe pseudoholomorphic curves close to the approximate solution. Pseudoholomorphic maps are zeroes of the section

$$\mathcal{F}_\nu: \mathcal{M}_\text{glue} \times \text{Map}(C^\nu, X^\nu)_{1,p} \to \Omega^{0,1}(C^\nu, u^*TX), \quad (j, u) \mapsto \overline{\partial}_j, j(u).$$

The first space in the domain is a moduli space of curves and it is finite dimensional. We give a careful description in order to obtain uniform bounds. Recall that the domain of $u$ is $C$, which is a curve of type $\Gamma$. Consider a trivialization of the universal curve $U_\Gamma \to \mathcal{M}_\Gamma$ of type $\Gamma$ in a neighbourhood

$$U_{\Gamma, C} \subset \mathcal{M}_\Gamma$$
of $C$. The trivialization yields a map

$$U_{\Gamma,C} \to \mathcal{J}(\Gamma), \quad m \mapsto j_C + j_\Gamma(m)$$

of complex structures on $C$ which are constant in neighbourhoods of nodes. This trivialization of the universal curve of type $\Gamma$ is lifted to a trivialization of the universal curve over $\mathcal{M}_\Gamma$ as follows. In a neighborhood $U_{\mathcal{M}_\Gamma} \subset \overline{\mathcal{M}}_\Gamma$ of $\mathcal{M}_\Gamma$, there is a projection map

$$\pi_\Gamma : U_{\mathcal{M}_\Gamma} \to \mathcal{M}_\Gamma$$

such that any curve $C' \in U_{\mathcal{M}_\Gamma}$ is obtained by gluing at interior nodes of the curve $\pi_\Gamma(C')$ using the coordinates (8.2). Since the sequence of curves $C_\nu$ converges to $C$, we can assume that $[C_\nu] \in U_{\mathcal{M}_\Gamma}$, and by construction $\pi_\Gamma(C_\nu) = C$. The subset of smooth curves $\mathcal{M}_\Gamma \cap U_{\mathcal{M}_\Gamma}$ is equipped with a metric (8.10)

$$g_{\Gamma,\text{glue}} : T(\mathcal{M}_\Gamma \cap U_{\mathcal{M}_\Gamma}) \to \mathbb{R}$$

that is cylindrical in the fibers of $\pi_\Gamma$. That is, each fiber of $\pi_\Gamma$ is isometric to a product of cylinders $\Pi_e(\mathbb{R} \times S^1)$ parametrized by logarithmic neck length parameters $(s_e, t_e)$. In a neighborhood $U_{C_\nu}$ of $C_\nu$, the universal curve $U_{\Gamma,\text{glue}} \to \mathcal{M}_\Gamma$ is trivialized so that in the resulting map

$$U_{C_\nu} \to \mathcal{J}(C_\nu), \quad m \mapsto j_\nu(m)$$

the function $j_\nu$ is a sum

$$j_\nu(m) = j_\nu + j_\bullet(m) + j_{\text{neck}}(m)$$

that satisfies the following:

- $j_\nu$ is the complex structure on the glued curve $C_\nu$ and is $\nu$-independent,
- the function $m \mapsto j_\bullet(m)$ is supported in the complement of the neck regions of $C_\nu$ and is equal to $j_\bullet(\pi_\Gamma(m))$,
- and $j_{\text{neck}}(m)$ is supported in the neck regions of $C_\nu$, and the support is contained in a uniformly bounded neighborhood the boundary of the neck, that is, there is a $\nu$-independent constant $L$ such that

$$\text{supp}(j_{\text{neck}}) \subset \bigcup_{e \in \text{Edge}_e(\Gamma)} \{ s : \frac{v_e}{2} - L \leq |s| \leq \frac{v_e}{2} \} \times S^1 \subset A(l_\nu) \subset C'_\nu.$$  

Further there is a $\nu$-independent constant $c$ such that on the neck region corresponding to any edge $e \in \text{Edge}_e(\Gamma)$ and for any $\nu$

$$\|j_{\text{neck}}(m)\|_{C^1} \approx c|n_e(m) - \nu l_e|,$$

where $n_e(m) \in \mathbb{R}_+ \times S^1$ is the length of the annulus in $C_\nu$ that replaced the node $w_e$ in the curve $\pi_\Gamma(m)$.

Such a choice of the trivialization of the universal curve ensures that there is a $\nu$-independent constant $c$ such that for any two curves represented by $m_1, m_2$

$$|j_\nu(m_2) - j_\nu(m_1)| \leq cd_{g_{\Gamma,\text{glue}}}(m_1, m_2).$$

These uniform estimates are used in the proof of the quadratic estimate in Section 8.5. We also allow variation of the length of treed segments in the domain curve.
This can be handled in a straightforward way by scaling the unit speed vector $\frac{d}{dt}$ on the treed segments. We drop this feature from the notation.

The second space in the domain of (8.8) is a space of maps $\text{Map}(C^\nu, X^\nu)_{1,p}$. It is the space of $W^1_{\text{loc}}$ maps from $C^\nu$ to $X^\nu$. The tangent space of $\text{Map}(C^\nu, X^\nu)_{1,p}$ at a map $u : C^\nu \to X^\nu$ is the space of sections

$$\Omega^0(C^\nu, u^*TX) = \Omega^0(S^\nu, (u|S)^*TX) \oplus \Omega^0(T, (u|T)^*TL).$$

As in Abouzaid [1, 5.38] this space is equipped with a weighted Sobolev norm based on the decomposition of the section into a part constant on the neck and the difference on the neck corresponding to each edge $e \in \text{Edge}_{\bullet}(\Gamma)$ described as follows. Denote by

$$(s_e, t_e) \in [-\nu_e/2, \nu_e/2] \times S^1$$

the coordinates on the neck region created by the gluing at the node corresponding to the edge $e \in \text{Edge}_{\bullet}(\Gamma)$. Let

$$\lambda \in (0, 1)$$

be a Sobolev weight. Define a Sobolev weight function

$$\kappa_\nu : C^\nu \to [0, \infty), \quad \kappa_\nu := \sum_{e \in \text{Edge}_{\bullet}(\Gamma)} \beta(\nu_e/2 - |s_e|)(\nu_e/2 - |s_e|).$$

Here, the function $\beta(\nu_e/2 - |s_e|)$ is extended by zero outside the neck region corresponding to $e$. As $\nu \to \infty$, $\kappa_\nu$ converges to the weight function $\kappa$ defined on the punctured curve $C - \{w_e : e \in \text{Edge}_{\bullet}(\Gamma)\}$ in (5.9). Given a section

$$\xi = (\xi_S, \xi_T) \in \Omega^0(C^\nu, u^*TX)$$

define

$$\|\xi\|_{1,p,\lambda} := \|\xi_S\|_{1,p,\lambda} + \|\xi_T\|_{1,p,\lambda}$$

$$\|\xi_S\|_{1,p,\lambda} := \left(\sum_e \|\xi_{S,e}(0,0)\|^p + \int_{C^\nu} (\|\nabla \xi_S\|^p + \|\xi_S - \sum_e \beta(\nu_e/2 - |s_e|)\mathcal{T}^{\nu}e \xi_{S,e}(0,0)\|^p) \exp(\kappa_\nu \lambda p) d\text{Vol}_C\right)^{1/p}$$

where $\xi_{S,e}$ is the restriction of $\xi_S$ to the neck region $[-\nu_e/2, \nu_e/2] \times S^1$ corresponding to the edge $e$, and $\mathcal{T}^{\nu}$ is parallel transport from $u^{\text{pre}}(0, t)$ to $u^{\text{pre}}(s, t)$ along $u^{\text{pre}}(s', t)$, $s' \in [0, s]$. Let $\Omega^0(C^\nu, u^*TX)_{1,p,\lambda}$ be the Sobolev completion of $W^1_{\text{loc}}$ sections with finite norm (8.14); these are sections whose difference from a covariant-constant section on the neck has an exponential decay behavior governed by the Sobolev constant $\lambda$.

The target space of (8.8) is a space of $(0,1)$-forms, which we equip with a weighted $L^p$ norm. For a $0, 1$-form $\eta \in \Omega^{0,1}(C^\nu, u^*TX^\nu)$ define

$$\|\eta\|_{0,p,\lambda} := \left(\int_{C^\nu} \|\eta\|^p \exp(\kappa_\nu \lambda p) d\text{Vol}_C\right)^{1/p}.$$

The implicit function theorem is applied on the $\overline{\partial}$ map pulled back by an exponential map. Pointwise geodesic exponentiation defines a map (using Sobolev
multiplication estimates)

\[(8.15) \quad \exp_{w^\text{pre}} : \Omega^0(C^\nu, (w^\text{pre}_v)^*TX^\nu)_{1,p,\lambda} \to \text{Map}_{1,p}(C^\nu, X^\nu)\]

where \(\text{Map}_{1,p}(C^\nu, X^\nu)\) denotes maps of class \(W^1_{1,p}\) from \(C^\nu\) to \(X^\nu\). We define

\[(8.16) \quad \mathcal{F}_\nu : U_{C^\nu} \times \Omega^0(C^\nu, (w^\text{pre}_v)^*TX^\nu)_{1,p,\lambda} \to \Omega^{0,1}(C^\nu, (\exp_{w^\text{pre}}(\xi))^*TX^\nu)_{0,p,\lambda},\]

\[ (m, \xi) \mapsto \mathcal{T}_{w^\text{pre}}^{-1} \mathcal{F}_j(m, \xi), \]

where

\[\mathcal{T}_{w^\text{pre}}(\xi) : \Omega^{0,1}(C^\nu, (w^\text{pre}_v)^*TX^\nu)_{0,p,\lambda} \to \Omega^{0,1}(C^\nu, (\exp_{w^\text{pre}}(\xi))^*TX^\nu)_{0,p,\lambda}\]

is the parallel transport defined using an almost-complex connection, and we recall that \(U_{C^\nu} \subset \mathcal{M}_{\text{glue}}\) is a neighborhood of \(C^\nu\).

In order to construct local models for moduli of adapted tree disks, we require that the treed disks \(C^\nu\) have a collection of interior leaves \(e_1, \ldots, e_{d(\bullet)}\) and

\[(\exp_{w^\text{pre}}(\xi))(e_i) \in D, \quad i = 1, \ldots, n.\]

Additionally we require matching conditions at boundary nodes and lifts of \(S^\nu \cap T^\nu\). Using notation from the proof of transversality (Theorem 5.22), these constraints may be incorporated into \(\mathcal{F}_\nu\) to produce a map

\[\mathcal{F}_\nu : U_{C^\nu} \times \Omega^0(C^\nu, (w^\text{pre}_v)^*TX^\nu)_{1,p} \to \Omega^{0,1}(C^\nu, (\exp_{w^\text{pre}}(\xi))^*TX^\nu)_{0,p} \times TX(\mathcal{M}_{\text{glue}}) / \Delta(\mathcal{M}_{\text{glue}}).\]

whose zeroes correspond to adapted pseudoholomorphic maps near the pre-glued map \(w^\text{pre}_v\).

### 8.3. Error estimate

We estimate the failure of the approximate solution to be an exact solution in the Banach norms of the previous section. To derive the estimate, we split the curve \(C^\nu\) into neck regions corresponding to nodes in \(C\), namely

\[\{(s_e, t_e) \in \left[-\frac{\nu_e}{2}, \frac{\nu_e}{2}\right] \times S^1\} \subset C^\nu, \quad \forall e \in \text{Edge}_\bullet(\Gamma);\]

and its complement

\[C^\nu_{\text{pre}} := C^\nu \setminus \cup_{e \in \text{Edge}_\bullet(\Gamma)} \left([\left[-\frac{\nu_e}{2}, \frac{\nu_e}{2}\right] \times S^1\right).\]

The one-form \(\mathcal{F}_\nu(0)\) has contributions created by the cutoff function as well as the difference between \(J_u\) and \(J_u^\text{pre}\):

\[(8.17) \quad ||\mathcal{F}_\nu(0)||_{L^p,\lambda(C^\nu)} = ||\overline{\mathcal{T}}_u^\text{pre} w^\nu_{\text{pre}}||_{L^p,\lambda(C^\nu)} + \sum_{e} ||\overline{\mathcal{T}} \exp_{\mathcal{E}(\nu_e)_{2, e}}(\beta(-s_e)\xi_e^- (s_e + \nu_e/2, t_e) + \beta(s_e)\xi_e^+ (s_e - \nu_e/2, t_e))||_{0,p,\lambda}.\]

The first term may not vanish because the almost complex structure is domain dependent: in the complement of the neck regions, the map \(w^\text{pre}_v\) is \(J(C, j^-)\)-holomorphic but not \(J(C^\nu, j^-)\)-holomorphic. For any metric \(d_{\mathcal{M}}\) on the compact manifold \(\overline{\mathcal{M}}_{\text{glue}}\), the distance between the domain curves is bounded as

\[d_{\mathcal{M}}((C^\nu, j^\nu), (C, j)) \leq c \max_{e \in \text{Edge}_\bullet(\Gamma)} \exp(-\nu_e).\]
Therefore, the distance between the domain-dependent almost complex structures has a similar bound. On the complement of the necks $u^\text{pre}_v$ is $J(C, j)$-holomorphic, so

\begin{equation}
\|\bar{\partial}_{J_{u^\text{pre}}}u^\text{pre}_v\|_{L^p, \lambda(C^\circ)} \leq c\|J(C, j) - J(C^\nu, j^\nu)\|_{L^\infty} \leq c \max_{e \in \text{Edge}_0(\Gamma)} \exp(-\nu l_e).
\end{equation}

The second term in the right-hand side of (8.17) is equal to

\[ \sum_{e}(D \exp_{e(s-t)}(s - s_e)\zeta_e^{-}) (s_e + \nu l_e/2, t_e) + d\beta(s_e)\zeta_e^+(s_e - \nu l_e/2, t_e)) + (\beta(-s_e) d\zeta_e^- (s_e + \nu l_e/2, t_e) + \beta(s_e) d\zeta_e^+(s_e - \nu l_e/2, t_e)))^{0.1}\|_{L^p, \lambda(C^\nu)} \]

On the neck regions, the almost complex structure is domain-independent. Holomorphicity of $u$ implies that the terms are non-zero only in the support of $d\beta_e$ which is contained in the interval $[-1, 1]$ in the neck region in $C^\nu$. Both $\zeta_e^\pm$ and its derivative decay at the rate of $e^{-s_e}$ on the cylindrical end $\mathbb{R}^+ \times S^1$ in $C^\circ$, see (8.5). As a result both the difference between $J_u$ and $J_{u^\text{pre}}$, and the terms containing $d\beta$ are bounded by $ce^{-l_e \nu/2}$ where $C$ is a constant independent of $\nu$. The Sobolev weight function (8.13) has a multiplicative factor of $e^{\nu l_e \nu/2}$, and therefore,

\begin{equation}
\|\mathcal{F}_\nu(0)\| \leq c\sum_{e \in \text{Edge}_0(\Gamma)} e^{-(1-\lambda)l_e \nu/2},
\end{equation}

with $c$ a constant independent of $\nu$. (See Abouzaid [1, 5.10]).

8.4. Uniform right inverse. We construct a uniformly bounded right inverse for the linearized operator of the approximate solution from the given right inverses of the pieces of the broken map. On the neck region for the edge $e \in \text{Edge}_0(\Gamma)$ in $C^\nu$, let $T_eu^\pm$ be parallel transport along the path

\[ \exp_{e(s-t)}(s, t) + (1 - \rho)\zeta_e^+(s, t), \quad \rho \in [0, 1]. \]

Given an element

\[ \eta = (\eta_S, \eta_T) \in \Omega^0,1(S^\nu, (u^\text{pre})^*TX^\nu)_{0,0,0} \oplus \Omega^1(T, u^\ast TL) \]

one obtains an element in the target space of the linearized operator $D_u$ of the broken map

\[ \tilde{\eta} = (\eta_v)_{v \in \text{Vert}(\Gamma)} \oplus \eta_T \in \bigoplus_{v \in \text{Vert}(\Gamma)} \Omega^0,1(S^\circ_v, u^\ast_v TX^\circ_v \pi_v) \]

as follows. The element $\tilde{\eta}$ is equal to $\eta$ in the tree components $T_e \subset C$ and in the complement of the neck region on the surface components $S_v \subset C$. On the neck region for an edge $e$, $\tilde{\eta}$ is defined by multiplication with the cutoff function and parallel transport:

\[ \eta_{e,+} = T_eu^+\beta(s - 1/2)\eta, \quad \eta_{e,-} = T_eu^-\beta(1/2 - s)\eta. \]

Since the broken map $u$ is regular and isolated, its linearized operator is bijective. We recall that the linearized operator is a map of Banach spaces (see (5.21))

\[ D_u : T_m\mathcal{M}_\Gamma \times \text{Map}(C, \mathcal{F})_{0,\lambda} \rightarrow \Omega^0,1(S, (u|S)^*TX) \oplus \Omega^1(T, (u|T)^*TL) \oplus ev_v^\ast TX/T\Delta. \]

Bijectivity of $D_u$ implies there is an inverse $(m, \xi)$ for the element $(\tilde{\eta}, 0) \in ev_v^\ast TX/T\Delta$. We write $\xi = ((\xi_v)_{v \in \text{Vert}(\Gamma)}, \xi_T)$. The vanishing of the last term in $D_u(m, \xi)$ means
that $\xi$ satisfies matching conditions at interior and boundary nodes, and the interior markings $z_i$ satisfy the divisor constraint:

$$\xi(z_i) \in T_{u_\pm(z_i)} D.$$  

The matching at interior nodes implies that for any interior edge $e = (v_+, v_-)$, the limit of $\xi_{v_+}$, $\xi_{v_-}$ at the cylindrical end $e$ is equal:

$$\xi_{v_+, e} = \xi_{v_-, e} =: \xi_e \in TX_{P(e)} \oplus t_{P(e), C}.$$  

We now define the approximate inverse by patching the inverse of the linearization of the broken map. Define $Q^\nu \eta$ equal to $\xi$ away from the neck regions $\cup_{e} [−\nu l_e/2, \nu l_e/2] \times S^1 \subset C^\nu$, and on the neck region for an edge $e$ by patching the solutions $\xi_{v_\pm}$ together using a cutoff function:

$$Q^\nu := \beta (−s + \frac{1}{4} \nu l_e) ((T_{e}^{u_\nu})^{-1} \xi_{v_\nu} (s + \nu l_e/4) − T^u \xi_e)$$

$$+ \beta (s + \frac{1}{4} \nu l_e) ((T_{e}^{u_\nu})^{-1} \xi_{v_+} (s − \nu l_e/4)) − T^u \xi_e + T^u \xi_e$$

$$\leq \Omega^0 (C^\nu, (u^\nu_{\text{pre}})^*TX)_{1,p,\lambda}.$$  

Next, we give an error estimate for the approximate inverse. We need to bound the quantity $D_{u^\nu_{\text{pre}}} Q^\nu \eta$ in the domain $C^\nu$. The second and third term arises from the derivative $d\beta$ of the cutoff function $\beta$. Of these we analyze the second term, which is supported in the interval $[−1/2, 1/2]$ in the neck. In this interval, the $\nu$-dependent constant $c$ such that

$$\|d\beta(s − \nu l_e/4)\xi_{v_-}\|_{L^p,\lambda}(C^\nu) < ce^{-\lambda \nu l_e/2}.$$
From (8.21), (8.22), (8.23) and a similar estimate for the third term, one obtains an estimate as in Fukaya-Oh-Ohta-Ono [29, 7.1.32], Abouzaid [1, Lemma 5.13]: For some constant $c > 0$, for any $\nu$

\[
(8.24) \quad \|D_{u^\nu}Q^\nu - \text{Id}\| < c \min_{e \in \text{Edge} \bullet (\Gamma)} (\exp(-\lambda \nu l_e/2), \exp(-(1-\lambda)\nu l_e/2)).
\]

It follows that for $\nu$ sufficiently large an actual inverse may be obtained from the Taylor series formula

\[
D_{u^\nu}^{-1} = Q^\nu (D_{u^\nu}Q^\nu)^{-1} = Q^\nu \sum_{k \geq 0} (I - Q^\nu D_{u^\nu})^k.
\]

The approximate inverse $Q^\nu$ is uniformly bounded for all $\nu$. For large enough $\nu$, (8.24) implies that $\|D_{u^\nu}Q^\nu - \text{Id}\| \leq \frac{1}{2}$, and so,

\[
(8.25) \quad \|D_{u^\nu}^{-1}\| \leq 2\|Q^\nu\| \leq c.
\]

8.5. Uniform quadratic estimate. We obtain a uniform quadratic estimate for the non-linear terms in the map cutting out the moduli space locally. We will prove that there exists a constant $c$ such that for all $\nu$

\[
(8.26) \quad \|D_{(m_1, \xi_1)} F_\nu (m_2, \xi_2) - D_{u^\nu} (m_2, \xi_2)\| \leq c \| (m_1, \xi_1) \|_{1,p,\lambda} \| (m_2, \xi_2) \|_{1,p,\lambda}.
\]

The quadratic estimate is proved using bounds on parallel transport. Let

\[
T^{\nu,x}_\xi (m, \xi) : \Lambda^{0,1}_{\nu} T^*_\xi C^\nu \otimes T_x X \to \Lambda^{0,1}_{\nu^\prime} T^*_\xi C^\nu \otimes T_{\exp(\xi)} X
\]

denote pointwise parallel transport. Consider its derivative

\[
DT^{\nu,x}_\xi (m, \xi, m_1, \xi_1; \eta) = \nabla_{\xi_{\xi_1}} T^{\nu,x}_\xi (m + tm_1, \xi + t\xi_1)\eta.
\]

For a map $u : C \to X$ we denote by $DT^\nu u$ the corresponding map on sections. By Sobolev multiplication (for which the constants are uniform because of the uniform cone condition on the metric on $C^\nu$ and uniform bounds on the metric on $X^\nu$) there exists a constant $c$ such that

\[
(8.27) \quad \|DT^\nu u (m, \xi, m_1, \xi_1; \eta)\|_{0,p,\lambda} \leq c \| (m, \xi) \|_{1,p,\lambda} \| (m_1, \xi_1) \|_{1,p,\lambda} \| \eta \|_{0,p,\lambda}.
\]

Differentiate the equation

\[
T^{\nu,x}_\xi (m, \xi) F_\nu (m, \xi) = \overline{\partial}_{\nu^\prime} (\exp_{u^\nu} (\xi))
\]

with respect to $(m_1, \xi_1)$ to obtain

\[
(8.28) \quad DT^{\nu,x}_\xi (m, \xi, m_1, \xi_1, F_\nu (m, \xi)) + T^{\nu}_\xi (m, \xi) (DF_\nu (m, \xi, m_1, \xi_1)) =
\]

\[
(D\overline{\partial})_{\nu^\prime} (\exp_{u^\nu} (\xi)) (Dj^\nu (m, m_1), D\exp_{\nu^\prime} (\xi_1)).
\]

Using the pointwise inequality

\[
|F_\nu (m, \xi)| < c |d\exp_{u^\nu} (\xi_1)| < c (|du^\nu| + |\nabla \xi|)
\]

for $m, \xi$ sufficiently small, the estimate (8.27) yields a pointwise estimate

\[
|T^{\nu,x}_\xi (m, \xi, m_1, \xi_1, F_\nu (m, \xi))| \leq c (|du^\nu| + |\nabla \xi|) \| (m, \xi) \| \| (m_1, \xi_1) \|.
\]
Hence

\[(8.29) \quad \|T_{u^\nu_{\text{pre}}}^{\nu}(\xi)^{-1}DT_{u^\nu_{\text{pre}}}^{\nu}(m, \xi, m_1, \xi_1, \mathcal{F}_\nu(m, \xi))\|_{0,p,\lambda} \leq c(1 + \|d\nu\|_{0,p,\lambda} + \|\nabla\xi\|_{0,p,\lambda})\|(m, \xi)\|_{L^\infty}\|(\xi_1, m_1)\|_{L^\infty}.
\]

It follows that

\[(8.30) \quad \|T_{u^\nu_{\text{pre}}}^{\nu}(\xi)^{-1}DT_{u^\nu_{\text{pre}}}^{\nu}(m, \xi, m_1, \xi_1, \mathcal{F}_\nu(m, \xi))\|_{0,p,\lambda} \leq c\|(m, \xi)\|_{1,p,\lambda}\|(m_1, \xi_1)\|_{1,p,\lambda}
\]
since the $W^{1,p}$ norm controls the $L^\infty$ norm by the uniform Sobolev estimates. Then, as in McDuff-Salamon [46, Chapter 10], Abouzaid [1] there exists a constant $c > 0$ such that for all $\nu$ sufficiently large, after another redefinition of $c$ we have

\[(8.31) \quad \|T_{u^\nu_{\text{pre}}}^{\nu}(\xi)^{-1}D\exp_{u^\nu_{\text{pre}}}^{\nu}(m_1, \xi_1) - D_{u^\nu_{\text{pre}}}^{\nu}(m_1, \xi_1)\|_{0,p,\lambda} \leq c\|(m, \xi)\|_{1,p,\lambda}\|(m_1, \xi_1)\|_{1,p,\lambda}.
\]

Combining these estimates and integrating completes the proof of claim (8.26).

8.6. Picard iteration. We apply the implicit function theorem to obtain an exact solution. We recall a version of the Picard’s lemma [46, Proposition A.3.4].

**Lemma 8.5.** Let $X$ and $Y$ be Banach spaces, $U \subset X$ be an open set containing 0, and $f : U \to Y$ be a smooth map. Suppose $df(0)$ is invertible with inverse $Q : Y \to X$. Suppose $c$ and $\epsilon > 0$ are constants such that $\|Q\| \leq c$, $B_\epsilon \subset U$, and

\[\|df(x) - df(0)\| \leq \frac{1}{2c} \quad \forall x \in B_\epsilon(0).
\]

Suppose $f(0) \leq \frac{\epsilon}{c}$. Then, there is a unique point $x_0 \in B_\epsilon$ satisfying $f(x_0) = 0$.

Picard’s Lemma and the estimates (8.19), (8.25), (8.26) imply the existence of a solution $(m(\nu), \xi(\nu))$ to the equation

\[\mathcal{F}_\nu(m(\nu), \xi(\nu)) = 0
\]

for each $\nu$. The map

\[u_\nu := \exp_{u^\nu_{\text{pre}}}^{\nu}(\xi(\nu))
\]
is a $(f(m(\nu)), J^\nu)$-holomorphic map to $X^\nu$. Additionally, there is a $\nu$-independent constant $\epsilon > 0$ such that $(m(\nu), \xi(\nu))$ is the unique zero of $\mathcal{F}_\nu$ in an $\epsilon$-neighbourhood of $((C^\nu, J^\nu), u^\nu_{\text{pre}})$ with respect to the $g_{\text{glue}}$-norm on $m(\nu)$ and the weighted Sobolev norm $W^{1,p,\lambda}$ on $\xi(\nu)$.

8.7. Surjectivity of gluing. We show that the gluing construction gives a bijection. Note that any family $[u^\nu : C^\nu \to X^\nu]$ converges to a broken map $u : C \to \mathfrak{X}$ by Theorem 7.1. To prove the bijection we must show that any such family of maps is in the image of the gluing construction. Since the implicit function theorem used to construct the gluing gives a unique solution in a neighbourhood, it suffices to show that the maps $u^\nu$ are close, in the Sobolev norm used for the gluing construction, to the approximate solution $u^\nu_{\text{pre}}$ defined by (8.7).

We first show that the domain curves of the converging sequence of maps are close enough to the domains of the approximate solution with respect to the cylindrical
metric \(g_{\Gamma_{\text{glue}}} \) from (8.10). In the definition of Gromov convergence, the convergence of domains implies that \(C_{\nu}' \to C \) in the compactified moduli space \(\overline{\mathcal{M}}_{\Gamma_{\text{glue}}} \). We additionally need to prove that the distance \(d_{g_{\Gamma_{\text{glue}}}}(C_{\nu}', C_{\nu}) \to 0 \) where the metric \(g_{\Gamma_{\text{glue}}} \) is cylindrical in the non-compact ends of \(\overline{\mathcal{M}}_{\Gamma_{\text{glue}}} \). The convergence in \(\overline{\mathcal{M}}_{\Gamma_{\text{glue}}} \) implies

\[
\pi_\Gamma(C_{\nu}') \to C \quad \text{in } \mathcal{M}_\Gamma.
\]

By assumption the limit map \(u \) does not have any tropical symmetry. Therefore, the translation sequence \(t_\nu \) is uniquely determined by the tropical graph of \(u \) and coincides with the translations used for gluing. By (Thin cylinder convergence), for any interior edge \(e = (v_+, v_-) \in \text{Edge}_\bullet(\Gamma) \), the logarithmic neck lengths \(l'_\nu(e) + i\theta'_\nu(e) \) of the curves \(C_{\nu}' \) satisfy

\[
\lim_{\nu \to \infty} \theta'_\nu(e) = 0, \quad \lim_{\nu \to \infty} (t_\nu(v_+) - t_\nu(v_-) - \mathcal{T}(e)l'_\nu(e)) = 0.
\]

The logarithmic neck length of \(C_{\nu} \) is \(\nu l_e \), which satisfies the relation

\[
t_\nu(v_+) - t_\nu(v_-) = \mathcal{T}(e)\nu l_e.
\]

Therefore, \(l'_\nu(e) - \nu l_e \to 0 \), and \(d_{g_{\Gamma_{\text{glue}}}}(C_{\nu}, C_{\nu}') \to 0 \) as \(\nu \to \infty \). In addition there is a biholomorphism

\[
(8.33) \quad \phi_\nu : (C_{\nu}, i'(C_{\nu}')) \to C_{\nu}'.
\]

Next, we show that the maps in the converging sequence are close enough to the approximate solutions. Via the identification (8.33), we view \(u_{\nu}' \) as a map on \(C_{\nu} \). We need to bound the section \(\xi_{\nu}' \in \Omega^0(C_{\nu}, (u_{\nu}^{\text{pre}})'TX_{\nu}) \) defined by the equation \(u_{\nu}' = \exp_{u_{\nu}^{\text{pre}}} \xi_{\nu}' \) in the weighted Sobolev norm (8.14). Consider a neck region in \(C_{\nu} \) corresponding to an edge \(e \) with coordinates

\[
(s_e, t_e) \in \left[ -\frac{\nu l_e}{2}, \frac{\nu l_e}{2} \right] \times \mathbb{R}/2\pi\mathbb{Z}.
\]

Denote the midpoint of the neck as

\[
0_e := \{(s_e, t_e) = (0, 0)\} \in C_{\nu}'.
\]

In the neck region, the maps \(u_{\nu}^{\text{pre}} \) and \(u_{\nu}' \) are equal to a vertical cylinder perturbed by a quantity that decays exponentially in the middle of the cylinder. The vertical cylinder is determined by \(u_{\nu}^{\text{pre}}(0_e) \) resp. \(u_{\nu}'(0_e) \). The sequence \(u_{\nu}^{\text{pre}}(0_e) \) converges to \(x_e \) because of the asymptotic decay of the sections \(\xi^\pm \). The sequence \(u_{\nu}'(0_e) \) converges to \(x_e \) by (Thin cylinder convergence). Indeed, since the complex structure \(j'(C_{\nu}) \) is standard on a truncation \([-\frac{\nu l_e}{2} - L, \frac{\nu l_e}{2} - L] \times S^1 \) of the neck (see (8.11)), and the mid point of the cylinder is preserved by the biholomorphism \(\phi \) in (8.33), (Thin cylinder convergence) is applicable with the coordinates \((s_e, t_e)\). On the neck region, the section \(\xi_{\nu}' \) and its derivatives decay exponentially:

\[
|D^k \xi_{\nu}'(s_e, t_e)| \leq c e^{-\nu l_e/2} |s_e|, \quad k \in \{0, 1\}.
\]

This inequality follows from the decay of the terms \(\xi_{\nu}^\pm \) in the definition of \(u_{\nu}^{\text{pre}} \), and the breaking annulus lemma applied to \(u_{\nu}' \). Consequently \(\|\xi_{\nu}'\|_{W^{1,p,\lambda}} \) can be made small enough by taking a large \(\nu \) and shrinking the neck by a fixed amount: that is, we decrease the cylinder length to \(\nu l_e - C \) where \(C \) is a constant independent of \(\nu \).
Next we consider the complement of the neck regions. Here, the sequences \( u'_\nu \) and \( u'^{\text{pre}}_\nu \) uniformly converge to \( u \). So, by taking \( \nu \) large enough the maps \( u'^{\text{pre}}_\nu \) and \( u'_\nu \) are \( W^{1,p} \)-close enough in the complement of the neck regions.

8.8. Tubular neighbourhoods. In this section we state a gluing result, without proof, for gluing pseudoholomorphic curves at disk nodes. This result can be viewed as a special case of the gluing theorem 8.1 where both components are in the same component of the broken manifold, but generalized to include Lagrangian boundary conditions.

**Theorem 8.6.** Let \( \underline{p} = (p_\Gamma)_\Gamma \) be a coherent regular perturbation datum for all types \( \Gamma \). Suppose \( \Gamma \) is a type of broken treed disks and \( \underline{x} \in \mathcal{I}(L)^{n+1} \) is a set of limits for boundary leaves such that \( i(\Gamma, \underline{x}) = 1 \).

(a) (Tubular neighbourhoods) If \( \Gamma \) is obtained from \( \Gamma' \) by collapsing an edge of \( \text{Edge}_{\circ}^-(\Gamma') \) or making an edge or weight finite/non-zero or by gluing \( \Gamma' \) at a breaking, then the stratum \( \mathcal{M}_{\Gamma'}(\underline{x}, L, D) \) has a tubular neighbourhood in \( \mathcal{M}_{\Gamma}(\underline{x}, L, D) \); and

(b) ( Orientations) there exist orientations on \( \mathcal{M}_{\Gamma}(\underline{x}, L, D) \) compatible with the morphisms (Cutting an edge) and (Collapsing an edge/Making an edge/weight finite/non-zero) in the following sense:

(i) If \( \Gamma \) is obtained from \( \Gamma' \) by (Cutting an edge) then the isomorphism \( \mathcal{M}_{\Gamma'}(\underline{x}, L, D) \to \mathcal{M}_{\Gamma}(\underline{x}, L, D) \) is orientation preserving.

(ii) If \( \Gamma \) is obtained from \( \Gamma' \) by (Collapsing an edge) or (Making an edge/weight finite/non-zero) then the inclusion \( \mathcal{M}_{\Gamma'}(\underline{x}, L, D) \to \mathcal{M}_{\Gamma}(\underline{x}, L, D) \) has orientation (using the decomposition

\[
T\mathcal{M}_{\Gamma}(\underline{x}, L, D)\big|\mathcal{M}_{\Gamma'}(\underline{x}, L, D) \cong \mathbb{R} \oplus T\mathcal{M}_{\Gamma'}(\underline{x}, L, D)
\]

and the outward normal orientation on the first factor) given by a universal sign depending only on \( \Gamma, \Gamma' \).
9. Broken Fukaya algebras

In this section, we describe $A_\infty$-algebras structures defined by counting treed holomorphic disks on broken and unbroken manifolds, and show that they are equivalent up to homotopy.

9.1. $A_\infty$ algebras. The set of treed holomorphic disks has the structure of an $A_\infty$-algebra. $A_\infty$-algebras were introduced by Stasheff [64] in order to capture algebraic structures on the space of cochains on loop spaces. We follow the sign convention in Seidel [61]. Let $g > 0$ be an even integer. A $\mathbb{Z}_g$-graded $A_\infty$ algebra consists of a $\mathbb{Z}_g$-graded vector space $A$ together with for each $d \geq 0$ a multilinear degree zero composition map $m_d : A^d \to A[2 - d]$ satisfying the $A_\infty$-associativity equations [61, (2.1)]

\[
0 = \sum_{j,k\geq 0, j+k\leq d} (-1)^{j+\sum_{i=1}^{j}[a_i]} m_{d-k+1}(a_1, \ldots, a_j, m_k(a_{j+1}, \ldots, a_{j+k}, a_{j+k+1}, \ldots, a_d))
\]

for any $d \geq 0$ and any tuple of homogeneous elements $a_1, \ldots, a_d$ with degrees $[a_1], \ldots, [a_d] \in \mathbb{Z}_g$. One of the first of these associativity relations is

\[
m_2(m_0, a) - (-1)^{|a|} m_2(a, m_0) + m_1(m_1(a)) = 0, \quad \forall a \in A.
\]

The signs are the shifted Koszul signs, that is, the Koszul signs for the shifted grading in which the structure maps have degree one as in Kontsevich-Soibelman [39]. The notation $[2 - d]$ denotes a degree shift by $2 - d$, so that without the shifting $m_1$ has degree 1, $m_2$ has degree 0 etc. The element $m_0(1) \in A$ (where $1 \in \Lambda$ is the unit) is called the curvature of the algebra. The $A_\infty$ algebra $A$ is flat if the curvature vanishes. The cohomology of a flat $A_\infty$ algebra $A$ is defined by

\[
H(m_1) = \frac{\ker(m_1)}{\im(m_1)}.
\]

The algebra structure on $H(m_1)$ is given by

\[
[a_1 a_2] = (-1)^{|a_1|} [m_2(a_1, a_2)].
\]

A strict unit for $A$ is an element $e_A \in A$ such that

\[
m_2(e_A, a) = a = (-1)^{|a|} m_2(a, e_A), \quad m_d(\ldots, e_A, \ldots) = 0, \forall d \neq 2.
\]

A strictly unital $A_\infty$ algebra is an $A_\infty$ algebra equipped with a strict unit. Cohomology can be defined for a curved strictly unital $A_\infty$-algebra if the curvature is a multiple of the unit : $m_0(1) \in \Lambda e_A$, because in this case, $(m_1)^2 = 0$ by (9.2). More generally, the cohomology exists for any solution to the projective Maurer-Cartan equation [29].

Similarly one has homotopy notions of algebra morphisms. Let $A_0, A_1$ be $A_\infty$ algebras. An $A_\infty$ morphism $\mathcal{F}$ from $A_0$ to $A_1$ consists of a sequence of linear maps

\[
\mathcal{F}^d : A_0^d \to A_1[1 - d], \quad d \geq 0
\]
such that the following holds:

\[
\sum_{i+j \leq d} (-1)^{i+j} \sum_{j=1}^{i} |a_j| \mathcal{F}^{d-j+1}(a_1, \ldots, a_i, m_{A_0}^j(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_d) = \sum_{i_1 + \ldots + i_m = d} m_{A_1}^m(\mathcal{F}^{i_1}(a_1, \ldots, a_{i_1}), \ldots, \mathcal{F}^{i_m}(a_{i_1+1} + \ldots + i_{m-1} + 1, \ldots, a_d))
\]

where the first sum is over integers \(i, j\) with \(i + j \leq d\), the second is over partitions \(d = i_1 + \ldots + i_m\). An \(A_\infty\) morphism \(\mathcal{F}\) is \textit{unital} if and only if

\[
\mathcal{F}^{1}(e_0) = e_1, \quad \mathcal{F}^{k}(a_1, \ldots, a_i, e_0, a_{i+2}, \ldots, a_k) = 0
\]

for every \(k \geq 2\) and every \(0 \leq i \leq k - 1\), where \(e_0\) resp. \(e_1\) is the strict unit in \(A_0\) resp. \(A_1\).

9.2. Composition maps. In this section, we describe an \(A_\infty\) algebra whose composition maps are given by treed holomorphic maps in a broken manifold, with boundary in a Lagrangian contained in the complement of the normal crossing divisor. The result is summarized in Theorem 9.4 at the end of the section.

We recall the count of pseudoholomorphic treed disks that defines the structure coefficients of the Fukaya algebra. Let \(q\) be a formal variable and \(\Lambda\) the universal Novikov field of formal sums with rational coefficients

\[
\Lambda = \left\{ \sum_{i} c_i q^{\alpha_i} \mid c_i \in \mathbb{C}, \alpha_i \in \mathbb{R}, \alpha_i \to \infty \right\}
\]

Denote by \(\Lambda_{\geq 0}\) resp. \(\Lambda_{> 0}\) the subalgebra with only non-negative resp. positive exponents. Denote by

\[
\Lambda^\times = \left\{ c_0 + \sum_{i > 0} c_i q^{\alpha_i} \subset \Lambda_{\geq 0} \mid c_0 \neq 0 \right\}
\]

the subgroup of formal power series with invertible leading coefficient.

Lagrangians will be equipped with additional data called \textit{brane structures}. We assume Lagrangians are compact, connected and oriented. A brane structure consists of a relative spin structure and a local system, which is an element

\[
y \in \mathcal{R}(L) = \text{Hom}(\pi_1(L), \Lambda^\times).
\]

When working with multiple Lagrangian submanifolds, one needs a grading on each of the Lagrangians, see Seidel [59]. We do not use gradings in this paper. For a Lagrangian \(L\) with a brane structure define the space of Floer cochains

\[
CF^{\text{geom}}(L) := \bigoplus_{d \in \mathbb{Z}_g} CF^d(L), \quad CF^d(L) := \bigoplus_{x \in \mathcal{I}_d(L)} \Lambda(x)
\]

where \(\mathcal{I}_d(L)\) is the set of index \(d\) critical points of the Morse function \(F : L \to \mathbb{R}\), see Definition ???. The composition maps are as follows.
Definition 9.1. (Composition maps) For admissible perturbation data \((p_\Gamma)\) on the manifold \(X\) define

\[ m_{d(\cdot)} : (CF_{\text{geom}}(L))^{\otimes d(\cdot)} \to CF_{\text{geom}}(L) \]

on generators by

\[ m_{d(\cdot)}(x_1, \ldots, x_{d(\cdot)}) = \sum_{x_0, u \in \tilde{M}(X, L, P_\gamma, x)_{d(\cdot)}} w(u) x_0 \]

where the sum is over all types \(\Gamma\) of maps with simple intersections with the stabilizing divisor and with \(d(\cdot)\) incoming boundary edges, and

\[ w(u) := (-1)^{\bigtriangledown}(d_\cdot(\Gamma)!)^{-1} y([\partial u]) \epsilon(u) q^{A(u)}. \]

The terms in (9.8) are as below:

(a) \(\bigtriangledown = \sum_{i=1}^{d(\cdot)} i |x_i|\),
(b) \(y([\partial u]) \in \Lambda^\times\) is the evaluation of the local system \(y \in \mathcal{R}(L)\) on the homotopy class of loops \([\partial u] \in \pi_1(L)\) defined by going around the boundary of each disk component in the treed disk once,
(c) \(d_\cdot(\Gamma)\) the number of interior markings on the map \(u\),
(d) \(\epsilon(u) \in \{\pm 1\}\) is the orientation sign, see [13, Definition 4.8].

The \(A_\infty\) relation follows from the description of the true boundary of the moduli space, see [13, Theorem 4.31].

Theorem 9.2. (\(A_\infty\) algebra for a Lagrangian) For any admissible perturbation system \(p = (p_\Gamma)_\Gamma\) on the manifold \(X\), the maps \((m_{d(\cdot)}), d(\cdot) \geq 0\) satisfy the axioms of a (possibly curved) \(A_\infty\) algebra \(CF_{\text{geom}}(L)\).

Composition maps on a broken manifold are defined analogously. Assume that \(L \subset \mathcal{X}\) is a Lagrangian submanifold that is contained in a single piece of \(\mathcal{X}\) and does not intersect boundary divisors. The Floer cochains and the brane structure on \(L\) are defined as in the unbroken case:

\[ CF_{\text{brok}}^d(L) := \bigoplus_{d \in \mathbb{Z}_g} CF_{\text{brok}}^d(L), \quad CF_{\text{brok}}(L) := \bigoplus_{x \in \mathcal{I}_d(L)} \Lambda(x). \]

Definition 9.3. (Composition maps for the broken Fukaya algebra) For admissible perturbation data \((p_\Gamma)\) define

\[ m_{d(\cdot)}^{\text{brok}} : (CF_{\text{brok}}^d(L))^{\otimes d(\cdot)} \to CF_{\text{brok}}^d(L) \]

on generators by

\[ m_{d(\cdot)}(x_1, \ldots, x_{d(\cdot)}) = \sum_{x_0, u \in \tilde{M}_{\text{brok}}(X, L, D, x)_{d(\cdot)}} w(u) x_0 \]

where the combinatorial type \(\Gamma\) of the broken map \(u\) ranges over all rigid types (see Definition 7.37) with \(d(\cdot)\) boundary inputs, and \(w(u)\) is as in (9.8).
Theorem 9.4. (A∞ algebra for a Lagrangian) For any admissible perturbation system $p = (p_\Gamma)$ on the broken manifold $X$, the maps $(m_{d(\sigma)}^{\text{break}})_{d(\sigma) \geq 0}$ satisfy the axioms of a (possibly curved) $A_\infty$ algebra $\mathcal{CF}^{\text{geom}}(L)$.

The $A_\infty$ relation follows from the description of the true boundary of the moduli space, see Proposition 7.39.

9.3. Homotopy units. In the Fukaya algebra constructed in the previous section, a homotopy unit construction can be applied to produce a strictly unital $A_\infty$-algebra. Recall that the Morse function $F: L \to \mathbb{R}$ used in the construction of $\mathcal{CF}^{\text{geom}}(L)$ is assumed to have a unique maximum point denoted $x^\bullet \in \text{crit}(F)$. In an idealized situation where domain-dependent perturbations are not required, $\langle x^\bullet \rangle$ is a strict unit for $\mathcal{CF}^{\text{geom}}(L)$. This is because a boundary marked point mapping to the unstable locus of $x^\bullet$ is an empty constraint, and such a marking can be forgotten without affecting the disk. In our setting, marked points can not be forgotten because domain-dependent perturbations depend on them. The homotopy unit construction is a way of enhancing the Fukaya algebra so that the perturbation system admits forgetful maps.

Theorem 9.5. There exists a strictly unital $A_\infty$ structure on the vector space

$$\mathcal{CF}(L) := \mathcal{CF}^{\text{geom}}(L) \oplus \Lambda x^\ast[1] \oplus \Lambda x^\circ,$$

with gradings $|x^\ast| = 0$, $|x^\circ| = -1$ and whose composition maps are such that $x^\circ$ is a strict unit, $\mathcal{CF}^{\text{geom}}(L) \subset \mathcal{CF}(L)$ is a $A_\infty$ sub-algebra, and

$$m_1(x^\ast) = x^\circ - x^\ast \mod \Lambda_{>0}.$$

The composition maps are defined by counts of weighted $p$-adapted disks, where the perturbation datum $p$ is an extension of the perturbation datum used to define $\mathcal{CF}^{\text{geom}}(L)$.

The condition that $x^\circ$ is a strict unit determines all $A_\infty$ structure maps involving occurrences of $x^\circ$. In the following geometric construction of a homotopy unit, the axioms are designed keeping this fact in mind.

Definition 9.6. (a) (Weightings) A weighting of a treed disk $C = S \cup T$ of type $\Gamma$ consists of a partition of the boundary semi-infinite edges

$$\text{Edge}^*(\Gamma) \sqcup \text{Edge}^\circ(\Gamma) \sqcup \text{Edge}^\ast(\Gamma) = \text{Edge}_{\infty\to}(\Gamma)$$

into weighted resp. forgettable resp. unforgettable, and a weight on semi-infinite edges $\rho : \text{Edge}_{\infty\to}(\Gamma) \to [0, \infty]$ satisfying

$$\rho(e) \in \begin{cases} \{0\} & e \in \text{Edge}^\ast(\Gamma) \\ [0, \infty] & e \in \text{Edge}^\circ(\Gamma) \\ \{\infty\} & e \in \text{Edge}^*(\Gamma). \end{cases}$$

The weighting $\rho$ satisfies the following axioms.
(i) (Infinite segment) An infinite segment can only have labels
\[ \vartriangleright \rightarrow \vartriangleright, \ \vartriangleright \rightarrow \vartriangleright, \ \text{or} \ \vartriangleright \rightarrow \vartriangleright. \]

In the first two cases, the input has weight \( \rho(e) = \infty \) resp. 0.

(ii) (Outgoing edge axiom) A disk output \( e_0 \in \text{Edge}_0(\Gamma) \) can be weighted only if the disk has exactly one weighted input \( e_1 \in \text{Edge}^\vartriangleright(\Gamma) \), all the other inputs \( e_i \in \text{Edge}(\Gamma), i \neq 1 \) are forgettable, and there are no interior leaves, \( \text{Edge}_\bullet(\Gamma) = \emptyset \). In this case, the output \( e_1 \) has the same weight \( \rho(e_1) = \rho(e_0) \) as the weighted input \( e_0 \). A disk output \( e_0 \) can be forgettable only if all the inputs are forgettable, and there are no interior leaves. In all the other cases, the output of a disk is unforgettable.

(b) (Stability) A weighted treed disk \( u : C \rightarrow X \) is stable if each component \( C_v \subset C \) is stable as a treed disk, or is an infinite segment with labels \( \vartriangleright \rightarrow \vartriangleright \) or \( \vartriangleright \rightarrow \vartriangleright \).

(c) (Isomorphism) Two weighted treed disks \( C \) and \( C' \) are isomorphic if there is an isomorphism of treed disks \( \phi : C \rightarrow C' \), the edge labels are identical, and the following is true.

(i) If the output edge is not weighted, then the weights on the inputs of \( C_1 \) and \( \phi(C_1) \) are equal;
(ii) if the output edge \( e_0 \) is weighted, then the weights on the inputs are equal up to scalar multiplication, i.e.

\[
\exists \lambda \in (0, \infty), \ \forall e \in \text{Edge}_0 \rightarrow (C) \setminus \{e_0\} \rho(e) = \lambda \rho(\phi(e)).
\]

Consequently, if the output edge is weighted, the weights do not matter.

The type of a weighted treed curve is given by the type of the treed curve, and the labels \{\( \vartriangleright, \vartriangleright, \vartriangleright \)\} at the inputs and outputs, and whether the weight at any vertex is zero, infinite or neither. The moduli space \( M_\Gamma \) of weighted treed disks can be identified with \( \left\{ M_{\Gamma'} \times [0, \infty)^{|\text{Edge}^\vartriangleright(\Gamma)|}, \text{if the output label is not} \vartriangleright \right\} M_{\Gamma'} \), if the output edge is \( \vartriangleright \),

where \( \Gamma' \) is the type of treed disk obtained by forgetting the weighting. If the type \( \Gamma \) is \( \vartriangleright \rightarrow \vartriangleright \) resp. \( \vartriangleright \rightarrow \vartriangleright \), then \( M_\Gamma \) is a point.

The (Cutting edges) morphism has some additional features for weighted treed disks. Given a type \( \Gamma \) of a weighted treed disk, suppose \( \Gamma_+, \Gamma_- \) (here \( \Gamma_+ \) contains the root of \( \Gamma \)) are the treed disk types produced by cutting an edge \( e \in \text{Edge}_{0-}(\Gamma) \) in \( \Gamma \), and \( e_\pm \in \text{Edge}_{0\rightarrow}(\Gamma_\pm) \) be the pair of new edges created by the cutting. The label \( \{\vartriangleright, \vartriangleright, \vartriangleright \} \) and the weight at \( e_\pm \) are same, and is determined by the (Outgoing edges axiom) applied to \( \Gamma_- \). In this case there is a new type of (Cutting edges) morphism:

(Cutting a weighted input edge) Suppose \( e \in \text{Edge}^\vartriangleright(\Gamma) \) is an input, and \( \rho(e) = 0 \) resp. \( \infty \). Cutting \( e \) produces two types: \( \Gamma_- \) is an infinite segment \( \vartriangleright \rightarrow \vartriangleright \) resp. \( \vartriangleright \rightarrow \vartriangleright \), and \( \Gamma_+ \) is \( \Gamma \) with \( e \) as an unforgettable resp. forgettable edge, see Figure 17.
Perturbation datum defined on the moduli space of weighted treed disks is coherent with respect to (Collapsing of edges, making an edge length/weight finite or non-zero) and (Cutting of edges) for boundary edges as earlier. There is one additional coherence condition.

(Forgetting edges) Suppose $e$ is an input edge in a weighted treed type $\Gamma$ that is either forgettable or weighted with infinite weight, and $\Gamma'$ is the type obtained by forgetting $e$. Then $p_\Gamma$ is the pullback of $p_{\Gamma'}$.

Let $p = (p_\Gamma)_{\Gamma}$ be a coherent perturbation datum for weighted treed disks. An adapted $p_\Gamma$-map is a map $u : C \to X$ if it is adapted in the sense of treed holomorphic disks, and additionally satisfies the following.

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where
\[ w(u) = (-1)\bigtriangledown (d(\bullet)!)^{-1} y(u) q A(u) \epsilon(u), \quad \bigtriangledown = \sum_{i=1}^{d(\bullet)} i |x_i|, \]
and the the sum ranges over all rigid types \( \Gamma \), and the weight for any weighted input is finite non-zero in the type \( \Gamma \).

**Proof of Theorem 9.5.** For a one-dimensional moduli space of weighted maps of rigid type, the true boundary strata contain one of the following configurations: a configuration with a weight 0 or \( \infty \) at a weighted input which is equivalent to the broken configuration in Figure 17, a boundary node with a broken segment, or a broken Morse trajectory. These configurations exactly correspond to the terms in the \( A_\infty \)-associativity relations, and so \( CF(L) \) is an \( A_\infty \)-algebra. The geometric part \( CF_{\text{geom}}(L) \) is a sub-algebra because if the inputs to a treed disk are unforgettable, the output is also unforgettable.

The element \( x^r \) is a strict unit for the following reasons. For \( d(\bullet) > 2 \), we have
\[ m_{d(\bullet)}(\ldots, x^r, \ldots) = 0 \]
because the input \( x^r \) is an empty constraint, and can be forgotten because the perturbation satisfies the (Forgetting edges) axiom. The term \( m_1(x^r) \) is also zero: any disk that is counted has interior markings, and therefore, placing the marked point \( x^r \) adds one to the dimension, and therefore the moduli space is not zero-dimensional. Finally, by the same argument, \( m_2(x^r, y) \) and \( m_2(y, x^r) \) do not count any disk with interior markings. The only contributions are from constant disks. We conclude that both terms are equal to \( \pm y \), for any generator \( y \). \( \square \)

**Remark 9.7.** (Leading order term in the first composition map) Constant trajectories \( \bullet \rightarrow x, \bullet \rightarrow \bullet \) contribute to the first composition map \( m_1(x^r) \), and the choice of orientation [13, Remark 4.23] implies
\[
(9.13) \quad m_1(x^r) = x^r - x^\circ + \sum_{x_0, [u] \in \mathcal{M}_\Gamma(L, D, x^r, x_0), E(u) > 0} (-1)\bigtriangledown (d(\bullet)!)^{-1} q E(u) \epsilon(u) y(u) x_0.
\]
This formula is similar to that in Fukaya-Oh-Ohta-Ono [29, (3.3.5.2)].

9.4. **Quilted disks.** Morphisms between Fukaya algebras are defined by counts of quilted holomorphic disks. The domains of these curves are *quilted disks*, which we describe in this section.

The terms in the \( A_\infty \) morphism axiom correspond to codimension one cells in a cell complex called the multiplihedron introduced by Stasheff [64]. Stasheff’s definition identifies the \( n \)-multiplihedron as the cell complex whose vertices correspond to total bracketings of \( x_1, \ldots, x_{d(\bullet)} \), together with the insertion of expressions \( f(\cdot) \) so that every \( x_j \) is contained in an argument of some \( f \). For example, the second multiplihedron is an interval with vertices \( f(x_1) f(x_2) \) and \( f(x_1 x_2) \). A geometric realization of this polytope was given by Boardman-Vogt [8] in terms of metric ribbon trees. A different realization of the multiplihedron is the moduli space of stable quilted disks in Ma’u-Woodward [44].
9.4.1. Quilted disks.

**Definition 9.8.** (Quilted disks and spheres)

(a) A **quilted disk** is a datum \((S, Q, x_0, \ldots, x_{d(0)}) \in \partial S\) consisting of a marked complex disk \((S, x_0, \ldots, x_{d(0)}) \in \partial S\) (the points are required to be in counterclockwise cyclic order) together with a circle \(Q \subset S\) (here we take \(S\) to be a ball in the complex plane, so the notion of circle makes sense) tangent to the 0-th marking \(x_0\). An **isomorphism** of quilted disks from \((S, Q, x_0, \ldots, x_{d(0)}) \rightarrow (S', Q', x_0', \ldots, x_{d(0)}')\) is an isomorphism of holomorphic disks \(S \rightarrow S'\) mapping \(Q\) to \(Q'\) and \(x_0, \ldots, x_{d(0)}\) to \(x_0', \ldots, x_{d(0)}'\).

(b) (Affine structures) An **affine structure** on a disk \(S\) with boundary point \(z_0 \in \partial S\) is an isomorphism with a half-space \(\phi : S - \{z_0\} \rightarrow \mathbb{H}\). Two affine structures \(\phi, \phi' : S - \{z_0\} \rightarrow \mathbb{H}\) are considered equivalent if \(\exists \zeta \in \mathbb{R}, \forall z \in S, \phi(z) = \phi'(z) + \zeta\).

A quilting is equivalent to an affine structure, by taking the quilting to be \(Q = \{\text{Im}(z) = 1\}\).

(c) (Quilted spheres) In this context the notion of quilted disk admits a natural generalization to the notion of a **quilted sphere**: a marked sphere \((C, (z_0, \ldots, z_{d(0)}))\) equipped with an isomorphism \(\phi : C - \{z_0\} \rightarrow \mathbb{C}\) to the affine line \(\mathbb{C}\). Again, two such isomorphisms \(\phi, \phi'\) are considered equivalent if they differ by a translation: \(\phi(z) = \phi'(z) + \zeta\) for some \(\zeta \in \mathbb{C}\).

(d) (Combinatorial types) The combinatorial type \(\Gamma\) of a quilted nodal marked disk \((S, Q, x)\) is defined similar to the combinatorial type of a nodal marked disk disk. The set of vertices \(\text{Vert}(\Gamma)\) has a distinguished subset \(\text{Vert}_1(\Gamma)\) of colored vertices corresponding to the quilted components. Thus the unique non-self-crossing path \(\gamma_e\) from the root edge \(e_0\) of the tree to any leaf \(e\) is required to pass through exactly one colored vertex \(v \in \text{Vert}_1(\Gamma)\) of colored vertices corresponding to the quilted components. Let \(\text{Vert}_0(\Gamma) = \text{Vert}(\Gamma) - \text{Vert}_1(\Gamma)\). A nodal quilted disk is **stable** if there are no non-trivial automorphisms \(\text{Aut}(S) - \{1\}\), or equivalently, each component \(S_v, v \in \text{Vert}(\Gamma)\) has no automorphisms. In the case of sphere components \(S_v, v \in \text{Vert}(\Gamma)\) has no automorphisms. In the case of combined

The moduli space of stable quilted disks with interior and boundary markings is a compact cell complex. As the interior and boundary markings go to infinity, they bubble off onto either quilted disks or quilted spheres. The case of combined
boundary and interior markings is a straight-forward generalization of the boundary and interior cases treated separately in [44].

9.4.2. Quilted treed disks. There is a combined moduli space which includes both quilted disk, spheres, and (possibly broken) treed segments.

**Definition 9.9.** A quilted treed disk $C$ is obtained from a quilted nodal disk $S$ of type $\Gamma$ by replacing each boundary marking $w_e, e \in \text{Edge}_{\gamma_v}(\Gamma)$ with a semi-infinite treed segment $T_e, e \in \text{Edge}_{\gamma_v}(\Gamma)$, and each boundary node $w_v, e \in \text{Edge}_{\gamma_v}(\Gamma(S))$ with a treed segment with finite end-points satisfying a balancing condition: The sum of edge lengths $\pm \ell(e)$ on the path $e \in \gamma_v$ connecting any two colored vertices $v \in \text{Vert}^1(\Gamma)$ vanishes, with sign $\pm$ at $e$ if the orientation of the path agrees with the orientation on the tree.

**Definition 9.10.** The combinatorial type $\Gamma(C)$ of a quilted treed disk $C = S \cup T$ is the combinatorial type $\Gamma(S)$ of the surface with the additional labellings of the number of breakings $b(e)$ of each edge $T_e$. A quilted treed disk $C$ is stable if

1. any unquilted surface component $S_v, v \in \text{Vert}^0(\Gamma)$ has at least three special points,
2. any quilted surface component $S_v, v \in \text{Vert}^1(\Gamma)$ has at least two special points,
3. and there are no infinite tree segments $T_e \cong \mathbb{R}, e \in \text{Edge}(\Gamma)$.

Let $\overline{M}_{d\downarrow, d\uparrow}$ denote the moduli space of stable marked quilted treed disks $u : C \to X$ with $n$ boundary leaves and $m$ interior leaves. See Figure 18 for a picture of $\overline{M}_{2,0}^d$. The quilted disks $S_v \subset S, v \in \text{Vert}^1(\Gamma)$ are those with two shadings; while the ordinary disks $S_v, v \notin \text{Vert}^1(\Gamma)$ have either light or dark shading depending on whether they can be connected to the zero-th edge without passing a colored vertex. The hashes on the line segments $T_e$ indicate breakings.

![Figure 18. Moduli space of stable quilted treed disks](image)

9.4.3. Orientations. Orientations of the moduli space of quilted treed disks are defined as follows. Each main stratum of $\overline{M}_{d\downarrow, d\uparrow}$ can be oriented using the isomorphism of the stratum made of quilted treed disks having a single disk with $\mathbb{R}$ times $\overline{M}_{d\downarrow, d\uparrow}$, the extra factor corresponding to the quilting parameter. The boundary
of the moduli space is naturally isomorphic to a union of moduli spaces:

\[(9.14) \quad \partial \overline{M}_{d(\bullet),d(\circ)} \cong \bigcup_{d(\circ)_1+d(\circ)_2=d(\circ)} \left( \overline{M}_{d(\bullet)-i+1,d(\circ)_1} \times \overline{M}_{i,d(\circ)_2} \right) \]

\[
\bigcup \bigcup_{i_1+\ldots+i_r=d(\circ)+\sum d(\circ)_j=d(\circ)} \left( \overline{M}_{r,m_0} \times \prod_{j=1}^r \overline{M}_{I_j,m_j}^q \right).
\]

By construction, for the facet of the first type, the sign of the inclusions of boundary strata are the same as that for the corresponding inclusion of boundary facets of \( \overline{M}_{d(\bullet),d(\circ)} \), that is, \((-1)^{(n-i-j)+j}\). For facets of the second type, the gluing map is

\[(9.15) \quad (0, \infty) \times M_{r,m_0} \times \bigoplus_{j=1}^r M_{I_j,m_j}^q \to M_{d(\bullet),d(\circ)}^q \]

given for boundary markings by

\[(\delta, x_1, \ldots, x_r, (x_{1,j} = 0, x_{2,j}, \ldots, x_{|I_j|,j})_{j=1}^r) \mapsto (x_1, x_1 + \delta^{-1}x_{2,1}, \ldots, x_1 + \delta^{-1}x_{|I_1|,1}, \ldots, x_r, x_r + \delta^{-1}x_{2,r}, \ldots, x_r + \delta^{-1}x_{|I_r|,r}).\]

This map changes orientations by \(\sum_{j=1}^r (r-j)(|I_j|-1)\); in case of non-trivial weightings, \(|I_j|\) should be replaced by the number of incoming markings or non-trivial weightings on the \(j\)-th component.

9.4.4. **Morphisms of types.** The combinatorial type of a quilted disk is the graph obtained as in the unquilted case by replacing each quilted disk component with its combinatorial tree (now having colored vertices), each unquilted disk or sphere component with its combinatorial tree, and each edge being identified as infinite, semi-infinite, finite non-zero or zero. We also wish to allow disconnected types. An unquilted component is labelled 1 resp. 0 if it is closer resp. further from the root than quilted components. of the unquilted components by \(\{0,1\}\). Morphisms of graphs (Cutting an edge, collapsing edges, making edge lengths finite or non-zero) induce morphisms of moduli spaces of stable quilted treed disks as in the unquilted case. The new feature is that (Cutting an edge) is done such that one of the pieces is quilted and the other unquilted. This implies that output edges of quilted disks are cut simultaneously, and therefore the output has a disconnected type.

For example, in Figure 19, in the left picture, one can cut the \(e\) at the breaking to obtain an unquilted disk labelled 0, and a quilted disk. In the picture to the right, the edges \(e_1\) and \(e_2\) get cut simultaneously to yield an unquilted disk labelled 1 and a disconnected type consisting of two quilted disks.

For any combinatorial type \(\Gamma\) of quilted disk there is a **universal quilted treed disk** \(\overline{U}_\Gamma \to \overline{M}_\Gamma\) which is a cell complex whose fiber over \(C\) is isomorphic to \(C\), and splits into surface and tree parts \(\overline{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_{\circ,\Gamma} \cup \mathcal{T}_{\bullet,\Gamma}\), where the last two sets are the boundary and interior parts of the tree respectively.
9.4.5. **Weightings.** Weights can be added to the inputs and output of a quilted treed disks as in the case of treed disks. We suppose there is a partition of the boundary markings

$$\text{Edge}^+(T) \sqcup \text{Edge}^-(T) \sqcup \text{Edge}^*(T) = \text{Edge}_{\to}^\to(T)$$

into *weighted* resp. *forgettable* resp. *unforgettable* edges as in the unquilted case. The outgoing edge axiom is same as in the unquilted case. In the quilted case, the trees in Figure 20 are stable. Isomorphism of weighted quilted disks is same as the unquilted case, and therefore, the moduli space with a single weighted leaf and no markings is a point.

9.5. **Quilted pseudoholomorphic disks.** The Fukaya algebra of a Lagrangian submanifold is independent of the choice of perturbation data up to homotopy equivalence. We outline the proof in the case that the degree of the stabilizing divisors is fixed. One considers two systems of perturbations and extends them to a set of perturbations for the moduli space of quilted treed disks. A morphism is defined between the two $A_\infty$-algebras by counts of quilted treed disks.

**Theorem 9.11.** Suppose $p^0, p^1$ are regular perturbation data that are defined using stabilizing pairs $(J^0, D^0)$ and $(J^1, D^1)$, which are connected by a path of stabilizing pairs $\{(J^t, D^t)\}_{t \in [0,1]}$. There exists a coherent perturbation datum $p^{01}$ which induces a unital $A_\infty$-morphism

$$\phi_{01} : CF(L, p^0) \to CF(L, p^1)$$

which is a homotopy equivalence and such that $\phi_{01}(1)$ has positive $q$-valuation.
In the next section, we use this result to show that the Fukaya algebra of a Lagrangian in a neck-stretched manifold is independent up to homotopy of the neck length parameter. The result is proved in [13]. We reproduce part of the proof because the machinery is used again in Section 10. In this section, we only consider unbroken manifolds. A broken version of quilted holomorphic disks will be encountered in Section 10.

**Definition 9.12.** (Distance from the seam function) In order to specify which divisor of the above family to use at a given point of a quilted domain, define a function as follows. For any point on the universal quilted disk $z \in U_\Gamma$, let

$$d(z) := \pm \sum_{\text{edge to seams}} \ell(e) \in [-\infty, \infty]$$

be the distance of $z$ to the quilted components of $C$ (with respect to the lengths of the edges) times 1 resp. $-1$ if $z$ is above resp. below the quilted components (that is, further from resp. closer to the root than the quilted components). Note that on any treed quilted disk $C \subset U_\Gamma$, $d$ is constant on surface components.

**Definition 9.13.** Given perturbation data $p^0$ and $p^1$ with respect to metrics $G^0, G^1 \in \mathcal{G}(L)$ over unquilted treed disks for $D^0$ resp. $D^1$, a perturbation morphism $p^{01}$ from $p^0$ to $p^1$ for the quilted combinatorial type $\Gamma$ consists of

(a) a smooth function $\delta_{01}^\Gamma: [-\infty, \infty] \to [0, 1]$ (to be composed with $d$ from (9.16)),

(b) a domain-dependent Morse function

$$F_{01}^\Gamma: \mathcal{T}_{0, \Gamma} \to \mathbb{R}$$

constant to $F^0$ resp. $F^1$ on the neighbourhood $\mathcal{T}_{0, \Gamma} - \mathcal{T}_{0, \Gamma}^c$ of the endpoints for which $d = -\infty$ resp. $d = \infty$ and equal to $F_{0, \Gamma}^0$ resp. $F_{1, \Gamma}^1$ on the (unquilted) treed disks components of type $\Gamma_0$, $\Gamma_1$ for which $d = -\infty$ resp. $d = \infty$, and

(c) a domain-dependent almost complex structure

$$J_{01}^\Gamma: \mathcal{S}_\Gamma \to \mathcal{J}_r(X)$$

with the property that on the curve associated to any point $m \in \mathcal{M}_\Gamma$, $J_{01}^\Gamma$ is constant on $\mathcal{S}_{\Gamma, m} - \mathcal{S}_{\Gamma, m}^c$, where the compact set $\mathcal{S}_{\Gamma, m}^c$ is as defined in Section 5.1. Further, $J_{01}^\Gamma$ is equal to the complex structures $J_{0, \Gamma}^0$ resp. $J_{1, \Gamma}^1$ on the (unquilted) treed disks components of type $\Gamma_0$, $\Gamma_1$ for which $d = -\infty$ resp. $d = \infty$.

To obtain a well-behaved moduli space of quilted holomorphic treed disks we impose a stability condition and quotient by an equivalence relation. A quilted treed disk $u: C \to X$ is stable if every (surface or tree) component of $C$ on which $u$ is non-constant is stable in the sense of weighted quilted disks.

Let $p = (p_\Gamma)_\Gamma$ be a perturbation datum coherent with respect to (Cutting of edges), (Making an edge length/weight finite or non-zero) and (Forgetting edges). Given a quilted treed disk $C$, we obtain a stable quilted treed disk by collapsing unstable surface and tree components. The result may be identified with a fiber.
of a universal curve of some type \( s(\Gamma) \). By pullback we obtain a triple on \( C \), still denoted \((\delta_{\Gamma}^{01}, J_{\Gamma}^{01}, F_{\Gamma}^{01})\). A stable holomorphic quilted tree disk is \textit{adapted} iff each the interior marking \( z_i \) maps to \( D_{\delta_{\Gamma}^{01},\text{od}(z_i)} \), and for each \( t \in [0,1] \), each component of \( u^{-1}(D^t) \cap (\delta_{\Gamma}^{01})^{-1}(t) \) contains a marking. We remark that the union

\[
D_{\Gamma}^{01} = \bigcup_{z_i \in \mathcal{S}_\Gamma} \left( \{z_i\} \times D_{\delta_{\Gamma}^{01},\text{od}(z_i)} \right)
\]

is an almost complex submanifold of \( \mathcal{S}_\Gamma \times X \), and the intersection multiplicity of \( u : C \to X \) with \( D_{\delta_{\Gamma}^{01}} \) at any marking \( z_i \) is positive. The intersection number of \( u \) with \( D_{\delta_{\Gamma}^{01}} \) is \( k\omega[u] \), where the divisor \( D_0 \) (and \( D_1 \)) is Poincaré dual to \( k\omega \).

For any combinatorial type \( \Gamma \) of quilted disks we denote by \( \overline{\mathcal{M}}_\Gamma(L, D) \) the compactified moduli space of equivalence classes of adapted quilted holomorphic treed disks. The moduli space of quilted disks breaks into components depending on the limits along the root and leaf edges. Denote by \( \mathcal{M}_\Gamma(L, D, x) \subset \overline{\mathcal{M}}_\Gamma(L, D) \) the moduli space of isomorphism classes of stable adapted holomorphic quilted treed disks with boundary in \( L \) and limits \( \overline{x} \) along the root and leaf edges, where \( \overline{x} = (x_0, \ldots, x_{d(\overline{\Gamma})}) \in \widehat{T}(L) \).

For a comeager subset of perturbation morphisms extending those chosen for unquilted disks, the uncrowded moduli spaces of expected dimension at most one are smooth and of expected dimension. For sequential compactness, it suffices to consider a sequence \( u_\nu : C_\nu \to X \) of quilted treed disks of fixed combinatorial type \( \Gamma = \Gamma_\nu \) for all \( \nu \). Coherence of the perturbation morphism implies the existence of a stable limit \( u : C \to X \) which we claim is adapted. If a sequence of markings \( z_{i,\nu} \in C_\nu \) converges to \( z_i \in C \), then, \( u(z_i) \in D_{\delta_{\Gamma}^{01},\text{od}(z_i)} \). Indeed the coherence of the parameter \( \delta_{\nu}^{01} \) implies that \( D_{\delta_{\Gamma}^{01},\text{od}(z_i)} \) is the limit of the divisors \( D_{\delta_{\Gamma_\nu}^{01},\text{od}(z_{i,\nu})} \). For types of index at most one, each component of \( u^{-1}(D_{\delta_{\Gamma}^{01},\text{od}(z_i)}) \) is a limit of a unique component of \( u_\nu^{-1}(D_{\delta_{\Gamma_\nu}^{01},\text{od}(z_{i,\nu})}) \), otherwise the intersection degree would be more than one which is a codimension two condition. Therefore, every marking in \( C \) is a transverse divisor intersection. There are no other divisor intersections because the intersection number with \( D_{\delta_{\Gamma}^{01}} \) is preserved in the limit for topological reasons.

The moduli space of quilted broken disks then has the same transversality and compactness property as in the unquilted case, by similar arguments.

Given a regular, stabilized and coherent perturbation morphism \( \overline{p}^{01} \) from \( \overline{p}^0 \) to \( \overline{p}^1 \), define

\[
\phi^d : CF(L; \overline{p}^0)^{\otimes d} \to CF(L; \overline{p}^1)
\]

\[
(x_1, \ldots, x_d) \mapsto \sum_{x_0, u \in \mathcal{M}_\Gamma(L, D, x_0, \ldots, x_d)} (-1)^{\sigma(w(u))} w(u)x_0
\]

where the weight \( w(u) \) is given by

\[
w(u) = \epsilon([u])(\sigma([u])!)^{-1} q^{E([u])} \text{Hol}_L(u) x_0
\]

the sum is over strata \( \Gamma \) of weighted treed quilted disks with a single surface component and whose input and output labels are compatible with \( (x_0, \ldots, x_d) \) in terms of the \( \text{(Label axiom)} \).

Remark 9.14. (Lowest energy terms) For any $x \in \text{crit}(F^0) \cup \{x^*, x^\circ\}$, the element $\phi^1(x)$ contains zero energy terms coming from the count of a quilted treed disk with no interior marking, that is, a treed disk with only one disk that is quilted and mapped to a point. The domain is one of those in Figure 20. If $x$ is $x^*$ resp. $x^\circ$ there is one such configuration whose output is weighted resp. forgettable. In the latter case, it will be the only term with a forgettable output.

Remark 9.15. (Codimension one boundary strata) The codimension one strata are of several possible types: either there is one (or a collection of) edge $e$ of length $\ell(e)$ infinity, there is one (or a collection of) edge $e$ of length $\ell(e)$ zero, or equivalently, boundary nodes, or there is an edge $e$ with zero or infinite weight $\rho(e)$. The case of an edge of zero or infinite weighting is equivalent to breaking off a constant trajectory, and so may be ignored. In the case of edges of infinite length(s), then either $\Gamma$ is

(a) (Breaking off an uncolored tree) a pair $\Gamma_1 \sqcup \Gamma_2$ consisting of a colored tree $\Gamma_1$ and an uncolored tree $\Gamma_2$ as in the left side of Figure 19; necessarily the breaking must be a leaf of $\Gamma_1$; or

(b) (Breaking off colored trees) a collection consisting of an uncolored tree $\Gamma_0$ containing the root and a collection $\Gamma_1, \ldots, \Gamma_r$ of colored trees attached to each of its $r$ leaves as in the right side of Figure 19. Such a stratum $\mathcal{M}_\Gamma$ is codimension one because of the (Balanced Condition) which implies that if the length of any edge between $e_0$ to $e_i$ is infinite for some $i$ then the path from $e_0$ to $e_i$ for any $i$ has the same property.

In the case of a zero length(s), one obtains a fake boundary component with normal bundle $R$, corresponding to either deforming the edge(s) to have non-zero length or deforming the node(s). This ends the Remark.

Proof of Theorem 9.11. The true boundary strata of one-dimensional moduli spaces of quilted holomorphic disks are those described in Remark 9.15 and correspond to the terms in the axiom for $A_\infty$ morphisms (9.5). The signs are similar to those in [13] and omitted. The assertion on the strict units is a consequence of the existence of forgetful maps for infinite values of the weights. By assumption the $\phi^{d(\cdot)}$ products involving $x^*$ as inputs involve counts of quilted treed disks using perturbation that are independent of the disk boundary incidence points of the first leaf marked $x^*$ asymptotic to $x_M \in X$. Since forgetting that semi-infinite edge gives a configuration of negative expected dimension, if non-constant, the only configurations contributing to these terms must be the constant maps. Hence

$$\phi^1(x_M^*) = x_M^*, \quad \phi^{d(\cdot)}(\ldots, x_M^*, \ldots) = 0, \quad n \geq 2.$$ 

In other words, the only regular quilted trajectories from the maximum, considered as $x_M^*$, being regular are the ones reaching the other maximum that do not have interior markings (i.e. non-constant disks). The proof of convergence is left to the reader. The proof of homotopy equivalence is via twice-quilted disks, and we refer to [13] for details. \qed

9.6. Homotopy equivalence: unbroken to broken. In this section, we show that the Fukaya algebra of the broken manifold $\mathfrak{X}$ is homotopy equivalent to the
unbroken one. To prove the homotopy equivalence between a broken and unbroken Fukaya algebra, we use a breaking perturbation datum on neck-stretched manifolds. We recall from Definition 5.10 that given a perturbation datum $p^\infty$ on the broken manifold $X$, by gluing on the neck we obtain a perturbation datum $\rho_0(p^\infty)$ on any $X^\nu$. By Proposition 4.12 there is a stabilizing pair $(J, D)$ consisting of a cylindrical almost complex structure $J$ on $X$ and a $J$-holomorphic cylindrical stabilizing divisor $D$, for which the glued family $(J^\nu, D^\nu)$ is a stabilizing pair on the neck-stretched manifold $X^\nu$ for all $\nu$. For the perturbation datum $p^\infty$ we use $D$ as the stabilizing divisor and $J$ as the base almost complex structure.

**Proposition 9.16.** Let $p^\infty$ be a regular perturbation datum on $X$. For any $E_0 > 0$, there exists $\nu_0(E_0)$ such that the following holds.

(a) There is a bijection between the zero-dimensional moduli spaces :

$$\tilde{\mathcal{M}}^{\text{brok}}(X, L, p^\infty)_0^{<E_0} \simeq \mathcal{M}(X^\nu, L, \rho_0(p^\infty))_0^{<E_0}, \ \nu \geq \nu_0.$$  

(b) For any $\nu \in \mathbb{Z}_{>0}$, there exists a regular perturbation datum $p^\nu$, and a perturbation morphism $p^{\nu+1}$ extending $p^\nu$ and $p^{\nu+1}$ such that for all $E_0 > 0$ and $\nu \geq \nu_0(E_0)$, the $A_\infty$ morphism induced by $p^{\nu+1}$

$$\phi_\nu : CF(X^\nu, L, p^\nu) \to CF(X^{\nu+1}, L, p^{\nu+1})$$

is identity modulo $q^{E_0}$.

**Proof.** The proof of bijection of moduli spaces is a consequence of the compactness and gluing theorems – Theorem 7.1 and Theorem 8.1. By the gluing Theorem 8.1, any regular $p^\infty$-disk $u : C \to X$ can be glued to yield regular disks $u_\nu : C_\nu \to X^\nu$. Conversely, any sequence $(u_\nu)_\nu$ of maps with area $\leq E_0$ converges to a broken disk $u_\infty$. By surjectivity of gluing in Theorem 8.1, for large enough $\nu$, $u_\nu$ is contained in the image of the gluing map for $u_\infty$. Since the moduli space $\tilde{\mathcal{M}}(X, L)_0^{<E_0}$ has a finite number of points, the constant $\nu_0(E_0)$ can be chosen as the minimum obtained from gluing each of the broken maps.

On neck-stretched manifolds, a regular perturbation datum is constructed by extending the breaking perturbation datum. For any $E_0 > 0$ and an integer $\nu > \nu_0(E_0)$, the perturbation datum $p^\nu_\Gamma$ is equal to $\rho_0(p^\infty_\Gamma)$ if $E(\Gamma) \leq E_0$. For the other strata, a regular perturbation $p^\nu_\Gamma$ can be chosen using the transversality result, Theorem 5.22.

For strata with low enough area, perturbation morphisms are constructed using the breaking perturbation data. For $\nu, \Gamma$ satisfying $\nu \geq \nu_0(E(\Gamma))$, the perturbation morphism $p^{\nu+1}_\Gamma = (\delta^{\nu+1}_\Gamma, J^{\nu+1}_\Gamma, F^{\nu+1}_\Gamma)$ is defined as follows. The function $\delta^{\nu+1}_\Gamma : \mathbb{R} \to [0, 1]$ is chosen arbitrarily. For any $z \in S_\Gamma$,

$$J^{\nu+1}_\Gamma(z) := (\rho_{\delta^{\nu+1}_\Gamma}(d(z))(J^\infty_\Gamma))(z),$$

where $d$ is the distance to seams function, $\Gamma'$ is the type of broken disk obtained by forgetting the quilting, and $p^\infty_\Gamma = (J^\infty_\Gamma, F^\infty_\Gamma)$. The Morse datum $F^{\nu+1}_\Gamma$ is defined as equal to $F^{\infty}_\Gamma$. The perturbation morphisms are extended to higher strata so that they are coherent and regular.
We claim this perturbation morphism satisfies (b). If not, there is a sequence \( u_\nu \) of quilted \( p^{\nu,\nu+1} \)-disks with bounded area. The sequence converges to a \( p^\infty \)-unquilted disk of index \(-1\). The existence of such a disk is a contradiction, since \( p^\infty \) is a regular perturbation. \( \square \)

**Proposition 9.17.** Suppose the breaking perturbation \( \{p^\nu\}_\nu \) is as in Proposition 9.16. Then, for any fixed \( \nu_0 \), there are strictly unital \( A_\infty \) morphisms

\[
\phi : CF(X^{\nu_0}, L, p^{\nu_0}) \to CF_{\text{brok}}(L, p^\infty), \quad \psi : CF_{\text{brok}}(L, p^\infty) \to CF(X^{\nu_0}, L, p^{\nu_0})
\]

such that \( \psi \circ \phi \) and \( \phi \circ \psi \) are homotopy equivalent to identity and \( \text{val}_q(\phi_0(1)) > 0 \)

**Proof.** The structure coefficients \( m_{n,\nu} \) of the Fukaya algebra on \( X^\nu \) limit to the structure coefficient \( m_{n,\infty} \) in the broken Fukaya algebra. Indeed, for any energy bound \( E \), the terms in \( m_{n,\nu} \) are equal to the terms in \( m_{n,\infty} \) modulo \( q^E \), because of the bijection of the moduli spaces of disks in Proposition 9.16 (a). The bijection of moduli spaces preserves the area \( \text{Area}(u) \) of the map as well as the homology class \([\partial u] \in H_1(L)\) of the restriction of the map \( u : C \to X \) to the boundary \( \partial C \), hence preserves the holonomies of the flat connection on the brane around the boundary of the disk. It follows from the gluing results in e.g. [65] that the bijection is orientation preserving.

The perturbation morphisms in Proposition 9.16 (b) induce \( A_\infty \) morphisms

\[
\phi_\nu : CF(X^\nu, L, p^\nu) \to CF(X^{\nu+1}, L, p^{\nu+1}), \quad \psi_\nu : CF(X^{\nu+1}, L, p^{\nu+1}) \to CF(X^\nu, L, p^\nu),
\]

for which the terms in \( \phi_\nu \) with coefficient \( q^{E(u)} < E \) vanish for sufficiently large \( \nu \) except for constant disk.

Therefore, there exist limits of the successive compositions: letting

\[
\phi_n := \phi_{\nu_0} \circ \phi_{\nu_0+1} \circ \ldots \circ \phi_{\nu_0+n} : CF(X^{\nu_0}, L, p^{\nu_0}) \to CF(X^{\nu_0+n+1}, L, p^{\nu_0+n+1}),
\]

the limit

\[
\phi = \lim_{n \to \infty} \phi_n : CF(X^{\nu_0}, L, p^{\nu_0}) \to CF_{\text{brok}}(L, p^\infty)
\]

exists and similarly for

\[
\psi = \lim_{n \to \infty} \psi_n, \quad \psi_n := \psi_\nu \circ \psi_{\nu+1} \circ \ldots \circ \psi_{\nu+n}.
\]

Since the composition of strictly unital morphisms is strictly unital, the composition \( \psi \) is strictly unital modulo terms divisible by \( q^E \) for any \( E \), hence strictly unital. The limit map \( \psi \) resp. \( \phi \) is an \( A_\infty \) morphism whose domain resp. target is \( CF_{\text{brok}}(L) \) because the composition maps in \( CF(X^\nu, L, p^\nu) \) converge to \( CF_{\text{brok}}(L, p^\infty) \).

We claim that \( \phi \) and \( \psi \) are homotopy equivalences. Let \( h_k, g_k \) denote the homotopies satisfying

\[
\phi_k \circ \psi_k - \text{Id} = m_1(h_k), \quad \psi_k \circ \phi_k - \text{Id} = m_1(g_k),
\]

from the homotopies relating \( \phi_\nu \circ \psi_\nu \) and \( \psi_\nu \circ \phi_\nu \) to the identities. In particular, \( h_{k+1}, g_{k+1} \) differ from \( h_k, g_k \) by expressions involving counting *twice-quilted* breaking disks. For any \( E > 0 \), for \( \nu \) sufficiently large all terms in \( h_{k+1} - h_k \) are divisible by \( q^E \). It follows that the infinite composition

\[
h = \lim_{k \to \infty} h_k, \quad g = \lim_{k \to \infty} g_k
\]
exists and gives a homotopy equivalence between $\phi \circ \psi$ resp. $\psi \circ \phi$ and the identities. □
10. Deforming the matching conditions

In this section we introduce a further deformation of broken maps by changing the matching conditions corresponding to tropical edges.

10.1. Deforming the edge matching condition.

10.1.1. Deformed maps. A deformed map is a version of a broken map where the edge matching condition for split edges is replaced by a deformed matching condition. Let \((X, \Phi, \mathcal{P})\) be symplectic manifold with a tropical Hamiltonian action as in Definition 2.13.

**Definition 10.1.** (Split edges) Let \(P_s \subset \mathcal{P}\) be the set of polytopes \(P\) for which

(a) the tropical moment map \(\Phi\) generates a \(T/T_P\)-action on \(X_P\) and this action makes \(X_P\) a toric manifold,
(b) and any torus-invariant divisor \(D\) of \(X_P\) is a boundary divisor. That is, there is a polytope \(Q \subset P\) such that \(X_Q = D\).

For a tropical graph \(\Gamma\), \(e \in \text{Edge}(\Gamma)\) is a split edge if \(P(e) \in P_s\). For a type \(\Gamma\) of broken maps, the set of split edges is denoted by \(\text{Edge}_s(\Gamma) \subset \text{Edge}(\Gamma)\).

We justify this definition of split edges later in the section, see Remark 11.38.

**Definition 10.2.** A deformation parameter for a tropical graph \(\Gamma\) is an element

\[ \eta = (\eta_e)_{e \in \text{Edge}_s(\Gamma)} \in \bigoplus_{e \in \text{Edge}(\Gamma)} \frac{t/T(e)}{t} =: t^\Gamma, \]

where \(t/T(e) \subset t\) is the linear span \((T(e))\) of the slope \(T(e) \in t_P(e)\) of the edge \(e\).

The quotient \(t/R T(e)\) is identified with a subspace \(t^\perp_{T(e)} \subset t\) via the non-degenerate pairing \(t \times t \rightarrow Z\) fixed in (2.24). A deformation parameter may be viewed as an element in \(t^\perp_{T(e)} \subset t\).

**Definition 10.3.** (Deformed map) Let \(\eta\) be a deformation parameter for a tropical graph \(\Gamma\). A \(\eta\)-deformed map of type \(\Gamma\) consists of a

(a) a treed curve \(C\) of type \(\Gamma\),
(b) a framing \(fr_e : T_{w_+}(e) \bar{C} \otimes T_{w_-}(e) \bar{C} \rightarrow \mathbb{C}\) for each \(e \in \text{Edge}_s(\Gamma)\),
(c) and maps

\[ u_v : C_v \rightarrow X_{T(v)}, \quad v \in \text{Vert}(\Gamma) \]

satisfying edge matching conditions as in Definition 3.4 of broken maps for all edges \(e \in \text{Edge}(\Gamma) \setminus \text{Edge}_s(\Gamma)\). On split edges the maps \((u_v)_v\) satisfy a deformed matching condition described as follows: For a node \(w = (w_+, w_-)\) corresponding to a split edge \(e = (v_+, v_-) \in \text{Edge}_s(\Gamma)\), and coordinates on the neighborhoods of \(w_\pm\) that respect the framing

\[ x_+ = e^{\eta_e} x_-, \quad \text{where } x_\pm := \text{ev}^{T(e)} u_{w_\pm}(w_\pm(e)) \in Z_{P(e), \mathcal{C}}. \]

Here \(T(e) \in t_{P(e), \mathcal{C}}\), \(\text{ev}^{T(e)}\) is the evaluation of the \(T(e)\)-order derivative as in (3.11), and \(\eta_e\) is seen as an element in \(t^\perp_{T(e)} \subset t\).
Remark 10.4. We take the deformation parameter to be in the quotient $t/t_T(e)$ (and not in $t$) because altering it in the $T(e)$-direction affects only the framing, not the map. For example, consider a node where the edge matching condition is satisfied for a broken map, and so, $x_+ = x_-$. Changing the domain trivialization has the effect of changing the matching to $x_+ = e^{T(e)z} x_-$ for some $z \in \mathbb{C}$.

10.1.2. Moduli spaces of deformed maps. We define moduli spaces of deformed maps with the deformation parameter varying smoothly with the domain curve. The compactification of the moduli space for any type of map may contain maps with additional vertices in the tropical graph. When this happens, it is necessary to remember the set of split edges since the newly formed edges are not split edges. The combinatorial type of the deformed map therefore must include the original tropical graph, and the set of split edges. The deformation datum is then defined as a smooth function on the moduli space of these curves with additional data. The reader might wish to keep in mind that these additional strata, whose types we denote by $\tilde{\Gamma}$, do not occur for generic perturbation, but this is a fortiori.

Definition 10.5. (Curve with base type) A curve with base type consists of

(a) a stable treed curve $C$ of type $\tilde{\Gamma}$,
(b) a base tropical graph $\Gamma$ that does not contain any edges $e$ with zero slope $T(e)$,
(c) and an edge collapse morphism $\kappa : \tilde{\Gamma} \to \Gamma$ that necessarily collapses all disk edges $e \in \text{Edge}_c(\tilde{\Gamma})$ and treed segments $T_e$, $e \in \text{Edge}_c(\tilde{\Gamma})$. (The map $\kappa$ is an edge collapse of graphs, and not of tropical graphs, because $\tilde{\Gamma}$ does not have a tropical structure.)

The type of such a curve consists of the datum $\tilde{\Gamma} \xrightarrow{\kappa} \Gamma$. The base tropical graph $\Gamma$ is suppressed in the notation. The moduli space of stable treed curves of type $\tilde{\Gamma}$ with base type $\Gamma$ is denoted by $\mathcal{M}_{\tilde{\Gamma}}$. This ends the Definition.

Remark 10.6. All curves in the compactification of the moduli space $\mathcal{M}_F$ have the same base tropical graph.

Definition 10.7. (Based graph morphisms) The morphisms (Collapsing edges) and (Making an edge length/weight finite/non-zero) are defined on types of curves with a base in the same way. In both cases, the base tropical graph is left unchanged by the morphism. In contrast the (Cutting edges) morphism affects the base tropical graph and is defined as follows.

(Cutting edges for types with base) A type with base $\tilde{\Gamma} \xrightarrow{\kappa} \Gamma$ is obtained by cutting an edge in $\tilde{\Gamma} \to \Gamma'$ if

- $\tilde{\Gamma}'$ is obtained by cutting an $e = (v_+, v_-) \in \text{Edge}_c(\tilde{\Gamma}')$ containing a breaking, and
- $\Gamma'$ is obtained by identifying the vertices $\kappa(v_+), \kappa(v_-)$ in $\Gamma$.

The deformation datum for a deformed map is domain-dependent and satisfies coherence conditions.
Definition 10.8. (Deformation datum) A coherent deformation datum $\eta = (\eta_{\tilde{\Gamma}})_{\tilde{\Gamma}}$ consists of a continuous map

$$\eta_{\tilde{\Gamma}} : \mathcal{M}_{\tilde{\Gamma}} \to t_\Gamma \simeq \bigoplus_{e \in \text{Edge}_s(\Gamma)} t^e_{\tilde{\Gamma}(e)}$$

for all types $\tilde{\Gamma}$ of curves with base, that are coherent under morphisms (Cutting edges), (Collapsing edges), (Making an edge length/weight finite/non-zero), (Forgetting edges) and the following (Marking independence) axiom: For any type $\tilde{\Gamma}$, there is a map $\eta'_{\tilde{\Gamma}} : [0, \infty]^\text{Edge}_o,-(\tilde{\Gamma}) \to t_\Gamma$ such that the map $\eta_{\tilde{\Gamma}}|\mathcal{M}_{\tilde{\Gamma}}$ factors as

(10.2) (Marking independence) $\eta_{\tilde{\Gamma}} = \eta'_{\tilde{\Gamma}} \circ f_{\tilde{\Gamma}}$, $\mathcal{M}_{\tilde{\Gamma}} \xrightarrow{f_{\tilde{\Gamma}}} [0, \infty]^\text{Edge}_o,-(\tilde{\Gamma}) \xrightarrow{\eta'} t_\Gamma$,

where $f_{\tilde{\Gamma}}([C])$ is equal to the edge lengths $\ell(e)$ of treed components at the boundary nodes of the treed curve $C$. The above morphisms are on types of stable treed curves with base.

A perturbation datum for deformed maps $p = (p_{\tilde{\Gamma}})_{\tilde{\Gamma}}$ consists of maps

$$p_{\tilde{\Gamma}} = (J_{\tilde{\Gamma}}, F_{\tilde{\Gamma}}), \quad J_{\tilde{\Gamma}} : \mathcal{S}_{\tilde{\Gamma}} \to J_{\Gamma}^\text{cvl}(\mathcal{X}), \quad F_{\tilde{\Gamma}} : \mathcal{T}_{\tilde{\Gamma}} \to C^\infty(L, \mathbb{R})$$

for all types $\tilde{\Gamma}$ of curves with base, with coherence conditions corresponding to morphisms of stable treed curves with base (same as the conditions for deformation datum).

Definition 10.9. (Adapted deformed maps)

(a) Let $\tilde{\Gamma} \to \Gamma$ be a type of based curve. Let $p_{\tilde{\Gamma}}$ be a perturbation datum and $\eta_{\tilde{\Gamma}}$ be a deformation datum. An adapted deformed map consists of

(i) a curve $C$ of type $\tilde{\Gamma}$,
(ii) a tropical structure on $\tilde{\Gamma}$ so that $\kappa : \tilde{\Gamma} \to \Gamma$ is a tropical edge collapse,
(iii) and a collection of maps

$$u : C \to \mathcal{X} = (u_v : C_v \to X_{\mathcal{F}(v)})_{v \in \text{Vert}(\tilde{\Gamma})}$$

that is $p_{\tilde{\Gamma}}$-adapted in the sense of a broken map, and satisfies the edges-matching condition as in Definition 3.4 for edges $e \in \text{Edge}(\tilde{\Gamma}) \setminus \text{Edge}_s(\Gamma)$. The split edge $e \in \text{Edge}_s(\Gamma)$ is $\eta_{\tilde{\Gamma}}(i_{\tilde{\Gamma}}([C]), e)$-deformed. Here $i_{\tilde{\Gamma}} : \mathcal{M}_{\tilde{\Gamma}} \to \mathcal{M}_\Gamma$ is the inclusion map on the moduli space of curves.

(b) (Type) The type of the adapted deformed map in (a) consists of the tropical graph $\tilde{\Gamma}$, the tropical edge collapse map $\tilde{\Gamma} \to \Gamma$, and the homology and tangency data for the map $u$ (as in the type of a broken map, see Definition 5.14). Whenever it is possible, the base tropical type $\Gamma$ is suppressed in the notation.

(c) (Rigidity) The type $\tilde{\Gamma} \to \Gamma$ of a deformed map is rigid if $\Gamma$ is a rigid tropical graph and $\tilde{\Gamma}$ is rigid as a type of broken map. Thus for a rigid type, the morphism $\tilde{\Gamma} \to \Gamma$ only collapses treed components and boundary nodes.
For a type $\tilde{\Gamma}$ of adapted deformed maps, let 

$$\tilde{M}_{\text{def}, \Gamma}(p, \eta)$$

denote the moduli space of adapted deformed maps with deformation parameter $\eta$ modulo domain reparametrizations. Let 

$$M_{\text{def}, \Gamma}(p, \eta) := \tilde{M}_{\text{def}, \Gamma}(p, \eta)/T_{\text{trop}}(\tilde{\Gamma})$$

denote the space of reduced moduli space of adapted deformed maps. If the type $\tilde{\Gamma}$ is rigid the first space $\tilde{M}_{\text{def}, \Gamma}(p, \eta)$ is a finite cover of the second $M_{\text{def}, \Gamma}(p, \eta)$, since the tropical symmetry group $T_{\text{trop}}(\tilde{\Gamma})$ is finite.

Remark 10.10. Compactified moduli spaces of adapted deformed maps of different base tropical types do not intersect. We give an example of two tropical types whose base is the same and whose compactified moduli spaces may intersect. Let $\tilde{\Gamma} \xrightarrow{\kappa} \Gamma$ be a type of domain curve that has a treed edge $T_e$ and no other boundary node. Suppose $\tilde{\Gamma}_0 \xrightarrow{\kappa} \Gamma$ is obtained from $\tilde{\Gamma}$ by making the length of $T_e$ zero, and suppose $\tilde{\Gamma}' \xrightarrow{\kappa} \Gamma$ is obtained from $\tilde{\Gamma}_0$ by collapsing the edge $e$. The compactifications $\tilde{M}_{\text{def}, \Gamma}(p, \eta), M_{\text{def}, \Gamma}(p, \eta)$ intersect because both of them contain curves of type $\tilde{\Gamma}_0$ in their codimension one boundary strata.

The following is an analogue of the theorems for broken (undeformed) maps, and the proofs are analogous.

Proposition 10.11. Given a coherent deformation datum $\eta$, there is a comeager set $P_{\text{reg}}$ of coherent regular perturbations for which the following holds.

(a) For any uncrowded type $\tilde{\Gamma}$ of adapted deformed maps, regular perturbation $p_\tilde{\Gamma} \in P_{\text{reg}}$, and for any disk output and inputs $x_0, \ldots, x_{d(c)} \in I(L)$ such that $i(\tilde{\Gamma}, x) \leq 1$, the moduli space $\tilde{M}_{\text{def}, \Gamma}(p, \eta, x)$ is a manifold of expected dimension.

(b) Any one-dimensional component of the moduli space of rigid deformed broken treed disks $\tilde{M}_{\text{def}, \Gamma}(p, \eta, x)$ admits a compactification as a topological manifold with boundary. The boundary is equal to the union of zero-dimensional strata whose domain treed disks $u : C \to X_P$ have a boundary edge $e$ that either has length zero or is broken (i.e. infinite length).

(c) For any $E > 0$, there are finitely many zero and one-dimensional components of the moduli space of rigid broken treed disks with energy $\leq E$.

As in the case of broken maps, for a one-dimensional component of the moduli space, the configurations with a boundary edge of length zero constitute a fake boundary, whereas those with a broken boundary edge constitute the true boundary of the moduli space. We define a deformed version of the Fukaya algebra by counting deformed broken maps:

Definition 10.12. (Deformed Fukaya algebra) Let $\eta$ be a coherent deformation datum, and let $p_\tilde{\Gamma} = (p_\tilde{\Gamma})_{\tilde{\Gamma}}$ be a coherent regular perturbation datum for all tropical
types $\tilde{\Gamma}$ whose base type $\Gamma$ is rigid. The deformed Fukaya algebra is the graded vector space

$$CF_{\text{def}}(L, \eta) := CF_{\text{geom}}(L) \oplus \Lambda x^\eta[1] \oplus \Lambda x^\gamma$$

equipped with composition maps

$$(10.3)\ m^\text{def}_d(x_1, \ldots, x_d) = \sum_{x_0, u \in \tilde{M}_{\text{def}, \Gamma}(\mathcal{P}, \eta)} w(u)x_0,$$

where $w(u)$ is as in (9.8), $u$ ranges over all rigid types with $d$ inputs (see Definition 10.9 (c) for rigidity).

**Proposition 10.13.** The composition maps in (10.3) satisfy the $A_\infty$ axioms, and the Fukaya algebra $CF_{\text{def}}(L, \eta)$ has curvature with positive $q$-valuation and is strictly unital.

The Fukaya algebras are independent of deformation parameters $\eta$ up to homotopy equivalence. The proof is by constructing an $A_\infty$ morphism by counts of a version of quilted disks which we now define. Perturbation morphisms contain all the data as in the unbroken case as in Definition 9.13, and additionally include a path of deformation data.

**Definition 10.14.** (Deformed quilted holomorphic disks)

(a) (Perturbation morphism) Let $\eta_0$, $\eta_1$ be deformation data, and let $p_k$ be a regular perturbation datum for $\eta_k$, $k = 0, 1$. A perturbation morphism $(p_0^1, \eta_0^1)$ consists of

(i) a path of deformation data $\eta^0_1 := \{\eta_t\}_{t \in [0,1]}$ connecting $\eta_0$ and $\eta_1$,

(ii) a perturbation morphism $p_0^1$ as in the unbroken case connecting $p_0$ and $p_1$.

(b) (Adapted deformed quilted disk) Suppose $(p_0^1, \eta_0^1)$ is a perturbation morphism for quilted deformed maps. A map $u : C \to \mathcal{X}_p$ is an adapted deformed quilted disk if

(i) it is $p_0^1$-adapted in the sense of a holomorphic quilted broken treed disk,

(ii) and the deformation parameter at the node $w_e$ corresponding to a split edge $e \in \text{Edge}_s(\Gamma)$ is $\delta^0_1(d(w_e))$, $\tilde{\Gamma}(C, e)$. Here we note that $w_e$ is a point in the universal curve $\mathcal{U}_\Gamma$, and so, $\delta^0_1(d(w_e))$ is in $[0, 1]$, see Definition 9.13 (a).

The following Proposition is analogous to Theorem 9.11, and the proof of that Theorem carries over.

**Proposition 10.15.** Given a path of deformation data $\{\eta_t\}_{t \in [0,1]}$, and regular perturbation data $p^0, p^1$ for the end-points, there is a comeager set of regular perturbation morphisms extending $p^0, p^1$. Any such perturbation morphism induces a unital $A_\infty$ morphism

$$(\phi_r)_{r \geq 0} : CF_{\text{def}}(L, p^0, \eta_0) \to CF_{\text{def}}(L, p^1, \eta_1)$$
that is a homotopy equivalence and has zero-th composition map \( \phi_0(1) \) with positive \( q \)-valuation.
11. Tropical Fukaya algebras

In this last step, we take the limit of the deformation in the previous section to obtain what we call split maps. In split maps, there is no edge matching condition in a subset of edges, called split edges. We construct a homotopy equivalence between the deformed Fukaya algebras and resulting tropical Fukaya algebras.

11.1. Split maps. A split map is a version of a broken map where there is no matching condition along split edges. Formally, the set of tropical symmetry orbits of deformed maps ranging over all deformation parameters may be viewed as a manifold, with an action of a torus $\mathbb{T}$ of deformation parameters from the previous section. The limit we study formally corresponds to taking the limit of the moduli space under a generic one-parameter subgroup of $\mathbb{T}$. In this picture tropical symmetry orbits of split maps correspond to fixed-point sets of the $\mathbb{T}$-action. We first define the notion of genericity that we require.

Definition 11.1. Let $V \simeq \mathbb{R}^n$ be a vector space equipped with a dense rational lattice $V_\mathbb{Q} \simeq \mathbb{Q}^n$.

(a) (Rational subspace) A rational subspace of $V$ is a linear subspace $W \subset V$ in which $W \cap V_\mathbb{Q}$ is dense.

(b) (Generic vector) A vector $\eta \in V$ is generic if it is not contained in any proper rational subspace of $V$.

(c) (Rational linear map) Suppose $W$ is a real vector space with a dense rational lattice $W_\mathbb{Q}$. A linear map $f : V \to W$ is rational if $f(V_\mathbb{Q}) \subseteq W_\mathbb{Q}$. The subspaces $\ker(f)$ and $\text{im}(f)$ are rational subspaces.

Remark 11.2. For any integral element $\mathcal{T}(e) \in t_\mathbb{Z}$ the projection

$$\pi^\perp_{\mathcal{T}(e)} : t \to t^\perp_{\mathcal{T}(e)}$$

preserves genericity because $\pi^\perp_{\mathcal{T}(e)}$ is a rational map. Indeed the projection map arises from the non-degenerate pairing $t_\mathbb{Z} \times t_\mathbb{Z} \to \mathbb{Z}$ from (2.24).

Definition 11.3. (Quasi-split tropical graph) A quasi-split tropical graph $\tilde{\Gamma}$ consists of

(a) a base tropical graph $\Gamma$ with no edges $e$ of zero slope $\mathcal{T}(e) = 0$,

(b) a graph $\tilde{\Gamma}$ with an edge collapse morphism $\kappa : \tilde{\Gamma} \to \Gamma$,

(c) and a tropical structure on each connected component of the graph $\tilde{\Gamma} \setminus \text{Edge}_e(\Gamma)$ so that the restricted map

$$\kappa : \tilde{\Gamma} \setminus \text{Edge}_e(\Gamma) \to \Gamma \setminus \text{Edge}_e(\Gamma)$$

is a tropical edge collapse. Consequently for any vertex $v \in \text{Vert}(\tilde{\Gamma})$, $P_\Gamma(v) \subset P_{\tilde{\Gamma}}(v)$.

Remark 11.4. In a quasi-split tropical graph $\tilde{\Gamma} \to \Gamma$ the base tropical graph $\Gamma$ does not have disk edges disk edges $e \in \text{Edge}_e(\Gamma)$ because it does not have any edges $e$ of zero slope $\mathcal{T}(e)$.
Definition 11.5. (Relative weights for a quasi-split graph) A relative weight for a quasi-split tropical graph $\Gamma \rightarrow \tilde{\Gamma}$ is a set of tropical weights $\mathcal{T}(v) \in \text{Cone}(\kappa, v) \subset t^\vee, \quad v \in \text{Vert}(\tilde{\Gamma})$ satisfying

\begin{equation}
\text{(Slope)} \quad \mathcal{T}(v_+) - \mathcal{T}(v_-) \in \begin{cases} 
\mathbb{R}_{\geq 0} \mathcal{T}(e), & e \in \text{Edge}(\tilde{\Gamma}) \setminus \text{Edge}(\Gamma), \\
\mathcal{T}(e), & e \in \text{Edge}(\Gamma) \setminus \text{Edge}_s(\Gamma).
\end{cases}
\end{equation}

There is no (Slope) condition on the split edges $e \in \text{Edge}_s(\Gamma)$. Let $\mathcal{W}(\tilde{\Gamma}, \Gamma) = \{ \mathcal{T}(v) | (11.1) \}$ be the set of relative tropical weights for the quasi-split tropical graph $\tilde{\Gamma} \rightarrow \Gamma$. Define

\begin{equation}
\text{Diff} := \text{Diff}_e : \mathcal{W}(\tilde{\Gamma}, \Gamma) \rightarrow t^\vee \simeq \bigoplus_{e \in \text{Edge}_s(\Gamma)} t / \langle \mathcal{T}(e) \rangle,
\end{equation}

as the amount by which split edges $e \in \text{Edge}_s(\Gamma) \subset \text{Edge}(\Gamma)$ fail to satisfy the (Slope) condition. The discrepancy cone for a quasi-split tropical graph is the image

$$\text{Cone}(\tilde{\Gamma}, \Gamma) := \text{Diff}(\mathcal{W}(\tilde{\Gamma}, \Gamma)) \subset \bigoplus_{e \in \text{Edge}_s(\Gamma)} t / \langle \mathcal{T}(e) \rangle.$$ 

The set $\text{Cone}(\tilde{\Gamma}, \Gamma)$ is indeed a cone, because $\mathcal{W}(\tilde{\Gamma}, \Gamma)$ is a cone. This ends the definition.

In order to define a split map, we also need an ordering of the split edges. An ordering relation on split edges

$$\prec := (\prec_{\tilde{\Gamma}})_{\tilde{\Gamma}}$$

consists of a total order $\prec_{\tilde{\Gamma}}$ on the set $\text{Edge}_s(\tilde{\Gamma})$ for each quasi-split tropical graph $\tilde{\Gamma}$. We will require that the ordering be preserved by the (Cutting edges) morphism. We remark that the based graph morphisms from Definition 10.7 are well-defined on quasi-split tropical graphs.

Lemma 11.6. (Ordering of split edges) There is an ordering $\prec := (\prec_{\tilde{\Gamma}})_{\tilde{\Gamma}}$ of split edges that is coherent under the (Cutting edges) morphism (from Definition 10.7).

Proof. For a treed curve with base type, there is a natural way to order the split edges $\text{Edge}_s(\Gamma)$ that depends only on the ordering of interior leaves (or markings). Consider a quasi-split tropical graph $\tilde{\Gamma}$, and two split edges $e_i, e_j \in \text{Edge}_s(\tilde{\Gamma})$. Let $\tilde{\Gamma}_i, \tilde{\Gamma}_j$ be the sub-trees containing $e_i$ resp. $e_j$ and all edges further away from the root. Let $i'$ resp. $j'$ be the least numbered edge corresponding to an interior marking that occurs in the subtrees $\tilde{\Gamma}_i$ resp. $\tilde{\Gamma}_j$. We declare $e_i \prec_{\tilde{\Gamma}} e_j$ if one of the following holds:

(a) either $i' < j'$,
(b) or $i' = j'$ (in which case one of the subtrees is contained in another), and the marked point $z_{i'}$ is closer to $w(e_i)$ than to $w(e_j)$ (in sequence of components connecting $z_{i'}$ to either of these nodes).

It is easy to verify that such a choice of $\preceq_{\tilde{\Gamma}}$ for all types $\tilde{\Gamma}$ satisfies coherence under (Cutting edges).

**Definition 11.7.** (Split tropical graph) Let $\eta_0 \in t^v$ be a generic element, which we call the *cone direction*. A split tropical graph $\tilde{\Gamma}$ is a quasi-split tropical graph $\tilde{\Gamma}$ (as in Definition 11.3) that satisfies the following cone condition: Let $(\pi_{T(e)}^{\perp}(\eta_0))_{e, e_i}$ denote the projection of $(\pi_{T(e)}^{\perp}(\eta_0))_e$ on the first $i$ edges in the ordering above, that is, $(\pi_{T(e)}^{\perp}(\eta_0))_{e_j, e_i}$ is equal to $(\pi_{T(e)}^{\perp}(\eta_0))_{e_j}$ if $j < i$ and vanishes otherwise. Let

$$\Upsilon_i \subset (R_{\geq 0})^{\text{Edges}(\Gamma)}$$

denote the face containing $(\pi_{T(e)}^{\perp}(\eta_0))_e, e_i$. Then

$$\text{(11.3) (Cone Condition)} \Upsilon_i \cap \text{Cone}(\tilde{\Gamma}, \Gamma) \neq \emptyset, \quad \forall i = 1, \ldots, |\text{Edges}(\Gamma)|.$$

**Remark 11.8.** The (Cone condition) (11.3) may be stated equivalently as follows: For any tuple $(c_e)_{e \in \text{Edges}(\Gamma), e_i} \in (R_+)^{\text{Edges}(\Gamma)}$

$$(c_e \pi_{T(e)}^{\perp}(\eta_0))_e \in \text{Cone}(\tilde{\Gamma}, \Gamma)$$

if the ratios $\left\{ \frac{c_{e_j}}{c_{e_i}} \right\}_{i < j}$ are small enough. Indeed, if this holds then such an element is clearly in the face $\Upsilon_j$ if $c_{e_k}$ vanishes for $k > j$. On the other hand, the intersection with $\Upsilon_j$ is non-empty then (11.4) holds for some collection $(c_e)$ and the ratios may be made arbitrarily small by taking convex combinations.

The cone condition can alternately be expressed in terms of increasing sequences. A sequence of tuples

$$(c_{e}(e))_{e \in \text{Edges}(\Gamma), e_i} \in (R_+)^{\text{Edges}(\Gamma)}$$

is *increasing* if $c_{e}(e) \to \infty$ for all split edges $e \in \text{Edges}(\Gamma)$ and for a pair of split edges $e_i \prec e_j, e_i, e_j \in \text{Edges}(\Gamma)$ (by the ordering fixed in Lemma 11.6)

$$\lim_{\nu} \frac{c_{e_j}(e)}{c_{e_i}(e)} = 0.$$

In the definition of split tropical graphs the cone condition (11.4) can alternately be stated as follows: for any increasing sequence of tuples $(c_{\nu}(e))_{e, \nu}$ there exists $\nu_0$ such that

$$\text{(11.6) } (c_{\nu}(e) \pi_{T(e)}^{\perp}(\eta_0))_e \in \text{Cone}(\tilde{\Gamma}, \Gamma) \quad \forall \nu \geq \nu_0.$$

The cone condition for a split tropical graph implies that the discrepancy cone is top-dimensional as we prove in the next result.

**Proposition 11.9.** Suppose $\tilde{\Gamma} \to \Gamma$ is a split tropical graph, with cone direction $\eta_0$. Then,

(a) (Dimension of discrepancy cone)

$$\text{(11.7) } \dim(\text{Diff}(W(\tilde{\Gamma}, \Gamma))) = |\text{Edges}(\Gamma)|(|\text{Dim}(t) - 1)|.$$
(b) (Weak cone condition) The cone condition is equivalent to the following:

There exists an increasing sequence of tuples \((c_\nu(e))_{\nu,e}\) such that

\[
(c_\nu(e)\pi_{T(e)}^{\perp}(\eta_0))_e \in \text{Cone}(\Gamma, \Gamma)
\]

for large enough \(\nu\).

The proof of Proposition 11.9 and the convergence result Proposition 11.37 use a less restrictive version of relative weights defined below.

**Definition 11.10.** (Unsigned relative weight) An unsigned relative weight for a quasi-split tropical graph \(\tilde{\Gamma} \to \Gamma\) is a set of tropical weights

\[
T(v) \in t_{\tilde{\Gamma}}^v(\tilde{\Gamma}) \subset t_{\tilde{\Gamma}}^\ast, \quad v \in \text{Vert}(\tilde{\Gamma})
\]

satisfying

\[
(11.8) \quad \text{(Slope)} \quad T(v_+) - T(v_-) \in \mathbb{R}T(e), \quad e \in \text{Edge}(\tilde{\Gamma}) \setminus \text{Edge}_s(\Gamma).
\]

The set of unsigned relative tropical weights for the quasi-split tropical graph \(\tilde{\Gamma} \to \Gamma\), denoted by

\[
W^\pm(\tilde{\Gamma}, \Gamma)
\]

is a vector space with a dense rational subspace

\[
(11.9) \quad W^\pm_Q(\tilde{\Gamma}, \Gamma) := \left\{ T \in W^\pm(\tilde{\Gamma}, \Gamma) \mid T(v_+) - T(v_-) \in \mathbb{Q}T(e) \quad \forall e \in \text{Edge}(\tilde{\Gamma}) \setminus \text{Edge}_s(\Gamma) \right\}.
\]

**Proof of Proposition 11.9.** We will assume that the quasi-split tropical graph \(\tilde{\Gamma}\) satisfies the (Weak cone condition) and prove that \(\tilde{\Gamma}\) also satisfies the cone condition, stated in its sequential form in Remark 11.4. The top-dimensionality of the discrepancy cone \(W(\tilde{\Gamma}, \Gamma)\) is proved along the way.

We first give a decomposition of the space of unsigned relative weights into non-trivial subspaces using an increasing deformation sequence. By the (Weak cone condition) there is a deformation sequence

\[
\eta_\nu := (c_\nu(e)\pi_{T(e)}^{\perp}(\eta_0))_{e \in \text{Edge}_s(\Gamma)} \in \text{Cone}(\tilde{\Gamma}, \Gamma),
\]

where \((c_\nu(e))_{\nu,e}\) is a sequence of increasing tuples. For each element in the sequence, there is a relative weight \(t_\nu \in W^{\pm}(\tilde{\Gamma}, \Gamma)\) satisfying

\[
t_\nu(v_+) - t_\nu(v_-) = \eta_\nu(e) \quad \forall e = (v_+, v_-) \in \text{Edge}_s(\Gamma).
\]

We now show for any split edge \(e \in \text{Edge}_s(\Gamma)\) the subspace

\[
W^\pm_e := \left\{ T \in W^{\pm}(\tilde{\Gamma}, \Gamma) \mid \text{Diff}_{e'}(\Gamma) = 0 \forall e' \in \text{Edge}_s(\Gamma) \setminus \{e\} \right\},
\]

is non-trivial via the following Claim.

**Claim 11.11.** The image \(\text{Diff}_{e'}(W^\pm_e)\) contains \(\eta_\nu(e)\) for large enough \(\nu\).
Proof. The proof is via an iteration, where in each step we subtract a sequence of fastest growing unsigned relative weights, similar to the proof of Lemma 7.35. The sequences subtracted at each step turn out to be linearly independent. We start the iteration with the collection of sequences

\[(11.10) \quad t^0_\nu(v) := t_\nu(v), \quad v \in \text{Vert}(\tilde{\Gamma}), \quad \eta^0_\nu(e) := \begin{cases} \eta_\nu(e) & e \in \text{Edge}_s(\Gamma), \\ 0 & e \in \text{Edge}(\tilde{\Gamma}) \setminus \text{Edge}_s(\Gamma). \end{cases}\]

The sequences at the end of \(i\)-th step are denoted by \(t^i_\nu(v), \eta^i_\nu(e)\), and we assume that they satisfy

\[(11.11) \quad (t^i_\nu(v^+) - t^i_\nu(v^-)) \mod T(e) = \eta^i_\nu(e).\]

After passing to a subsequence, the set of sequences \(\{t^i_\nu(v) \cup \{\eta^i_\nu(e)\}_e\} \) from (11.10) has a fastest growing sequence. If one of the fastest growing sequences is \(\eta^i_\nu(e_0)\) for a split edge \(e_0 \in \text{Edge}_s(\Gamma)\), then we set \(r_\nu := |\eta^i_\nu(e_0)|\). Otherwise, we define \(r_\nu\) to be a fastest growing translation sequence \(|t^i_\nu(v)|\). Next, we define an unsigned relative weight

\[\Delta t^i_\nu(v) := r_\nu \lim_\nu t^i_\nu(v) r_\nu\]

whose discrepancy at a split edge \(e \in \text{Edge}_s(\Gamma)\) is

\[\Delta \eta^i_\nu(e) := r_\nu \lim_\nu \eta^i_\nu(e) r_\nu.\]

The sequences for \((i + 1)\)-st iterative step are

\[(11.12) \quad t^{i+1}_\nu(v) := t^i_\nu(v) - \Delta t^i_\nu(v), \quad \eta^{i+1}_\nu(e) := \eta^i_\nu(e) - \Delta \eta^i_\nu(e).\]

We observe that \(t^{i+1}_\nu\) satisfies (11.11). The iterative process terminates when all the edge sequences vanish. The process is guaranteed to terminate because in every step at least one vertex sequence \(t^i_\nu(v)\) or an edge sequence \(\eta^i_\nu(e)\) is changed to zero. Further, if a vertex or an edge sequence is zero, it cannot become non-zero in subsequent iterations. In any step of the above iteration an edge sequence \(\eta^i_\nu(e)\) is either left unchanged or it is turned to zero, since no two edge sequences \(|\eta_\nu(e)|, |\eta_\nu(e')|\) grow at the same rate: That is, \(\lim_\nu \frac{|\eta_\nu(e)|}{|\eta_\nu(e')|}\) is not finite and non-zero for any pair of split edges \(e \neq e'\). Therefore, for any split edge \(e\), there is an iteration step \(j(e) \in \mathbb{N}\) such that

\[\Delta \eta^j_\nu(e) = \eta_\nu(e), \quad \Delta \eta^i_\nu(e) = 0, \quad \forall i \neq j(e).\]

The sequence of unsigned relative weights \(\Delta t^j_\nu(e)\) is in \(\mathcal{W}_e^\pm\). This proves the Claim because \(\text{Diff}_e(\Delta t^j_\nu(e)) = \eta_\nu(e)\). \hfill \square

We now prove that the discrepancy cone is top-dimensional. The space \(\mathcal{W}_e^\pm\) of unsigned weights has a dense rational subspace \(\mathcal{W}_e^\pm_{\mathbb{Q}}\), see (11.9). The subspace \(\mathcal{W}_e^\pm\) is rational in \(\mathcal{W}_e^\pm\), and the map \(\text{Diff}_e\) is rational. Therefore, \(\text{Diff}_e(\mathcal{W}_e^\pm)\) is a rational subspace of \(t^j_\nu(e)\). The subspace \(\text{Diff}_e(\mathcal{W}_e^\pm)\) contains \(\pi^\pm_\nu(e)\), which is a generic element. We conclude that \(\text{Diff}_e(\mathcal{W}_e^\pm)\) is equal to \(t^j_\nu(e)\). By the definition of \(\mathcal{W}_e^\pm\), part (a) of the Proposition follows.
The cone of relative tropical weights \( \mathcal{W}(\hat{\Gamma}, \Gamma) \) is a subset of the vector space of unsigned relative weights \( \mathcal{W}^\pm(\hat{\Gamma}, \Gamma) \) cut out by the condition that each of the edge lengths is positive. Since \( \text{Diff}(\mathcal{W}) = \text{Cone}(\hat{\Gamma}, \Gamma) \) and \( \text{Diff} \) is a linear map, the boundary of \( \text{Cone}(\hat{\Gamma}, \Gamma) \) in \( \mathfrak{t}_\Gamma \) also consists of a finite number of hyperplanes. Therefore, the same is true of \( \mathcal{W} : = \{ (c_e)_{e} \in (\mathbb{R}_+)^{\text{Edge}_s(\Gamma)} \mid (c_e \pi_{T(e)}^{\perp}(\eta_0))_e \in \text{Cone}(\hat{\Gamma}, \Gamma) \} \subset \mathbb{R}^{|\text{Edge}_s(\Gamma)|} \), since it is the intersection of the cone \( \text{Cone}(\hat{\Gamma}, \Gamma) \) with the subspace \( \{ (c_e \pi_{T(e)}^{\perp}(\eta_0))_e : c_e \in \mathbb{R} \} \) of \( \mathcal{W}^\pm \). The (Cone condition) (11.3) now follows from the (Weak cone condition) and the following claim applied to \( \mathcal{W} \). The proof of the claim is left the reader.

**Claim 11.12.** Let \( \mathcal{C} \subset \mathbb{R}^n \) be a cone whose boundary consists of a finite number of hyperplanes. Suppose there exists an increasing sequence of tuples \( (c_\nu) := (c_1, \nu, \ldots, c_n, \nu) \) in \( \mathcal{C} \). Then for any increasing sequence of tuples \( (c'_\nu) \) in \( \mathbb{R}^n \), \( c'_\nu \in \mathcal{C} \) for large enough \( \nu \).

**Definition 11.13.** (Split map) Given a split tropical graph \( \hat{\Gamma} \rightarrow \Gamma \), a **split map** is a collection of maps 

\[
u : C \rightarrow \mathfrak{X} = (u_v : C_v \rightarrow X_{\bar{F}(v)})_{v \in \text{Vert}(\hat{\Gamma})}
\]

whose domain \( C \) is a treed disk of type \( \hat{\Gamma} \), and framing isomorphisms for all **non-split** edges

\[
\text{fr} : T_{w_+}(e) \hat{\mathcal{C}} \otimes T_{w_-}(e) \hat{\mathcal{C}} \rightarrow \mathcal{C}, \quad e \in \text{Edge}(\hat{\Gamma}) \setminus \text{Edge}_s(\Gamma), \quad \mathcal{T}(e) \neq 0,
\]

such that the edge-matching condition (as in Definition 3.4) is satisfied for edges \( e \in \text{Edge}(\hat{\Gamma}) \setminus \text{Edge}_s(\Gamma) \). At the split edges \( e \in \text{Edge}_s(\Gamma) \), the edge matching condition is dropped.

The moduli spaces decompose into strata indexed by combinatorial types, as before. The **type** of a split map is given by the split tropical graph \( \hat{\Gamma} \rightarrow \Gamma \) and the tangency and homology data on \( \hat{\Gamma} \). Thus the type of a split map is same as the type of a deformed map in Definition 10.9 (b) with the only difference that the tropical graph \( \hat{\Gamma} \) does not satisfy the (Slope) condition (7.26) on the split edges. The type of a split map is denoted by \( \hat{\Gamma} \) when the base tropical type \( \Gamma \) is clear from the context.

We wish to count rigid split maps to define the tropical Fukaya algebra. Rigidity, meaning that the expected dimension of the moduli space is zero, is a condition depending only on the combinatorial type:

**Definition 11.14.** The type \( \hat{\Gamma} \rightarrow \Gamma \) of a split map is **rigid** if

(a) \( \Gamma \) is a rigid tropical graph,

(b) in \( \hat{\Gamma} \) the only edges \( e \in \text{Edge}(\Gamma) \) with non-zero slope \( \mathcal{T}(e) \neq 0 \) are boundary edges \( e \in \text{Edge}_s(\Gamma) \) with finite non-zero length \( \ell(e) \in (0, \infty) \),

(c) for all interior markings \( z_e, e \in \text{Edge}_\bullet(\hat{\Gamma}) \), the intersection multiplicity with the stabilizing divisor is 1,
(d) and the cone of tropical weights \( \mathcal{W}(\tilde{\Gamma}, \Gamma) \) has dimension
\[
\dim(\mathcal{W}(\tilde{\Gamma}, \Gamma)) = |\text{Edge}_s(\Gamma)|(|\dim(t) - 1|).
\]

The definition of tropical symmetry in the case of split maps is adjusted to reflect the lack of matching condition at split edges.

**Definition 11.15.** A tropical symmetry on a split map with graph \( \tilde{\Gamma} \to \Gamma \) consists of a translation \( g_v \in T_{P(v), \mathbb{C}} \) for each vertex \( v \in \text{Vert}(\tilde{\Gamma}) \), and a framing translation \( z_e \in \mathbb{C}^\times \) for each non-split edge \( e = (v_+, v_-) \in \text{Edge}(\tilde{\Gamma}) \setminus \text{Edge}_s(\Gamma) \) that satisfies
\[
g(v_+)g(v_-)^{-1} = z^T(e).
\]

Denote the group of tropical symmetries as
\[
T_{\text{trop}}(\tilde{\Gamma}, \Gamma) = \{(g_v)_{v} \in T_{P(v), \mathbb{C}} | v \in \text{Vert}(\Gamma), (z_e)_{e} \in \mathbb{C} \times | \; e \in \text{Edge}(\tilde{\Gamma}) \setminus \text{Edge}_s(\Gamma) \text{, satisfying (11.14)} \}.
\]

For a split tropical graph \( \tilde{\Gamma} \to \Gamma \), the space of relative weights \( \mathcal{W}(\tilde{\Gamma}, \Gamma) \) generates a subgroup of tropical symmetries (see (7.27)).

**Remark 11.16.** (Dimension of the symmetry group) For a split tropical graph \( \tilde{\Gamma} \to \Gamma \), the space of relative weights \( \mathcal{W}(\tilde{\Gamma}, \Gamma) \) generates a subgroup of \( T_{\text{trop}}(\tilde{\Gamma}, \Gamma) \), and by (11.7) \( \mathcal{W}(\tilde{\Gamma}, \Gamma) \) is a cone of dimension at least \( |\text{Edge}_s(\Gamma)|(|\dim(t) - 1|) \). If \( \tilde{\Gamma} \) is rigid, then (11.16) is an equality.

**Remark 11.17.** (Splitting of the tropical symmetry group) For a split tropical graph \( \tilde{\Gamma} \to \Gamma \), the group \( T_{\text{trop}}(\tilde{\Gamma}, \Gamma) \) is a product
\[
T_{\text{trop}}(\tilde{\Gamma}, \Gamma) = T_{\text{trop}}(\tilde{\Gamma}_1, \Gamma_1) \times \cdots \times T_{\text{trop}}(\tilde{\Gamma}_s, \Gamma_s)
\]

of tropical symmetry groups of the connected subgraphs
\[
\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_s \subset \tilde{\Gamma} \setminus \text{Edge}_s(\Gamma), \; \Gamma_i := \kappa(\tilde{\Gamma}_i) \subset \Gamma.
\]

Indeed by definition there are no matching conditions (11.14) on split edges.

To relate deformed maps to split maps, we will need a variation of split maps where there is a matching condition on split nodes also. Such maps should be thought of as lying on a slice of the \( T_{\text{trop}}(\tilde{\Gamma}, \Gamma) \)-action.

**Definition 11.18.** (a) (Framed split map) A framed split map is a split map \( u \) together with an additional datum of framings on split edges
\[
\text{fr}_e : T_{w_+(e)} \tilde{\mathcal{C}} \otimes T_{w_-(e)} \tilde{\mathcal{C}} \to \mathbb{C}, \; e \in \text{Edge}_s(\Gamma)
\]
such that the matching condition is satisfied at the split edges.

(b) (Tropical symmetry for framed split maps) A tropical symmetry for a framed split map with graph \( \tilde{\Gamma} \to \Gamma \) is a tuple
\[
((g_v \in T_{P(v), \mathbb{C}})_{v \in \text{Vert}(\Gamma)}, (z_e \in \mathbb{C}^\times)_{e \in \text{Edge}(\tilde{\Gamma}) \setminus \text{Edge}_s(\Gamma)})
\]
satisfying a matching condition on all interior edges:
\[
g(v_+)g(v_-)^{-1} = z^T(e) \; \forall e \in \text{Edge}(\tilde{\Gamma}).
\]
The group of tropical symmetries for framed split maps is denoted by
\[(11.19)\quad T_{trop, fr}(\tilde{\Gamma}, \Gamma) = \{(g_v, z_e) \mid (11.18)\}.
\]

(c) (Multiplicity of a split tropical graph) For a rigid split tropical graph \(\tilde{\Gamma} \to \Gamma\), the group \(T_{trop, fr}(\tilde{\Gamma})\) is finite, and
\[
\text{mult}(\tilde{\Gamma}) := |T_{trop, fr}(\tilde{\Gamma})|
\]
is called the multiplicity of \(\tilde{\Gamma}\).

Remark 11.19. A framed split map is not a broken map, because the underlying tropical graph \(\tilde{\Gamma}\) does not satisfy the (Slope) condition on split edges.

Remark 11.20. (Relating the framed and unframed symmetry groups) For a split tropical graph \(\tilde{\Gamma}\), there is an exact sequence
\[(11.20)\quad 0 \to Z_{fr}(\Gamma) \to T_{trop, fr}(\tilde{\Gamma}) \overset{f}{\to} T_{trop}(\tilde{\Gamma}) \overset{ev}{\to} \prod_{e \in \text{Edges}(\Gamma)} T_{C}/T_{T(e), \mathbb{C}} \to 0,
\]
where
\[(11.21)\quad Z_{fr}(\tilde{\Gamma}) := \{(g, z) \in T_{trop, fr}(\tilde{\Gamma}) : g_v = \text{Id} \ \forall v, z_e = \text{Id} \ \forall e \notin \text{Edges}(\Gamma)\}
\]
is a finite group, the map \(f\) forgets \(z_e\) for all split edges \(e\), and the map \(ev\) sends \((g, z)\) to \(g_v, g_e^{-1}\) for each split edge \(e = (v_+, v_-)\).

11.2. Examples of split maps. We give further examples of split maps beyond the case of codimension one splitting.

Example 11.21. In this example, we present two rigid split tropical graphs with the same base tropical graph and cone direction. The two graphs give different splittings of the tropical symmetry group. Suppose a broken manifold \(X\) is obtained by cutting a symplectic manifold \(X\) along two transversely intersecting hypersurfaces \(Z_0, Z_1\). Broken maps are modelled on tropical graphs in the dual complex \(B^\vee\), which is a square as in Figure 21. The torus \(T\) is \((S^1)^2\). Since \(t/\mathbb{R}T(e)\) is one-dimensional, any \(\eta_0 \in t^\vee\) satisfying \(\langle \eta_0, (1, -1) \rangle \neq 0\) is (trivially) a generic cone direction. There are two non-equivalent choices of cone direction. We choose \(\eta_0\) such that \(\langle \eta_0, (1, -1) \rangle > 0\). In Figure 21, \(\tilde{\Gamma}_1 \to \Gamma\) and \(\tilde{\Gamma}_2 \to \Gamma\) are two of the possible split tropical graphs compatible with \(\eta_0\).

Figure 21. Split tropical graphs \(\tilde{\Gamma}_1 \to \Gamma, \tilde{\Gamma}_2 \to \Gamma\) with split edge \(e\).
(a) A relative weight $T \in \mathcal{W}(\tilde{\Gamma}, \Gamma)$ satisfies

$$T(v_+) = (2,1)t \quad \text{for some } t \geq 0, \quad T(v_-) = (0,0).$$

The discrepancy cone is

$$\pi^\perp_{T(e)}\{(T(v_+) - T(v_-)) : T \in \mathcal{W}(\tilde{\Gamma}, \Gamma)\} = \mathbb{R}_+\langle \pi^\perp_{T(e)}\eta_0 \rangle.$$

For the connected component $\tilde{\Gamma}_1^+ \subset \tilde{\Gamma}_1 \setminus \{e\}$ containing $v_\pm$, the component of the tropical symmetry group (see (11.17)) is

$$T_{\text{trop}}(\tilde{\Gamma}_1^+) = \exp(\mathbb{C}(2,1)), \quad T_{\text{trop}}(\tilde{\Gamma}_1^-) = \{\text{Id}\}.$$

(b) A relative weight $T \in \mathcal{W}(\tilde{\Gamma}, \Gamma)$ satisfies

$$T(v_+) = (0,0) \quad \text{and } T(v_-) = -(1,0)t \quad \text{for some } t \geq 0,$$

and the discrepancy cone is

$$\pi^\perp_{T(e)}\{(T(v_+) - T(v_-)) : T \in \mathcal{W}(\tilde{\Gamma}, \Gamma)\} = \mathbb{R}_+\langle \pi^\perp_{T(e)}\eta_0 \rangle.$$

In $\tilde{\Gamma}_2$, the tropical symmetry group splits as

$$T_{\text{trop}}(\tilde{\Gamma}_2^+) = \{\text{Id}\}, \quad T_{\text{trop}}(\tilde{\Gamma}_2^-) = \exp(\mathbb{C}(1,0)).$$

---

**Figure 22.** $\tilde{\Gamma}_2 \to \Gamma$ is a split tropical graph, and $\tilde{\Gamma}_1 \to \Gamma$ is not a split tropical graph, see Example 11.22

**Example 11.22.** In this example, we show that if the $\rho$-increasing condition is dropped in the cone condition for a split tropical graph, then the dimension of the tropical symmetry group may be lower than expected. We continue with the broken manifold of example 11.21, but consider a different base tropical graph $\Gamma$ shown in Figure 22. Genericity of a cone direction $\eta \in \mathfrak{t}^\vee \simeq \mathbb{R}^2$ only implies $\pi^\perp_{T(e)}(\eta) \neq 0$ for both split edges in $\Gamma$. Since $\mathfrak{t}/\langle T(e) \rangle$ is one-dimensional for any split edge $e$, a choice of cone direction is equivalent to choosing a half plane for each split edge: we choose $\eta$ such that

$$\langle \eta, (1,-1) \rangle < 0, \quad \langle \eta, (1,0) \rangle < 0.$$

(a) In the quasi-split graph $\tilde{\Gamma}_1 \to \Gamma$, the set of relative weights

$$\mathcal{W}(\tilde{\Gamma}_1, \Gamma) = \{-2,1) : t \geq 0\},$$
satisfies the cone condition individually for both split edges, i.e. the discrepancy cone
\[ \text{Diff}_{e_1}(\mathcal{W}) = \mathbb{R}_+ \langle \pi_{(1,1)}^{\perp}(-2,1)t \rangle, \text{ resp. } \text{Diff}_{e_2}(\mathcal{W}) = \mathbb{R} \langle \pi_{(1,0)}^{\perp}(-2,1)t \rangle \]
is top-dimensional in \( t/t_{\mathcal{T}(e_1)} \) resp. \( t/t_{\mathcal{T}(e_2)} \), and therefore contains \( \pi_{\mathcal{T}(e_1)}^{\perp}(\eta) \) resp. \( \pi_{\mathcal{T}(e_2)}^{\perp}(\eta) \). However, the cone condition requires that
\[ (c_1 \pi_{\mathcal{T}(e_1)}^{\perp}(\eta_0), c_2 \pi_{\mathcal{T}(e_2)}^{\perp}(\eta_0)) \in \text{Diff}(\mathcal{W}(\hat{\Gamma}_1, \Gamma_1)) \subset \oplus_{i=1,2} t/t_{\mathcal{T}(e_i)} \]
whenever \( c_1/c_2 \) is large enough. This condition is satisfied only if \( \text{Diff}(\mathcal{W}(\hat{\Gamma}_1, \Gamma_1)) \) is two-dimensional, and therefore fails in this example.

(b) In the graph \( \hat{\Gamma}_2 \to \Gamma \), a relative weight \( \mathcal{T} \) is given by
\[ \mathcal{T}(v_+) = (2,1)t_1 + (1,0)t_2 \text{ for some } t_1, t_2 > 0, \quad \mathcal{T}(v) = 0, v \neq v_+, \]
and it can be verified that the cone condition is satisfied.

![Figure 23. A split tropical graph \( \hat{\Gamma} \to \Gamma \) with split edge \( e \)](image)

*Example* 11.23. In the split tropical graphs in Example 11.21 and 11.22, the product splitting of the tropical symmetry group (as in (11.17)) has only one non-trivial component. In this example, there are two non-trivial components. Consider a broken manifold \( \mathcal{X}_P \) obtained by three transverse cuts applied to a symplectic manifold \( X \). Thus the structure torus is \( \mathcal{T} = (S^1)^3 \), and the dual complex \( B^\vee \subset t^\vee \) is a cube. Consider the quasi-split tropical graph \( \hat{\Gamma} \to \Gamma \) in Figure 23 with a single split edge \( e \) with slope \( \mathcal{T}(e) = (1,1,1) \). We take the transverse subspace \( t_{\mathcal{T}(e)}^\perp \) to be \( \{(*,*,0)\} \), and show that \( \hat{\Gamma} \) satisfies the cone condition for the cone direction \( \eta_0 = (r,1,0) \) where \( \frac{1}{2} < r < 2 \) is irrational. For any relative weight \( \mathcal{T} \in \mathcal{W}(\hat{\Gamma}, \Gamma) \),
\[ \pi_{\mathcal{T}(e)}^{\perp}(\mathcal{T}(v_+)) = (2,1,0)t_1, \quad t_1 \geq 0; \quad \pi_{\mathcal{T}(e)}^{\perp}(\mathcal{T}(v_-)) = -(1,2,0)t_2, \quad t_2 \geq 0. \]

Their difference is a cone
\[ \pi_{\mathcal{T}(e)}^{\perp}(\mathcal{T}(v_+)) - \pi_{\mathcal{T}(e)}^{\perp}(\mathcal{T}(v_-)) = \{ (2,1,0)t_1 + (1,2,0)t_2 : t_1, t_2 \geq 0 \} \]
which contains the cone direction \( (r,1,0) \). So, \( \hat{\Gamma} \to \Gamma \) is a split tropical graph. Let \( \hat{\Gamma}_\pm \subset \hat{\Gamma}\setminus\{e\} \) be the connected component containing \( v_\pm \). The group of tropical symmetries is a product
\[ T_{\mathcal{C}}/T_{\mathcal{T}(e), \mathcal{C}} = T_{\text{trop}}(\hat{\Gamma}^+) \times T_{\text{trop}}(\hat{\Gamma}^-), \]
where

\[ T_{\text{trop}}(\tilde{\Gamma}^+) = \exp(\mathbb{C}(2,1,0)), \quad T_{\text{trop}}(\tilde{\Gamma}^-) = \exp(\mathbb{C}(1,2,0)). \]

**Example 11.24.** (Framed tropical symmetry group) The tropical graph \( \tilde{\Gamma} \to \Gamma \) in Figure 23 is rigid, and consequently the framed tropical symmetry group \( T_{\text{trop,fr}}(\tilde{\Gamma}) \) is finite. However, this group is non-trivial. An element \((g, z) \in T_{\text{trop,fr}}(\tilde{\Gamma})\) satisfies the equations

\[ g_v = g_{v_1} = \text{Id}, \quad g_{v_0} g_v^{-1} = z_{e_+}^{(2,1,0)}, \quad g_{v_+} g_{v_1}^{-1} = z_{e_+}^{(1,1,1)}, \quad g_{v_-} g_{v_1}^{-1} = z_{e_-}^{(1,2,0)}. \]

There are 3 elements in \( T_{\text{trop,fr}}(\tilde{\Gamma}) \) given by \( z_{e_+} = z_{e_-} = \omega \) where \( \omega \) is a cube root of unity.

**Example 11.25.** A broken map becomes a quasi-split map if we forget the edge matching conditions on split edges. The following is an example where such a quasi-split map satisfies the (Cones condition) \((11.3)\) and so, is a split map. Let \( u \) be a rigid broken map of type \( \Gamma \) whose tropical graph is given on the left side of Figure 24. Let \( u_{\text{split}} \) be the quasi-split map the object obtained by dropping the edge matching condition for \( u \) at the split edge \( e := (0,0) - (\frac{1}{2}, \frac{1}{2}) \).

We claim that \( u_{\text{split}} \) is a split map for any choice of cone direction \( \eta_0 \in t/\langle T(e) \rangle \). Indeed, let \( \tilde{\Gamma} \to \Gamma \) be the quasi-split tropical graph obtained by forgetting the (slope) condition \((7.26)\) on \( e \). In the graph \( \tilde{\Gamma} \), the vertex at \( (\frac{1}{2}, \frac{1}{2}) \) can be moved along the line \( x_1 + x_2 = 1 \). The space of relative weights \( \mathcal{W}(\tilde{\Gamma}, \Gamma) \) is one-dimensional and the image of \( \text{Diff}_{\tilde{\Gamma}} \) spans the line \( t/\langle T(e) \rangle \). Any cone direction \( \eta_0 \) is contained in the image of \( \text{Diff}_{\tilde{\Gamma}} \), and so, \( \tilde{\Gamma} \to \Gamma \) is a split tropical graph for any choice of \( \eta_0 \).

**Example 11.26.** (Twenty-one disks) We continue Example 1.2 from the Introduction. In this case the homology classes of the disks of Maslov index two that are torus-invariant up to isomorphism must consist of a single Blaschke product and a collection of boundary divisors of Chern number zero. Here a Blaschke product is one of obtained by viewing the toric variety as a quotient of some vector space \( \mathbb{C}^k \), the torus as a quotient of the standard torus in \( \mathbb{C}^k \), in which case the disks bounding the torus are of the form

\[(11.22) \quad u : C \to \mathbb{C}^k, \quad z \mapsto \left( \zeta_i \prod_{j=1}^{d_i} \frac{z - a_{i,j}}{1 - z \bar{a}_{i,j}} \right)_{i=1, \ldots, n}. \]
for some constants $a_{i,j}, \zeta_i$; a Maslov index two Blaschke disks has a single non-vanishing degree $d_i$ equal to 1. Consider a multiple cut as shown in Figure 4. After cutting, the possible configurations include Blaschke products of either two (hitting the “long boundary divisors”) or four (hitting the corner of the interior piece). As stable disks, the resulting configurations are not regular, since the spheres of index one are rigid. However, the edges connecting the spheres may be split exactly as in the previous example, so that the split matching conditions are regular and each vertex of the allowable tropical graphs contribute 1 to the product. Thus

$$s(\tilde{\Gamma}) \prod_{i=1}^{s(\tilde{\Gamma})} m_{\tilde{\Gamma}_i}^0(1) = \prod_{i=1}^{s(\Gamma)} 1 = 1$$

for each tropical graph contributing. Since there are twenty-one such graphs we obtain

$$m_{\text{split}}^0(1) = \sum_{\tilde{\Gamma}, d_c(\tilde{\Gamma}) = d} m_{\text{split}, \tilde{\Gamma}}^0 = \sum_{\tilde{\Gamma}, d_c(\tilde{\Gamma}) = d} 1 = 21$$

as claimed, c.f. [63, Appendix B.2]; note that one does not expect this torus to generate the Fukaya category, but rather the sub-category corresponding to the “small eigenvalue” in Sheridan’s language in [63].

11.3. **Tropical Fukaya algebras.** We define moduli spaces of split maps and use them to define composition maps for tropical Fukaya algebras. To define the moduli spaces, we fix a generic cone direction $\eta_0 \in \mathbb{t}^\vee$ and perturbation data consisting of almost complex structures and Morse functions on the Lagrangian submanifold. Perturbation data are functions on the domain curve, which is a treed curve with base type. That is, for a type $\tilde{\Gamma}$ of treed curves with base type, the perturbation datum is a pair $p_{\tilde{\Gamma}} = (J_{\tilde{\Gamma}}, F_{\tilde{\Gamma}})$, where

$$J_{\tilde{\Gamma}} : S_{\tilde{\Gamma}} \to \mathcal{J}(X), \quad F_{\tilde{\Gamma}} : T_{\tilde{\Gamma}} \to C^\infty(L, \mathbb{R}).$$

Coherence conditions for domain-dependent perturbation data for types of split maps are exactly same as types of deformed maps. After fixing a coherent perturbation datum $p_{\tilde{\Gamma}} = (J_{\tilde{\Gamma}}, F_{\tilde{\Gamma}})$, adapted maps are defined in a standard way.

**Definition 11.27.** The moduli space of split maps of type $\tilde{\Gamma}$ modulo the action of domain reparametrizations is denoted

$$\widetilde{\mathcal{M}}_{\text{split}, \tilde{\Gamma}}(L, p_{\tilde{\Gamma}}, \eta).$$

Quotienting by the action of the tropical symmetry group defines the reduced moduli space

$$\mathcal{M}_{\text{split}, \tilde{\Gamma}}(L, \eta_0) := \widetilde{\mathcal{M}}_{\text{split}, \tilde{\Gamma}}(L, \eta)/T_{\text{trop}}(\tilde{\Gamma}).$$

The moduli space of framed split maps of type $\tilde{\Gamma}$ modulo the action of domain reparametrizations is denoted

$$\widetilde{\mathcal{M}}_{\text{split, fr}, \tilde{\Gamma}}(L, p_{\Gamma}, \eta).$$

The quotient $\widetilde{\mathcal{M}}_{\text{split, fr}, \tilde{\Gamma}}/T_{\text{trop, fr}}(\tilde{\Gamma})$ is equal to the reduced moduli space $\mathcal{M}_{\text{split, fr}, \tilde{\Gamma}}(L, \eta_0)$. This ends the definition.
Remark 11.28. If the split tropical graph $\tilde{\Gamma}$ is rigid then $T_{\text{trop, fr}}(\tilde{\Gamma})$ is a finite group, and $\tilde{M}_{\text{split, fr}}(\tilde{\Gamma}, L, \eta)$ is a finite cover of the reduced moduli space.

Remark 11.29. The expected dimension of strata of split maps can be computed as follows. Let $(\tilde{\Gamma}, \Gamma)$ be a type of split map. Let $\Gamma \neq 0$ be the broken map type obtained by collapsing edges $e$ in $\tilde{\Gamma}$ that do not appear in $\Gamma$ and which have non-zero slope $T(e)$. That is, the edges $\{e \in \text{Edge}_*(\tilde{\Gamma}) \setminus \text{Edge}_*(\Gamma) : T(e) \neq 0\}$ are collapsed to produce $\Gamma \neq 0$. Then the moduli space of split maps $\tilde{M}_{\text{split}}(\tilde{\Gamma}, L, \eta)$ with boundary end points $x \in \mathcal{I}(L)^{d(\omega)}$ has expected dimension

\[ \tilde{i}_{\text{split}}(\tilde{\Gamma}, x) = i(\Gamma \neq 0, x) + 2 |\text{Edge}_*(\Gamma)| |(\dim(t) - 1)|. \]

Here $i$ is the index function for types of broken maps defined in (5.15). Indeed dropping the toric matching condition adds $2 |\text{Edge}_*(\Gamma)| |(\dim(t) - 1)|$ dimensions for each split edge. Gluing along edges $e$ with non-zero slope $T(e) \neq 0$ doesn’t change dimension as proved in Proposition 5.18. For a rigid split map, the tropical symmetry group $T_{\text{trop}}(\tilde{\Gamma}, \Gamma)$ has dimension $2 |\text{Edge}_*(\Gamma)| |(\dim(t) - 1)|$. The expected dimension of the quotiented moduli space is then

\[ i_{\text{split}}(\tilde{\Gamma}, x) = \tilde{i}_{\text{split}}(\tilde{\Gamma}, x) - 2 |\text{Edge}_*(\Gamma)| |(\dim(t) - 1)| = i(\Gamma \neq 0, x). \]

This ends the Remark.

Proposition 11.30. (Moduli spaces of split maps) Let $\eta_0 \in t'$ be a generic element. There is a comeager subset $\mathcal{P}_{\text{reg}}^{\tilde{\Gamma}} \subset \mathcal{P}_{\tilde{\Gamma}}$ of coherent perturbations for split maps such that if $p_{\tilde{\Gamma}} \in \mathcal{P}_{\text{reg}}^{\tilde{\Gamma}}$ then the following holds. Let $\tilde{\Gamma} \to \Gamma$ be a type of an uncrowded split map, and let $x \in \mathcal{I}(L)^{d(\omega)}$ be boundary end points for which the index $i_{\text{split}}(\tilde{\Gamma} \to \Gamma, x)$ is $\leq 1$. Then

(a) the moduli space of split maps $\tilde{M}_{\text{split, fr}}(p_{\tilde{\Gamma}}, L, \eta_0)$, framed split maps $\tilde{M}_{\text{split, fr}}(p_{\tilde{\Gamma}}, L, \eta_0)$ and the reduced moduli space $M_{\text{split, fr}}(p_{\tilde{\Gamma}}, L, \eta_0)$ are manifolds of expected dimension.

(b) The moduli space $M_{\text{split, fr}}(p_{\tilde{\Gamma}}, L, \eta_0)$ is compact if $i_{\text{split}}(\tilde{\Gamma} \to \Gamma, x) = 0$. If this index is 1, then the compactification $\overline{M}_{\text{split, fr}}(p_{\tilde{\Gamma}}, L, \eta_0)$ consists of codimension one boundary points which contain a boundary edge $e \in \text{Edge}_*(\tilde{\Gamma})$ with $\ell(e) = 0$ or $\ell(e) = \infty$.

Proof of Proposition 11.30. The proof of the first statement is by a Sard-Smale argument similar to the case of broken maps. The only variation is that there is no matching condition on the split edges.

We prove the compactness result for the moduli space of framed split maps since that is a finite cover over the reduced moduli space. The compactness statement for framed split maps can be proved in exactly the same way as the corresponding proof for broken maps. In fact, given a sequence of framed split maps $u_\nu$ of the same type, the limit is obtained by translations $\{t_v(\nu)\}_{v \in \text{Vert}(\tilde{\Gamma})}$ that satisfy the slope condition (7.26) for all edges, including split edges. Consequently, the limit is a framed split
map $u$. The codimension one boundary strata are as required in the Proposition via the arguments in the proof of Proposition 7.39. The finiteness of the tropical symmetry group used in the proof of Proposition 7.39 is replaced by the finiteness of the framed symmetry group $T_{trop,fr}(\tilde{\Gamma})$ of the split tropical graph $\tilde{\Gamma}$. \hfill \Box

**Proposition 11.31.** Given $E > 0$, there are finitely many types of split maps that have energy $\leq E$.

The proof is identical to the corresponding result Proposition 7.33 for broken maps, and is omitted.

Next we define the tropical Fukaya algebra whose composition maps are defined using counts of rigid split maps. In order to obtain homotopy equivalence with other versions of Fukaya algebras, the definition must use framed split maps. We instead use a count of split maps in the reduced moduli space weighted by the size of the discrete tropical symmetry group, since the latter is a finite quotient of the former.

**Definition 11.32.** Let $\eta_0 \in t^\vee$ be a generic cone direction, and let $p$ be a regular perturbation datum for all base tropical types. The *tropical Fukaya algebra* is the graded vector space

$$CF_{trop}(L, \eta_0) := CF_{geom}(L) \oplus \Lambda x^\ast[1] \oplus \Lambda x^\nu$$

equipped with composition maps

$$(11.23) \quad m_{d(c)}(x_1, \ldots, x_{d(c)}) = \sum_{x_0, u \in M_{\Gamma,split}(L, D), x_0} \text{mult}(\tilde{\Gamma})w(u)x_0$$

where $w(u)$ is as in (9.8), the combinatorial type $\tilde{\Gamma}$ of the split map $u$ ranges over all rigid types with $d(c)$ inputs (see Definition 11.14 for rigidity), and $\text{mult}(\tilde{\Gamma}) := |T_{trop,fr}(\tilde{\Gamma})|$ is the multiplicity of the split map.

By Proposition 11.30, the $A_\infty$-axioms are satisfied, and so, $CF_{trop}(L, p, \eta_0)$ is an $A_\infty$-algebra. The $A_\infty$-algebra is convergent by Proposition 11.31 and compactness of moduli spaces in Proposition 11.30.

**Remark 11.33.** The composition map $m_d^{trop}$ can be expressed as a sum of products. The sum is over rigid types $\tilde{\Gamma}$ of split maps, and the product is over the connected components $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_{s(\tilde{\Gamma})}$ of the graph $\tilde{\Gamma} \setminus \text{Edge}_s(\Gamma)$. Such a decomposition is possible because split maps do not have any edge matching conditions. Therefore, for any type $\tilde{\Gamma}$ of a rigid tropical map, the reduced moduli space $M_{\Gamma,split}(L, \eta)$ is a product of reduced moduli spaces:

$$M_{\Gamma,split}(L, \eta) = \prod_{i=1}^{s(\tilde{\Gamma})} M_{\tilde{\Gamma}_i}.$$ 

Here, $M_{\tilde{\Gamma}_i}$ is the quotient

$$M_{\tilde{\Gamma}_i} := \tilde{M}_{\tilde{\Gamma}_i} / T_{trop}(\tilde{\Gamma}_i, \Gamma_i),$$
of the moduli space $\mathcal{M}_{\tilde{\Gamma}_i}$ of maps modelled on the subgraph $\tilde{\Gamma}_i \subset \tilde{\Gamma}_i$ by the component of the tropical symmetry arising from $\tilde{\Gamma}_i$ as in (11.17). Thus the composition map decomposes as

$$m^d_{\text{split}} = \sum_{\tilde{\Gamma} : d(c(\tilde{\Gamma})) = d} m^d_{\text{split}, \tilde{\Gamma}}, \quad m^d_{\text{split}, \tilde{\Gamma}} = (-1)^{\varnothing \text{mult}(\tilde{\Gamma}) \frac{s(\tilde{\Gamma})}{d(\tilde{\Gamma})!}} \prod_{i=1} m_{\tilde{\Gamma}_i},$$

where

- the sum ranges over all rigid types $\tilde{\Gamma}$ of split maps with $d$ inputs
- and

$$m_{\tilde{\Gamma}_i} := \sum_{u \in \mathcal{M}_{\tilde{\Gamma}_i}(L, \eta_0)} y(\partial u)\epsilon(u)q^{A(u)}$$

is a weighted count of rigid elements in $\mathcal{M}_{\tilde{\Gamma}_i}$, where $y$, $\epsilon$ are as in (9.8).

11.4. **Homotopy equivalence: deformed to split.** Consider a sequence of rigid deformed maps of a fixed type, whose deformation parameters go to infinity in the complex torus $T_{\Gamma_0} := \prod_{e \in \text{Edge}_s(\Gamma_0)} T_{e} / T_{(e)}$. If the deformation parameters approach infinity along a generic direction, we will prove that a subsequence converges to a split map. The cone direction of the split map is the direction in which the deformation parameters approach infinity. Deformed maps in the sequence have index zero, whereas the index of the split map is at least the dimension of its tropical symmetry group, which is at least $\dim(T_{\Gamma_0} \mathcal{C})$. This seems to conflict with the fact that gluing edges $e$ of non-zero slope $T(e) \neq 0$ produces a type with the same index $\Gamma'$ (Proposition 5.18). The conflict is resolved when we realize that, after accounting for multiplicities, gluing an equivalence class of split maps produces deformed maps for all deformation parameters close to infinity in a neighbourhood of the cone direction. Thus in this section, we produce a cobordism from the framed moduli space of split maps to the space of $\tau$-deformed maps for large deformation parameters $\tau$.

Given a cone direction we define a notion of compatibility for a sequence of deformation data, to ensure that the deformed maps converge to a split map with the given cone direction.

**Definition 11.34.** (Compatible deformation data for a cone direction) Given a generic cone direction $\eta_0 \in \hat{t}^{\vee}$, a sequence of deformation data

$$\eta_{\Gamma, \nu} : \mathcal{M}_{\Gamma} \to \hat{t}_\Gamma \simeq \bigoplus_{e \in \text{Edge}_s(\Gamma)} t_{\tilde{T}(e)}^+, \quad c_{\Gamma, \nu} : \pi^+_{\tilde{T}(e)}(\eta_0),$$

for a curve type $\tilde{\Gamma}$ with base $\Gamma$ is compatible with the cone direction $\eta_0$ if

$$c_{\Gamma, \nu} : \mathcal{M}_{\tilde{\Gamma}} \times \text{Edge}_s(\Gamma) \to \mathbb{R}_+$$
is a continuous function such that for any converging sequence \( m_\nu \to m \) in \( \overline{\mathcal{M}}_\Gamma \), 
\( c_{\Gamma, \nu}(m_\nu, \cdot) \) is an increasing tuple (as in Definition 11.7 of split graphs).

For the gluing proof, we need a more restrictive class of deformation data.

**Definition 11.35.** (Uniformly continuous deformation data) A sequence of deformation data \( \eta \) compatible with a cone direction \( \eta_0 \) is uniformly continuous if for any split tropical graph \( \tilde{\Gamma} \) there is a constant \( k \) such that for any two curves \( C, C' \) of type \( \tilde{\Gamma} \)
\[
|\eta_{\tilde{\Gamma}, \nu}([C]) - \eta_{\tilde{\Gamma}, \nu}([C'])| \leq k \sum_{e \in \text{Edge}_{\tilde{\Gamma}, \cdot}} |\ell(T_e, [C]) - \ell(T_e, [C'])|
\]
for all \( \nu \). We recall that \( \ell(T_e, [C]) \) is the length of the treed segment \( T_e \) in the curve \( C \).

**Lemma 11.36.** Given a generic cone direction \( \eta_0 \in t' \), there exists a compatible sequence \( \{\eta_\nu\}_\nu \) of uniformly continuous coherent deformation data.

**Proof.** We first describe connected components of the moduli space of based curves. A connected component is determined by a tuple \( \gamma := (\Gamma, m, n, \gamma_0) \) consisting of the base tropical graph \( \Gamma \), the number \( m \) of boundary markings, the number \( n \) of interior markings, and a function \( \gamma_0 : \{1, \ldots, n\} \to \text{Vert}(\Gamma) \) that maps a marked point \( z_i \) on a curve component \( C_v \subset C \) to \( \kappa(v) \in \text{Vert}(\Gamma) \), and we denote the component by \( \mathcal{M}_\gamma \). The sequence of functions
\[
c_{\gamma, \nu} : \mathcal{M}_\gamma \times \text{Edge}_\gamma(\Gamma) \to \mathbb{R}
\]
is constructed by induction on \( m, n, |\text{Vert}(\Gamma)| \). We consider a connected component \((\Gamma, n, m, \gamma)\), and assume that a sequence of coherent data has been constructed for smaller types. The data on the smaller types determines \( c_{\gamma, \nu} \) on the true boundary of \( \mathcal{M}_\gamma \), which consists of curves with at least a single broken boundary edge.

The deformation sequence on \( \mathcal{M}_\gamma \) is constructed via a partition
\[
\mathcal{M}_\gamma = \bigcup_{i \geq 0} \mathcal{M}_\gamma^i,
\]
where \( \mathcal{M}_\gamma^i \) consists of treed curves with exactly \( i \) boundary edges. On \( \mathcal{M}_\gamma^0 \), for any split edge \( e \) we fix \( c_{\gamma, \nu}(e) \) to be a sequence of constant functions whose values satisfy the increasing condition (11.5). Let
\[
C_\nu := \sup \{ c_\nu(e, m) : m \in \mathcal{M}_\gamma^0 \cup \partial \mathcal{M}_\gamma \}.
\]

We extend the deformation sequence to all of \( \mathcal{M}_\gamma \) by interpolating between the codimension one strata \( \partial \mathcal{M}_\gamma^0 \) and \( \partial \mathcal{M}_\gamma \). The boundary \( \partial \mathcal{M}_\gamma^0 \) partitions into sets
\[
\partial \mathcal{M}_\gamma^0 = \bigcup_{i \geq 1} \mathcal{M}_\gamma^{0,i}, \quad \mathcal{M}_\gamma^{0,i} := \overline{\mathcal{M}_\gamma^{0,i}} \cap \mathcal{M}_\gamma^i,
\]
and \( \mathcal{M}_\gamma^i \) is a product
\[
(11.25) \quad \mathcal{M}_\gamma^i = \mathcal{M}_\gamma^{0,i} \times [0, \infty)^i,
\]
such that projection to the second factor is equal to the edge length function. We first consider $M^1_\gamma$. The function $c_{\nu,e}$ is fixed to be locally constant on the boundary $M^0_\gamma \times \{0, \infty\}$. Under the splitting (11.25) we define

\begin{equation}
(11.26)
c_{\nu,e}(m,t) := (1 - \tau_{\nu,e}(t))c_{\nu,e}(m,0) + \tau_{\nu,e}(t)c_{\nu,e}(m,\infty), \quad (m,t) \in M^0_\gamma \times [0, \infty]
\end{equation}

where $\tau_{\nu,e} : [0, \infty] \to [0, 1]$ is a diffeomorphism. In (11.26), $c_{\nu,e}$ satisfies the (Marking independence) property (10.2) because on each connected component it factors through the projection $M^1_\gamma \to [0, \infty]$. Indeed $c_{\nu,e}(m,\infty)$ is locally constant on $\partial M^1_\gamma \cap M^1_\gamma$ by the (Marking independence) property (10.2) property for smaller strata. We obtain uniform continuity on $c_{\nu,e}$ by requiring that the derivative of $\tau_{\nu,e}$ is bounded by $C^{-1}_{\nu}$ for all $\nu$. This can be arranged by allowing $d\tau_{\nu,e}$ to be supported in an interval of length $C_{\nu}$. In a similar way, the functions $c_{\nu,e}^j$ are extended to $M^i_\gamma$, assuming that its value on the boundary $\partial M^i_\gamma$ factors through the projection to $[0, \infty]^i$.

\begin{proposition}
(Convergence) Let $\eta_0 \in \mathcal{T}'$ be a generic cone direction, and let $\{\eta_{\nu}\}_\nu$ be a coherent sequence of deformation data compatible with $\eta_0$ (as in Definition 11.34). Let $u_\nu : C_\nu \to X_P$ be a sequence of $\eta_{\nu}$-deformed maps with uniformly bounded area. Then, a subsequence converges to a framed split map $u_\infty : C \to X_P$ of type $(\tilde{\Gamma}_{\infty}, \Gamma)$. The limit map is unique up to the action of the identity component of the framed tropical symmetry group $T_{\text{trop, fr}}(\tilde{\Gamma}_{\infty}, \Gamma)$ (see (11.19)). If the maps $u_\nu$ have index 0, and the perturbation datum for the limit is regular, then, the split tropical graph $\tilde{\Gamma}_{\infty}$ is rigid.
\end{proposition}

If the limit split tropical graph is rigid, the limit is unique, because the framed symmetry group $T_{\text{trop, fr}}(\tilde{\Gamma}_{\infty}, \Gamma)$ is finite.

\begin{proof}[Proof of Proposition 11.37] As in the convergence of broken maps, the first step is to find the component-wise limit map. By Proposition 7.33, there are finitely many types of deformed maps that satisfy an area bound. Therefore, after passing to a subsequence, we can assume that the maps $u_\nu$ have a $\nu$-independent type $\tilde{\Gamma} \to \Gamma$, with deformation datum

\begin{equation}
\eta_{\nu}(e) := \eta_{\tilde{\Gamma}, \nu}([C_\nu], e) \in t_{\tilde{\Gamma}_\nu}(e), \quad \forall e \in \text{Edge}_s(\Gamma).
\end{equation}

We apply Gromov convergence for broken maps on each connected component of $\tilde{\Gamma} \setminus \text{Edge}_s(\Gamma)$. The collection of the limit maps, denoted by $u_\infty$, is modelled on a quasi-split tropical graph $\tilde{\Gamma}_{\infty} \to \Gamma$, equipped with a tropical edge collapse morphism $\kappa : \tilde{\Gamma}_{\infty} \setminus \text{Edge}_s(\Gamma) \to \tilde{\Gamma} \setminus \text{Edge}_s(\Gamma)$. The translation sequence

\begin{equation}
t_{\nu}(v) \in \text{Cone}(\kappa, v), \quad v \in \text{Vert}(\tilde{\Gamma}_{\infty})
\end{equation}

satisfies the (Slope) condition for all non-split edges $e \in \text{Edge}(\tilde{\Gamma}_{\infty}) \setminus \text{Edge}_s(\Gamma)$.

The limit map $u_\infty$ is a quasi-split map, and it remains to show that the cone constraint (11.6) is satisfied. The sequence of relative weights required by the cone
condition will be a refinement of the translation sequences $t_\nu$. For any split edge $e$, the sequences

$$(11.27) \quad (t_\nu(v_+) - t_\nu(v_-)) \mod T(e) - \eta_\nu(e), \quad e = (v_+, v_-) \in \text{Edge}_s(\Gamma)$$

are uniformly bounded. This fact can be proved in the same way as the proof of boundedness of the sequences in (7.30) as part of the proof of convergence of broken maps. The next step is to adjust the translation sequences by a uniformly bounded amount so that the discrepancy at the split edges is equal to the deformation parameters, i.e. the sequences in (11.27), vanish. The adjustment to the translation sequences is by an iterative process described in the proof of Proposition 11.9. We run the iteration on the sequences $\{t_\nu(v)\}_{v, \nu}$ and $\{\eta_\nu(e)\}_{e, \nu}$ and in each step a fastest growing sequence is subtracted. We use the same notation as in the proof of Proposition 11.9, and recall that the resulting sequence in the $i$-th step is denoted by $\{t^i_\nu(v)\}_{v}$, $\{\eta^i_\nu(e)\}_{e}$. The iteration runs till the sequences corresponding to all edges and vertices are uniformly bounded, which happens after, say, $k$ steps. The edge sequence $\eta^k_\nu(e)$ vanishes for all split edges $e$. The iterative step (11.12) is such that an unsigned relative weight $\Delta t^i_\nu$ is subtracted at each step, and therefore

$$\bar{t}_\nu := t_\nu - t^k_\nu = \sum_{i=1}^{k} \Delta t^i_\nu$$

is an unsigned relative weight. Since $t^k_\nu$ is uniformly bounded for all $\nu$, $\bar{t}_\nu$ is a relative weight for large $\nu$. Further,

$$(11.28) \quad \text{Diff}(\bar{t}_\nu) = \eta_\nu(e), \quad e \in \text{Edge}_s(\Gamma).$$

Since for any vertex $v$ the translation sequences $t_\nu(v)$, $\bar{t}_\nu(v)$ differ by a uniformly bounded amount, a subsequence $e^{-\bar{t}_\nu_{\nu}}$ converges to a limit $\bar{u}_{\infty,v}$. The limit map $\bar{u}_{\infty} := (\bar{u}_{\infty,v})_v$ satisfies the matching condition on all edges, including split edges, because by (11.28) the translations $\bar{t}_\nu$ compensate for the deformation parameters. Finally $\bar{\Gamma} \to \Gamma$ is a split graph because

$$\text{Diff}(\bar{t}_\nu) = \eta_\nu \implies \eta_\nu \in \text{Cone}(\bar{\Gamma}, \Gamma) \quad \forall \nu.$$

We have shown that $\bar{u}_{\infty}$ is a framed split map. Uniqueness of limits is same as in the convergence of broken maps. The last statement about the rigidity of the limit split map follows from a dimensional argument, see Remark 11.29.

**Remark 11.38.** We justify our definition of split edges in Section 10.1. Consider a sequence of deformed maps converging to a split map. For a deformed node lying in a space $X_P$, one of the lifts of the node will collapse into a torus-invariant divisor $D \subset X_P$ in the limit split map. Firstly we want $D$ to be a boundary divisor. This ensures that in the limit there is bubbling in the sense of broken maps leading to the required amount of tropical symmetry. Secondly assuming $D = X_Q$ for a polytope $Q \subset P$, we would like the same torus actions on the thickened spaces $X_P$ and $X_Q$, justifying our requirement that $X_P$ possess a $T/T_P$-action.

Gluing a rigid split map with a cone direction $\eta_0$ yields a sequence of deformed maps for any sequence of uniformly continuous deformation data compatible with $\eta_0$. As in the set-up of the gluing construction for broken maps (Theorem 8.1), we
assume that the perturbation datum for the deformed maps is given by gluing the perturbation datum of the split map.

**Proposition 11.39.** (Gluing a split map) Suppose $u$ is a regular rigid framed split map of type $\tilde{\Gamma} \to \Gamma$ with a generic cone direction $\eta_0 \in t'$. Let $\eta_{\nu} = \{\eta_{\tilde{\nu}}\}_{\tilde{\Gamma}}$ be a sequence of uniformly continuous coherent deformation data compatible with $\eta_0$ (as in Definition 11.34). Then, there exists $\nu_0$ such that for $\nu \geq \nu_0$

(a) (Existence of glued family) there is a regular rigid $\eta_{\nu}$-deformed map $u_{\nu}$ such that the sequence $\{u_{\nu}\}_\nu$ of deformed maps converges to $u$.

(b) (Surjectivity of gluing) For any sequence of $\eta_{\nu}$-deformed maps $u'_{\nu}$ that converge to the framed split map $u$, for large enough $\nu$ the map $u'_{\nu}$ is contained in the glued family constructed in (a).

**Proof.** The type of the deformed map that occurs in the glued family is denoted by $\tilde{\Gamma}_d$, and is obtained by tropically collapsing all interior edges in $\tilde{\Gamma}$ that do not occur in the base tropical type $\Gamma$. The edge collapse map $\kappa$ factors as

$$\tilde{\Gamma} \xrightarrow{\kappa_0} \tilde{\Gamma}_d \xrightarrow{\kappa_1} \Gamma.$$ 

The tree part of $\tilde{\Gamma}_d$ is same as $\tilde{\Gamma}$, and the interior edges in $\tilde{\Gamma}_d$ are same as that of $\Gamma$. We will construct a family of deformed maps of type $\tilde{\Gamma}_d \to \Gamma$.

The gluing construction is very similar to the gluing of broken maps in Theorem 8.1, so we only point out the differences at each step of the proof.

**Step 1: Construction of an approximate solution:** An approximate solution is constructed using relative weights corresponding to a deformation parameter. The domain $C'_{\nu}$ of the approximate solution is constructed by gluing some of the interior nodes in $C$, and therefore the edge lengths for treed segments $T_e, e \in \text{Edge}_{\nu}(\tilde{\Gamma})$ in $C'_{\nu}$ are the same as $C$. By the (Marking independence property) of deformation data, $\eta_{\nu,\tilde{\Gamma}_d}(C) = \eta_{\nu,\tilde{\Gamma}_d}(C')$, and we denote

$$\eta_{\nu} := \eta_{\nu,\tilde{\Gamma}_d}(C) \in t_\Gamma.$$

By Proposition 11.9, for large enough $\nu$, there is a relative weight $\mathcal{T}_\nu$ satisfying

$$\pi_{\tilde{\Gamma}(e)}(\mathcal{T}_\nu(v_+) - \mathcal{T}_\nu(v_-)) = \eta_{\nu}(e), \quad e = (v_+, v_-) \in \text{Edge}_{\nu}(\tilde{\Gamma}).$$

Since the split tropical graph $\tilde{\Gamma}$ is rigid, the relative weight $\mathcal{T}_\nu \in \mathcal{W}(\tilde{\Gamma}, \Gamma)$ is uniquely determined by (11.29). Indeed, if there is another solution $\mathcal{T}'$ of (11.29), the difference $\mathcal{T} - \mathcal{T}'$ satisfies the slope condition on split edges and therefore generates a non-trivial subgroup $\exp((\mathcal{T} - \mathcal{T}')(\cdot))$ in $T_{\text{trop,fr}}(\tilde{\Gamma})$ contradicting the rigidity of $\tilde{\Gamma}$. For future use, we point out that the inverse

$$\text{Diff}^{-1}_{\tilde{\Gamma}} : t_\Gamma \to \mathcal{W}^\pm(\tilde{\Gamma}, \Gamma),$$

which maps deformation parameters to unsigned relative weights, is a well-defined linear map. For any edge $e = (v_+, v_-)$ that is collapsed by $\tilde{\Gamma} \to \tilde{\Gamma}_d$, we can assign a length $l_\nu(e) > 0$ satisfying

$$\mathcal{T}_\nu(v_+) - \mathcal{T}_\nu(v_-) = l_\nu(e)\mathcal{T}(e).$$
The domain and target spaces for the approximate solution are as follows. The domain is a nodal curve \( C^v \) of type \( \tilde{\Gamma}_d \) which is obtained from \( C \) by replacing each node corresponding to an edge \( e \in \text{Edge}_e(\tilde{\Gamma}) \setminus \text{Edge}_e(\tilde{\Gamma}_d) \) by a neck of length \( l_\nu(e) \). A component of the glued curve corresponds to a vertex \( v \) of \( \tilde{\Gamma}_d \). The map \( u|C_{\kappa_0^{-1}(v)} \) is a broken map with relative marked points, whose pieces map to \( X_{\tilde{\Gamma}} \), \( Q \subseteq P(v) \).

Our goal is to glue at the nodes of \( u|C_{\kappa_0^{-1}(v)} \) to produce a curve lying in \( X_{\tilde{\Gamma}(v)} \).

We extend the definition of the deformation parameter to all edges in \( \Gamma \) by defining
\[
\eta_\nu(e) = 0 \quad \forall e \in \text{Edge}_e(\Gamma) \setminus \text{Edge}_s(\Gamma).
\]
Thus by definition a \( \eta_\nu \)-deformed map satisfies the edge matching condition for non-split edges of \( \Gamma \).

Next, we construct the approximate solution which will shown to be \( \eta_\nu \)-deformed. Since \( u \) is a framed split map it satisfies a matching condition on the split edges: For any edge \( e = (v_+, v_-) \) in \( \tilde{\Gamma}_d \)
\[
(11.31) \quad \operatorname{ev}_{w_+}(\pi_+^{\perp}(e) \circ u_{v_+}) = \operatorname{ev}_{w_-}(\pi_+^{\perp}(e) \circ u_{v_-})
\]
For a vertex \( v \) in \( \tilde{\Gamma}_d \), and a vertex \( v' \) in the collapsed graph \( \kappa_0^{-1}(v) \), the translation \( T_\nu(v') \) gives an identification (see (2.33))
\[
\nu^T_{\nu}(v') : X^o_{\tilde{\Gamma}(v')} \to X^o_{\tilde{\Gamma}(v)},
\]
where \( X^o_{\tilde{\Gamma}(v')} \subset X_{\tilde{\Gamma}(v)} \) is the complement of the boundary divisors of \( X_{\tilde{\Gamma}(v')} \). The translated map
\[
(11.32) \quad u_{v', \eta_\nu} := \nu^T_{\nu}(v')u_{v'} : C^o_{\nu} \to X^o_{\tilde{\Gamma}(v)}
\]
is well-defined on the complement of nodal points on \( C^\nu \). For any uncollapsed interior edge \( e = (v_+, v_-) \) in \( \tilde{\Gamma}_d \), the matching condition (11.31) and the translation in (11.32) together imply a deformed matching condition
\[
(11.33) \quad \operatorname{ev}_{w_+}(\pi_+^{\perp}(e) \circ u_{v_+; \eta_\nu}) = \nu_\nu(e) \operatorname{ev}_{w_-}(\pi_+^{\perp}(e) \circ u_{v_-; \eta_\nu}),
\]
because \( \eta_\nu(e) = T_\nu(v_+) - T_\nu(v_-) \). As in the proof of Theorem 8.1, for each vertex \( v \) translated maps \( (u_{v'; \eta_\nu})_{v' \in \kappa_0^{-1}(v)} \) can be glued at the nodal points to yield an approximate solution for the holomorphic curve equation which is denoted by
\[
u_\nu^\text{pre} = \left( u_{v', \eta_\nu} \right)_{v \in \text{Vert}(\tilde{\Gamma}_d)}, \quad \nu_\nu^\text{pre} : C_{\nu; v} \to X_{\tilde{\Gamma}(v)}.
\]
Since the patching does not alter the maps \( u_{v, \eta_\nu} \) away from the collapsed nodes of \( \tilde{\Gamma} \to \tilde{\Gamma}_d \), we conclude by (11.33) that the pre-glued map \( u_\nu^\text{pre} \) is \( \eta_\nu \)-deformed.

**Step 2: Fredholm theory:** The Sobolev norms carry over entirely from the proof of Theorem 8.1. On the curves \( C^\nu \), we use Sobolev weights on both neck regions created by gluing, and on nodal points corresponding to interior edges in \( \tilde{\Gamma}_d \). The map \( \mathcal{F}_\nu \) in Theorem 8.1 incorporated the holomorphicity condition, marked points mapping to divisors and matching at disk nodes. Now we additionally require a deformed matching condition on interior nodes, given by
\[
\operatorname{ev}_e(u) \in \Delta_e \subset (Z_P(e) \cap T_P(e))/2, \quad e \in \text{Edge}_e(\tilde{\Gamma}_d),
\]
where $\Delta_e$ is the diagonal and $e v_e$ is defined as

$$\text{ev}_e : M_{\Gamma_d} \times \text{Map}(C^\nu, X)_{1,p,\lambda} \to (Z_{P(e),C}/TP(e),C)^2,$$

$$(m, u) \mapsto (\pi_{1,T(e)}(\text{ev}_e u, w_+(w_+(e)))) \exp(\eta_{v-e}(m, e)) \pi_{1,T(e)}(\text{ev}_e w_-(w_-(e))).$$

Incorporating these conditions we obtain a map

$$\mathcal{F}_\nu : M_{\Gamma_d}^\nu \times \Omega^0(C, (u_{\nu}^{\text{pre}})^*TX)_{1,p} \to \Omega^0(C, (u_{\nu}^{\text{pre}})^*TX)_{0,p} \oplus \text{ev}_1^* TX(\tilde{\Gamma}_d)/\Delta(\tilde{\Gamma}_d)$$

whose zeros correspond to $\eta_{\nu}$-deformed pseudoholomorphic maps near the approximate solution $u_{\nu}^{\text{pre}}$, and where the notations $X(\tilde{\Gamma}_d)$ and $\Delta(\tilde{\Gamma}_d)$ are defined.

Step 3: Error estimate: The error estimate for the approximate solution is produced in the same way as Theorem 8.1. Indeed the only contribution to the error estimate is from the failure of holomorphicity, as the approximate solution satisfies the matching conditions at boundary and interior nodes.

The next few steps of construction of a uniformly bounded right inverse, quadratic estimates, and application of the Picard iteration are same as in Theorem 8.1.

Step 7: Surjectivity of gluing: Compared to the corresponding step in Theorem 8.1, in the gluing of split map one additionally has to deal with deformation parameters. Suppose $u'_{\nu} : C'_{\nu} \to X$ is a sequence of $\eta_{\nu}$-deformed maps that converges to the split map $u$. To prove that the maps $u'_{\nu}$ lie in the image of the gluing map of $u$, it is enough to show that $u'_{\nu}$ is close enough to the pre-glued map $u_{\nu}^{\text{pre}} : C_{\nu} \to X$. Since $C'_{\nu} \to C$ in the compactified moduli space $\mathcal{M}_{\Gamma_d}$, the edge lengths $\ell(C'_{\nu}, T_e)$ on the treed segments converge to a finite limit $\ell(C, T_e)$ for all boundary edges $e \in \text{Edge}_{\nu}(\tilde{\Gamma}_d)$. Therefore the differences converge

$$|\ell(C'_{\nu}, T_e) - \ell(C_{\nu}, T_e)| \to 0.$$ 

The deformation parameters for $u'_{\nu}$ are $\eta'_{\nu} := \eta_{\Gamma_d}(C'_{\nu})$. On a fixed stratum $\tilde{\Gamma}_d$ of curves, the deformation parameter depends only on the lengths of the treed edges by the (Marking independence) property (10.2), see (10.2). Together with the uniform continuity of the deformation datum, we conclude $|\eta'_{\nu} - \eta_{\nu}'| \to 0$. By the proof of convergence of deformed maps, the translation sequence $\{\nu_{\nu}(v)\}_{\nu,v}$ for the convergence of $\{u'_{\nu}\}_\nu$ can be chosen to satisfy

$$t_{\nu}(v_+) - t_{\nu}(v_-) = \eta'_{\nu}(e), \quad \forall e = (v_+, v_-),$$

see (11.28). The inverse $\text{Diff}_{\tilde{\Gamma}}^{-1}$ is a well-defined map (see (11.30)) that maps

$$\eta_{\nu} \mapsto \mathcal{F}_\nu, \quad \eta'_{\nu} \mapsto t_{\nu}.$$

Hence $\mathcal{T}_\nu - t_{\nu} \to 0$. As in Theorem 8.1 the closeness of relative weights $\mathcal{T}_\nu$, $t_{\nu}$ implies that the difference between the logarithmic neck lengths $\nu_{\nu}(e)$ of $C_{\nu}, C'_{\nu}$ converge to 0. The rest of the proof is same as Theorem 8.1.

Proposition 11.40. Let $\eta_0 \in \mathcal{V}$ be a generic cone direction and let $\{\eta'_{\nu}\}_\nu$ be a sequence of uniformly continuous deformation data compatible with $\eta_0$. For any set of disk input/outputs $x$ and energy level $E > 0$, there exists $\nu_0$ such that for $\nu \geq \nu_0$ there is a bijection between the moduli space of rigid framed split disks
Proposition 11.41. Let $\eta_0$ be a generic cone direction, and let $\eta_\nu$ be a compatible sequence of deformation data. Then, for any $\nu$, there is an $A_\infty$-homotopy equivalence
\[ CF_{trop}(L, \eta_0) \to CF_{\text{def}}(L, \eta_\nu). \]

Proof. The proof is exactly the same as the proof of homotopy equivalence between unbroken and broken Fukaya algebras, see Proposition 9.17. The homotopy equivalence is constructed as a limit of homotopy equivalences using the bijection in Proposition 11.40 for lower order terms. □

Example 11.42. We continue Example 11.25. For the broken map $u$ in that example, we showed that forgetting the edge matching condition at the split edge $e$ yielded a split map denoted by $u^\text{split}$ for any choice of cone direction $\eta_0 \in t^\nu$. We now show that the split map $u^\text{split}$ can also be obtained as a limit of deformed maps that are homotopic to $u$. Two cone directions describe the same condition if one is a positive multiple of another. Therefore, in this example, there are two unique choices of cone condition: namely the two ends of the line $t/\langle T(e) \rangle \simeq \mathbb{R}^2/\langle (1,1) \rangle$, which we represent by $(1,-1), (-1,1) \in t^\perp_{T(e)}$. For $\tau_\nu \to \infty$, the sequence of deformation parameters
\[ \eta_\nu := (\tau_\nu, -\tau_\nu) \in t^\perp_{T(e)} \]

is compatible with the cone direction $(1,-1)$. We write the broken map $u$ as $(u_+, u_-)$ where $u_+, u_-$ are the restrictions to the connected components of $\tilde{\Gamma} \setminus \{e\}$. Then,
\[ u_\nu := (u_+, e^{-\tau_\nu \tau_\nu} u_-) \]
is an $\eta_\nu$-deformed map homotopic to $u$. As $\nu \to \infty$, the sequence converges to the split map $u^\text{split}$ with translation sequence $0$ on $v_+$, and $(\tau_\nu, -\tau_\nu)$ on the other two vertices.

11.5. The cone displacement formula. We relate split maps in the case of a codimension one edge splitting to the degeneration of the diagonal in a toric variety described by Fulton-Sturmfels [28]. Let
\[ \Delta_P \subset X_P \times X_P \]
denote the diagonal of a toric variety $X_P$ corresponding to a polytope $P$. Standard topology shows that the subvarieties $X_Q$ for $Q \subset P$ form a spanning set for the homology of $X_P$:
\[ \text{span}\{[X_Q], Q \subseteq P\} = H(X_P). \]
By the Künneth formula, the homology class of the diagonal admits a decomposition
\[ [\Delta_P] = \sum_{Q_-, Q_+ \subset P} n(Q_-, Q_+) [X_{Q_-}] \times [X_{Q_+}]. \]
Of course, there are many such decompositions since the classes \([X_Q]\) are linearly dependent. If \(X_P\) admits an invariant Morse-Smale pair, then such a decomposition may be obtained by Morse theory for a component of the moment map: In this case \(n(Q_-, Q_+)=1\) if \(X_{Q_-}\) and \(X_{Q_+}\) represent closures of stable resp. unstable manifolds of the Morse function, and zero otherwise. However, for most toric varieties, a generic component of the moment map is not part of a Morse-Smale pair. Regardless, Fulton-Sturmfels \([28]\) give one such formula for each choice of generic vector \(\eta\) \in \(t\) for any toric variety \(X_P\): Let \(\text{Cone}(Q_-) \oplus X_P\times X_Q^+\) = \(\text{pt}\) for an arbitrary \(\eta\). We write \(\text{Cone}(Q_-) \oplus X_P\times X_Q^+\) = \(\text{pt}\) if the intersection between the (displaced cones) is a transversely-cut-out point, and if so, let

\[
n(Q_-, Q_+) = \frac{(\text{span Cone}(Q_-) \cap t_Z) + (\text{span Cone}(Q_+) \cap t_Z)}{t_Z}
\]

the quotient of lattices in \(t\), necessarily finite.

**Theorem 11.43.** (Cone displacement formula, \([28]\)) There exists an equivalence in the Chow group

\[
\Delta \sim \sum_{n(Q_-, Q_+)=\{\text{pt}\}} n(Q_-, Q_+)(X_{Q_-} \times X_{Q_+}).
\]

**Example 11.44.** In the case of projective space one easily obtains the usual K"unneth decomposition

\[
[\Delta_{\mathbb{P}^n}] = \sum_{i=1}^{n} [\mathbb{P}^i] \times [\mathbb{P}^{n-i}].
\]

Note that our direction is not chosen to be generic across edges, but rather we require the ratios between the vectors for different split edges to be arbitrarily small. In its current form, our condition appears stronger than the Fulton-Sturmfels formula. It would be interesting to know whether there is a general formula for a “generic direction” in the tropical symmetry group.

**Remark 11.45.** Suppose that the set of split edges consists of a single split edge \(e\), \(P(e)\) is a polytope of codimension one with \(X_{P(e)}\) toric with polytope \(P(e)\). In this case, \(\tilde{P}(v_+), \tilde{P}(v_-)\) are faces of \(P(e)\), by definition any discrepancy cone \(\text{Diff}(W(\tilde{P}, \Gamma)))\) is contained in the cone difference. \(\text{Cone}\tilde{P}(v_+)(P(e)) - \text{Cone}\tilde{P}(v_-)(P(e))\). Thus the sum in our splitting formula is a refinement of the splitting formula in Fulton-Sturmfels \([?]\).

To see how this plays out in practice, consider a situation where the deformation parameter is taken to infinity and the deformed map develops additional components. Suppose that \(\tilde{P}(v_\pm)\) is the intersection of \(P(e)\) with facets \(Q_1^\pm, \ldots, Q_k^\pm\). Locally a map near \(X_{Q_\pm}^j\) is given by a section of the normal bundle. Consider a sequence \(u_{\pm, \nu} : C_{v_\pm} \to X_{P(v_\pm)}\) of components corresponding to the vertices \(v_\pm\) as stable maps. After passing to a subsequence we may assume the existence of a limit. If the limiting stable maps \(u_{\pm, \infty}\) is not contained in \(X_{Q_\pm}^j\) for any \(j\) then the
additional components in the split limit correspond to roots of sections of the normal bundle to $X_{P(v_{\pm})}$ falling into the divisor $X_{P(e)}$. Maximal dimensionality of the discrepancy cone shows that at least $k_+ + k_-$ such additional bubbles must form, and then a dimension count shows that a single root of some $Q_j$ is contained in each additional component. Indeed, more than one root occurs on some component then the configuration obtained by removing all components after $u_{\pm}$ would be of negative expected dimension. The corresponding tropical graph has edge vectors $\nu_1, \pm \nu_{k_\pm}, \pm \nu_1, \pm \nu_{k_\pm} - 1, \pm, \nu_1, \pm$ where $\nu_1, \ldots, \nu_k$ are the normal vectors to $Q_1, \ldots, Q_{k_\pm}$, given in the order that the roots fall into the face $Q_{\pm}$. Summing over all possible orders one sees that the union of these normal cones is the difference of cones appearing in the Fulton-MacPherson formula. The case that $u_{\pm}$ falls into some $X_{Q_j}$ of $X_{P(v_{\pm})}$, for which no additional components in the split limit arise from roots of the normal bundle of $X_{Q_j}$, is similar.

We extend the splitting formula to the case that not all divisors are boundary divisors as follows. We first recall the contrasting case of a single cut from [13], assuming that the boundary divisor $X_0 \subset X_+, X_-$ is a $T$-toric variety. We deform the matching condition by the $T$-action in a generic direction $\eta \in \mathfrak{t}^\vee$, which may alternately be viewed as deformation by the Morse function $\Phi_\eta := \langle \Phi, \eta \rangle$. In the limit the evaluations of split nodes are connected by broken Morse trajectories in $X_0$. This setting is an example where none of the toric divisors of $X_{P(e)}$ are boundary divisors. We study the “mixed” case where limits of Morse trajectories are either broken trajectories, or one of the ends of the trajectory maps to a boundary submanifold and contributes to the tropical symmetry group of the limit map.

For our generalization of split maps, we focus on the special case where $\dim(X) = 4$ and consequently

$$X_Q \simeq \mathbb{P}^1 \forall Q \in \mathcal{P}, \text{codim}(Q) = 1.$$ 

Such an $X_Q$ has two toric divisors, both points. If a toric divisor $Y \subset X_Q$ is not a boundary divisor, we call it a Morse submanifold, because it is either the stable or unstable submanifold of a critical point $p \in X_Q^\circ$ of the toric moment map.

**Definition 11.46. (Mixed split map)** Suppose $X$ is 4-dimensional. The mixed split map $u$ of base type $\Gamma$ consists of a partition of split edges

$$\text{Edge}_s(\Gamma) = \text{Edge}_s^{\text{Morse}}(u) \cup \text{Edge}_s^{\text{trop}}(u)$$

into Morse-type and tropical-type edges, satisfying

$$e \in \text{Edge}_s^{\text{Morse}}(u) \implies \text{codim}(P(e)) = 1.$$ 

The cone condition is a condition on the set of tropical-type edges in the following sense:

(a) The set of split edges underlying the tropical graph $\tilde{\Gamma} \to \Gamma$ of $u$ is $\text{Edge}_s^{\text{trop}}(u)$. In the statement of the (Cone condition) (11.3) the set of split edges $\text{Edge}_s(\Gamma)$ is replaced by the set of tropical-type split edges $\text{Edge}_s^{\text{trop}}(u)$.

(b) If $e \in \text{Edge}_s^{\text{Morse}}(u)$ is Morse-type, then for one of the lifts of the node $w_e$, the evaluation $u(w_{\pm}(e))$ lies on a Morse submanifold of $X_Q$. 
The proof of convergence of deformed maps to a split map extends to the case of mixed split maps. Given a sequence of deformed maps for an edge $e$ with $\text{codim}(P(e)) = 1$, the Morse trajectories connecting the lifts of the node $w_e$

- either converge to a broken trajectory in $X^P_{P(e)}$, in which case $e$ is a Morse-type edge in the limit split map,
- or at least one of the lifts $w_+(e), w_-(e)$ converges to a boundary divisor of $X_{P(e)}$, in which case, the analysis in the proof of Proposition 11.37 applies to prove the (Cone condition) (11.3).

We expect that mixed split maps can be defined even when $\dim(X) > 4$. The combinatorics will be more complicated since a split edge can simultaneously be both a Morse-type and a tropical-type edge.

![Figure 25. A long divisor $D$ and short divisors $E_1, E_2, F_1, F_2$. The short divisor $E_i$ is cut into spheres $E_i', E_i''$ of Chern number 1.](image)

**Example 11.47.** (Spheres in the cubic surface) We return to the Example 1.1 from the introduction, and consider the moduli space of pseudoholomorphic spheres of Chern number 1 in the toric surface shown. This corresponds to the space of lines in the cubic surface, since these are the spheres of minimal Chern number. Since the torus acts on the space of spheres in the toric surface, each homology class may be identified with a union of boundary divisors. One sees easily that Chern number one classes are those consisting of one “long” boundary divisor and an arbitrary number of “short” boundary divisors.

The basic result in Bryan-Leung [11] is that the only classes with non-zero Gromov-Witten invariant are those for which the short boundary divisors form a non-self-crossing path, in which case the Gromov-Witten invariant is one. Using the viewpoint of multiple cuts, the same result is a consequence of the following claim.

**Claim 11.48.** A broken map $u : C \to \mathcal{X}$ can not have the following components:

- (a) a sphere $u : C_0 \to X_{P_0}$ of class $D + kE_1$, $k > 1$,
- (b) a sphere $u : C_1 \to X_{P_1}$ of class $E_1' + kE_2$, $k > 1$,
- (c) a sphere $u : C_1 \to X_{P_1}$ of class $kE_1'' + E_2'$, $k > 1$.

**Proof of Claim.** The proof of the Claim is a homological argument. We consider a small perturbation $J$ of the standard almost complex structure. Suppose $u$ is a $J$-holomorphic curve as in (a). Then $[u].E_1 = 1 - k < 0$. By a Gromov-Witten argument there is an exceptional $J$-holomorphic sphere $u_1 : \mathbb{P}^1 \to X_{P_0}$ of homology class $E_1$. Since $[u].E_1 = 1 - k < 0$, the image of $u$ is contained in $\text{im}(u_1)$. This
is clearly a contradiction: For example, \( [u].F_1 = 1 \) but the image of \( u_1 \) does no intersect the divisor \( F_1 \). Part (b) resp. (c) can be proved by applying the same argument in \( X_{P_1} \), and by showing that the image of \( u \) is contained in the image of a \( J \)-holomorphic representative of \( E_2' \) resp. \( E_1'' \). □

It follows that the path of “short boundary divisors” is non-self-crossing. Each such non-self-crossing path corresponds to a regular rigid broken holomorphic sphere as follows. Firstly we observe that a nodal sphere \( u_v : C_v \to X_{P(v)} \), whose components \( C_v = C_v' \cup C_v'' \) are spheres of self-intersection \(-1\), is not regular. However, \( u_v \) may be deformed to a sphere of self-intersection \(0\), and these deformations sweep out the component \( X_{P(v)} \). It follows that this moduli space, with a point constraint in any of the divisors corresponding to adjacent each edge, is transversally cut out. We remark that the standard almost complex structure needs to be perturbed to achieve transversality. Otherwise for a component of homology class \( E_1'' \cup E_2 \) (for example), the point constraint lies on the exceptional sphere \( E_1'' \). Other maps occurring as components of the broken map are also regular: the map corresponding to a terminal vertex in a non-self-crossing path is rigid and regular. Finally the component homologous to \( D + E_1 + F_1 \) in \( X_{P_1} \) with generic point constraints on two relative divisors is also regular since the Chern number is 3. Thus each tropical graph shown in Figure 3 corresponds to a single broken map.

![Figure 26](image-url)

**Figure 26.** Split graphs that are degenerations of the given tropical graph in the top row, middle column. A red dot at an end of an edge \( e \) indicates that the corresponding nodal lift lies on a Morse submanifold and that \( e \) is a Morse-type edge.
Each of the rigid holomorphic spheres on the cubic surface deforms to a mixed split map. We list the possible split graphs corresponding to one of the twenty seven broken maps, see Figure 26. Suppose we split the edge joining $v$ with $v'$. By blowing down say $E_{j+1}$ one sees that the moduli space of spheres in this homology class is smooth and corresponds to a ruling of $\mathbb{P}^1 \to \mathbb{P}^1$ by fibers of the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. The possible cones in the cone displacement formula are a vertex together with the entirety of $P$. The only possible degeneration assigns to $u_v$ no constraint at $w_-$, and to $u_{v'}$ a generic point constraint at $w_+$. Only the graph in Figure 26 with four Morse splittings corresponds to a mixed split map, since the $-1$ curves are rigid. With this splitting, the configurations $u_v, u_{v'}$ becomes regular, and the multiplicity associated to the vertex is one. By similar arguments, the contribution of the piece corresponding to the “long divisor” is also one. Thus the Gromov-Witten invariant, in analogy with (11.24), is

\begin{equation}
(11.34) \quad m_{\text{split}} = \sum_{\tilde{\Gamma}} m_{\text{split,}\tilde{\Gamma}}, \quad m_{\text{split,}\tilde{\Gamma}} = \frac{(-1)^{\text{mult}(\tilde{\Gamma})}}{d_{\tilde{\Gamma}}!} \prod_{i=1} s(\tilde{\Gamma}) \prod_{i=1} m_{\tilde{\Gamma}_i},
\end{equation}

where each $\tilde{\Gamma}_i$ has a single vertex and

$$\prod_{i=1} s(\tilde{\Gamma}) \prod_{i=1} m_{\tilde{\Gamma}_i} = 1$$

for each tropical graph contributing. Since there are twenty-seven such graphs we obtain

$$m_{\text{split}} = \sum_{\tilde{\Gamma} : d_\lambda(\tilde{\Gamma}) = d} m_{\text{split,}\tilde{\Gamma}} \sum_{\tilde{\Gamma} : d_\lambda(\tilde{\Gamma}) = d} 1 = 27.$$  

We emphasize that the computation in, for example, Mumford [47, Section 8D], is much shorter.

11.6. Unobstructedness. We apply the tropical Fukaya algebra to the question of obstructedness. We introduce the following notation. Let $P_0 \in \mathcal{P}$ be a top-dimensional polytope and let $L \subset X_{P_0} \subset \mathcal{X}_\mathcal{P}$ be an embedded Lagrangian submanifold.

**Definition 11.49.** The Lagrangian $L$ is called a tropical moment fiber if

- $X_{P_0}$ is a compact toric manifold, and the tropical moment map $\Phi_{P_0} : X_{P_0} \to t^\vee$ is an honest moment map, and
- $L = \Phi_{P_0}^{-1}(\lambda)$ for a point $\lambda$ in the interior of the polytope $P_0$.

**Theorem 11.50.** (Unobstructedness of Lagrangian fiber in toric piece) Suppose $L \subset \mathcal{X}_\mathcal{P}$ is a tropical torus contained in $X_{P_0}$, where $P_0 \in \mathcal{P}$ and $\text{codim}_\mathcal{P}(P_0) = 0$, and all the facets of $P_0$ are elements of $\mathcal{P}$. Then the Fukaya algebra $\mathcal{CF}(L)$ is weakly unobstructed: For any local system $y \in \mathcal{R}(L)$, there exists a projective Maurer-Cartan solutions $b(y) \in \mathcal{MC}(L)$.

**Proof.** We prove unobstructedness for the tropical Fukaya algebra $\mathcal{CF}_{\text{trop}}(L)$. The tropical Fukaya algebra is homotopy equivalent to the unbroken Fukaya algebra.
$CF(L)$ by Theorem 1.3. Unobstructedness is preserved under homotopy equivalence, see [13, Lemma 5.2].

We first describe the perturbation datum for the moduli space of split maps. Let $J_0$ be the standard almost complex structure on $X_{\hat{\Gamma}}$ for all $Q \subseteq P_0$. By Proposition 5.23, all $J_0$-holomorphic disks in $X_{P_0}$ are regular. Since all the torus-invariant divisors $D_Q \subset X_{P_0}$ correspond to polytopes $Q \in \mathcal{P}$, spheres occurring in the compactification of the moduli space intersect the divisors $D_Q$ at isolated points. Consequently, $J_0$-holomorphic spheres are also regular. Therefore, there is a coherent regular perturbation datum $\mathfrak{p}$ on $X$ for which the almost complex structure on $X_{P_0}$ is standard.

Next, we show that $m_0(1)$ is a multiple of the geometric unit $x^\bullet$. Consider a split disk $[u]$ of type $\hat{\Gamma} \rightarrow \Gamma$ contributing to $m_0(1)$, and whose boundary output asymptotes to $x_0 \in \mathcal{I}(L)$. Let

$$\hat{\Gamma}_1 \subset \hat{\Gamma} \setminus \text{Edge}_s(\Gamma)$$

be the connected component containing the disk components, and let $u_1 := u|_{\hat{\Gamma}_1}$. Let $u'$ be the split disk obtained by forgetting the boundary output leaf, and let $\hat{\Gamma}' \rightarrow \Gamma$ be the type of $u'$. For any torus element $t \in T$, the split map

$$(u'_t)_v := \begin{cases} tu'_v, & v \in \text{Vert}(\hat{\Gamma}_1), \\ u_v, & v \in \text{Vert}(\hat{\Gamma}) \setminus \text{Vert}(\hat{\Gamma}_1). \end{cases}$$

is not contained in the $T_{\text{trop}}(\hat{\Gamma}, \Gamma)$-orbit of $u'$. Indeed the tropical symmetry group has a trivial action on $X_{P_0}$, which contains the disk components. By the same reason, for any pair $t_1 \neq t_2 \in T$, the maps $u_{t_1}, u_{t_2}$ lie in distinct $T_{\text{trop}}(\hat{\Gamma}, \Gamma)$-orbits. The regularity of $u$ implies $u'_t$ is regular for all $t \in T$, Thus $\{u'_t\}$ is a $\dim(T)$-dimensional family in the reduced moduli space, and therefore.

$$\dim(\mathcal{M}_{\hat{\Gamma} \rightarrow \Gamma, \infty}(L)) \geq \dim(T)$$

Therefore, for the reduced index of $u$ to be 0, the output is necessarily the geometric unit $x^\bullet$. So, $m_0(1) = Wx^\bullet$ for some $W \in \Lambda$.

The existence of a solution to the projective Maurer-Cartan equation now follows. We first claim that $m_1(x^\bullet)$ only has zero order terms. If not, let $u$ be a split map with non-zero area contributing to $m_1(x^\bullet)$. By the locality axiom, and the standardness of the perturbation datum on $X_{P_0}$, we conclude that forgetting the input in $u$ produces a regular split disk $u'$ with no inputs. Since the index of $u'$ can not be negative, we conclude $u$ can not exist for dimension reasons. Indeed, the index of $u$ is two more than the index of $u'$: one from the choice of a boundary incoming marking, and one from the weight on the incoming leaf. Therefore,

$$m_1(x^\bullet) = x^\bullet - x^\bullet.$$ 

By a similar argument,

$$m_d(x^\bullet, \ldots, x^\bullet) = 0 \quad d \geq 2,$$

and consequently, $Wx^\bullet$ is a solution of the projective Maurer-Cartan equation. □
The unobstructedness result in tropical Fukaya algebras gives an alternate proof of the Fukaya-Oh-Ohta-Ono result \cite{30} on unobstructedness of toric Lagrangians in toric manifolds.

**Corollary 11.51.** (Unobstructedness in a toric manifold) Suppose $X$ is a symplectic toric manifold with an action of a compact torus $T$, and moment map $\Phi : X \to t^\vee$. Then any Lagrangian moment fiber in $X$ is weakly unobstructed.

**Proof.** Let $X$ be a $T$-toric variety whose moment polytope is
\[ \Delta := \Phi(X) = \{ x \in t^\vee : \langle \mu_i, x \rangle \leq c_i, \ i = 1, \ldots, N \} \]
where $\mu_i \in t$ and $c_i \in \mathbb{R}$. To show that a moment fiber $\Phi^{-1}(\lambda)$ is unobstructed, we consider a broken manifold $X_{P}$ obtained by the following cuts
\[ \langle x, \mu_i \rangle = c_i - \epsilon_i, \ i = 1, \ldots, N \]
where $\epsilon_i > 0$ is a small constant. In particular, we assume that $\langle \lambda, \mu_i \rangle \leq c_i - \epsilon_i$ and the piece $\{ \langle x, \mu_i \rangle \geq c_i - \epsilon_i \}$ is a $\mathbb{P}^1$-fibration for all $i$. The top-dimensional $P_0 \in P$ containing the fiber $\Phi^{-1}(\lambda)$ satisfies the hypothesis of Theorem 11.50, because a toric invariant divisor of $X_{P_0}$ is of the form
\[ \{ x | \langle x, \mu_i \rangle = c_i - \epsilon_i \}, \]
and is a boundary divisor. Therefore, the Lagrangian $\Phi^{-1}(\lambda)$ is weakly unobstructed. \hfill \Box

**11.7. Tropical disk potentials.** For a toric Lagrangian in a toric manifold leading order terms in the Batyrev-Givental potential \cite{6, 32} correspond to index two Maslov disks. In the notation of the proof, the Batyrev-Givental potential for the Lagrangian $L := \Phi^{-1}(\lambda)$ is given by
\[ W_{BG}(y) = \sum_{i=1}^{N} \langle y, \mu_i \rangle q^{\langle \lambda, \mu_i \rangle - c_i}. \]

**Proposition 11.52.** (Leading order terms of the potential) Let $P$ be the polytopal decomposition of the toric manifold $X$ in the proof of Corollary 11.51. The leading order terms of the potential of $CF_{trop}(X_{P}, L)$ coincide with the leading order terms of the Batyrev-Givental potential.

**Proof.** We first describe the polytopes in $P$ and the dual complex. Polytopes in $P$ are indexed by pairs of faces of $\Delta$ where one is contained in the other:
\[ P := \{(P_1, P_2) : P_1 \subseteq P_2 \subseteq \Delta \}, \quad \dim_P(P_1, P_2) = n - \text{codim}_{P_2}(P_1) \]
with the containment relation
\[ (P_1, P_2) \subseteq (P'_1, P'_2) \quad \text{iff} \quad P_1 \subseteq P'_1 \subseteq P'_2 \subseteq P_2. \]
Thus top-dimensional pieces in $X$ are of the form $(P, P)$ and correspond to the faces of $\Delta$. We use short hand notation for some elements in $P$:
\[ X_0 := X_{(\Delta, \Delta)}, \quad X_i := X_{(D_i, D_i)}, \quad X_{0i} := X_{(D_i, \Delta)}. \]
Figure 27. Proposition 11.52: The tropical graph of the broken and split map in the dual polytope. The edge $e$ is a split edge.

where $D_i$ is a facet of $\Delta$. The dual polytope $B^\vee$ consists of a top-dimensional polytope for every corner in $\Delta$, and a zero-dimensional cell for every face of $\Delta$. For example, if $\Delta$ is a three-dimensional cube, the dual polytope $B^\vee$ is as in Figure 27, where the grey dots are 0-cells, and the dotted lines bound one of the top-dimensional cells.

Corresponding to every toric divisor of $X$ there is a broken disk whose glued type has Maslov index two, which we now describe. The perturbation datum used to define the broken Fukaya algebra on $X_P$ is such that the almost complex structure on $X_0$ is standard, and the pieces $X_1, \ldots, X_N$ are fibrations

\[(11.35) \quad \pi_i : X_i \to D_i\]

whose fibers are holomorphic spheres, each of which intersects the boundary divisor $X_{0i} \simeq X_i \cap X_0$ at a single point. The index two disk incident on the $i$-th toric divisor has two components: $u_{\text{brok}} = (u_+, u_-)$. In this pair $u_+ : D \to X_0$ is an index two Maslov disk intersecting the boundary divisor $X_{0i}$ at $p_i \in X_{0i}$, and $u_-$ maps to $X_i$ and is a fiber of $\pi_i$ in (11.35).

The broken disk $u_{\text{brok}}$ is homotopic to a family of deformed disks

\[ T_C/T_{T(e), C} \ni \tau \mapsto u^\tau, \quad u_0 = u. \]

Here $u^\tau := (u_+, u_-^\tau)$ is a $\tau$-deformed map, and $u_-^\tau$ is a sphere in $X_{P_i}$ homotopic to $u_-$ that satisfies

\[(11.36) \quad p_i := \text{ev}_{X_{0i}}(u_+) = e^\tau \text{ev}_{X_{0i}}(u_-^\tau), \]

where $\text{ev}_{X_{0i}}$ is the ordinary evaluation map at the lift of the nodal point $w_+(e)$ mapping to $X_{0i}$. The intersection multiplicity at the node is

\[ m(u_+, X_{0i}) = m(u_-, X_{0i}) = 1. \]

It follows that the matching condition at $e$ is a purely horizontal condition given by (11.36). The component $u_+$ stays constant under variation of $\tau$ because it is a disk of Maslov index two in the toric variety $X_0$, and is therefore rigid.

The split map is the limit of a sequence of deformed broken maps whose deformation parameters $\tau$ approach the infinite end of the torus in a generic direction. The sequence of points $e^{-\nu} p_i \in X_{0i}$ approaches a $T$-fixed point of $p \in X_{0i}$ as $\nu \to \infty$. \[\]
Consequently, the sequence $u^\tau$ converges to a split map $u^\infty = (u_+, u^\infty_-)$ where $u^\infty_-$ maps to the neck piece $X_{(p,D_i)}$ where $p \subset D_i$ is a vertex of the facet $D_i$. The tropical graph underlying $u^\infty$ satisfies the cone condition for the following reason. The quasi-split tropical graph $\tilde{\Gamma}$ underlying $u^\infty$ has vertices $v_+, v_-$ corresponding to $u_+, u^\infty_-$, which are connected by a split edge $e$. The vertex $v_-$ is free to move in the dual polytope $P^\vee_{(p,D_i)}$ which is $n - 1$-dimensional (see Figure 27). Thus the set of relative weights $W(\tilde{\Gamma}, \Gamma)$ is a $(\dim(t) - 1)$-dimensional cone. The area and boundary holonomy of $u^\infty$ are equal to the area and boundary holonomy of $u$, and therefore, $u^\infty$ makes the expected contribution to $m_0(1)$ in $CF_{\text{split}}(\hat{X}_\tilde{p}, L)$. \hfill \Box

**Example 11.53.** Continuing the example of the cubic surface in 11.26, we may compute from the description there the homology classes of the split Maslov index two disks to obtain the potential

$$W(y_1, y_2) = y_1 y_2 + y_1^2/y_2 + y_2^2/y_1 + 3(y_1 + y_1/y_2 + y_2 + y_2/y_1 + 1/y_1 + 1/y_2)$$

originally obtained by Pascaleff-Tonkonog [48] using mutations. This formula is equivalent to the one in their paper by the change of variables

$$z_1^3 = y_1^2/y_2, z_2^3 = y_2^2/y_1$$

after which the formula becomes

$$W(z_1, z_2) = (1 + z_1 + z_2)^3/z_1 z_2 - 6$$

which is the formula obtained in Galkin-Usnich [31]; c.f. [5, Table 5.1].

**References**


