# Tropical Fukaya algebras

Sushmita Venugopalan and Chris T. Woodward

30th December 2024

# Contents

We thank Jonny Evans, Mohammad F. Tehrani, and Nick Sheridan for discussions and helpful emails. C.W. was partially supported by NSF grant DMS 2105417. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Chapter 1

# Statement of results

In this monograph, we study the behavior of holomorphic disks under a multiple symplectic cut. In particular, we study the behavior of the Fukaya algebra associated to a Lagrangian submanifold of a symplectic manifold. The Fukaya algebra is an $A_\infty$ algebra defined using counts of holomorphic disks whose boundaries lie on the Lagrangian submanifold. The main result of the book is a $A_\infty$ homotopy equivalence between two versions of the Fukaya algebra: the standard version defined on the symplectic manifold and the *broken* version obtained by cutting the manifold. The Lagrangian submanifold is assumed to be disjoint from the cuts.

We use the homotopy equivalence to compute the disk potentials of Lagrangians in various examples. An important application is the weak unobstructedness of almost toric Lagrangians, which we carry out in a sequel [83] which was split off from the original manuscript. In this introductory chapter, we give a low-tech tour of the paper, interwoven with motivations, context, and also limitations, of our results.

## 1.1 Multiple cuts

The *cut* operation introduced by Lerman [48] cuts a symplectic manifold along a regular level set of a moment map for a circle action, and then quotients the boundary by the circle action to produce a smooth symplectic manifold. The inverse operation is called a *symplectic sum*. The *multiple cut* is a generalization where the symplectic manifold is cut along intersecting hypersurfaces. The set-up for a multiple cut is that of a tropical Hamiltonian action, which we now define.

**Definition 1.1.** (Polyhedral decomposition) Let $n > 0$ be an integer and $T \simeq (S^1)^n$ a torus with Lie algebra $\mathfrak{t} \cong \mathbb{R}^n$. A *simplicial polyhedral decomposition* of $\mathfrak{t}^\vee$ is a collection

$$\mathcal{P} = \{P \subset \mathfrak{t}^\vee\}$$

of simple polytopes[1] such that

(a) (Covering property) the interiors $P^\circ$ of the polytopes $P \in \mathcal{P}$ cover $\mathfrak{t}^\vee$; that is,

$$\mathfrak{t}^\vee = \cup_{P \in \mathcal{P}} P^\circ;$$

[1]By a polytope we mean a finite intersection of half-planes, as in Definition 3.2. As such, our polytopes are closed but not necessarily compact.

(b) (Face property) and for any $\sigma_1, \dots, \sigma_k \in \mathcal{P}$, the intersection $\sigma_1 \cap \dots \cap \sigma_k$ is a polytope in $\mathcal{P}$ that is a face of each of the polytopes $\sigma_1, \dots, \sigma_k$.

Any polytope $P$ in a simplicial polyhedral decomposition $\mathcal{P}$ corresponds to a subtorus

$$T_P \subseteq T, \quad \text{defined by} \quad \mathfrak{t}_P := \operatorname{ann}(TP). \tag{1.1}$$

Thus $P$ and $T_P$ have complementary dimensions, and $T_P = \{\mathrm{Id}\}$ for top-dimensional polytopes $P \in \mathcal{P}$.

**Definition 1.2.** (Tropical Hamiltonian action) A *tropical Hamiltonian action* $(X, \mathcal{P}, \Phi)$ consists of

(a) a simplicial polyhedral decomposition $\mathcal{P}$ of $\mathfrak{t}^\vee$, and

(b) a compact symplectic manifold $(X, \omega_X)$ with a *tropical moment map*

$$\Phi : X \to \mathfrak{t}^\vee,$$

such for any $P \in \mathcal{P}$ there is a neighborhood $U_P \subset X$ of $\Phi^{-1}(P)$ on which the projection

$$U_P \to \mathfrak{t}^\vee \to \mathfrak{t}_P^\vee$$

is a moment map for a free Hamiltonian action of a torus $T_P$.

Given a tropical Hamiltonian action $X$ the output of a multiple cut is a collection of *cut spaces*

$$X_P := \Phi^{-1}(P^\circ), \tag{1.2}$$

where $P^\circ$ is the complement of the faces of $P$; the cut space $X_P$ compactifies to

$$\overline{X}_P = \Phi^{-1}(P)/\sim,$$

where $\Phi^{-1}(P)$ is a manifold with corners and the equivalence relation quotients any codimension one boundary $\Phi^{-1}(Q) \subset \Phi^{-1}(P)$, $\operatorname{codim}_P(Q) = 1$ by the action of $S^1 \simeq T_Q/T_P$. The spaces $\overline{X}_P$, $P \in \mathcal{P}$ are orbifolds, whose local structure is given by an iterative application of Lerman's construction [48]. In a multiple cut with polyhedral decomposition $\mathcal{P}$, for any pair of facets $Q_1, Q_2 \in \mathcal{P}$ of a polytope $P \in \mathcal{P}$, $\overline{X}_{Q_1}$ and $\overline{X}_{Q_2}$ are embedded as divisors in the compactified cut space $\overline{X}_P$, called *relative divisors*, that intersect each other normally along $\overline{X}_{Q_1 \cap Q_2}$. See Figure 1.2. The space $X_P$ is the complement of the relative divisors in $\overline{X}_P$.

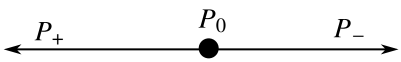



**Figure 1.1.** A single cut.

*Example* 1.3. For a single cut, the polyhedral decomposition $\mathcal{P}$ partitions $\mathfrak{t}^\vee \simeq \mathbb{R}$ into two semi-infinite lines intersecting at a point as in Figure 1.1. The cut spaces are $X_{P_+}$, $X_{P_-}$ and $X_{P_0}$, of which the first two are top-dimensional. The simplest example of a multiple cut consists of two single cuts along hypersurfaces

$$Z_1 := \Phi^{-1}(P_{12} \cup P_{30}), \quad Z_2 := \Phi^{-1}(P_{01} \cup P_{23}).$$

The hypersurfaces are $S^1$-bundles that along $\Phi^{-1}(P_\cap)$ which is a $(S^1)^2$ bundle over $\overline{X}_{P_\cap}$. See Figure 1.2. The cut spaces are

$$X_{P_i}, \quad i = 1, \dots, 4; \quad X_{P_{ij}}, \quad i = j \mod 4; \quad X_{P_\cap}.$$

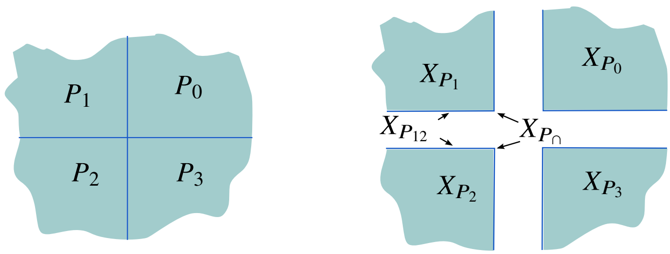



**Figure 1.2.** A multiple cut of $\mathbb{R}^2$.

*Example* 1.4. (On orbifolds) Although, the definition of a tropical Hamiltonian action requires torus actions to be free in neighborhoods of cut loci, cut spaces may be orbifolds. For example, the multiple cut on the right side of Figure 1.3 (which is a part of the multiple cut of a cubic surface in Figure 2.10) is allowed, although two of the points $X_{Q_1}$, $X_{Q_2}$ in the cut space $\overline{X}_{P_2}$ are $A_1$-singularities. However, the single cut in the left side of Figure 1.3 is not allowed since the circle action corresponding to the cut is not free. Requiring a free torus action in the neighborhoods of cut loci is needed to ensure the transversality of moduli spaces of broken maps as explained in page 17.

Our definition of tropical Hamiltonian actions is similar to Gross-Siebert [39]. A limitation of our set-up is that we require integral affine singularities to lie away from the cut locus. We expect that this requirement can be weakened by replacing a single cut passing through a singularity by two parallel cuts straddling the singularity, as explained below in Section 1.6. On the other hand, our definitions include manifolds that are not integral affine, because we require a toric structure only in the neighborhood of the cut locus, rather than globally.

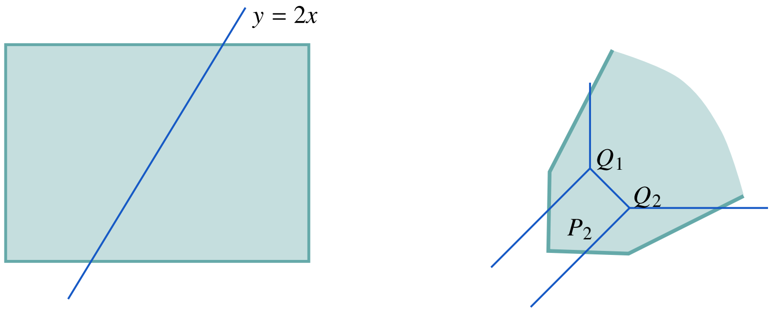



**Figure 1.3.** The cut in the left is disallowed, since the torus action is not free; the cut on the right is allowed.

## 1.2 Neck-stretching

To build a correspondence between curves in the manifold and curves in the cut spaces, we construct a sequence of "large almost complex structures", in the language of Kontsevich-Soibelman [47], which degenerate to almost complex structures in the cut spaces. We call this limiting process *neck stretching*, because it amounts to modifying the complex structure in the neighborhoods of the cut loci, which we call *necks*.

As an example of neck-stretching, consider a single cut. Let $\Phi : X \to \mathbb{R}$ be a moment map for a Hamiltonian circle action, so that the zero level set

$$Z := \Phi^{-1}(0)$$

is a separating hypersurface. Choose a tubular neighborhood

$$U_Z \subset X, \quad U_Z \cong Z \times I$$

where $I \subset \mathbb{R}$ is an interval. For a sequence $\{J^\nu\}_\nu$ of *neck-stretching* almost complex structures on $X$, the fibers of the projection $U_Z \to Z/S^1$ are $J_\nu$-holomorphic cylinders $[\frac{-\nu}{2}, \frac{\nu}{2}] \times S^1$ for any $\nu \in \mathbb{N}$, as shown in Figure 1.4.

To define neck-stretching for a multiple cut, we need an additional datum of a *dual complex*, which encodes the proportion in which the neck is stretched in different directions. The dual complex consists of a complementary dimensional polytope denoted $P^\vee$ for every $P \in \mathcal{P}$. Curve counts will depend on the choice of dual complex, but as with other choices (such as tamed almost complex structures), the curve counts corresponding to any two dual complexes will produce Fukaya algebras that are $A_\infty$ homotopy equivalent.

*Example* 1.5. For the single cut in Figure 1.1, the dual complex is a line segment and is shown in Figure 1.5. For the multiple cut consisting of two single cuts in Figure

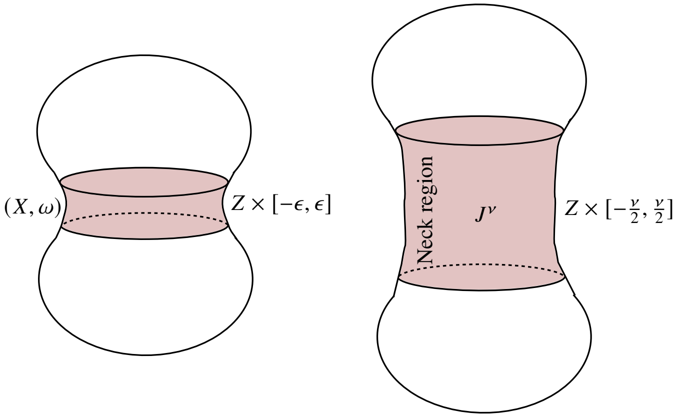



**Figure 1.4.** Neck-stretching for a single cut.

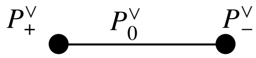



**Figure 1.5.** The dual complex for a single cut.

1.2, the dual complex is a rectangle shown in Figure 1.6 with side lengths, say $l_1$, $l_2$. In the neck-stretched almost complex manifold $(X, J^\nu)$, a neighborhood $U_{P_\cap}$ of $\Phi^{-1}(P_\cap)$ fibers over $X_{P_\cap}$. The fibers of $U_{P_\cap} \to X_{P_\cap}$ are biholomorphic to the product of cylinders $((\frac{-\nu l_1}{2}, \frac{\nu l_1}{2}) \times S^1) \times ((\frac{-\nu l_2}{2}, \frac{\nu l_2}{2}) \times S^1)$, as shown in Figure 1.7. The ratio $\frac{l_1}{l_2}$ is not required to be rational.

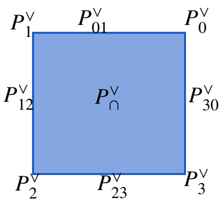



**Figure 1.6.** Dual complex $B^\vee$ for the cut in Figure 1.2.

A *broken manifold* corresponding to a multiple cut $(X, \mathcal{P})$ is a disjoint union of thickenings of of cut spaces

$$\mathfrak{X} = \bigsqcup_{P\in\mathcal{P}} \mathfrak{X}_P. \tag{1.3}$$

For top-dimensional polytopes $P \in \mathcal{P}$, $\mathfrak{X}_P := X_P$ is the cut space from (1.2), and for lower dimensional polytopes $P$, $\mathfrak{X}_P$ is a toric fibration over the cut space $X_P$ and is called a *neck piece*. The manifold $\mathfrak{X}_P$ has a compactification $\overline{\mathfrak{X}}_P$ obtained by adding relative divisors to $\mathfrak{X}_P$.

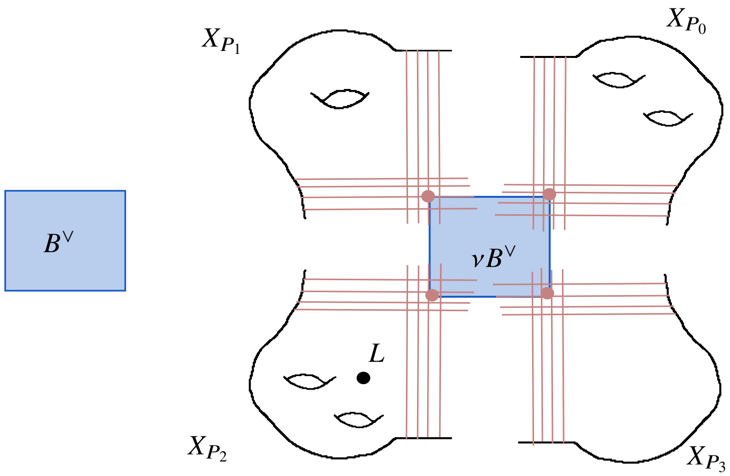



**Figure 1.7.** Neck-stretched manifold $(X, J_\nu)$ for the double cut in Figure 1.2 using the dual complex $B^\vee$.

For example, in the case of a single cut as in Figure 1.1, the neck piece denoted by $\mathfrak{X}_{P_0}$ arises as the limit of the neighborhoods of the separating hypersurface and corresponds to a zero-dimensional polytope $P_0 \in \mathcal{P}$. The space $\mathfrak{X}_{P_0}$ is a $(\mathbb{R} \times S^1)$-bundle over the relative divisor $X_{P_0}$, and its compactification $X_{P_0}$ is a $\mathbb{P}^1$-bundle over $X_{P_0}$. The analog for multiple cuts is the following: For a polytope $P \in \mathcal{P}$, the neck piece $\mathfrak{X}_P$ is a torus bundle

$$T_{P,\mathbb{C}} \to \mathfrak{X}_P \to X_P$$

over the cut space $X_P$. The fiber of the bundle is isomorphic to the complexification $T_{P,\mathbb{C}}$ of the compact torus $T_P$. This torus is part of the data of the tropical Hamiltonian action $X$, and was defined in (1.1); see Figure 1.8. The compactification of $\mathfrak{X}_P$ is a fibration $\overline{\mathfrak{X}}_P \to \overline{X}_P$ with toric orbifold fibers.

We will study the behavior of holomorphic disks bounding a Lagrangian submanifold under a family of neck-stretching almost complex submanifolds. To simplify the situation somewhat, we assume that the Lagrangian submanifold $L \subset (X, \omega)$ is disjoint from the cuts. The Lagrangian therefore descends to a Lagrangian submanifold, also denoted by $L$, in a cut space $X_P$ for a top-dimensional polytope $P$.

Broken manifolds have orbifold components, and the reader may be concerned that these present additional technical issues. However, we consider maps on punctured curves whose images lie in the uncompactified spaces $\mathfrak{X}_P$, which are manifolds with cylindrical ends. Broken maps have extensions over punctures, with puncture points mapping to relative divisors, but the compactification is just a useful technical tool in some places and no analysis on orbifolds is actually used.

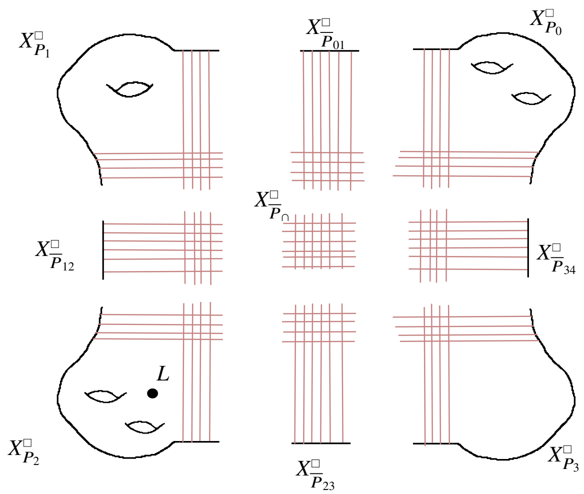



**Figure 1.8.** Broken manifold arising from the neck-stretching in Figure 1.7.

## 1.3 Broken maps

A broken map is modelled on a graph and consists of a "map part" and a "tropical part". For the map part, the domain is the normalization of a nodal curve $C$ whose irreducible components denoted

$$C_v, v \in \mathrm{Vert}(\Gamma)$$

correspond to vertices of a graph $\Gamma$, and whose nodes denoted $w_e \in C$ correspond to edges $e \in \mathrm{Edge}(\Gamma)$ of $\Gamma$. The broken map is a collection of holomorphic maps on punctured curves

$$u_v : C_v^\circ \to \mathfrak{X}_{P(v)}, \quad v \in \mathrm{Vert}(\Gamma),$$

satisfying certain matching conditions (explained later in the paragraph) on the lifts of nodal points. Each of the domain components $C_v^\circ$ is an irreducible curve component $C_v \subset C$ (possibly with boundary) punctured at interior nodal points, that is,

$$C_v^\circ := C_v \backslash \{\text{interior nodes}\},$$

the target space $\mathfrak{X}_{P(v)}$ is a piece of the broken manifold corresponding to the polytope

$$P(v) \in \mathcal{P},$$

and the punctures in the domain are removable singularities when $u_v$ is viewed as a map to the compactification $\overline{\mathfrak{X}}_{P(v)}$.[2] Thus a broken map can be evaluated at nodal lifts. In particular, in a broken map $u$ modelled on a graph $\Gamma$, consider a node $w$ corresponding to an edge $e = (v_+, v_-) \in \mathrm{Edge}(\Gamma)$ which lifts to $w^\pm \in C_{v_\pm}$ in the normalization $\tilde{C}$ of $C$. The map $u$ can be evaluated at the nodal lift $w^+ \in C_{v_+}$, and the point $u(w^+)$ lies on the intersection

$$Y_+ = \cap_{i=1}^k D_i^+$$

of a collection of relative divisors $D_1^+, \ldots, D_k^+$ in $\overline{X}_{P(v_+)}$. Assume that each $D_i^+$ is the fixed point set of a one-dimensional torus generated by $\mu_i \in \mathfrak{t}$, and $n_i$ is the intersection multiplicity of the map $u_{v_+}$ at $w^+$. Then the sum

$$\mathcal{T}(w^+) := \sum n_i \mu_i \in \mathfrak{t}_{\mathbb{Z}}$$

lies in the integer lattice $\mathfrak{t}_{\mathbb{Z}} \subset \mathfrak{t}$. Define $\mathcal{T}(w^-) \in \mathfrak{t}_{\mathbb{Z}}$ for the lift $w^-$ similarly. For any holomorphic coordinate $z_\pm$ on a neighborhood of $w^\pm$ in $C_{v_\pm}$, the limit

$$\lim_{z_\pm \to 0} z_\pm^{-\mathcal{T}(w^\pm)} u$$

exists. (See Section 4.3 for more details.) The *matching condition* at the node $w$ says that

- $\mathcal{T}(w^+) = \mathcal{T}(w^-)$, and
- there exist holomorphic coordinates $z_+$, $z_-$ in neighborhoods of the nodal lifts $w_+$, $w_-$ such that

$$\lim_{z_+ \to 0} z_+^{-\mathcal{T}(w^+)} u = \lim_{z_- \to 0} z_-^{-\mathcal{T}(w^-)} u. \tag{1.4}$$

The quantity $\mathcal{T}(w^+) = \mathcal{T}(w^-)$ is called the *slope of the node* $w$ or the *slope of the edge* $e$ in $\Gamma$ corresponding to the node $w$; and is denoted by

$$\mathcal{T}(e). \tag{1.5}$$

The quantities in the left-hand side and right-hand side of (1.4) are called the *tropical evaluations* at $w^+$ and $w^-$ respectively. An equivalent formulation of the matching condition states that, for any node, the evaluations of the lifts are equal modulo the action of the torus generated by the slope of the node: For a node $w \in C$ corresponding to an edge $e$, the evaluations of the lifts $w^\pm$ lie in $\mathfrak{X}_{P(e)}$ which has the structure of a $T_{P(e),\mathbb{C}}$-bundle, where

$$P(e) := P(v_+) \cap P(v_-) \in \mathcal{P},$$

[2] Here for the sake of exposition we assume that $\mathfrak{X}_{P(v)}$ has a manifold compactification. The general case is treated in Chapter 4, where the quantity $\mathcal{T}(e)$ is defined using the asymptotic behavior of the map near the nodal point.

is the polytope assigned to the edge $e = (v_+, v_-)$. The slope of the node $\mathcal{T}(w)$ lies in the lattice $\mathfrak{t}_{P(e),\mathbb{Z}}$ and generates a one-dimensional torus $T_{\mathcal{T}(e),\mathbb{C}}$. The matching condition is then that the images of the evaluations match in the base of the $T_{\mathcal{T}(e),\mathbb{C}}$-fibration:

$$(\pi^{\perp}_{\mathcal{T}(e)} \circ u)(z_+) = (\pi^{\perp}_{\mathcal{T}(e)} \circ u)(z_-) \in \mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}, \tag{1.6}$$

where $\pi^{\perp}_{\mathcal{T}(e)} : \mathfrak{X}_{P(e)} \to \mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}$ is the projection to the quotient. The quantities in the left-hand side and right-hand side of (1.6) are called *projected tropical evaluations*. Thus the space $\mathfrak{X}_{P(e),\mathbb{C}}/T_{\mathcal{T}(e),\mathbb{C}}$ in which the matching condition of an edge $e$ is defined is dependent on $e$. The matching condition is simpler in the special case of a single cut, because the space $\overline{\mathfrak{X}}_{P(e),\mathbb{C}}/T_{\mathcal{T}(e),\mathbb{C}}$ is the relative divisor $X_{P_0}$ (using notation from Figure 1.1). The matching condition at a node $w$ is

$$u_{v_+}(w^+) = u_{v_-}(w^-) \in X_{P_0}.$$

A *broken map* is a collection of maps $u_v : C_v^\circ \to \mathfrak{X}_{P(v)}$ described above together with a tropical structure on the graph $\Gamma$ underlying the domain nodal curve. A *tropical structure* on a graph $\Gamma$ is a collection of edge slopes

$$\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}} \subset \mathfrak{t} \simeq \mathfrak{t}^\vee,$$

and polytope assignments $P(v) \in \mathcal{P}$ for vertices $v \in \mathrm{Vert}(\Gamma)$ so that the graph is realizable in the dual complex $B^\vee$ of the neck-stretching in the following sense: There exist *tropical positions* of the vertices in the dual complex

$$\mathcal{T} : \mathrm{Vert}(\Gamma) \to B^\vee \subset \mathfrak{t}^\vee, \quad \mathcal{T}(v) \in P(v)^\vee$$

that satisfy the following: For any edge $e = (v_+, v_-)$ the slope of the line segment joining $\mathcal{T}(v_+)$ to $\mathcal{T}(v_-)$ is equal to the slope $\mathcal{T}(e)$ of the node $w$ corresponding to $e$. Recall that the slope $\mathcal{T}(e)$ is given by the intersection multiplicities of the map with relative divisors at the node defined in (1.5). That is,

$$\mathcal{T}(e) \in \mathbb{R}_{>0}(\mathcal{T}(v_+) - \mathcal{T}(v_-)). \tag{1.7}$$

The image of $\mathcal{T}(\Gamma)$ under the map to $B^\vee$ induced by $v \mapsto \mathcal{T}(v)$ is called an *a realization of a tropical graph*, and the underlying graph equipped with just the edge slopes $\{\mathcal{T}(e)\}_e$ and vertex polytopes $\{P(v)\}_v$ is called a *tropical graph*. Thus, changing the tropical vertex positions $\{\mathcal{T}(v)\}_v$ produces a different realization of the same tropical graph; an example is shown in Figure 4.3. Broken maps may also contain nodes corresponding to the standard nodal degeneration encountered in Gromov-Witten theory. The edges corresponding to such nodes do not appear in the tropical graph.

Broken maps arise naturally as limits of sequences of holomorphic maps in neck-stretched manifolds. A converging sequence of maps consists of pockets of high symplectic area separated by long cylinders in neck regions. Each such sequence $u_\nu$ of long cylinders maps to a neck piece

$$u_\nu : \left[\frac{-\nu}{2}, \frac{\nu}{2}\right] \times S^1 \to \mathfrak{X}_P$$

for some $P \in \mathcal{P}$ in the polytopal decomposition, and is asymptotically close to a *trivial cylinder*. A trivial cylinder is a holomorphic cylinder which lies in a fiber of the projection

$$T_{P,\mathbb{C}} \to \mathfrak{X}_P \to X_P,$$

and is thus diffeomorphic to a sub-torus $T_{\mu,\mathbb{C}} \subset T_{P,\mathbb{C}}$ generated by a rational element $\mu \in \mathfrak{t}_{P,\mathbb{Z}}$. The pockets of high area converge to spheres or disks, and (roughly speaking) the long cylinders connecting these pockets converge to trivial cylinders. We drop the rational cylinders from our description of the "map part" of a broken map, and instead encode them as edges of the tropical graph, as shown in Figure 1.9. The generator

$$\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}}$$

of the trivial cylinder corresponding to an edge $e$ is the slope of the edge in the tropical graph. In particular, each edge slope is integral. Every component $C_v$, $v \in \mathrm{Vert}(\Gamma)$ of the limit curve has a natural position $\mathcal{T}(v) \in B^\vee$ in the dual complex as follows: For a limit curve component mapping to a neck piece $\mathfrak{X}_P$, $P \in \mathcal{P}$ with fibers $T_{P,\mathbb{C}} \simeq (\mathbb{C}^\times)^n$, the convergence is modulo $\mathbb{R}^n$-translation in target space $\mathfrak{X}_P$. More specifically, for a limit curve component $u : C \to \mathfrak{X}_P$ there is a sequence $t_\nu \in \nu P^\vee \subset \mathbb{R}^n \subset \mathfrak{t}_{P,\mathbb{C}}$ such that $u$ is the limit of the translated maps $e^{-t_\nu} u_\nu$. Thus, the component $u$ inherits a *tropical coordinate*

$$\mathcal{T}(v) := \lim_{\nu\to\infty} \frac{t_\nu}{\nu} \in P^\vee.$$

For an edge $e = (v_+, v_-)$, the line segment connecting $\mathcal{T}(v_+)$ and $\mathcal{T}(v_-)$ has slope $\mathcal{T}(e)$.

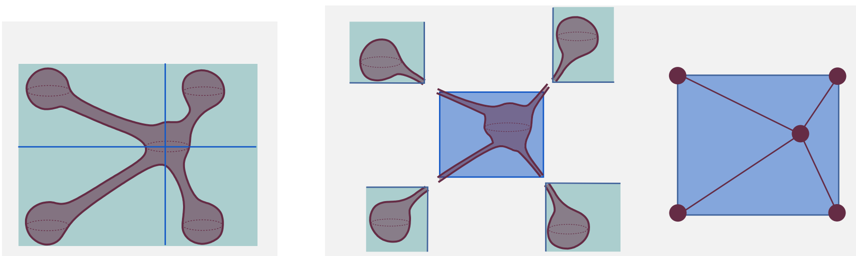


**Figure 1.9.** Left : Map in a neck-stretched manifold. Right : A broken map and its tropical graph.

The realization of the broken map as a limit of maps in neck-stretched manifolds also explains the matching condition at nodes. A node $w$ corresponding to an edge $e$

in a limit broken map arises from the convergence of a sequence of long cylinders

$$u_\nu : S^1 \times [-l_\nu, l_\nu] \to (X, J_\nu), \quad l_\nu \to \infty$$

with small area, each of which is asymptotically close to a trivial cylinder

$$u_{\text{triv}} : S^1 \times [-l_\nu, l_\nu] \to (X, J_\nu), \quad (s,t) \mapsto e^{\mathcal{T}(e)(s+it)} x_0, \quad x_0 \in X$$

generated by an integral element $\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}}$. As a result, the evaluation of both lifts of the node $w_e^\pm$ are equal modulo the action of the one-dimensional complex torus $T_{\mathcal{T}(e),\mathbb{C}}$. This is a re-statement of the matching condition (1.6).

The set of broken maps of a fixed type $\Gamma$ has a free action of a *tropical symmetry* group $T_{\text{trop}}(\Gamma)$ arising out of the torus action on neck pieces. The tropical symmetry group $T_{\text{trop}}(\Gamma)$ is generated by the degrees of freedom of the tropical graph $\mathcal{T}$: Each element of the real part of this group corresponds to ways of moving the vertex positions $\{\mathcal{T}(v)\}_v$ without changing edge slopes $\{\mathcal{T}(e)\}_e$, as shown in Figure 4.5. In particular, the symmetry group $T_{\text{trop}}(\Gamma)$ is finite if in the tropical graph vertex positions $\mathcal{T}(v)$ are uniquely determined by the edge slope $\mathcal{T}(e)$. Such tropical graphs are called *rigid*. In the second graph $\Gamma_2$ in Figure 4.5, there is one degree of freedom moving the edges of the tropical graph, and this generates a one-dimensional complex torus $T_{\text{trop}}(\Gamma_2)$. The quotient of the moduli space $\mathcal{M}_\Gamma^{\text{brok}}(\mathfrak{X}, L)$ of broken maps of type $\Gamma$ by the action of the tropical symmetry group $T_{\text{trop}}(\Gamma)$ is called the *reduced moduli space* (see (6.10)), and is denoted by

$$\mathcal{M}_{\Gamma,\text{red}}^{\text{brok}}(\mathfrak{X}, L) := \mathcal{M}_\Gamma^{\text{brok}}(\mathfrak{X}, L)/T_{\text{trop}}(\Gamma).$$

A feature of broken maps is that "nodes do not lower the index of a map", which is in contrast to the behavior of stable maps in smooth manifolds. Here, the *index* of a broken map $u$,[3] denoted by $i^{\text{brok}}(u)$, refers to the expected dimension of the moduli space component containing $u$, and is given by the index of the linearization of the Fredholm operator cutting out the moduli space. The index is equal to the actual dimension

$$i^{\text{brok}}(u) = \dim T_u \mathcal{M}(\mathfrak{X}, L)$$

if the moduli space $\mathcal{M}(\mathfrak{X}, L)$ is regular. Nodes do not contribute negatively to the expected dimension formula for broken maps because the matching condition at an edge $e \in \text{Edge}(\Gamma)$ has codimension $(\dim X - 2)$, since the matching condition is a condition on the quotient of $\mathfrak{X}_{P(e)}$ by a complex one-dimensional torus $T_{\mathcal{T}(e),\mathbb{C}}$ as in (1.6). In contrast, the codimension of the matching condition for ordinary stable maps is $\dim(X)$. Therefore, whereas in nodes for stable maps occur only in strata

[3] We use the notation $i(u)$ for the index of a map $u$, which is equal to the expected dimension of the moduli space containing $u$; and the notation $I(u)$ for Maslov index of a map $u$ with disk as domain with Lagrangian boundary conditions.

whose expected codimension is at least two, broken maps $u$ with components in neck pieces may have index zero, $i^{\text{brok}}(u) = 0$, though that can happen only if the tropical graph is rigid. Indeed, if for a type $\Gamma$ the tropical graph is not rigid, then the tropical symmetry group $T_{\text{trop}}(\Gamma)$ is at least two-dimensional. Since $T_{\text{trop}}(\Gamma)$ has a free action on the moduli space $\mathcal{M}_{\Gamma}^{\text{brok}}(\mathfrak{X}, L)$ of maps of type $\Gamma$, the dimension of the moduli space $\mathcal{M}_{\Gamma}^{\text{brok}}(\mathfrak{X}, L)$ in this case must be at least 2.

The Gromov limit of a sequence of broken disks of type $\Gamma$ may have configurations with sphere and disk bubbling, and may also have configurations with a different tropical graph $\Gamma'$. Such a graph $\Gamma$ can be recovered from $\Gamma'$ by collapsing a subset of edges (Theorem 8.3). The tropical graph $\Gamma'$, when it is distinct from $\Gamma$, necessarily has a larger tropical symmetry group. By the previous paragraph, the dimensions of the moduli spaces $\mathcal{M}_{\Gamma'}^{\text{brok}}(\mathfrak{X}, L)$ and $\mathcal{M}_{\Gamma}^{\text{brok}}(\mathfrak{X}, L)$ are the same, and therefore, the dimension of the quotient $\mathcal{M}_{\text{red},\Gamma'}^{\text{brok}}(\mathfrak{X}, L)$ is at least two lower than the dimension of $\mathcal{M}_{\text{red},\Gamma}^{\text{brok}}(\mathfrak{X}, L)$. Thus, the moduli space $\mathcal{M}_{\text{red},\Gamma}^{\text{brok}}(\mathfrak{X}, L)$ has a compactification consisting of codimension one strata made up of configurations with broken treed segments, and strata of codimension at least two with additional tropical nodes, as in Remark 6.25. In this monograph, we only deal with moduli spaces of broken maps of expected dimension zero and one; and therefore codimension-two strata do not occur. Reduced moduli spaces of broken maps consisting of tropical symmetry orbits of broken maps are the usual objects of study in the literature on relative and log Gromov-Witten theory [2], [18], [33]. However, we choose to use the unquotiented moduli space $\mathcal{M}_{\Gamma}^{\text{brok}}(\mathfrak{X}, L)$ because in the zero-dimensional strata, it has a bijection to unbroken maps, as we explain in page 16 following the statement of Theorem 1.6.

Symplectic cuts have been used in the literature to give formulas for Gromov-Witten invariants in works of many different authors. The case of a single cut was studied by Ionel-Parker [45] and, in the algebro-geometric context, J. Li [50]. They obtained a symplectic sum formula for Gromov-Witten invariants on $X$ in terms of relative Gromov-Witten invariants of the cut spaces, relative to the relative divisor. Their work is a special case of symplectic field theory of Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [11], in which the analogs of broken maps are known as "holomorphic buildings". Eleny Ionel [44] first studied compactifications of moduli spaces of maps relative to a normal crossing divisor. She used a generalization of holomorphic buildings with levels and chambers to describe the maps appearing in the moduli space. Brett Parker [63–71] used the tropical approach to study this problem. Parker defined a category of exploded manifolds, which combined the map part and tropical part into a single space, in a way that convergence of broken maps is continuous. The corresponding Gromov-Witten invariants in algebraic geometry are studied in, for example, Abramovich-Chen-Gross-Siebert [2]. Tehrani gives an alternate compactification of holomorphic curves relative to a normal crossing symplectic divisor [32,34] and uses it to give a degeneration formula [33] for Gromov-Witten invariants in the almost Kähler category. Our approach is essentially a version of Parker's, except that we use Morse chain models rather than de Rham theory and work on the chain level to define counts of pseudoholomorphic maps with Lagrangian boundary.

## 1.4 Broken Fukaya algebras

Counts of disks in the broken symplectic manifold lead to a definition of a broken Fukaya algebra. We use Cieliebak-Mohnke perturbations [26] to regularize the various moduli spaces occurring in the paper. These are domain-dependent perturbations; the domain of the map is stabilized by treating intersection points of the map with a Donaldson-type divisor as marked points. In order to construct a Donaldson-type divisor, which we call a *stabilizing divisor*, we assume the Lagrangian $L \subset X$ and the symplectic manifold $(X, \omega)$ are compact, connected, and *rational*. Rationality means that $X$ admits a line bundle whose curvature is the symplectic form and some tensor power of the line bundle is flat over $L$. The perturbation datum is a collection

$$\underline{\mathfrak{p}} = \{\mathfrak{p}_\Gamma = (J_\Gamma, F_\Gamma)\}_\Gamma$$

of domain-dependent perturbations $\mathfrak{p}_\Gamma$ for each type $\Gamma$ of treed disk, each consisting of a domain-dependent almost complex structure $J_\Gamma$ and Morse function $F_\Gamma$ on the Lagrangian $L$.

Using these perturbations, we construct the geometric Fukaya algebra $CF^{\mathrm{geom}}(L)$ of the Lagrangian $L$ using the Morse model. The constructions are similar to those of Seidel [76] in the exact case and Charest-Woodward [17] in the rational case. Let $\Lambda$ denote the universal Novikov field in a formal variable $q$. The underlying vector space of Floer cochains is generated by critical points of the Morse function $f : L \to \mathbb{R}$ so that

$$CF^{\mathrm{geom}}(L, \underline{\mathfrak{p}}) = \bigoplus_{x \in \mathrm{crit}(f)} \Lambda x,$$

where we sometimes drop the perturbation data $\underline{\mathfrak{p}}$ from the notation. The structure maps

$$m^d : (CF^{\mathrm{geom}}(L))^{\otimes d} \to CF^{\mathrm{geom}}(L)$$

are defined by counting rigid elements in the moduli space $\mathcal{M}(L)$ of holomorphic treed disks

$$\mathcal{M}(L) = \{u : C \to X : \quad u(\partial C) \subset L, u \text{ is a holomorphic treed disk}\}/\sim .$$

Here $C$ is a union of disks, spheres, and line segments; the map $u$ is pseudoholomorphic on the disk and sphere components and satisfies a gradient flow equation on the segments. See Figure 1.10 for a representation and Definition 4.5 for the details.

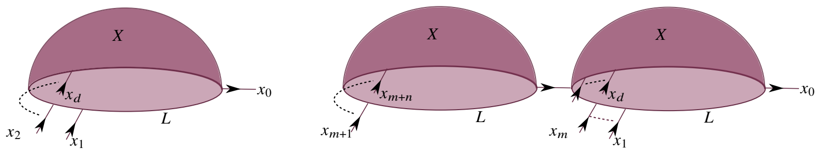



**Figure 1.10.** Holomorphic treed disks counted by the composition map $m^d$. Here, segments with arrows are gradient flow lines of the Morse function $f : L \to \mathbb{R}$, and $x_0, \dots, x_d \in \operatorname{crit}(f)$.

The geometric Fukaya algebra $CF^{\text{geom}}(L)$ is enlarged to an enhanced space of cochains $CF(L)$ by a *homotopy unit construction* so that a strict unit $1_L \in CF(L)$ exists, similar to the construction in Fukaya-Oh-Ohta-Ono [36, (3.3.5.2)] (see Section 10.3 for an exposition on the lines of [17]). Analogously, counts of zero-dimensional components of the moduli space of broken disks in $\mathfrak{X}$ with boundary in $L$ leads to a broken Fukaya algebra, denoted $CF_{\text{brok}}(L)$, with $A_\infty$ structure maps denoted

$$m^d_{\text{brok}} : CF_{\text{brok}}(L)^{\otimes d} \to CF_{\text{brok}}(L), \quad d \geq 0;$$

see Chapter 10. In defining the composition maps $m^d_{\text{brok}}$, we only need to consider broken maps with rigid tropical graphs, since otherwise tropical symmetry orbits are at least two-dimensional. The single cut version of this $A_\infty$ algebra has been constructed by Charest-Woodward [17]. Our main result is the following, proved in Chapter 10:

**Theorem 1.6.** *For a rational Lagrangian submanifold $L \subset X$ and polyhedral decomposition $\mathcal{P}$ as above, or more generally in Definition 3.15, the unbroken Fukaya algebra $CF(L)$ admits a curved $A_\infty$ homotopy equivalence to the broken Fukaya algebra $CF_{\text{brok}}(L)$.*

It follows, in particular, that the broken Fukaya algebras associated to different polyhedral decompositions are homotopy-equivalent. The proof of Theorem 1.6 is completed in Proposition 10.33. The ingredients of the proof are a convergence result and its converse, which is a gluing result. The convergence result is a generalization of compactness in symplectic field theory ([11], [25]) for a single cut and is proved in Chapter 8: Given a sequence of maps $u_\nu : C \to X$ that are holomorphic with respect to almost structures $J_\nu$ that are stretched along multiple necks, there is a subsequence of $u_\nu$ that converges to a broken map. The limit is unique up to the action of the tropical symmetry group. The gluing result in Chapter 9, which is proved only for broken maps of index zero, states that an index zero regular broken map gives rise to a family of $J^\nu$-holomorphic maps $u_\nu : C \to X$ in the unbroken manifold.

We emphasize that the bijection produced by the gluing construction involves broken maps, and not tropical symmetry orbits of broken maps. The distinction is significant even for rigid maps, because rigid broken maps may have a finite non-trivial tropical symmetry group, as in Example 4.42. The distinction is relevant even in case of a single cut: If at a node, a map intersects the relative divisor with multiplicity $k$, then the node has $k$ different framings, where *framing* is part of the data of a broken map in Definition 4.14. Thus there are $k$ broken maps, which differ from each other only in their framing, but they each produce a distinct unbroken map when glued at the neck. This set of $k$ broken maps lie in the same orbit of the tropical symmetry group.

To regularize the moduli spaces of broken maps, we require that the torus actions in the neighborhood of cut loci (in Definition 1.2 (b)) be free. In the absence of this condition, we would need to allow orbifold domain components for broken maps, and the node matching condition (4.22) would not be transversally cut out, since the evaluation map at the orbifold points would be constrained to lie on a strata of $\mathfrak{X}_{P(e)}/T_{P(e),\mathbb{C}}$ that has positive codimension.

## 1.5 Unobstructedness, disk potentials, and cohomology

Homotopy equivalence of $A_\infty$ algebras preserves the Floer cohomology and disk potentials, when each of these is interpreted as a function on the Maurer-Cartan moduli space.

The cohomology $H(A)$ of an $A_\infty$ algebra $A$ is well-defined if the first order composition map $m^1$ satisfies $(m^1)^2 = 0$. The condition $(m^1)^2 = 0$ may fail to hold if the *curvature* $m^0(1) \in A$ is not a $\Lambda$-multiple of the unit $1_L$, which is an "obstruction" to the definition of Floer cohomology. *Weak unobstructedness* is a more general condition under which the Floer cohomology of a Lagrangian brane can be defined. A Lagrangian brane $L$ is *weakly unobstructed* if the projective Maurer-Cartan equation

$$m^0(1) + m^1(b) + m^2(b, b) + \dots \in \Lambda_{>0} 1_L \tag{1.8}$$

has an odd solution $b \in CF(L)$; here, $\Lambda_{>0}$ is the subspace of $\Lambda$ with positive $q$-valuation. Such a solution $b$ is called a *weakly bounding cochain*, and the set of all the odd solutions is denoted $MC(L)$. Given a weakly bounding cochain, the Fukaya algebra $CF(L)$ may be "deformed" by $b$ (see (10.6)) to yield an $A_\infty$ -algebra $CF(L, b)$ with composition maps $(m_b^n)_{n\geq 0}$ satisfying $m_b^0 \in \Lambda_{>0} 1_L$. As a consequence, $(m_b^1)^2 = 0$, and the Floer cohomology

$$H(CF(L, b)) := \ker(m_b^1)/\operatorname{im}(m_b^1)$$

is well-defined.

The homotopy equivalence of Theorem 1.6 gives an isomorphism of Floer cohomology in the following sense. The $A_\infty$ homotopy equivalence in Theorem 1.6 is given by curved $A_\infty$ functors

$$\mathcal{F} : CF(L) \to CF_{\text{brok}}(L), \quad \mathcal{G} : CF_{\text{brok}}(L) \to CF(L), \tag{1.9}$$

The functors $\mathcal{F}, \mathcal{G}$ induce maps on the spaces of Maurer-Cartan solutions

$$\mathcal{F} : MC(CF(L)) \to MC(CF_{\mathrm{brok}}(L)), \quad \mathcal{G} : MC(CF_{\mathrm{brok}}(L)) \to MC(CF(L))$$

and maps on Floer cohomology, that are isomorphisms

$$H(\mathcal{F}) : H(CF(L, b_0)) \to H(CF_{\mathrm{brok}}(L, \mathcal{F}(b_0)),$$
$$H(\mathcal{G}) : H(CF_{\mathrm{brok}}(L, b_1)) \to H(CF(L, \mathcal{G}(b_1)))$$

for any Maurer-Cartan solution $b_0 \in MC(CF(L))$, $b_1 \in MC(CF_{\mathrm{brok}}(L))$. See [17, Section 5.1] for details.

The *disk potential* of a Lagrangian brane $L$ is defined as a count of disks with no input and a single output, and is a function on the space of local systems on $L$ defined as

$$W : \mathrm{Hom}(\pi_1(L), \mathbb{C}^\times) \to \Lambda, \quad y \mapsto \sum_u \epsilon(u) y([\partial u]) q^{\omega(u)}, \tag{1.10}$$

where $\epsilon(u) \in \pm 1$ is an orientation sign, $q$ is the Novikov formal variable, $\omega(u)$ is the area, and the sum ranges over all disk maps $u$ with Maslov index 2 that pass through a fixed (output) point on the boundary.

The homotopy equivalence between Fukaya algebras does not preserve the disk potential, but preserves a *generalized disk potential* defined by Fukaya-Oh-Ohta-Ono [36] as a function on the space of solutions of the projective Maurer-Cartan equation

$$W_{\mathrm{gen}} : MC(L) \to \Lambda, \quad b \mapsto W_{\mathrm{gen}}(b),$$

where the quantity $W_{\mathrm{gen}}(b)$ is given by the Maurer-Cartan equation

$$m^0(1) + m^1(b) + m^2(b, b) + \ldots = W_{\mathrm{gen}}(b) 1_L.$$

The potential is preserved by the homotopy equivalence: For any $b \in MC(CF(L))$, $F(b)$ is a solution of the Maurer-Cartan equation on $CF_{\mathrm{brok}}(L)$, that is, $\mathcal{F}(b) \in MC(CF_{\mathrm{brok}}(L))$, and $W_{\mathrm{gen}}(b) = W_{\mathrm{gen}}(\mathcal{F}(b))$. If $b = 0$ is a solution of the Maurer-Cartan equation, we say that $W_{\mathrm{gen}}(0)$ is the *naive disk potential* as in (1.10). The naive potential is not preserved by neck-stretching because the $A_\infty$ functors $\mathcal{F}, \mathcal{G}$ occurring in the homotopy equivalence are curved, that is, $\mathcal{F}^0 = \mathcal{G}^0 = 0$; and therefore, $\mathcal{F}(b = 0) \neq 0$. See example 2.4.

For all our applications, we use the definition of the disk potential as a naive disk count, in spite of the limitations of this definition. We consider various examples in Chapter 2; in all of these $b = 0$ is a solution of the Maurer-Cartan equation, [4]

[4]Technically speaking, once the Fukaya algebra is enhanced with homotopy units, in the examples of Chapter 2, we actually obtain that the naive disk count is a multiple of $x^{\blacktriangledown}$, which implies weak unobstructedness by Lemma 10.10.

and therefore the disk potential is meaningful. In the monotone case, the $A_\infty$ functors are not curved, and in the semi-Fano case, we work with perturbations for which $A_\infty$ functors are flat, so that in both cases, disk potentials are preserved by neck-stretching. In upcoming work [83], we use neck-stretching and a further degeneration to show weak unobstructedness of toric Lagrangians and to compute disk potentials for almost toric manifolds.

## 1.6 Almost toric manifolds and toric degenerations

We relate our results to those for toric degenerations as in Gross-Siebert ([40], [41]) and to computation of disk potentials in almost toric manifolds. Both of these involve generalizing broken manifolds by allowing the moment map to take values in singular tropical affine manifolds. A *tropical affine manifold* $B$ is a topological manifold with coordinate charts whose transition functions take values in $\mathbb{R}^n \ltimes GL(n, \mathbb{Z})$.

*Almost toric manifolds* in the sense of Symington [81] are a class of examples in dimension four; these are fibrations over tropical affine two-manifolds with dimension zero singular loci whose fibers are Lagrangian tori. Thus, they are generalizations of toric manifolds, where the completely integrable system is allowed to have isolated focus-focus singularities, with the monodromy of the Lagrangian torus fibration around the singular point given by a shear map; see [31] for recent exposition. An almost toric manifold is equipped with a *base diagram* which is the image of the tuple of Hamiltonians, with the additional data of the locations of the singularities on the base diagram, and the directions of the eigenvector of the monodromy indicated by a dotted line emanating from the singular value, as in Figure 1.11. The complement of the singularities and boundary has a torus fibration that projects to the complement of the dotted line. The affine structure is glued along the dotted line by the map of tori induced by transpose of the shear above. The fiber above the singular point is a pinched torus.

As an example, for integers $d > 1$ the resolution of an $A_d$-singularity $\mathbb{C}^2/\mathbb{Z}_d$ has an almost toric structure whose base diagram is as in Figure 1.11. The fibration may be deformed by a "nodal slide" operation (without changing the total space, as a symplectic manifold) that pushes the singular values in the base diagram to a vertex; see [81, Theorem 6.5]. In the limit when the singular value in the base diagram coincides with a vertex $v$, the resulting fibration has a non-toric singularity at the vertex. The fiber over the vertex $v$ is a path of $d-1$ Lagrangian spheres.[5] Via a deformation of

[5]For $t \in (0,1]$ the hypersurface $\mathcal{P}_t := \{z_1 z_2 + P_t(z_3) = 0\} \subset \mathbb{C}^3$ (where $P_t$ is a polynomial of degree $d$ with distinct real roots for $t > 0$ and $P_0(z) = z^d$) has an almost toric structure with $d$ focus-focus singularities ([31, Section 7.3]). There is a path of $d-1$ Lagrangian spheres,

the symplectic form, the Lagrangian spheres may be deformed to symplectic spheres with self-intersection number $-2$, as in the rightmost polytope in Figure 1.11. We compute the disk potentials of resolutions of $A_1$ and $A_2$-singularities using multiple cuts in Sections 2.1 and 2.2. $A_2$-singularities occur in toric degenerations of cubic surfaces. Similar multiple cuts to the one in Section 2.2 can presumably be used to analyze spheres and disks in resolutions of $A_d$-singularities for $d \geq 2$, which would give alternate proofs of results in Chan-Lau [15]. In upcoming work [82], we compute disk potentials of resolutions of cyclic quotient $T$-singularities, which are quotients of $A_d$-singularities by finite groups, and which model del Pezzo surfaces locally.

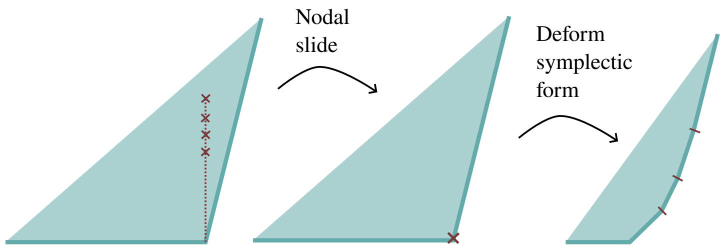



**Figure 1.11.** Resolution of the $A_{d-1}$-singularity $\mathbb{C}^2/\mathbb{Z}_d$ for $d = 4$. Left and middle polytope: $\{y \geq 0, y \geq dx\}$. In the middle figure $\Phi^{-1}(0)$ is a path of $(d-1)$ Lagrangian spheres. The right figure is a toric smoothing of the $A_{d-1}$-singularity whose moment polytope has additional facets $y = kx + \epsilon_k$, $k = 1, \ldots, (d-1)$.

Our results also apply to some toric degenerations. A *toric degeneration*, as in Gross and Siebert ([40], [41]), is a flat family

$$\pi : \mathcal{X} \to \mathbb{C}$$

whose fiber

$$X_t := \pi^{-1}(t)$$

is regular and diffeomorphic to an irreducible projective variety $X$ for $t \in \mathbb{C} \setminus \{0\}$, and whose central fiber $X_0$ is a union of toric varieties glued pair-wise along torus-invariant divisors. The central fiber $X_0$ has a tropical moment map $\Phi : X_0 \longrightarrow B$ to a singular integral affine manifold $B$, with a singular set $\Delta \subset B$ of codimension 2, known as the *discriminant locus*. The integral structure of $B$ has a non-trivial monodromy in $SL(n, \mathbb{Z})$ around points in the discriminant locus. In the complement of the discriminant locus, the tropical Hamiltonian action $X_0$ looks exactly like a union of cut spaces

---

each passing through a pair of the singular points ([31, Remark 7.7]). The path of spheres gets collapsed to an $A_{d-1}$-singularity in the limit $t \longrightarrow 0$.

obtained from a multiple cut. Because of the discriminant locus, the gluing of the toric varieties in the degeneration along torus-invariant divisors is not toric.

Conjecturally, Gross-Siebert's toric degenerations may be viewed as multiple cuts where the cuts are allowed to pass through singular Lagrangian fibers in an almost toric manifold, as can be seen from the toric degeneration of a cubic surface: Given a generic homogeneous polynomial $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ of degree 3, the toric degeneration of $\{f = 0\} \subset \mathbb{P}^2$ is given by the family

$$Y := \{tf + x_0x_1x_2 = 0 : t \in \mathbb{C}\} \overset{\pi}{\mapsto} t \in \mathbb{C}.$$

The fibers of $\pi$ are smooth except for the central fiber $X_0 := \pi^{-1}(0)$, which is the union of 3 copies of $\mathbb{P}^2$, namely $\{x_i = 0\}, i = 0, 1, 2$, glued pairwise along lines $\{x_i = x_j = 0\}$, $i \neq j$. The line $\{x_i = x_j = 0\}$ has singular points at its intersection points with $\{f = 0\}$. Thus, the central fiber $X_0$ has the structure of a tropical Hamiltonian action, where any two copies of $\mathbb{P}^2$ are glued along a line containing three points in the discriminant locus, and the underlying integral affine manifold $B$ is as shown in the left side of Figure 1.12.

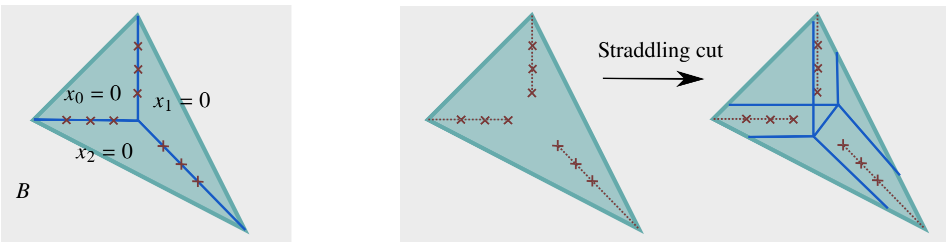



**Figure 1.12.** Left: The integral affine manifold $B$ underlying $X_0$, the central fiber in the toric degeneration of the cubic surface. Right: A straddling multiple cut on the almost toric base diagram of the cubic surface.

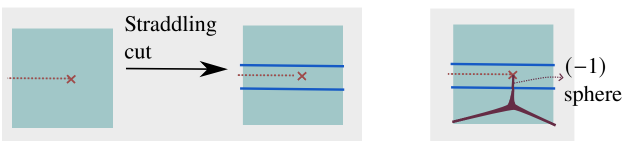



**Figure 1.13.** Left: A straddling cut. Right: Part of a broken sphere.

In some cases, we may apply our results to these kinds of singular toric degenerations if we replace a single cut by two parallel non-singular cuts *straddling* the singular point as shown in Figure 1.13. The "non-toric gluing" in toric degenerations

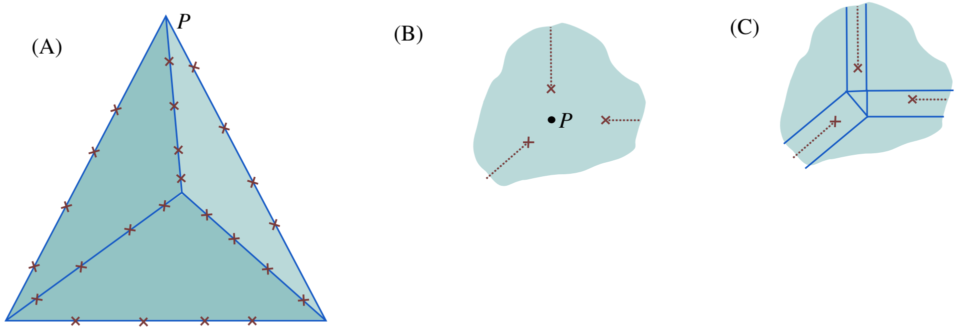



**Figure 1.14.** (a) Integral affine manifold corresponding to the toric degeneration $X_0$ of a quartic surface. There are 24 singular points on the cut locus. (b) Base diagram for the almost toric fibration on the gluing $X_0^{\text{glue}}$ in a neighborhood of the vertex $P$. (c) A part of the straddling cut near $P$.

is thus replaced by an extra cut space that is non-toric. Multiple singular cuts may also be replaced by straddling cuts, as in Figure 1.12 for the example of the cubic surface. Straddling cuts have the feature that (for a carefully chosen multiple cut), a holomorphic curve component intersecting the singular point is necessarily a cover of a fixed $(-1)$-curve. We use such cuts to compute disk potentials of resolutions of $T$-singularities in [82].

*Example* 1.7. A quartic surface in $\mathbb{P}^3$ can be degenerated in a similar way to the cubic surface, as explained in Gross [41, Section 2.1]. The degenerated variety consists of four copies of $\mathbb{P}^2$ glued pairwise along toric divisors, and there are 4 singular points along each of the glued divisors. See Figure 1.14. Analogously to the example of the cubic surface, via a straddling cut we obtain a multiply cut manifold whose (symplectic) sum is conjecturally symplectomorphic to the quartic. Our set-up assumes that the polyhedral decomposition in the tropical Hamiltonian action is embedded in a vector space, while in the quartic example the topological manifold $B = \cup_{P \in \mathcal{P}} P^\circ$ underlying the polyhedral decomposition is a two-sphere, and has the structure of an affine manifold with singularities as defined in Definition 3.12 below. It would be interesting to compute the disk potential of a Lagrangian using broken disks. This ends the Example.

# Chapter 2

# Applications to disk counting

In this Chapter, we describe several examples of disk potential and sphere count computations that the reader might keep in mind during the reading of the theory. Computationally, the easy applications of the theory are in cases where $(X, L)$ is monotone. A pair $(X, L)$ is *monotone* if there is a constant $\tau > 0$ for any disk $u$ in $X$ with boundary on $L$, the Maslov index of $u$ is equal to the multiple $\tau\omega(u)$ of the area $\omega(u)$. For a monotone pair $(X, L)$, the Fukaya algebra is weakly unobstructed since any holomorphic disk has Maslov index $\geq 2$. The potential $m^0(1)$ is independent of the choice of almost complex structure. Indeed, for any two almost complex structures $J_0$ and $J_1$, and a generic path $\mathcal{J} := \{J_t\}_t$ connecting them, the one-dimensional component of the moduli space of $\mathcal{J}$-holomorphic disks with a single boundary constraint does not have any disk bubbling in its compactification, since a non-constant disk has Maslov index $\geq 2$. As a consequence, the signed count of $J_0$ and $J_1$-holomorphic disks is equal. In other words, for a monotone pair $(X, L)$ for any $J$, $b = 0$ is a solution of the Maurer-Cartan equation.

The disk potential in (1.10) is preserved by the neck-stretching for a monotone pair $(X, L)$, because the functors $\mathcal{F}, \mathcal{G}$ in the homotopy equivalence in (1.9) are not curved, that is, $\mathcal{F}^0 = \mathcal{G}^0 = 0$. Indeed, $\mathcal{F}^0$, $\mathcal{G}^0$ are given by the count of disks of Maslov index 0 that are pseudoholomorphic with respect to some $J_t$ in the family $\{J_t\}_{t\in[0,\infty]}\}$ of neck-stretched almost complex structures, and there are no such disks by monotonicity. In Section 2.3 we work with flag varieties, which are monotone, and we compute the disk potential of a monotone Lagrangian torus.

In all the curve counting examples that we consider, the problem is reduced to curve counting in toric manifolds with a toric Lagrangian. For the convenience of the reader, we recall the classification result for holomorphic disks to a toric manifold $X$ with boundary on a toric Lagrangian $L$ from Cho-Oh [22]. For the standard complex structure on the toric manifold, such disks are regular by Cho-Oh [22]. We view the toric manifold as a geometric invariant theory (git) quotient of a vector space $\mathbb{C}^N$ where each hyperplane $\{z_i = 0\}$ projects to a torus-invariant divisor in $X$. The Lagrangian torus $L \subset X$ is a quotient of the standard torus $\{|z_i| = 1, 1 \leq i \leq N\}$ in $\mathbb{C}^N$. Then the holomorphic disks in $X$ bounding $L$ are projections of Blaschke disks in $\mathbb{C}^N$; each such disk is a product, and is of the form

$$u : \mathbb{D} \to \mathbb{C}^N, \quad z \mapsto \left( \zeta_i \prod_{j=1}^{d_i} \frac{z - a_{i,j}}{1 - z\overline{a_{i,j}}} \right)_{i=1,\dots,N}. \tag{2.1}$$

for some constants $a_{i,j}, \zeta_i \in \mathbb{C}, |a_{i,j}| \leq 1, |\zeta_i| = 1$. The moduli space of Blaschke disks of Maslov index two whose boundary passes through a fixed point on the Lagrangian $L$ has dimension zero. There is one such disk intersecting the $i$-th torus-invariant divisor of $X$, corresponding to the index two Blaschke disk in $\mathbb{C}^N$ with single non-vanishing degree $d_i$ equal to 1.

Blaschke disks of Maslov index two account for all disks of Maslov index two in the case when $X$ is Fano (that is, all torus-invariant spheres have positive Chern class). This includes the case when $X$ is monotone; and there are no disks of Maslov index less than two. As a result the Fukaya algebras of toric Lagrangians in Fano toric manifolds are weakly unobstructed. In general, one also needs to consider disk classes that are sums of Blaschke disk classes and sphere classes.

In the first two sections, we consider semi-Fano toric surfaces. A toric manifold $X$ is *semi-Fano* if for any torus invariant sphere of positive symplectic area $S \subset X$, the first Chern number $c_1(TX|S)$ is non-negative. Toric resolutions of $A_n$-singularities $\mathbb{C}^2/\mathbb{Z}_{n+1}$ are semi-Fano toric surfaces, since the spheres added by the resolution have Chern number 0.

In a semi-Fano toric surface, the standard torus-invariant almost complex structure is not regular. We show that any regular almost complex structure arbitrarily close to the torus-invariant complex structure, the Fukaya algebra of a toric Lagrangian is weakly unobstructed, and the disk count is independent of the exact choice of the almost complex structure. We prove a limited result (Proposition 11.15) describing multiple-cuts for which neck-stretching does not alter the disk count. For an almost complex structure close to the standard one, the sum of an index two Blaschke disk class and a sphere class of Chern number zero has Maslov index two. The sphere class may be replaced by its multiples without affecting the expected dimension of the moduli space, so it is not clear at the outset which classes contribute to the disk potential.

## 2.1 Counting disks in the second Hirzebruch surface

In this Section, we compute the potential of a toric Lagrangian in the second Hirzebruch surface $F_2$ using a cut that splits up $F_2$ into Fano surfaces. Here, $F_2$ is the toric symplectic manifold with moment map $\Phi : F_2 \to \mathbb{R}^2$ whose image is the polytope

$$\Delta_{F_2} := \{(x, y) \in \mathbb{R}^2 : -1 \leq y \leq 1, -1 - t + y \leq x \leq 1 + t - x\}. \tag{2.2}$$

for a fixed $t > 0$. The neighborhood of the divisor $\Phi^{-1}(\{y = 1\})$ in the second Hirzebruch surface is the toric smoothing of an $A_1$-singularity, as in Figure 1.11. The space $F_2$ is a semi-Fano toric surface and the sphere $\Phi^{-1}(\{y = 1\})$ has Chern number 0. This example is a warm-up for the calculation of the disk potential of the cubic surface which contains smoothings of $A_2$-singularities.

**Proposition 2.1.** *(Disk potential of $F_2$) Let $X := F_2$ be the second Hirzebruch surface with a Lagrangian $L$. Let $\mathfrak{X}_{\mathcal{P}}$ be the broken manifold with a single cut $\mathcal{P}$ that splits $X$ into two copies of $\mathrm{Bl}_{pt}\,\mathbb{P}^2$ as in Figure 2.1. In both $(X, L)$ and $(\mathfrak{X}_{\mathcal{P}}, L)$, the Lagrangian is weakly unobstructed and the naive disk potential of $L$ (see (1.10)) is (taking $q = 1$)*

$$W(y_1, y_2) = y_2 y_1^{-1} + y_2 y_1 + 2y_2 + y_2^{-1}.$$

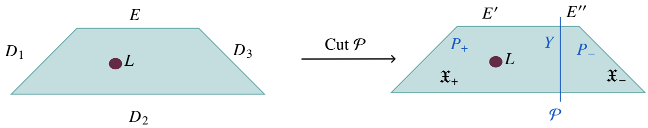



**Figure 2.1.** A cut $\mathcal{P}$ on the second Hirzebruch surface.

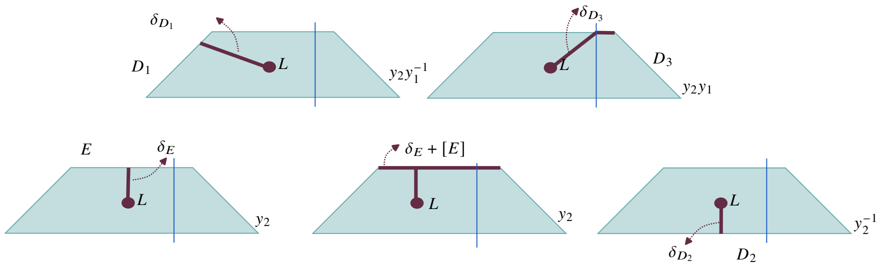



**Figure 2.2.** Broken disks with Maslov index 2 on the second Hirzebruch surface, along with their contributions to the potential.

*Prelude to the proof of Proposition 2.1.* We describe the perturbation used for counting broken spheres and broken disks on the multiply cut cubic surface. The perturbation depends both on the domain curve and the tropical graph. Such perturbations, called *split perturbations*, are defined rigorously in Section 11.1. In Section 11.1 we also show that split perturbations yield the same Gromov-Witten invariants and $A_\infty$ homotopy equivalent Fukaya algebras as coherent domain dependent almost complex structures. We recall that a domain-dependent perturbation has an underlying *background almost complex structure* (see Definition 6.1 (b)) which is perturbed in a compact subset of the complement of the nodal points in the domain. In a split perturbation, the background almost complex structure is allowed to vary across components $C_v$, $v \in \mathrm{Vert}(\Gamma)$ of the domain. For a tropical graph $\Gamma$ we denote the background almost complex structures by

$$J_v^b, \quad v \in \mathrm{Vert}(\Gamma),$$

where $J_v^b$ is a compatible almost complex structure on $\overline{\mathfrak{X}}_{P(v)}$ that is cylindrical in the neighborhoods of relative divisors. For our example of the second Hirzebruch surface, in cases where $P(v)$ is top-dimensional (that is, $P(v) = P_\pm$), $J_v^b$ is defined as

$$J_v^b := \phi_v^* J_{\mathrm{std}}, \tag{2.3}$$

where $J_{\mathrm{std}}$ is the standard almost complex structure on the toric manifold $\overline{\mathfrak{X}}_{P(v)}$; and $\phi_v : \overline{\mathfrak{X}}_{P(v)} \to \overline{\mathfrak{X}}_{P(v)}$ is a Hamiltonian diffeomorphism that is supported away from the relative divisors. If $P(v)$ is not top-dimensional, define $J_v^b := J_{\mathrm{std}}$. Observe that the tropical graph provides a necessary condition for a map component to be homologous to a multiple of a $(-1)$-curve, such as $E'$, $E''$, in Figure 2.1. For example, a map component $u_v$ with $P(v) = P_+$ (see Figure 2.10 for notation) is homologous to $kE'$ only if the sum of multiplicities of the edges incident on $v$ is $k$. We call a map component $u_v$ *univalent*, if in the tropical graph there is at most one edge incident on $v$, and the multiplicity of the edge is 1. Choose perturbations that

(Constant on univalent components) vanish on components where the map is univalent. (2.4)

That is, on such a component $C_v$ the perturbed almost complex structure is equal to the background $J_v^b$. On other components, where the map may be a multiple cover of a $(-1)$-sphere, the background $J_v^b$ is perturbed in a domain-dependent way. The proof of the disk count is carried out below. ■

*Notation* 2.2. (Pseudoholomorphic $(-1)$-spheres) Suppose $\overline{\mathfrak{X}}_P$ is the cut space corresponding to one of the top-dimensional polytopes $P \in \mathcal{P}$. (In the current case $\overline{\mathfrak{X}}_P$ is either $\mathfrak{X}_+$ or $\mathfrak{X}_-$.) Suppose $E \subset \overline{\mathfrak{X}}_P$ is a $(-1)$-sphere that is $J_{\mathrm{std}}$-holomorphic. For any tamed almost complex structure $J$ on $\overline{\mathfrak{X}}_P$, there is a $J$-holomorphic sphere homologous to $E$ which we denote by

$$E_J \subset \overline{\mathfrak{X}}_P. \tag{2.5}$$

Observe that in this notation $E$ is the same as $E_{J_{\mathrm{std}}}$.

*Remark* 2.3. (There are no broken $(-2)$-spheres) A split perturbation as above ensures that, with respect to the background almost complex structure, there are no pseudoholomorphic spheres whose self-intersection number is $-2$ by the following reason: We need to rule out spheres that are homologous to the torus-invariant divisor $E$ whose broken version has components $E' \subset \overline{\mathfrak{X}}_+$, $E'' \subset \overline{\mathfrak{X}}_-$. For any tropical graph $\Gamma$, and $e = (v_0, v_1) \in \mathrm{Edge}(\Gamma)$, with $P(v_0) = P_+$, $P(v_1) = P_-$, for generic background almost complex structures $J_{v_0}^b$, $J_{v_1}^b$ the spheres $E'_{J_{v_0}}$ and $E''_{J_{v_1}}$ (using notation from (2.5)) intersect $Y$ at different points. Therefore there is no broken pseudoholomorphic sphere homologous to $E' \cup E''$ for the background almost complex structures. We may assume the domain-dependent perturbations of $J^b_{v_0}, J^b_{v_1}$ are small enough that the evaluations $\mathrm{ev}_{w_e^-}(u_{v_0}), \mathrm{ev}_{w_e^+}(u_{v_1}) \in Y$ lie in disjoint neighborhoods of the points $E'_{J_{v_0}} \cap Y$ and $E''_{J_{v_1}} \cap Y$, thereby ruling out perturbed $(-2)$-spheres also.

*Proof of Proposition* 2.1. The cut $\mathcal{P}$ satisfies the hypothesis of Proposition 11.15 and therefore, the potential of $L$ in the broken manifold $\mathfrak{X}_{\mathcal{P}}$ is the same as the potential in the unbroken manifold $X$. The set of disk classes in $(X, L)$ that have Maslov index 2 is

$$\delta_E, \quad \delta_{D_i}, \quad i = 1, \ldots, 3, \quad \delta_E + k[E], \quad k \geq 1.$$

Indeed, any disk class in $(X, L)$ is a sum $\sum_D \delta_D + \sum_{D'} [D']$, where $D$, $D'$ range over torus-invariant divisors of $X$, the first summation is non-empty; and the Maslov index of such a disk class is $\sum_D 2 + \sum_{D'} c_1(D')$. We will work in the broken manifold $\mathfrak{X}_{\mathcal{P}}$ to show that these classes are the only relative homology classes that contain a regular broken holomorphic disk. The classes $\delta_E$, $\delta_{D_1}$, $\delta_{D_2}$ are each represented by a perturbation of a Blaschke disk in $\overline{\mathfrak{X}}_+$, see (2.1). The class $\delta_{D_3}$ is represented by a broken disk as shown in Figure 2.2. Next, we consider a broken disk $u$ in $\mathfrak{X}_{\mathcal{P}}$ whose gluing has homology class $\delta_E + k[E]$ for some $k \geq 1$. We will show that $k = 1$ is the only possibility by the following steps.

Step 1: *The disk component $u_+ : C_{v_+} \to \overline{\mathfrak{X}}_+$ of $u$ has homology class $\delta_E + [E']$, and has a simple intersection with the relative divisor $Y$ at a node $w$.* The possibility that $[u_+] = \delta_E + m[E']$, $m > 1$, is ruled out as follows: We use a split perturbation described above in the proof of Propositions 2.6 and 2.8. The condition (Constant on univalent components) in (2.4) implies that $u_+$ is holomorphic with respect to a domain-independent almost complex structure $J_{v_+}$ on $\overline{\mathfrak{X}}_+$. Using notation in (2.5) we denote by $E_{J_{v_+}}$ the $J_{v_+}$-holomorphic sphere in $\overline{\mathfrak{X}}_+$ that is homologous to the $(-1)$-sphere $E'$. If $m > 1$, the intersection number $[u_+].[E_{J_{v_+}}]$ is negative. This is not possible since $[u_+]$ can only have positive intersections with $E_{J_{v_+}}$.

Step 2: *The component $u_- : C_{v_-} \to \overline{\mathfrak{X}}_-$ that is incident on the node $w$ is homologous to $E''$.* Since $[u] = \delta_E + k[E]$ and $[u_+] = \delta_E + [E']$, we have $[u_-] = m[E'']$ for some $m \geq 1$. By the matching condition at the node $w$, $u_-$ has a simple intersection with $Y$ at the node $w$. If $m > 1$, $u_-$ has another intersection with $Y$ at a node $w' \neq w$. If $u' : C_{v'} \to \overline{\mathfrak{X}}_+$ is the other component incident at the node $w'$ then $[u']$ is a multiple of $[E']$. Indeed, that this is so is forced by homology requirements since $[u] = \delta_E + k[E]$. Since both $[u_-]$ and $[u']$ are multiples of $(-1)$-spheres the matching condition at the node $w'$ is not satisfied for our choice of split perturbation. A generic choice of background almost complex structures $J_{v_-}, J_{v'}$ ensures that the $(-1)$-spheres $E_{J_{v_-}} \subset \overline{\mathfrak{X}}_-$, $E_{J_{v'}} \subset \overline{\mathfrak{X}}_+$ do not intersect $Y$ at the same point; and for small enough domain-dependent perturbations the points $\mathrm{ev}_{w'}(u_-) \in Y$ and $\mathrm{ev}_{w'}(u') \in Y$ are distinct. Therefore we conclude that $k = 1$.

The five disks of Maslov index 2 yield the potential claimed by the Proposition. ■

*Example* 2.4. We give an example where a multiple cut changes the potential of a Lagrangian brane. Consider the second Hirzebruch surface $X := F_2$ with two single cuts as in Figure 2.3 and a toric Lagrangian $L$. The potential for the broken manifold is

$$W_2(y_1, y_2) = y_2 y_1^{-1} + y_2 + y_2^{-1}.$$

It does not have disks in two out of the five classes of disks of Maslov index two, namely $\delta_{D_3}$ and $\delta_E + [E]$. For example, if $\mathfrak{X}_{P_2}$ were to contain a regular broken disk $u$ in the class $\delta_{D_1} + [D_1]$, the homology classes of its pieces would be as shown in Figure 2.3, and $u$ would satisfy a matching condition along the relative divisor $Y_2$ between two curves of self-intersection number $-1$. Indeed, for a generic perturbation the two $(-1)$-curves will not be incident at the same point on $Y_2$. By the same argument, broken disks of class $\delta_{D_4}$ are also ruled out in the broken manifold $\mathfrak{X}_{P_2}$. The discrepancy between the disk potentials of $(X, L)$ and $(\mathfrak{X}_{\mathcal{P}}, L)$ is explained by the fact that there is a $J_t$-holomorphic disk of Maslov index zero for some $J_t$ in the family of neck-stretched almost complex structures. See Remark 11.16.

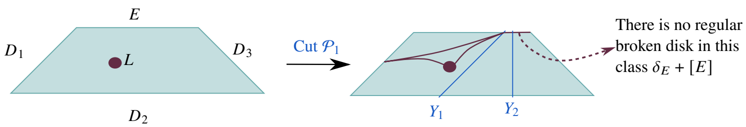



**Figure 2.3.** Moving a cut in the second Hirzebruch surface changes the disk count.

## 2.2 Counting disks in cubic surfaces

We use broken maps to give a proof of the classic result that there are twenty seven lines on the cubic surface, originally proved by Salmon and Cayley, see Mumford [60, Section 8D]. We also compute its "open" analog, namely the disk potential of a toric Lagrangian in the cubic surface. The number of disks is independent of perturbations because the Lagrangian is monotone. We show that there are twenty one disks contributing to the potential as conjectured by Sheridan [78, Appendix B]. This example also appears in Pascaleff-Tonkonog [72] and can also be addressed using the method of Chan-Lau-Cho-Tseng [14]. The paper Bardwell-Evans-Hong-Lin [6] describes the disks in terms of scattering diagrams.

The cubic surface is a monotone almost toric manifold equipped with a monotone Lagrangian torus. Let

$$(X_0, \omega_0, J_0) \subset \mathbb{P}^3$$

be a non-singular cubic surface with the Fubini-Study symplectic form. Since any cubic surface is biholomorphic to the del Pezzo surface $\mathrm{Bl}_6\,\mathbb{P}^2$, we view $(X_0, \omega_0)$ as $\mathrm{Bl}_6\,\mathbb{P}^2$ with a monotone symplectic form, which is $(\mathbb{P}^2, \omega_{FS})$ blown up at six points with a blow-up parameter of 1. McDuff [54, Corollary 1.5] shows that up to symplectic isotopy, there is exactly one way of performing such a blow-up. By Vianna [84], $(X_0, \omega_0)$ has an almost toric structure whose base diagram

$$\Delta_0 := \mathrm{hull}\{(-1,-1),(0,1),(1,0)\} \tag{2.6}$$

is given in Figure 2.4. Using a multiple cut, we count the number of holomorphic spheres with Chern number 1, which is the same as the number of $(-1)$-curves in $\mathrm{Bl}_6\,\mathbb{P}^2$, and which is the same as the number of lines in the cubic surface. Using the same multiple cut, we will also compute the number of rigid disks (Proposition 2.8) bounding the monotone Lagrangian

$$L_0 := \Phi^{-1}(0,0). \tag{2.7}$$

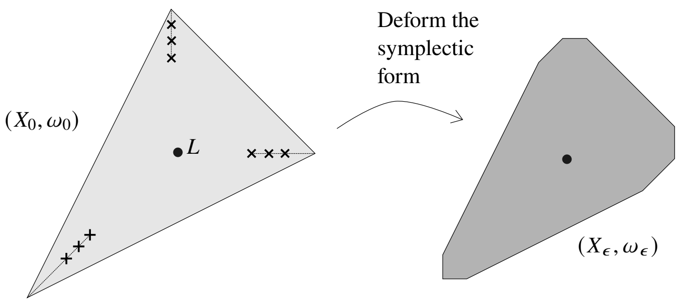



**Figure 2.4**

To perform curve counts on the cubic surface, we symplectically deform it to a semi-Fano toric surface (defined in page 24). In the following result (Proposition 2.5) we show that the cubic surface $(X_0, \omega_0)$ is symplectically deformation equivalent to a toric manifold $(X_\epsilon, \omega_\epsilon)$ whose moment polytope is

$$\Delta_\epsilon := \{(x, y) \in \mathbb{R}^2 : 2x - y + 1, -x + 2y + 1, -x - y + 1 \geq \epsilon, \\ -1 \leq x, y, y - x \leq 1\}, \tag{2.8}$$

for any small $\epsilon > 0$. Note that $(X_\epsilon, \omega_\epsilon)$ is a semi-Fano toric surface.

We observe that $(X_\epsilon, \omega_\epsilon)$ is obtained by blowing up the toric monotone del Pezzo $M := \mathrm{Bl}_3\,\mathbb{P}^2$ at three points with blow-up parameter $1 - \epsilon$, see Figure 2.5. Here, the moment polytope of $M$ is $\Delta_M := \{-1 \leq x, y, y - x \leq 1\}$ and the blow-up is performed at the torus fixed points corresponding to $(1,1),(0,-1),(-1,0)\in\Delta_M$. The torus-invariant divisors of $X_\epsilon$ corresponding to the facets

$$y=\pm 1,\quad x=\pm 1,\quad x-y=\pm 1 \tag{2.9}$$

of $\Delta_\epsilon$ are called *short divisors*, and those corresponding to

$$x+y=1+\epsilon,\quad y-1-\epsilon=2x,\quad x-1-\epsilon=2y. \tag{2.10}$$

are called *long divisors*. The short resp. long divisors are denoted by $E_i$ resp. $D_i$ in Figure 2.9. Note that the short facets of $\Delta_\epsilon$ are collapsed to points in the moment polytope $\Delta_0$ of the almost toric $X_0$. However this is not a proof of the fact that $X_0$ is symplectic deformation equivalent to $X_\epsilon$, $\epsilon>0$, because in Figure 2.5 the blow-up parameter can not be equal to 1. So in Proposition 2.5 we take a more indirect approach to show the symplectic deformation equivalence.

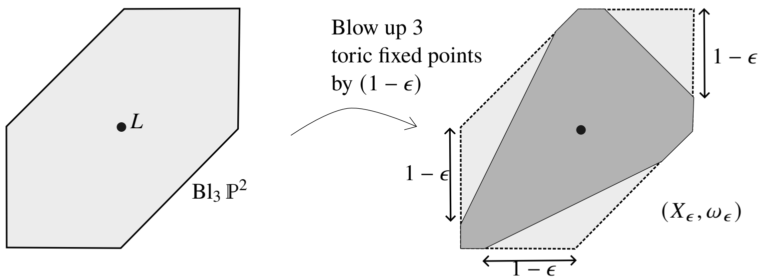



**Figure 2.5**

**Proposition 2.5.** *(Symplectic deformation equivalence) Let $(X_0,\omega_0)$ be the cubic surface with an almost toric structure (2.6), and for any $\epsilon>0$, let $(X_\epsilon,\omega_\epsilon)$ be the toric manifold (2.8). There is a continuous family of diffeomorphisms $\phi_\epsilon : X_0\to X_\epsilon$ with $\phi_0=\mathrm{Id}_{X_0}$, so that $\phi_\epsilon^*\omega_\epsilon$ is a continuous family of symplectic forms on $X_0$ and $L_\epsilon := \phi_\epsilon(L_0)$ is a toric Lagrangian for all $\epsilon>0$.*

*Proof.* To prove that $X_0$ and $X_\epsilon$ are symplectic deformation equivalent, it is enough to show the same for the manifolds with moment polytopes in Figure 2.6. The first space in Figure 2.6 is an almost toric manifold whose base diagram is a polytope $\Delta_0' := \{-x+2y+1,-x-y+1\geq 0\}$ and three focus-focus singularities along $y=0$. This space is the smoothing of the $A_2$ singularity, see [31, Example 7.6]. The right side $Y_\epsilon$ in Figure 2.6 is a toric manifold with moment polytope

$$\Delta_\epsilon' := \{-x+2y,-x-y\geq 0,x,x-y\leq -\epsilon\}.$$

The deformation is carried out by the series of steps in Figure 2.7. All the steps except

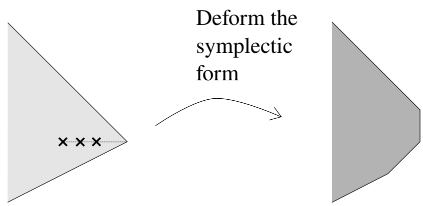



**Figure 2.6**

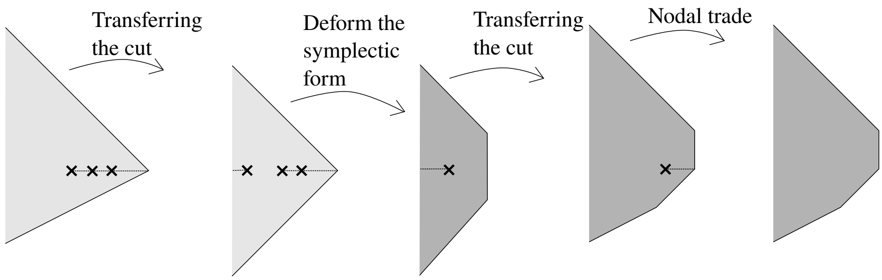



**Figure 2.7**

the second one leave the symplectic form unchanged. In the second step of "deforming the symplectic form", the resolution of an $A_1$-singularity is replaced by the total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$. To see that such a deformation exists recall that the $A_1$-singularity can be smoothed to $T^*\mathbb{P}^1$ [31, p138]. The symplectic form on the total space $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2))$ is given by $\omega_{T^*\mathbb{P}^1} + \epsilon\pi^*\omega_{\mathbb{P}^1}$ where $\pi : \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2)) \to \mathbb{P}^1$ is the projection map and $\omega_{P^1}$ is the Fubini-Study form on $\mathbb{P}^1$. We leave the details of constructing the map $\phi_\epsilon$ to the reader. ■

As a second step in the curve count, we fix $\epsilon > 0$ and apply a multiple cut on

$$X := (X_\epsilon, \omega_\epsilon) \tag{2.11}$$

that splits it into Fano pieces as shown in Figure 2.8. The resulting broken manifold is called $\mathfrak{X}$. We count broken maps consisting of disks and spheres in $\mathfrak{X}$, and prove that the count is the same as the unbroken count in the cubic surface. Under the multiple cut in Figure 2.8, the homology class of any short sphere degenerates to the homology class of a union of two spheres in the pieces $X_{P_i}$, each of which has self-intersection number equal to $-1$.

### 2.2.1 Spheres in the cubic surface

In the following result, we show that a holomorphic sphere in $X := X_\epsilon$ with Chern number one consists of one long divisor and a non-self crossing path of short divisors.

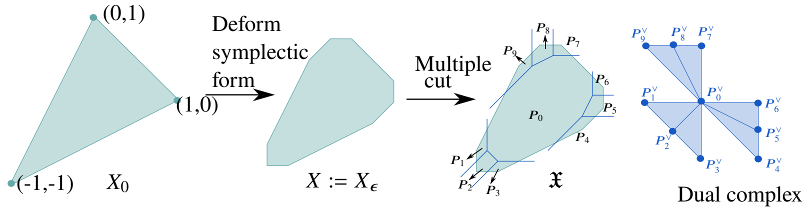



**Figure 2.8.** Multiple cut on the cubic surface.

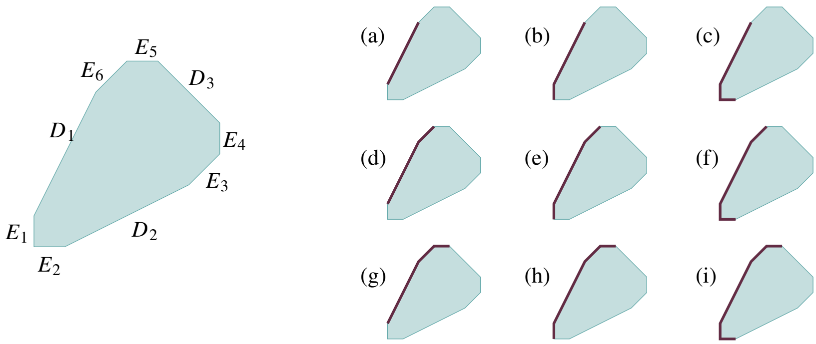



**Figure 2.9.** Nine of the twenty-seven lines on a (deformation of) a cubic surface, in the homology classes $[D_1] + \sum_{i=1,2,5,6} n_i [E_i]$.

There is one configuration corresponding to each of the pictures in Figure 2.9. By symmetry there are twenty-seven such configurations in total.

**Proposition 2.6.** *(Twenty seven lines in the cubic) Any sphere of index zero in the manifold $X := X_\epsilon$ is of the form shown in Figure 2.9, and each such broken sphere has multiplicity one. Therefore, a cubic has 27 lines.*

*Remark* 2.7. (Disks versus spheres) We point out that the results in this paper are about Fukaya algebras (defined using counts of disks), whereas Proposition 2.6 uses analogous results in Gromov-Witten theory (defined using counts of spheres). Broken Gromov-Witten invariants are well-defined in cases when markings are constrained to lie on homology classes that are represented by cycles that are contained in a single top-dimensional cut space. In the case of lines in the cubic surface there are no markings. Whenever the broken Gromov-Witten invariants are well-defined, they are equal to the ordinary GW invariants on the unbroken manifold. The proofs are easier than the $A_\infty$ counterpart because the coherence condition on the perturbation datum is weaker. We elaborate on this difference in Remark 6.8.

We now prove the result on spheres on the cubic surface.

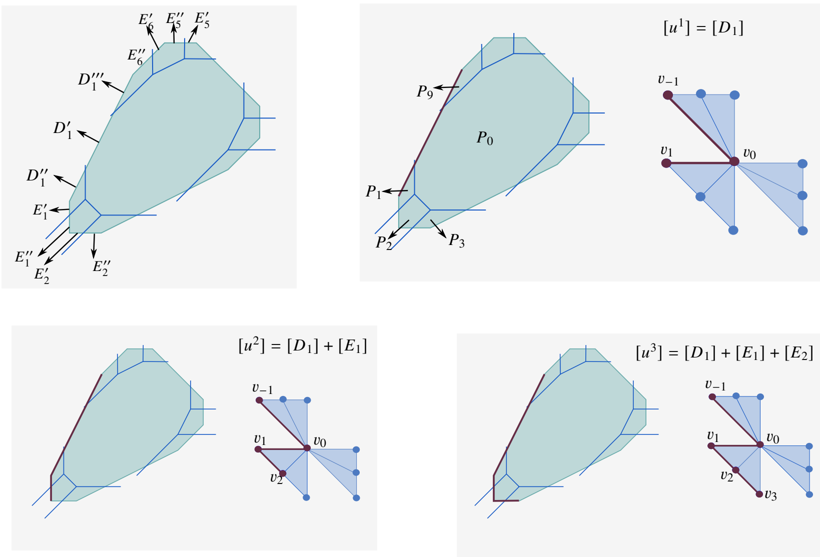



**Figure 2.10.** Top left: Broken versions of some divisors in the cubic surface. Other figures: Broken maps $u^1$, $u^2$, $u^3$ whose gluing is homologous to $[D_1] + k_1[E_1] + k_2[E_2]$ for some $k_1, k_2 \geq 0$.

*Proof of Proposition* 2.6. We recall from the discussion at the beginning of the section that the homology class of a sphere of index zero in $X$ is the sum of a long divisor class $[D_i]$ and an arbitrary number of short divisor classes. Consider a broken sphere $u$ such that its gluing has homology class

$$[u_{\text{glue}}] = [D_1] + n_1[E_1] + n_2[E_2] + n_5[E_5] + n_6[E_6],$$

where $D_1$, $E_i$ are defined in Figure 2.9. We aim to prove that $u$ is of the form shown in Figure 2.9. We first assume that that $n_5 = n_6 = 0$, and so,

$$[u_{\text{glue}}] = [D_1] + n_1[E_1] + n_2[E_2]. \tag{2.12}$$

We will show that a regular configuration is one of the types (a)-(c) in Figure 2.9. The conclusion for the general case $(n_5, n_6 \neq 0)$ follows in a similar way. Indeed, the arguments given below for the "$E_1$ -end" of the long divisor $D_1$ can be applied to the "$E_6$-end" also.

Given a broken map $u$ with homology class as in (2.12) and a tropical graph $\Gamma$, we will show that $n_1, n_2 \leq 1$. The proof is carried out in steps, where in every step, we uncover the possible maps $u_v$ corresponding to a vertex $v$ of the tropical graph. In the following analysis we refer to Figure 2.10 for notation.

STEP 1 ($v_0$): Since the long divisor is a summand in the homology class represented by the map $u$, there is a component of $u_{v_0}$ of the broken map that maps to $D_1'$ (see Figure 2.10 for notations). By the assumption that $n_5 = n_6 = 0$, $v_0$ has

- an edge to $v_{-1}$ with $P(v_{-1}) = P_9$, and the homology class of the map $u_{v_{-1}}$ is $[D_1''']$; and
- an edge to $v_1$ with $P(v_1) = P_1$;

and no other edges.

STEP 2 ($v_1$): So far, we have shown that the broken map has vertices $v_0$, $v_{-1}$ and further, $v_0$ has an edge of multiplicity one to a vertex $v_1$ with $P(v_1) = P_1$. Given that the homology class of the gluing of $u$ is given by (2.12), the homology class of $u_{v_1}$ is

$$[D_1''] + k[E_1'] \quad \text{for some } k \geq 0. \tag{2.13}$$

We proceed to show that $k = 0$ or 1: Recall that $E'_{1,J_{v_1}}$ is a $J_{v_1}$-holomorphic sphere in $\overline{X}_{P_1}$ that is homologous to the $(-1)$-sphere $E_1' \subset \overline{X}_{P_1}$, using the notation in (2.5). Since the map $u_{v_1}$ intersects the relative divisor $\overline{X}_{P_{01}}$, the homology class $[u_{v_1}]$ can not be a multiple of the class $[E_1']$. The property (Constant on univalent components) in (2.4) then implies that the domain-dependent almost complex structure for $u_{v_1}$ is equal to the constant $J_{v_1}$. In (2.13) if $k > 1$, then $[u_{v_1}].[E'_{1,J_{v_1}}] < 0$. This implies that the image of $u_{v_1}$ is contained in $E'_{1,J_{v_1}}$, contradicting the fact that $u_{v_1}$ intersects the relative divisor $\overline{X}_{P_{01}}$. In case $k = 0$, the broken map $u$ is $u^1$ represented in Figure 2.9 (a) and Figure 2.10 (a), $[u^1_{\text{glue}}] = [D_1]$ and there are no more vertices in the tropical graph. If $k = 1$, $v_1$ has an edge to a vertex $v_2$ with $P(v_2) = P_2$.

STEP 3 ($v_2$): So far, we have shown that for a broken map $u$ whose gluing has homology class (2.12), either $[u_{\text{glue}}] = [D_1]$ or the tropical graph of $u$ has vertices $v_{-1}, v_0, v_1$ and $v_1$ has an edge of multiplicity 1 to a vertex $v_2$ with $P(v_2) = P_2$. In the latter case, the homology class (2.12) of the glued map $[u_{\text{glue}}]$ implies that the homology of $u_{v_2}$ is

$$[u_{v_2}] = k_1[E_1''] + k_2[E_2'],$$

and since $v_2$ has an edge to $v_1$, $k_1 \geq 1$.

First, consider the case $k_2 = 0$. If $k_1 = 1$, then there are no more vertices, $u = u^2$, and the map is represented in Figure 2.9 (b) and Figure 2.10 (b). Now suppose $k_2 = 0$ and $k_1 > 1$. Then $u_{v_2}$ is a multiple cover of the sphere $E''_{1,J_{v_2}}$ (using notation from (2.5)). Besides the edge $e$ to $v_1$, $v_2$ has at least one more edge to a vertex, say $\overline{v}_3$, with $P(\overline{v}_3) = P_1$, see Figure 2.11. Indeed, at the node $w_{e'}$ corresponding to the edge $e' := (v_1, v_2)$, the map $u_{v_2}$ has an intersection of order 1 with the relative divisor $\overline{X}_{P_{12}}$. More edges are required to account for the $k$ intersections that $u_{v_2}$ has with $\overline{X}_{P_{12}}$. The homology class of $u_{\text{glue}}$ in (2.12) implies that the homology class of the map $u_{\overline{v}_3}$ is a multiple of $[E'_1]$. For generic background almost complex structures $J^b_{v_2}, J^b_{\bar{v}_3}$ and small enough domain-dependent perturbations, the evaluations

$$\mathrm{ev}_{w_e}(u_{v_2}), \quad \mathrm{ev}_{w_e}(u_{\bar{v}_3}) \in \overline{X}_{P_{12}}$$

will not agree; see Remark 2.3 which explains a similar result. Therefore, matching conditions can not be simultaneously satisfied at the node $(v_1, v_2)$ and $(v_1, \bar{v}_3)$. This argument rules out the possibility that $k_1 > 1$ when $k_2 = 0$.

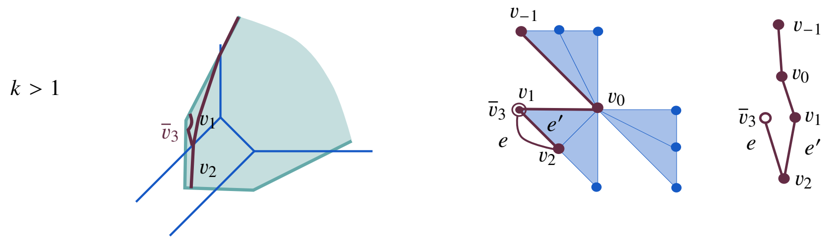



**Figure 2.11.** A hypothetical broken map (ruled out in proof of Proposition 2.6).

Next, assuming $k_2 > 0$, we show that $k_1 = k_2$. The argument is similar to the one used in Step 2. For example, if $k_2 > k_1$, then $[u_{v_2}].[E'_2] < 0$. It follows that the image of $u_{v_2}$ lies in the $J_{v_2}$-holomorphic sphere homologous to $E'_2$, contradicting the fact that $k_1 \geq 1$. The case $k_1 > k_2$ is ruled out by a similar argument.

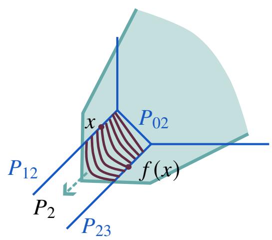



**Figure 2.12.** The space $\overline{X}_{P_2} \backslash (\overline{X}_{P_{02}} \cup E''_{1,J_{v_2}} \cup E'_{2,J_{v_2}})$ is a $\mathbb{P}^1$-fibration. The map $f$ maps an end-point of a $\mathbb{P}^1$-fiber to its other end-point.

Next, we rule out the case $k_1 = k_2 > 1$ using the genericity of $J$. Gluing the $(-1)$-spheres $E''_{1,J_{v_2}}$ and $E'_{2,J_{v_2}}$ produces a $J_{v_2}$-holomorphic sphere with a trivial normal bundle in $\overline{X}_{P_2}$. In this case, there is a fibration

$$\pi : \overline{X}_{P_2} \backslash (\overline{X}_{P_{02}} \cup E''_{1,J_{v_2}} \cup E'_{2,J_{v_2}}) \to \mathbb{C}^\times$$

where the fibers are $J_{v_2}$-holomorphic spheres. When $k_1 = k_2$, the map $u_{v_2}$ is a cover of one of the fibers of $\pi$. We define a diffeomorphism

$$f : \overline{X}_{P_{12}} \backslash \overline{X}_{P_{012}} \to \overline{X}_{P_{23}} \backslash \overline{X}_{P_{023}}$$

by the condition that $x$ and $f(x)$ lie on the same fiber of $\pi$, see Figure 2.12. As we argued in the previous paragraph, the inequality $k_1 > 1$ implies that $v_2$ has an edge to a vertex, say, $\overline{v}_3$ with $P(\overline{v}_3) = P_1$ and $u_{\overline{v}_3}$ is a cover of the sphere $E'_{1,J_{\overline{v}_3}}$ (see Figure 2.13). Since $k_2 > 0$, $v_2$ also has an edge to a vertex, say, $\overline{v}_4$ with $P(\overline{v}_4) = P_3$ and $u_{\overline{v}_4}$ is a cover of the sphere $E''_{2,J_{\overline{v}_4}}$. For generic background almost complex structures $J_{\overline{v}_3}$, $J_{\overline{v}_4}$,

$$f(E'_{1,J_{\overline{v}_3}} \cap X_{P_{12}}) \neq E''_{2,J_{\overline{v}_4}} \cap X_{P_{23}}.$$

The domain-dependent perturbations are small enough that the evaluations $\mathrm{ev}_{w_{e''}}(u_{v_2})$, $\mathrm{ev}_{w_{e''}}(u_{\overline{v}_4}) \in \overline{X}_{P_{23}}$ lie in disjoint neighborhoods of $f(E'_{1,J_{\overline{v}_3}} \cap \overline{X}_{P_{12}})$, $E''_{2,J_{\overline{v}_4}} \cap \overline{X}_{P_{23}}$. As a consequence, there is no broken map $u$ for which $[u_{\mathrm{glue}}] = D_1 + k(E_1 + E_2)$, $k > 1$.

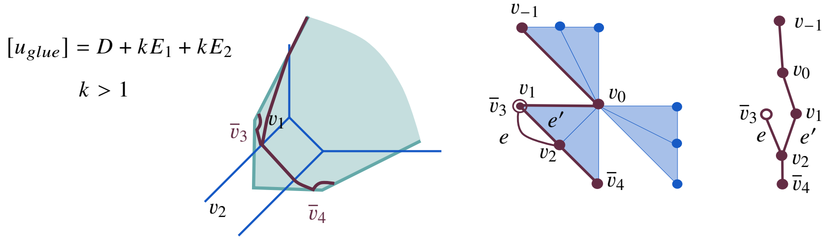



**Figure 2.13.** A hypothetical broken map (ruled out in proof of Proposition 2.6).

Finally, we are left with the case $k_1 = k_2 = 1$. In this case $u_{v_2}$ is a simple map to a fiber of $\pi$, and $v_2$ has an edge to a vertex $v_3$ with $P(v_3) = 0$.

Step 4 ($v_3$) : So far, we have shown that if a broken map $u$ whose gluing has homology class (2.12), then either $[u_{\mathrm{glue}}] = [D_1]$ or $[u_{\mathrm{glue}}] = [D_1] + [E_1]$ or the tropical graph of $u$ has vertices $v_{-1}, v_0, v_1, v_2$, and $v_2$ has an edge of multiplicity 1 to a vertex $v_3$ with $P(v_3) = P_3$. In this last case, the only possibility is that $u_{v_3}$ is a simple map of class $[E''_2]$. The resulting broken map, of class $[D_1] + [E_1] + [E_2]$ is shown in Figure 2.9 (c) and Figure 2.10 (c). Multiple covers of $[E''_2]$ are ruled out using arguments similar to those in Step 2: If $u_{v_3}$ is a multiple cover of $E''_2$ then there is a path $P$ in $\Gamma$ emanating out of $v_3$ (which does not contain $v_2$) whose end-point is a $(-1)$-curve. Let $e \ni v_3$ be the first edge in the path $P$. Since the path ends in a $(-1)$-curve, there is a point constraint on $u_{v_3}$ at the node $w_e$ corresponding to $e$. For a generic background $J$ and a small enough perturbation, the point constraint will be distinct from $E''_{2,J_{v_3}} \cap \overline{X}_{P_{23}}$, contradicting the existence of such a multiple cover $m[E''_2]$ in the broken map.

Finally, we show that each tropical graph in Figure 2.9 contributes +1 to the curve count. Since the sphere $u_v$ corresponding to each vertex $v \in \mathrm{Vert}(\Gamma)$ is pseudoholomorphic with respect to an integrable almost complex structure $J^b_v$ (see (2.3)), and matching conditions are cut out by diagonals on the relative divisors, each of the broken maps has positive orientation in the moduli space. In each of these cases, there is no automorphism of the tropical graph. Therefore, a broken map with $n$ interior markings contributes $\frac{1}{n!}$ to the curve count (see (10.16)). Assuming that the components of the broken map have $n_1, \ldots, n_m$ markings (that add up to $n$), permuting the labellings of marked points within each component gives $(n_1! \ldots n_m!)$ curves. Finally, there are $\frac{n!}{n_1! \ldots n_m!}$ ways of assigning interior markings to components – each of these would give rise to a different almost complex structure. Our proof shows that in all the cases, the number of broken maps corresponding to each tropical graph is unaffected. This finishes the proof of the curve count in broken maps. Since Gromov-Witten invariants are not altered by neck-stretching and deformation of the symplectic form, we have shown that there are 27 lines on a cubic surface. ■

### 2.2.2 Disks in the cubic surface

In the next result, we compute the disk potential of the monotone Lagrangian torus $L_0$ (see (2.7)) in the cubic surface. The monotone Lagrangian, with its trivial local system split-generates the sub-category of the Fukaya category corresponding to the "small eigenvalue" in Sheridan's language in [78]. To compute the potential of the Lagrangian $L_0 \subset X_0$, we work in $(X_\epsilon, \omega_\epsilon)$ which is symplectic deformation equivalent to $(X_0, \omega_0)$, and compute the potential of the Lagrangian torus $L_\epsilon \subset X_\epsilon$ (see Proposition 2.5). We show that a disk in the deformed cubic surface $X_\epsilon$ with boundary in $L_\epsilon$ and of Maslov index $I(u) = 2$ either

(a) intersects a long divisor $D_i$ (there are 3 such disks), or

(b) is a nodal disk consisting of a disk $u_0 : S \to X$ intersecting a short divisor $E_i$ and a path of spheres $u_1, \ldots, u_k : \mathbb{P}^1 \to X$, each mapping to a short divisor $E_j$, and thus having Maslov index zero. The maps $u_i, u_{i+1}$ intersect at a nodal point; and the maps $u_i$, $u_j$ do not intersect if $j - i \geq 2$. There are 18 such nodal disks. See Figure 2.14.

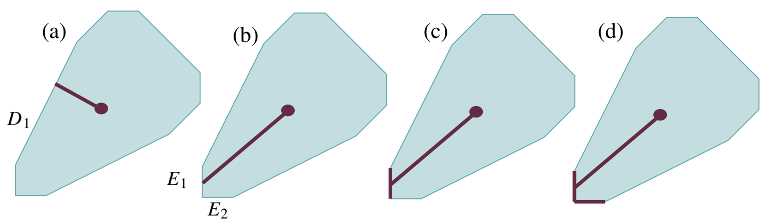



**Figure 2.14.** There are twenty-one Maslov-index-two-disks on a cubic surface: 3 of type (a), and 6 each of type (b), (c) and (d).

**Proposition 2.8.** *(Twenty one disks in the cubic surface) Any holomorphic disk in $X := X_\epsilon$ bounding the Lagrangian $L_\epsilon$ with Maslov index two is of the form shown in Figure 2.14; there are twenty one such disks and each disk has a multiplicity of one.*

*The cubic surface $(X_0, \omega_0)$ has twenty one disks of Maslov index two that bound the monotone Lagrangian $L_0 \subset X_0$ (see (2.7)). The disk potential of $(X_0, L_0)$ is*

$$W(y_1, y_2) = y_1 y_2 + y_1/y_2^2 + y_2/y_1^2 + 3(y_1 + y_1/y_2 + y_2 + y_2/y_1 + 1/y_1 + 1/y_2) \quad (2.14)$$

*Remark* 2.9. The disk potential of $L_0$ in the cubic surface was originally obtained by Pascaleff-Tonkonog [72] using mutations. This formula is equivalent to the one in [72, Table 1] by the change of variables

$$z_1 = y_1^{-1}, z_2 = y_2^{-1}$$

after which the formula becomes

$$W(z_1, z_2) = (1 + z_1 + z_2)^3 / z_1 z_2 - 6.$$

The formula is also obtained in Galkin-Usnich [59]; c.f. [7, Table 5.1].

*Proof of Proposition* 2.8. As in the proof of the sphere count, we first count the number of broken disks of Maslov index 2 in the broken manifold, and then argue that the disk count is the same for the cubic surface. Let $\mathfrak{X}_\epsilon$ be the broken manifold obtained by first deforming the symplectic form on the cubic surface to produce a toric manifold $X^\epsilon$, and then by multiple cutting.

The homology classes of the disks of Maslov index two in $X$ (equipped with a torus-invariant almost complex structure) must consist of a single Blaschke product (i.e. a disk having exactly one intersection with one of the toric divisors) and a collection of short divisors, each of which have Chern number zero:

$$H_2(X_\epsilon, L) \ni [u] = \begin{cases} \delta_{\text{long}} & \text{or} \\ \delta_{\text{short}} + \sum_i a_i [E_i], \end{cases} \quad (2.15)$$

where $\delta_{\text{long}}$ resp. $\delta_{\text{short}} \in H_2(X, L)$ is the class of a disk which is a single Blaschke product intersecting a long resp. short divisor, $a_i \in \mathbb{Z}_+$, and any $E_i \in H_2(X)$ is the homology class of a short divisor.

A semi-Fano toric surface has a well-defined disk potential in the following sense: The standard torus-invariant complex structure $J_\epsilon$ on $X_\epsilon$ is not regular, but for any (domain-dependent) almost complex structure $J'_\epsilon$ that is sufficiently close to $J_\epsilon$, the disk count is independent of $J'_\epsilon$, see Proposition 11.13. Furthermore, the potential is unchanged under neck-stretching if the broken manifold, with the standard almost complex structure (componentwise torus-invariant) $\mathfrak{J}_\epsilon$, does not have disks of Maslov index zero, as explained in Proposition 11.15. This condition is indeed satisfied in the multiple cut we consider on the cubic surface, as explained in Proposition 11.17. Therefore, for any regular domain-dependent almost complex structure $J'_\epsilon$ resp. $\mathfrak{J}'_\epsilon$ close enough to $J_\epsilon$ resp. $\mathfrak{J}_\epsilon$, the potential on $(X_\epsilon, J'_\epsilon)$ and $(\mathfrak{X}_\epsilon, \mathfrak{J}'_\epsilon)$ is the same. In particular, the contribution to the potential from any disk homology class $\beta \in H_2(X_\epsilon, L)$ is the same for $(X_\epsilon, J'_\epsilon)$ and $(\mathfrak{X}_\epsilon, \mathfrak{J}'_\epsilon)$. Thus we may assume that the homology class of any broken disk in $\mathfrak{X}_\epsilon$ is as in (2.15).

Next, we show that out of all the homology classes, only twenty one classes, corresponding to $a_i = 0, 1$ are represented by $\mathfrak{J}'_\epsilon$-holomorphic broken disks, with a single disk in each class. Let $u$ be a broken disk in $\mathfrak{X}_\epsilon$ whose gluing has relative homology class

$$[u_{\text{glue}}] = [\delta_{E_1}] + \sum_i n_i [E_i] \in H_2(X_\epsilon, L),$$

where $[\delta_{E_1}]$ is the class of a Maslov index two disk in $(X_\epsilon, L_\epsilon)$ that intersects the short divisor $E_1$ (using notation in Figure 2.14). The other cases where $\delta_{E_1}$ is replaced by $\delta_{E_i}, i = 2, \ldots, 6$, can be analyzed similarly. Now we need to show that the gluing of $u$ is either (b), (c) or (d) in Figure 2.14. Since the class $[\partial u] \in H_1(L_\epsilon)$ is preserved by neck stretching, we conclude that the disk component of $u$ is a Maslov index two disk in $\overline{X}_{P_0}$ that intersects the relative divisor $\overline{X}_{P_{01}}$. Indeed $\overline{X}_{P_0}$ is a toric manifold; and no other positive integral combination of outward normal vectors of facets adds up to the outward normal $\nu(\overline{X}_{P_{01}}) = (-1, 0)$, besides $(-1, 0)$ itself. The rest of the proof that $u$ is of the expected form is exactly same as Steps 2, 3, 4 in the proof of the sphere count (Proposition 2.6). Each of the tropical graphs contributes +1 to the curve count: Using the standard spin structure, isolated Blaschke disks have a positive orientation sign (see [22, p22]), the maps $u_v$ are pseudoholomorphic with respect to an integrable almost complex structure $J^b_v$ (see (2.3)), matching conditions are cut out by diagonals on the relative divisors, and each of the broken maps has positive orientation in the moduli space. The combinatorics involving marked points carries over from the sphere case.

So far, we have proved that for any regular perturbation $\underline{\mathfrak{p}}_0$ of the standard almost complex structure $J_0$ on the toric manifold $X$, the potential of $CF(L, \underline{\mathfrak{p}}_0)$ has twenty one terms. It remains to show that the potential is the same for the cubic surface $(X_0, \omega_0)$. We take an indirect route to construct the bijection of disks in $X_\epsilon$ and $X_0$, namely, we use a different multiple cut $\mathcal{P}_1$ (Figure 2.15) than the one above, and show that there is a bijection between the disks contributing to the potential in the broken manifolds $\mathfrak{X}_{\mathcal{P}_1}$ and $\mathfrak{X}_{0,\mathcal{P}_1}$. Thus we need to prove the three bijections of disks, corresponding to the degenerations

$$X \longleftrightarrow \mathfrak{X}_{\mathcal{P}_1} \longleftrightarrow \mathfrak{X}_{0,\mathcal{P}_1} \longleftrightarrow X_0. \tag{2.16}$$

The first of these bijections between the set of disks in $X$ and $\mathfrak{X}_{\mathcal{P}_1}$ is a consequence of Proposition 11.15. To prove the second bijection, denote by

$$\mathcal{M}^{<E_0}_{\beta,\mathfrak{J}}(\mathfrak{X})$$

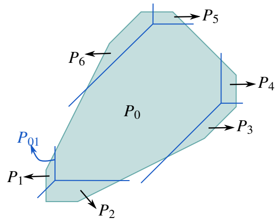



**Figure 2.15.** The cut $\mathcal{P}_1$.

the zero-dimensional component of the moduli space of broken disks with symplectic area $\leq E_0$ and $[\partial u] = \beta$ with a point constraint on the boundary, assuming that the broken disks are defined with respect to a perturbation $\mathfrak{p}$ of $\mathfrak{J}$. Let $\{\tilde{\mathfrak{J}}_t\}_{t\in[0,\epsilon]}$ be a generic path of almost complex structures with end-points $\tilde{\mathfrak{J}}_\epsilon := \mathfrak{J}'_\epsilon$ and $\tilde{\mathfrak{J}}_0$ compatible with the monotone form $\omega_0$. Also, let $\{\mathfrak{p}_t\}_{t\in[0,\epsilon]}$ be a family of perturbations of $\{\tilde{\mathfrak{J}}_t\}_t$. Finally, the moduli space

$$\mathcal{M}^{<E_0}_{\beta,\tilde{\mathfrak{J}}}(\mathfrak{X}) := \cup_t \mathcal{M}^{<E_0}_{\beta,\tilde{\mathfrak{J}}_t}(\mathfrak{X})$$

is one-dimensional whose end-points are $\mathcal{M}^{<E_0}_{\beta,\tilde{\mathfrak{J}}_0}(\mathfrak{X})$ and $\mathcal{M}^{<E_0}_{\beta,\tilde{\mathfrak{J}}_\epsilon}(\mathfrak{X})$. Indeed, $\mathcal{M}^{<E_0}_{\beta,\tilde{\mathfrak{J}}}(\mathfrak{X})$ there is no disk bubbling, because each $t$, the torus-invariant divisors of $\overline{X}_{P-0}$ are $\tilde{\mathfrak{J}}_t$-holomorphic; and for any broken map in the moduli space, the disk component has Maslov index two and intersects the divisor $\overline{X}_{P_{01}}$. Therefore, the boundary strata of $\mathcal{M}^{<E_0}_{\beta,\tilde{\mathfrak{J}}}(\mathfrak{X})$ consist only of $\mathcal{M}^{<E_0}_{\beta,\tilde{\mathfrak{J}}_0}(\mathfrak{X})$ and $\mathcal{M}^{<E_0}_{\beta,\tilde{\mathfrak{J}}_\epsilon}(\mathfrak{X})$. It follows that these last two moduli spaces are bijective. This proves the second bijection in (2.16). The last bijection in (2.16) follows from the fact that neck-stretching does not alter the potential of a monotone symplectic manifold, since there are no disks of Maslov index zero in the family of neck-stretched manifolds, see Page 23. This finishes the proof of the Proposition. ∎

## 2.3 Counting disks in flag varieties

In this Section, we use our results to compute the disk potential of partial flag varieties. The Gelfand-Cetlin system [42] is a collection of functions on a partial flag variety that forms a completely integrable system. The image of these functions is the *Gelfand-Cetlin* polytope $\Delta$, and the toric variety corresponding to this polytope is called the *Gelfand-Cetlin* toric variety. In the complement of faces of $\Delta$ of codimension $\geq 2$, the flag variety and Gelfand-Cetlin toric variety are symplectomorphic. We use this observation to compute the disk potential of the monotone Lagrangian torus in Theorem 2.17; we use a multiple cut consisting of a collection of intersecting single cuts (Figure 2.19) each of which is parallel to, and very close to a facet of the polytope $\Delta$. The original approach of computing disk potentials in flag varieties by Nishinou-Nohara-Ueda [61] and Nohara-Ueda [62] uses small resolutions.

We start by defining partial flag varieties. Given a tuple $\underline{n} := (n_1, \ldots, n_r)$ with $\sum_i n_i = n$, a *flag variety* $\mathrm{Fl}(n_1, \ldots, n_r)$ is the set of flags

$$\mathrm{Fl}(\underline{n}) := \mathrm{Fl}(n_1, \ldots, n_r) := \{\{0\} \subset V_{n_1} \subset V_{n_1+n_2} \subset \ldots V_{n_1+\cdots+n_r} = \mathbb{C}^n \\ : \dim(V_i) = i \ \ \forall i\}. \quad (2.17)$$

A flag variety may be identified with a coadjoint orbit of the action of $U(n)$ on $u(n)^\vee$. In particular,

$$\mathrm{Fl}(\underline{n}) \simeq \mathcal{O}_\Lambda \subset \mathfrak{u}(n)^\vee$$

is the coadjoint orbit of an element $\Lambda$ defined as

$$\Lambda := \mathrm{diag}(\sqrt{-1}(\underbrace{\Lambda_1, \ldots, \Lambda_1}_{n_1 \text{ times}}, \underbrace{\Lambda_2, \ldots, \Lambda_2}_{n_2 \text{ times}}, \ldots, \underbrace{\Lambda_r, \ldots, \Lambda_r}_{n_r \text{ times}})) \in \mathfrak{u}(n)^\vee, \quad (2.18)$$

where

$$\Lambda_i \in \mathbb{R}, \quad \Lambda_1 > \cdots > \Lambda_r,$$

and $\mathfrak{u}(n)$ is identified to its dual via the $\mathrm{Ad}_{U(n)}$-invariant inner product

$$(\cdot, \cdot) : \mathfrak{u}(n) \times \mathfrak{u}(n) \to \mathbb{R}, \quad (A, B) \mapsto \mathrm{trace}(A^* B).$$

Indeed, an element of (2.17) corresponds to an orthogonal decomposition

$$u(n)^\vee = W_{n_1} \oplus \cdots \oplus W_{n_r}$$

into subspaces $W_{n_i}$ of dimension $n_i$ for all $i = 1, \ldots, r$; and such a decomposition gives a unique element in the coadjoint orbit of (2.18) for which the $\sqrt{-1}\Lambda_i$-eigenspace is $W_{n_i}$ for each $i$.

The symplectic form $\omega_\Lambda$ on $\mathrm{Fl}(\underline{n})$ is given by the Kostant-Kirillov form: A tangent vector at a point $x \in \mathfrak{u}(n)$ is of the form $\mathrm{ad}_\xi(x)$ for some $\xi \in \mathfrak{u}(n)$ and the symplectic form at $x$ is given by $(\omega_\Lambda)_x(\mathrm{ad}_{\xi_1}(x), \mathrm{ad}_{\xi_2}(x)) := \langle x, [\xi_1, \xi_2]\rangle$.

We recall the construction of the Gelfand-Cetlin system on a flag variety from Guillemin-Sternberg [42]. The *Gelfand-Cetlin system* is a collection of functions that forms a completely integrable system on an open set of $\mathrm{Fl}(\underline{n})$. For any $m = 1, \ldots, n$, let

$$\pi_m : \mathrm{Fl}(\underline{n}) \to \mathfrak{u}(m)^\vee \quad (2.19)$$

be a map sending a matrix $A$ to its upper-left $m \times m$ submatrix $\pi_m(A)$. Let

$$\Phi : \mathrm{Fl}(\underline{n}) \to \mathfrak{t}^\vee := (\sqrt{-1}\mathbb{R})^{n(n-1)/2} \quad (2.20)$$

be a collection of maps $\Phi = \{\sqrt{-1}\Phi_{i,k}\}_{1\le k\le i\le n}$ where for any $m \le n$ and $A \in \mathrm{Fl}(\underline{n})$,

$$\Phi_{m,1}(A) \ge \Phi_{m,2}(A) \ge \ldots \ge \Phi_{m,m}(A),$$

and $\sqrt{-1}\Phi_{m,k}$, $1 \le k \le m$ are the eigenvalues of $\pi_m(A)$. For any $i$, the eigenvalue functions $\Phi_{i,j}$ are smooth functions whenever they are distinct and are continuous everywhere. By the min-max theorem for Hermitian matrices [85, Theorem 8.10], the eigenvalues satisfy the interlacing inequalities

$$\Phi_{i+1,k} \le \Phi_{i,k} \le \Phi_{i,k+1}, \quad \forall i, k < n$$

and they fit into the following diagram:

$$\begin{array}{ccccccccccc}
\Lambda_1 & & \Lambda_2 & & \Lambda_3 & \cdots & \Lambda_{n-1} & & \Lambda_n \\
& \rotatebox{-45}{$\ge$} \; \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\ge$} \; \rotatebox{45}{$\le$} & & & & \rotatebox{-45}{$\ge$} \; \rotatebox{45}{$\le$} & \\
& \Phi_{n-1,1} & & \Phi_{n-1,2} & & & & \Phi_{n-1,n-1} & \\
& & \rotatebox{-45}{$\ge$} \; \rotatebox{45}{$\le$} & & & & \rotatebox{45}{$\le$} & & \\
& & \Phi_{n-2,1} & & & & \Phi_{n-2,n-2} & & \\
& & & \rotatebox{-45}{$\ge$} & & \rotatebox{45}{$\le$} & & & \\
& & & \ddots & & \iddots & & & \\
& & & & \rotatebox{-45}{$\ge$} \; \rotatebox{45}{$\le$} & & & & \\
& & & & \Phi_{1,1} & & & &
\end{array} \tag{2.21}$$

We give a proof of the min-max theorem for completeness.

**Lemma 2.10.** *(Interlacing inequality) Let $A$ be an $(n+1)\times(n+1)$ Hermitian matrix with eigenvalues $a_1 \ge \cdots \ge a_{n+1}$. Suppose the upper left $n \times n$ submatrix $\pi_n(A)$ has eigenvalues $b_1 \ge \cdots \ge b_n$. Then,*

$$a_1 \ge b_1 \ge a_2 \ge \cdots \ge a_{n-1} \ge b_n \ge a_{n+1}.$$

*Proof.* It is enough to prove the result assuming that $b_1 > \cdots > b_n$, since the general result follows by continuity. Since $b_1, \ldots, b_n$ are the eigenvalues of $\pi_n(A)$, the $\mathrm{Ad}_{U(n)}$-orbit of $A$ contains a matrix

$$A' := \begin{pmatrix} b_1 & & 0 & \bar{z}_1 \\ & \ddots & & \vdots \\ 0 & & b_n & \bar{z}_n \\ z_1 & \ldots & z_n & r_{n+1}, \end{pmatrix}.$$

where $z_i \in \mathbb{C}$, $r_{n+1} \in \mathbb{R}$. Then,

$$\det(A - t\,\mathrm{Id}) = \det(A' - t\,\mathrm{Id}) = \prod_{i=1}^{n}(b_i - t) f(t), \quad f(t) = (r_{n+1} - t - \sum_{j=1}^{n} \frac{|z_j|^2}{b_j - t}).$$

The function $f(t)$ is decreasing in $t$ and discontinuous at $b_1, \dots, b_n$. Since the real numbers $b_j$'s are distinct, the function $f(t)$ has exactly one root each in the intervals $(-\infty, b_n)$, $(b_{i+1}, b_i)$, $i = 1, \dots, n-1$, $(b_1, \infty)$, and the roots are $a_{n+1}, \dots, a_1$. ■

For a partial flag manifold, certain values in the Gelfand-Cetlin system are fixed by the interlacing inequalities. In particular, for $k \leq i \leq n-1$,

$$\Phi_k = \Phi_{n+k-i} = \Lambda_j \implies \Phi_{i,k} = \Lambda_j.$$

The Gelfand-Cetlin system thus consists of exactly

$$N := \sum_{1 \leq i < j \leq r} n_i n_j$$

variables, which is half the dimension of $\mathrm{Fl}(\underline{n})$. See Figure 2.16. We denote the set of variables in the Gelfand-Cetlin system of the partial flag variety $\mathrm{Fl}(\underline{n})$ by

$$\mathrm{Free}(\underline{n}) := \{(i,j) : \Phi_{i,j} \text{ is not a constant on } \mathrm{Fl}(\underline{n})\}.$$

The image $\Phi(\mathrm{Fl}(\underline{n}))$ is contained in a polytope

$$\Delta_{\mathrm{Fl}(\underline{n})} \subset (\sqrt{-1}\mathbb{R})^N$$

cut out by the Gelfand-Cetlin inequalities involving free variables $\{\Phi_{i,j} : (i,j) \in \mathrm{Free}(\underline{n})\}$, which we call the *Gelfand-Cetlin polytope of* $\mathrm{Fl}(\underline{n})$.

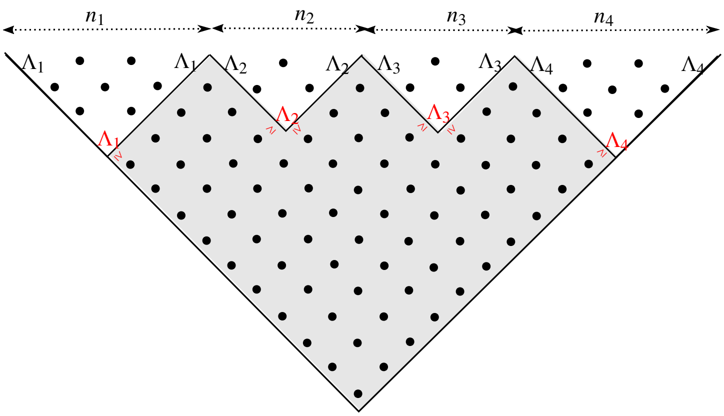



**Figure 2.16.** For a partial flag, the variables of the Gelfand-Cetlin system lie in the shaded region. Each of the unshaded triangles is a block of non-free variables. In any block, all the entries are equal to $\Lambda_i$ for some $i$.

*Remark* 2.11. In the Gelfand-Cetlin system corresponding to the flag variety $\mathrm{Fl}(n_1, \ldots, n_r)$, the non-free variables can be partitioned into $r$ disjoint sets

$$\{(i,k) : 1 \leq k \leq i \leq n-1\} \setminus \mathrm{Free}(\mathrm{Fl}(\underline{n})) = N_1 \cup \ldots N_r$$

where each $N_j$ consists of the collection of indices for maps $\Phi_{i,k}$ which are fixed to be equal to $\Lambda_j$ by the interlacing inequalities. We call $N_j$ a *block* of non-free variables. A block $N_j$ is non-empty if $n_j > 1$. Each non-empty block is triangular and has a *lowest* element whose location in the Gelfand-Cetlin diagram is the lowest. The lowest element in the block $N_j$ is

$$\Phi_{\mathfrak{i}_j,\mathfrak{k}_j}, \quad \text{where } (\mathfrak{i}_j, \mathfrak{k}_j) = \left( n - n_j + 1, \sum_{\ell=1}^{j} n_\ell + 1 \right). \tag{2.22}$$

In Figure 2.16 the lowest position in each block is in red.

The face structure of the Gelfand-Cetlin polytope can be read off from ladder diagrams [3]. For our purposes, we explicitly describe the faces of codimension zero and one. Let $\Delta_{\mathrm{Fl}(\underline{n})}$ be the Gelfand-Cetlin polytope of the flag variety $\mathrm{Fl}(\underline{n})$.

(a) (Interior) A point $\Phi \in \Delta_{\mathrm{Fl}(\underline{n})}$ is an interior point exactly if every Gelfand-Cetlin inequality involving at least one free variable $\Phi_{i,j}$ is strict. We denote the inverse image of the interior by

$$F^{(0)} := \Phi^{-1}(\Delta^{\circ}_{\mathrm{Fl}(\underline{n})}) \subset \mathrm{Fl}(\underline{n}).$$

(b) (Facet) A subset of the polytope $\Delta_{\mathrm{Fl}(\underline{n})}$

$$\mathcal{F}_{i,k,\diagdown} := \{\Phi_{i,k} = \Phi_{i+1,k}\} \quad \text{resp.} \quad \mathcal{F}_{i,k,\diagup} := \{\Phi_{i,k} = \Phi_{i+1,k+1}\} \tag{2.23}$$

is a facet of $\Delta_{\mathrm{Fl}(\underline{n})}$ if either both variables in the equality are free or the non-free variable is the lowest variable in a block of non-free variables (as defined in Remark 2.11). We remark that if the non-free variable in (2.23) were not the lowest in the block, then the codimension of the subset would be more than 1, because the interlacing inequalities would fix the values of some free variables not occurring in (2.23). We denote the set of indices of facets by

$$\mathrm{Facets}(\Delta_{\mathrm{Fl}(\underline{n})}) \subset \mathbb{Z}^2_+ \times \{\diagdown, \diagup\}.$$

A point $\Phi \in \mathcal{F}_{i,k,\diagdown}$ resp. $\mathcal{F}_{i,k,\diagup}$ is in the interior of the facet if all the other Gelfand-Cetlin inequalities are strict. The set of points in $\mathrm{Fl}(\underline{n})$ that map to the interiors of facets is denoted by

$$F^{(1)} := \cup_{I \in \mathrm{Facets}(\Delta(\mathrm{Fl}(\underline{n})))} \Phi^{-1}(\mathcal{F}_I).$$

(c) (Regular points) We denote by

$$F^{\mathrm{reg}} \subset \mathrm{Fl}(\underline{n}) \tag{2.24}$$

the set of points $x$ for which for every $m < n$ the set of eigenvalues of $\pi_m(x)$ has the maximum cardinality. In other words, $x \in F^{\mathrm{reg}}$ exactly when the following holds: For all positive integers $i \leq n-1$, $k < i$,

$$\Phi_{i,k}(x) = \Phi_{i,k+1}(x) \Leftrightarrow (i,k),(i,k+1) \notin \mathrm{Free}(\underline{n}).$$

The Gelfand-Cetlin map is smooth on $F^{\mathrm{reg}}$ (Proposition 2.15).

*Remark* 2.12. (Non-simplicial points of the Gelfand-Cetlin polytope) The Gelfand-Cetlin polytope contains non-simplicial points. A face $\mathcal{F}$ of a polytope $\Delta$ is *non-simplicial* if it is the intersection of $n_{\mathcal{F}}$ facets where $n_{\mathcal{F}} > \mathrm{codim}(\mathcal{F})$. A point $A \in \mathrm{Fl}(\underline{n})$ is mapped to a non-simplicial point the polytope $\Delta_{\mathrm{Fl}(\underline{n})}$ if it satisfies a loop of equalities (see [61, Example 3.8]),

$$\begin{array}{ccccc} & & \Phi_{i+1,k+1} & & \\ & /\!/ & & \backslash\!\backslash & \\ \Phi_{i,k} & & & & \Phi_{i,k+1} \\ & \backslash\!\backslash & & /\!/ & \\ & & \Phi_{i-1,k} & & \end{array} \tag{2.25}$$

and each inequality in the loop corresponds to a facet. Indeed, the loop of equalities in (2.25) gives a subset $S$ of codimension three, whereas $S$ is the intersection of four facets corresponding to each of the equalities in the loop. Observe that $S$ is the intersection of four faces of codimension two, each given by a pair of equalities as in the diagram of inequalities below:

$$\begin{array}{ccccc} & & \Phi_{i+1,k+1} & & \\ & \rotatebox{45}{$\leq$} & & \backslash\!\backslash & \\ \Phi_{i,k} & & & & \Phi_{i,k+1} \\ & \backslash\!\backslash & & \rotatebox{45}{$\leq$} & \\ & & \Phi_{i-1,k} & & \end{array} \quad \begin{array}{ccccc} & & \Phi_{i+1,k+1} & & \\ & /\!/ & & \rotatebox{-45}{$\leq$} & \\ \Phi_{i,k} & & & & \Phi_{i,k+1} \\ & \rotatebox{-45}{$\leq$} & & /\!/ & \\ & & \Phi_{i-1,k} & & \end{array} \quad \begin{array}{ccccc} & & \Phi_{i+1,k+1} & & \\ & \rotatebox{45}{$\leq$} & & \backslash\!\backslash & \\ \Phi_{i,k} & & & & \Phi_{i,k+1} \\ & \rotatebox{-45}{$\leq$} & & /\!/ & \\ & & \Phi_{i-1,k} & & \end{array} \quad \begin{array}{ccccc} & & \Phi_{i+1,k+1} & & \\ & /\!/ & & \rotatebox{-45}{$\leq$} & \\ \Phi_{i,k} & & & & \Phi_{i,k+1} \\ & \backslash\!\backslash & & \rotatebox{45}{$\leq$} & \\ & & \Phi_{i-1,k} & & \end{array}$$

*Example* 2.13. The full flag $F^{(3)} = \mathrm{Fl}(1,1,1)$ resp. the Grassmannian $\mathrm{Gr}(2,4) = \mathrm{Fl}(2,2)$ has a singular point resp. a singular line given by a loop of equalities (see Figure 2.17)

$$\begin{array}{ccccc} & & \Lambda_2 & & \\ & /\!/ & & \backslash\!\backslash & \\ \Phi_{2,1} & & & & \Phi_{2,2} \\ & \backslash\!\backslash & & /\!/ & \\ & & \Phi_{1,1} & & \end{array} \quad \text{resp.} \quad \begin{array}{ccccc} & & \Phi_{3,2} & & \\ & /\!/ & & \backslash\!\backslash & \\ \Phi_{2,1} & & & & \Phi_{2,2}. \\ & \backslash\!\backslash & & /\!/ & \\ & & \Phi_{1,1} & & \end{array} \tag{2.26}$$

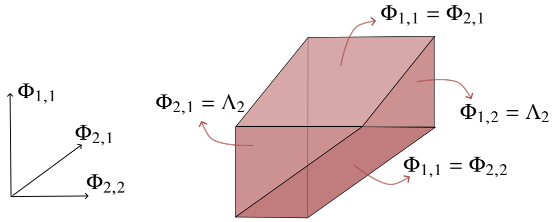



**Figure 2.17.** There is a non-simplicial point in $\Phi(\mathrm{Fl}(1,1,1))$ corresponding to a loop of equalities.

*Remark* 2.14. (Gelfand-Cetlin toric variety) For any tuple $\underline{n}$, the toric variety $M_{\underline{n}}$ corresponding to the polytope $\Phi(\mathrm{Fl}(\underline{n}))$, called the *Gelfand-Cetlin toric variety*, is a toric degeneration of the flag manifold $\mathrm{Fl}(\underline{n})$ [8]. The Gelfand-Cetlin toric variety has conifold singularities of codimension three (see Batyrev et al. [8]) corresponding to the non-simplicial points in the polytope $\Delta_{\mathrm{Fl}(\underline{n})}$. The fibers of $\Phi|\,\mathrm{Fl}(\underline{n})$ over non-simplicial points has larger dimension than the corresponding fibers in the toric variety. For example, the inverse image of the singular point in $\Phi(\mathrm{Fl}(1,1,1))$ in Figure 2.17 is a Lagrangian $S^3$ [61, Example 3.8]. A detailed description of the singular fibers in partial flag varieties, and their comparison with the Gelfand-Cetlin toric variety is carried out in [24].

The following result shows that on the regular locus $F^{\mathrm{reg}} \subset \mathrm{Fl}(\underline{n})$ (defined in (2.24)), the Gelfand-Cetlin system is smooth and is a moment map for a torus action. As a symplectic toric manifold, this set is isomorphic to the corresponding subset in the Gelfand-Cetlin toric variety. The regular locus $F^{\mathrm{reg}}$ includes $F^{(0)} \cup F^{(1)}$, that is, the inverse image of the interior and the facets of the Gelfand-Cetlin polytope.

**Proposition 2.15.** *([42]) Let $\Phi : \mathrm{Fl}(\underline{n}) \to \mathbb{R}^N$ be the Gelfand-Cetlin system on the flag variety.*

(a) *The restriction $\Phi|F^{\mathrm{reg}}$ is smooth and is a moment map for the action of the torus $\mathbb{T}^N$, where $F^{\mathrm{reg}} \subset \mathrm{Fl}(\underline{n})$ is defined in (2.24).*

(b) *The map $\Phi$ surjects onto the polytope $\Delta_{\mathrm{Fl}(\underline{n})}$.*

(c) *As a symplectic toric manifold, $F^{\mathrm{reg}} \subset \mathrm{Fl}(\underline{n})$ is isomorphic to the corresponding subset in the Gelfand-Cetlin toric variety. Furthermore, the inverse image $F^{(0)} \cup F^{(1)}$ of the interior and the facets of $\Delta_{\mathrm{Fl}(\underline{n})}$ is contained in $F^{\mathrm{reg}}$.*

(d) *Consequently, for any point $\lambda$ in the interior of $\Delta_{\mathrm{Fl}(\underline{n})}$, $\Phi^{-1}(\lambda)$ is a Lagrangian torus.*

*Proof.* The map $\Phi$ is smooth on the set of regular points, namely $F^{\mathrm{reg}}$. Indeed, the map $A \mapsto \lambda_i(A)$ that sends a symmetric $m \times m$ matrix to its $i$-th eigenvalue in non-increasing order is smooth at a point if $\lambda_i(A) \neq \lambda_{i+1}(A), \lambda_{i-1}(A)$. Any pair of functions in the Gelfand-Cetlin system Poisson commute :

$$\{\Phi_{i,j}, \Phi_{k,l}\} = 0$$

by [42, Section 5], [61, Lemma 3.3]. So, the system $\{\Phi_{i,k}\}_{i,k}$ generates a Hamiltonian $\mathbb{R}^n$-action on $\mathrm{Fl}(\underline{n})$. Any non-constant Hamiltonian trajectory generated by $\Phi_{i,k}$ has a period of $2\pi$ by [42, Lemma 3.4]. Consequently, the system $\{\Phi_{i,k}\}_{i,k}$ generates an action of the torus $(\mathbb{R}/2\pi\mathbb{Z})^N$ on $F^{\mathrm{reg}}$.

The map $\Phi$ surjects onto $\Delta_{\mathrm{Fl}(\underline{n})}$ by [42, Section 5]. The proof is by an inductive application of Lemma 2.16, which constructs a matrix $A \in \mathfrak{u}(n)$ for a prescribed collection of eigenvalues $\{\Phi_{i,j}\}_{i,j}$. The inductive construction gives a map $\{\Phi_{i,j}\}_{i,j} \mapsto A$ that is smooth strata-wise. Since this map is a right inverse of $\Phi$, we have shown that $\mathrm{d}\Phi$ has the maximum rank on every stratum $S$ of $F^{\mathrm{reg}}$,[1] that is, $\mathrm{rank}(\mathrm{d}\Phi) = \frac{1}{2}\dim(S)$. Therefore, on each stratum $S \subset F^{\mathrm{reg}}$, $\{\Phi_{i,j}\}_{i,j}$ is a complete integrable system; proving the first statement in (c). The definition of $F^{\mathrm{reg}}$ implies that $F^{(0)} \cup F^{(1)} \subset F^{\mathrm{reg}}$. By the Arnol'd-Liouville theorem (proved in, for example, Duistermaat [30]), $\Phi|F^{(0)}$ is a Lagrangian torus fibration. This finishes the proof. As an aside, we remark that the rank of $\mathrm{d}\Phi$ is smaller than half the dimension of the stratum, in case of a stratum mapping to a non-simplicial point of $\Delta_{\mathrm{Fl}(\underline{n})}$; see Example 2.13. ■

**Lemma 2.16.** *Let*

$$\Delta := \{(a,b) = ((a_1,\dots,a_n),(b_1,\dots,b_{n-1})) \in \mathbb{R}^{2n-1} : a_1 \geq b_1 \geq \dots \geq b_{n-1} \geq a_n\}.$$

*For any $(a,b) \in \Delta$, there is a unique tuple*

$$r = (r_1,\dots,r_{n+1}) \in (\mathbb{R}_{\geq 0})^n \times \mathbb{R}$$

*such that any matrix $A \in \sqrt{-1}\mathfrak{u}(n)$ with eigenvalues $(a_1,\dots,a_n)$, and for whom $\pi_{n-1}(A)$ (defined in (2.19)) is the diagonal matrix with entries $b_1,\dots,b_n$ is of the form*

$$A := \begin{pmatrix} b_1 & & 0 & \bar{z}_1 \\ & \ddots & & \vdots \\ 0 & & b_n & \bar{z}_n \\ z_1 & \dots & z_n & r_{k+1} \end{pmatrix}$$

*with $|z_i|^2 = r_i$ for $i = 1,\dots,n-1$.*

(a) *The map*

$$r : \Delta \to (\mathbb{R}_{\geq 0})^{n-1} \times \mathbb{R}, \quad (a,b) \mapsto (r_1,\dots,r_n)$$

*is continuous.*

[1] The partial flag manifold $\mathrm{Fl}(\underline{n})$ is stratified on the basis of which of the Gelfand-Cetlin inequalities are equalities.

(b) *For any i,* $r_i = 0$ *exactly when either* $b_i = a_i$ *or* $b_i = a_{i+1}$.

(c) *For any subset* $I \subseteq \{1, \ldots, 2n\}$*, let* $\Delta_I \subset \Delta$ *be the set of tuples* $(a_1, b_1, \ldots, b_{n-1}, a_n)$ *with equalities at the positions in I, and strict inequalities everywhere else. Then,* $r|\Delta_I$ *is smooth.*

*Proof.* We first consider the open stratum of $\Delta$, that is, $I = \emptyset$ and

$$a_1 > b_1 > a_2 > \cdots > a_{n-1} > b_{n-1} > a_n.$$

Observe that

$$\det(A - t\,\mathrm{Id}) = \left(\prod_{i=1}^{n-1}(b_i - t)\right)\left(r_n - t - \sum_{j=1}^{n-1}\frac{|z_j|^2}{b_j - t}\right). \tag{2.27}$$

We set

$$r_n := \sum_i a_i - \sum_i b_i,$$

and solve the system of equations

$$\det(A - a_i\,\mathrm{Id}) = 0 \quad \text{for } \ i = 1, \ldots, n-1 \tag{2.28}$$

for $z_1, \ldots, z_{n-1}$. Substituting $|z_i|^2 = r_i$, the system (2.27) is linear in $r_1, \ldots, r_{n-1}$, and the coefficient matrix for the variables $r_i$, $i = 1, \ldots, n-1$ is

$$\left(\prod_{1 \le k \le n : k \ne j}(b_k - a_i)\right)_{i,j}. \tag{2.29}$$

The determinant of (2.29) is

$$\det((2.29)) = \pm \prod_{i<j}(a_i - a_j)\prod_{i<j}(b_i - b_j) \ne 0.$$

Therefore, we can uniquely solve (2.27) for $(r_1, \ldots, r_{n-1})$.

It remains to show that $r_1, \ldots, r_{n-1}$ are non-negative. Firstly, we observe that $r^{-1}(\partial((\mathbb{R}_{\ge 0})^{n-1} \times \mathbb{R})) \subset \partial\Delta_b$. Indeed, in the matrix $A$, if $z_i = 0$ then $b_i$ is an eigenvalue, and so $b_i$ is equal to either $a_i$ or $a_{i+1}$. Secondly, by the interlacing property the image $r(\Delta_b^\circ)$ of the interior $\Delta_b^\circ$ of $\Delta_b$ intersects the interior of $(\mathbb{R}_{\ge 0})^{n-1} \times \mathbb{R}$. We may then conclude that $r(\Delta_b^\circ)$ is in fact contained in $(\mathbb{R}_+)^{n-1} \times \mathbb{R}$.

Finally, we define the map $r$ on lower dimensional strata of $\Delta$. If $b_i = a_i$ resp. $b_i = a_{i+1}$, then we set $r_i = 0$. Define $\hat{a} := (a_1, \ldots \hat{a}_i, \ldots, a_n)$ resp. $(a_1, \ldots \hat{a}_{i+1}, \ldots, a_n)$, and $\hat{b} := (b_1, \ldots \hat{b}_i, \ldots, b_n)$ to be the tuples deleted in each. The above construction applied to $(\hat{a}, \hat{b})$ produces $\hat{r} := (r_1, \ldots, \hat{r}_i, \ldots, r_{n+1})$. The map $(\hat{a}, \hat{b}) \mapsto \hat{r}$ is smooth, which shows that the map $(a, b) \mapsto r$ is smooth strata-wise. ∎

In the rest of the Section, we prove the following result, which was originally proved by Nishinou-Nohara-Ueda [61].

**Theorem 2.17.** *Let $X := \mathrm{Fl}(\underline{n})$ be a flag variety. There is a symplectic form on $X$ for which the Gelfand-Cetlin system of $X$ has a unique monotone Lagrangian fiber $L_0$. The $A_\infty$ algebra $CF(L_0)$ is weakly unobstructed and the naive potential of $CF(L_0)$ consists of one term for every facet of the Gelfand-Cetlin polytope of $X$.*

As part of the proof of Theorem 2.17, we compute the disk potential (defined in (1.10)) of $\mathrm{Fl}(\underline{n})$ as

$$W(y_{i,k})_{i,k} = \left( \sum_{k \le i : (i+1,k) \in \mathrm{Free}(\mathrm{Fl}(\underline{n}))} \frac{y_{i+1,k}}{y_{i,k}} + \sum_{k \le i : (i+1,k+1) \in \mathrm{Free}(\mathrm{Fl}(\underline{n}))} \frac{y_{i,k}}{y_{i+1,k+1}} \right.$$
$$\left. + \sum_{1 \le j \le r-1} y_{\mathfrak{i}_j, \mathfrak{k}_j} + \sum_{2 \le j \le r} \frac{1}{y_{\mathfrak{i}_j, \mathfrak{k}_j}} \right) q. \quad (2.30)$$

Here the index $(\mathfrak{i}_j, \mathfrak{k}_j)$ is the position of the lowest element in the $j$-th block, see (2.22). In the above expression (2.30) for the potential, each term corresponds to a facet (2.23) of the Gelfand-Cetlin polytope – the first resp. second summation corresponds to facets $\mathcal{F}_{i,k,\searrow}$ resp. $\mathcal{F}_{i,k,\swarrow}$ where both variables in the equality defining the facet are free, the third resp. fourth summation corresponds to facets $\mathcal{F}_{i,k,\searrow}$ resp. $\mathcal{F}_{i,k,\swarrow}$ where one of the variables in the defining equality is non-free and is the lowest element of the $j$-th block. The exponent of the Novikov formal variable $q$ is 1, because the symplectic form (given by a choice of $(\Lambda_i)_i$) and Lagrangian torus $L_0$ are chosen so that the pair $(\mathrm{Fl}(\underline{n}), L_0)$ is monotone and the area of any disk of Maslov index 2 is 1. We will show for every facet, there is a single holomorphic disk of Maslov index two intersecting the corresponding divisor in $\mathrm{Fl}((\underline{n}))$.

*Example* 2.18. The Gelfand-Cetlin inequalities of the Grassmannian $\mathrm{Gr}(2,4) = \mathrm{Fl}(2,2)$ are given by

$$\begin{array}{ccccccccccccc}
\Lambda_1 & & & & \Lambda_1 & & & & \Lambda_2 & & & & \Lambda_2 \\
& \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & \\
& & \Lambda_1 & & & & \Phi_{3,2} & & & & \Lambda_2 & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & & \Phi_{2,1} & & & & \Phi_{2,2} & & & & \\
& & & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & & & \\
& & & & & & \Phi_{1,1} & & & & & &
\end{array} \quad (2.31)$$

and the inequalities in grey correspond to facets. The potential is

$$W(y) = \left( \left( \frac{y_{2,1}}{y_{1,1}} + \frac{y_{3,2}}{y_{2,2}} \right) + \left( \frac{y_{1,1}}{y_{2,2}} + \frac{y_{2,1}}{y_{3,2}} \right) + \left( \frac{1}{y_{2,2}} \right) + \left( y_{2,1} \right) \right) q,$$

where the terms are parenthesized to reflect the summation groupings from (2.30).

*Remark* 2.19. In the case of a full flag variety ($r = n$), the least order terms of the potential in (2.30) coincide with the potential introduced by Givental in [38]. In the case of the Grassmannian ($r = 2$), the analysis of the potential (2.30) by Castronovo in [13] implies that the Lagrangian $L_0$ has non-trivial Floer homology.

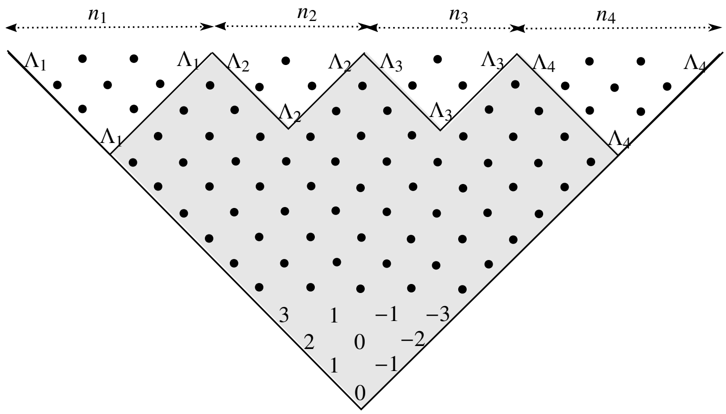



**Figure 2.18.** Center of the Gelfand-Cetlin polytope $\Delta_{\mathrm{Fl}(\underline{n})}$.

*Proof of Theorem* 2.17. First, we describe the monotone symplectic form and the monotone Lagrangian in the flag variety $\mathrm{Fl}(\underline{n})$. Let $\mathrm{Fl}(\underline{n})$ be the coadjoint $U(n)$-orbit of the element

$$\mathrm{diag}(\sqrt{-1}(\underbrace{\Lambda_1, \dots, \Lambda_1}_{n_1 \text{ times}}, \underbrace{\Lambda_2, \dots, \Lambda_2}_{n_2 \text{ times}}, \dots, \underbrace{\Lambda_r, \dots, \Lambda_r}_{n_r \text{ times}})) \in \mathfrak{u}(n)^\vee,$$

where

$$\Lambda_j := \mathfrak{i}_j - 1 - 2(\mathfrak{k}_j - 1),$$

and $(\mathfrak{i}_j, \mathfrak{k}_j)$ is the index of the lowest element in the $j$-th block of non-free variables, see (2.22). The Kostant-Kirillov form on $\mathrm{Fl}(\underline{n})$ is monotone (see [61, p653-654]) satisfying

$$c_1(T\,\mathrm{Fl}(\underline{n})) = [\omega_\Lambda]. \tag{2.32}$$

The Gelfand-Cetlin polytope $\Delta_{\mathrm{Fl}(\underline{n})}$ has a center $\lambda$ (see Figure 2.18) given by

$$\lambda_{i,k} = (i-1) - 2(k-1)$$

with coordinates satisfying

$$\lambda_{i,k} - \lambda_{i,k-1} = 1, \quad \forall i, k.$$

The fiber

$$L_\lambda := \Phi^{-1}(\lambda)$$

is a monotone Lagrangian with monotonicity constant $\frac{1}{2}$, that is,

$$\forall u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (X, L_\lambda) \;\; \omega(u) = \tfrac{1}{2} I(u). \tag{2.33}$$

Indeed, using (2.32) and $\pi_2(X, L_\lambda)/\pi_2(X) = \pi_1(L_\lambda)$ from the homotopy long exact sequence, it is enough to verify (2.33) on a set $S$ of disks $u$ bounding $L_\lambda$ for which $\{[\partial u] : u \in S\}$ generates $\pi_1(L_\lambda)$; and for a disk of Maslov index two that has a single intersection with a divisor corresponding to a facet of the Gelfand-Cetlin polytope, the symplectic area is 1. Cho-Kim [23, Theorem B] give an alternate proof of the monotonicity of the Lagrangian torus $L_\lambda$.

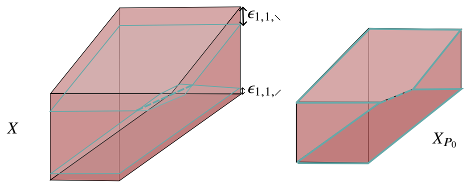



**Figure 2.19.** Left : Two of the cuts in the multiple-cut of $X := \mathrm{Fl}(1,1,1)$. There is a similar cut for each of the facets of the Gelfand-Cetlin polytope. Right: The Delzant polytope of the cut space $X_{P_0}$.

We consider a multiple cut $\mathcal{P}$ of the flag variety given by a collection of transversally intersecting single cuts along the hypersurfaces

$$Y_{i,k,\nearrow} := \{\Phi_{i,k} - \Phi_{i+1,k+1} = \epsilon_{i,k,\nearrow}\} \quad \text{resp.} \quad Y_{i,k,\searrow} := \{\Phi_{i,k} - \Phi_{i+1,k} = \epsilon_{i,k,\searrow}\} \tag{2.34}$$

for each $(i, k, \nearrow)$ resp. $(i, k, \searrow) \in \mathrm{Facets}(\mathrm{Fl}(\underline{n}))$, where the parameters

$$(\epsilon_{i,k,\searrow})_{i,k}, (\epsilon_{i,k,\nearrow})_{i,k}$$

are positive, generic and small enough that the cuts in (2.34) bound a polytope $P_0$ such that

- $P_0$ contains the Lagrangian $L_\lambda$; and
- the facets of $P_0$ are bijective to facets of the Gelfand-Cetlin polytope $\Delta_{\mathrm{Fl}(\underline{n})}$, with each cut being parallel to the corresponding facet of $\Delta_{\mathrm{Fl}(\underline{n})}$.

The cuts are indeed well-defined because the hypersurfaces in (2.34) lie in $F^{\text{reg}}$, where the torus action is well-defined. The polytope $P_0$ is a Delzant polytope (see Example 2.20 below) and the cut space $\overline{X}_{P_0}$ is a toric manifold. For any $\theta \in \text{Facets}(\text{Fl}(\underline{n}))$, let

$$P_\theta \in \mathcal{P} \tag{2.35}$$

denote the top-dimensional polytope whose intersection with $P_0$ is the facet parallel to $\theta$. We also denote $P_{0\theta} := P_0 \cap P_\theta \in \mathcal{P}$. For a generic tamed almost complex structure, the component $\overline{X}_{P_\theta}$ is a fibration

$$\mathbb{P}^1 \to \overline{X}_{P_\theta} \xrightarrow{\pi_\theta} \overline{X}_{P_{0\theta}} \tag{2.36}$$

by $J$-holomorphic spheres, which we justify below after Example 2.20. The map $\pi_\theta$ is such that for any $x \in \overline{X}_{P_{0\theta}}$, $\pi^{-1}(x)$ is the unique $J$-holomorphic sphere through $x$ of minimal area, for cuts along hyperplanes sufficiently close to the facets. Indeed, the fan for the polytope $P_\theta$ fibers over the fan for $P_{0\theta}$, with fiber the fan for $\mathbb{P}^1$.

*Example* 2.20. The genericity of the cutting parameters $\{\epsilon_{i,k,\searrow}, \epsilon_{i,k,\nearrow}\}$ ensures that the moment polytope of the cut space $X_{P_0}$ containing $L_\lambda$ is Delzant, that is, it does not have any non-simplicial faces. For example, in Fl(1, 1, 1) the non-simplicial corner corresponds to the simultaneous solution of

$$\Lambda_2 = \Phi_{2,1}, \quad \Lambda_2 = \Phi_{2,2}, \quad \Phi_{2,1} = \Phi_{1,1}, \quad \Phi_{2,2} = \Phi_{1,1},$$

(see Example 2.13). The genericity of $\{\epsilon_{i,k,\searrow}, \epsilon_{i,k,\nearrow}\}$ implies that the set of equations

$$\Lambda_2 - \Phi_{2,1} = \epsilon_{2,1,\nearrow}, \quad \Lambda_2 - \Phi_{2,2} = \epsilon_{2,2,\searrow}, \quad \Phi_{2,1} - \Phi_{1,1} = \epsilon_{1,1,\searrow}, \quad \Phi_{2,2} - \Phi_{1,1} = \epsilon_{1,1,\nearrow}$$

do not have a solution. If, for example, we assume $\epsilon_{2,1,\nearrow}, \epsilon_{2,2,\searrow} \ll \epsilon_{1,1,\searrow}, \epsilon_{1,1,\nearrow}$, then the polytope $P_0$ is as in Figure 2.19.

*Remark* 2.21. Note that $\overline{X}_{P_{0\theta}}$ is not the inverse image of the facet $\theta$ of the Gelfand-Cetlin polytope. Rather, the space $\overline{X}_{P_{0\theta}}$ is the inverse image of a facet of the toric manifold $\overline{X}_{P_0}$, which is a toric smoothing of the Gelfand-Cetlin toric variety. The subspace $\Phi^{-1}(\theta) \subset \text{Fl}(\underline{n})$ is, in general, not a smooth divisor.

We continue the proof of Theorem 2.17. Unobstructedness is a consequence of the monotonicity of the Lagrangian: Let $J^\nu$ be a family of neck-stretching almost complex structures on the flag manifold $X = \text{Fl}(\underline{n})$. Let $u_\nu$ be a $J^\nu$-holomorphic disk with a single output $x \in \text{crit}(F : L_\lambda \to \mathbb{R})$. By monotonicity of $(X, L_\lambda)$ the disk part of $u_\nu$ has Maslov index $I(u_\nu) \geq 2$. The index of $u_\nu$ (including the treed trajectory) is

$$\dim(L_\lambda) + 1 - (\dim(L_\lambda) - i(x)) + I(u_\nu) - \text{Aut}(\mathbb{D}^2) = 0.$$

Here $i(x)$ is the Morse index of the critical point $x$ of a Morse function $F : L_\lambda \to \mathbb{R}$ on the Lagrangian. The only possibility then is $I(u_\nu) = 2$ and $i(x) = \dim(L_\lambda)$, that is,

$x^\blacktriangledown$ is the maximum point of $F$. Therefore

$$\mathfrak{m}^0_{CF(L_\lambda, J_\nu)} = Wx^\blacktriangledown, \quad \text{for some } W \in \Lambda_{>0}.$$

Weak unobstructedness now follows from Lemma 10.10. Note that since $(X, L_\lambda)$ is monotone, the count of disks contributing to $\mathfrak{m}^0$ is the same for all $J^\nu$. The count also stays the same in the limit broken almost complex structure $\mathfrak{J}$ on $\mathfrak{X}_{\mathcal{P}}$.

We compute the potential $W$ by counting $\mathfrak{J}$-holomorphic broken disks in $\mathfrak{X}$. For any $\theta \in \mathrm{Facets}(\mathrm{Fl}(\underline{n}))$, there is a broken holomorphic disk $u_\theta$ of Maslov index 2, that intersects the divisor $\Phi^{-1}(\theta)$. Indeed,

- there is a single holomorphic sphere $u_{\theta,1} : \mathbb{P}^1 \to \overline{X}_{P_\theta}$ through $p$ of area $\epsilon_\theta$, where $\epsilon_\theta$ is the area of the fibers in the fibration in (2.36); and
- there is a disk $u_{\theta,0}$ in the toric manifold $\overline{X}_{P_0}$ of the form in Cho-Oh [22], with $I(u_{\theta,0}) = 2$, $\omega(u_{\theta,0}) = 1 - \epsilon_\theta$, and that intersects the relative divisor $\overline{X}_{P_{0\theta}}$ at a single point, say $p$.

Therefore $u_\theta = (u_{\theta,0}, u_{\theta,1})$ is a broken disk. See Figure 2.20. Each of these disks contributes $+1$ to the potential, since the corresponding unbroken disk is similar to a Blaschke disk in a toric variety, which for the appropriate choice of spin structure on $L_\lambda$, has a positive orientation sign, see [22, p22].

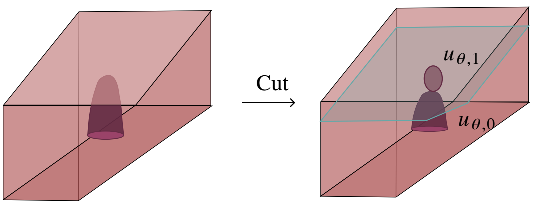



**Figure 2.20.** A broken disk of Maslov index two formed by cutting the complete flag variety $\mathrm{Fl}(1, 1, 1)$.

Next, we show that there are no other broken disks whose gluing has Maslov index two by using the fact that the symplectic area of the broken disk is half the Maslov index of its gluing. Let $u$ be any broken disk whose gluing has Maslov index 2. The disk component $u_0 : \mathbb{D}^2 \to \overline{X}_{P_0}$ has Maslov index two, otherwise the area of the disk is at least $2 - \epsilon_{\theta_1} - \epsilon_{\theta_2}$ for some $\theta_1, \theta_2 \in \mathrm{Facets}(\mathrm{Fl}(\underline{n}))$. Assuming $\epsilon_{\theta_1}, \epsilon_{\theta_2} \ll 1$, we get $\omega(u) \geq \omega(u_{\theta_0}) > 1$, which contradicts monotonicity (2.33). If $u_0$ intersects the relative divisor $\overline{X}_{P_{0\theta}}$, then the sphere in $\overline{X}_{P_\theta}$ has to be $u_{\theta,1}$ as above. Indeed, if $\epsilon_\theta$ is small enough, all other $\mathfrak{J}$-holomorphic spheres in $\overline{X}_{P_\theta}$ have larger area, and then $\omega(u) > 2$. This contradicts monotonicity.

The disk potential of $CF_{\mathrm{brok}}(L_\lambda)$ on the broken manifold $\mathfrak{X}$ is

$$W(y_{i,k})_{i,k} = \left( \sum_{k\le i:(i+1,k)\in \mathrm{Free}(\mathrm{Fl}(\underline{n}))} \frac{y_{i+1,k}}{y_{i,k}} + \sum_{k\le i:(i+1,k+1)\in \mathrm{Free}(\mathrm{Fl}(\underline{n}))} \frac{y_{i,k}}{y_{i+1,k+1}} \right.$$
$$\left. + \sum_{1\le j\le r-1} y_{\mathfrak{i}_j,\mathfrak{k}_j} + \sum_{2\le j\le r} \frac{1}{y_{\mathfrak{i}_j,\mathfrak{k}_j}} \right) q. \quad (2.37)$$

Here, we recall that $\mathrm{Free}(\mathrm{Fl}(\underline{n}))$ is the set of variables in the Gelfand-Cetlin system of $\mathrm{Fl}(\underline{n})$ that are not fixed by the interlacing inequalities, and for any $j$, $(\mathfrak{i}_j, \mathfrak{k}_j)$ is the lowest element in the $j$-th block of non-free variables in the Gelfand-Cetlin system, as in (2.22). This finishes the proof of Theorem 2.17. ∎

## 2.4 Counting curves in the plane

Mikhalkin's tropical curve counting [58] is the first example of applying tropical techniques to solving an enumerative problem in algebraic geometry. A tropical curve in [58] is a map from a graph to $\mathbb{R}^2$ that satisfies a balancing condition at nodes; and these curves correspond to holomorphic curves in $\mathbb{P}^2$. In this section, we show a correspondence between Mikhalkin's tropical curves and broken maps. We consider a degeneration of $\mathbb{P}^2$ by a multiple cut so that Mikhalkin's tropical curves correspond to tropical graphs of broken maps. In particular, Mikhalkin's curve represents the part of the tropical graph that lies in the dual polytope $P_0^\vee$ of a zero-dimensional polytope $P_0$ occurring in the multiple cut. In a sequel, we extend the formula to counts in almost toric manifolds.

**Definition 2.22.** A *tropical curve* is a map

$$h : \Gamma \to \mathbb{R}^2$$

from a graph $\Gamma$ (some of whose edges $e \in \mathrm{Edge}(\Gamma)$, called *leaves*, are incident on just one vertex $v \in \mathrm{Vert}(\Gamma)$ instead of two) to $\mathbb{R}^2$ such that

- any edge $e \in \mathrm{Edge}(\Gamma)$ maps to a line parallel to a vector $\mathcal{T}(e) \in \mathbb{Z}^2$ which is called the *slope of the edge*; and
- at any vertex $v \in \mathrm{Vert}(\Gamma)$ a *balancing condition* is satisfied, namely that the sum of the slopes $\mathcal{T}(e)$ of the edges $e \ni v$ emanating out of $v$ is equal to 0 :

$$\sum_{v\in e} \mathcal{T}(e) = 0. \quad (2.38)$$

The *multiplicity* $\mu_e$ *of an edge* $e$ is a positive integer such that $\frac{\mathcal{T}(e)}{\mu_e}$ is a primitive vector $w_e \in \mathbb{Z}^2$, which is called the *primitive slope*. [2] If $e \in \mathrm{Edge}(\Gamma)$ is a leaf, $h(e)$ is a semi-infinite line, otherwise $h(e)$ is a line segment.

We introduce basic terminology for tropical curves in the plane. Let $\Delta \subset \mathbb{R}^2$ be a simple polytope. A tropical curve $h : \Gamma \to \mathbb{R}^2$ is *adapted to* $\Delta$ if the slope $\mathcal{T}(e)$ of any leaf $e \in \mathrm{Edge}(\Gamma)$ is an outward normal of a facet of $\Delta$. The discussion in Mikhalkin focuses on $\mathbb{P}^2$ with moment polytope $\Delta_{\mathbb{P}^2}$ whose outward normals are $\nu = (-1, 0), (0, -1), (1, 1)$. For a tropical curve generically adapted to $\Delta_{\mathbb{P}^2}$, the *degree* of the curve is defined as the number of leaves (counted with multiplicity) that are normal to a fixed facet. By the balancing condition, the number is the same for any of the three facets. The *genus* of a tropical curve is the first Betti number $b_1(\Gamma)$ of the domain graph $\Gamma$.

**Definition 2.23.** A set of points $x_1, \ldots, x_k$ is *tropically generic* if any genus $g$ tropical curve $h : \Gamma \to \mathbb{R}^2$ whose image $h(\Gamma)$ contains $x_1, \ldots, x_k$,

(a) all the vertices of $\Gamma$ are trivalent,

(b) the images of vertices $h(\mathrm{Vert}(\Gamma))$ are disjoint from $x_1, \ldots, x_k$, and

(c) the slope $\mathcal{T}(e)$ of any leaf $e \in \mathrm{Edge}(\Gamma)$ is primitive.

**Definition 2.24.** (Multiplicity of Mikhalkin's graphs) Let $\Gamma$ be a Mikhalkin graph with only trivalent vertices, and let $v$ be a vertex in $\Gamma$ whose incident edges have slopes $\mu_1$, $\mu_2$, $\mu_3$. The multiplicity of the vertex $v$ is

$$\mathrm{mult}(v) := |\det(\mu_1 \mu_2)|,$$

which is the area of the parallelogram spanned by the vectors $\mu_1, \mu_2$. [3] The multiplicity of the graph $\Gamma$ is

$$\mathrm{mult}(\Gamma) := \prod_{v \in \mathrm{Vert}(\Gamma)} \mathrm{mult}(v).$$

Mikhalkin [58] shows that the number of curves in $\mathbb{P}^2$ of degree $d$ and genus $g$ passing through $3d - 1 + g$ tropically generic points can be computed by counting tropical curves with multiplicity.

The following result, Proposition 2.26 below, says that in the genus zero case, there is a bijective correspondence between Mikhalkin graphs adapted to $\Delta_{\mathbb{P}^2}$ and broken maps in $\mathbb{P}^2$ with respect to the multiple cut shown in Figure 2.21. This multiple cut is not "allowed" because the torus action corresponding to the cut $P_1$ is not free at the

[2] In Mikhalkin's notation in [58], the primitive slope $w_e$ is called *slope*.

[3] Any two of the three vectors $\mu_1$, $\mu_2$, $\mu_3$ may be used for the definition. The result is the same because of the balancing condition $\sum_i \mu_i = 0$.

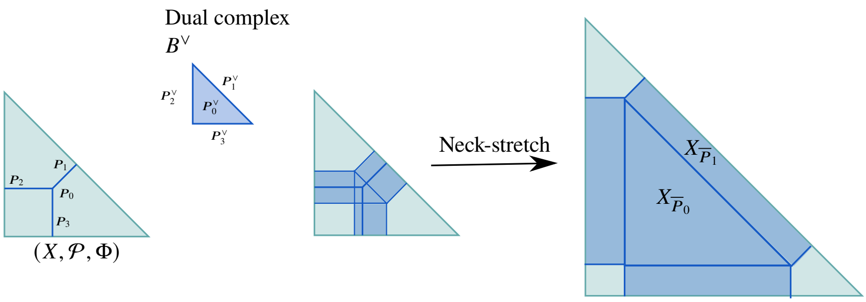



**Figure 2.21.** A multiple cut of $\mathbb{P}^2$.

$S^1$-orbit where $P_1$ intersects the toric divisor; a similar example was given in Figure 1.3. However, the broken maps corresponding to Mikhalkin graphs produced in our proof are not affected by this issue, as we later explain in Remark 2.27.

Before stating the bijection, we describe it intuitively and introduce notation for point constraints in a broken manifold. The degeneration corresponding to $\mathcal{P}$ has the effect of enlarging the almost complex structure in the neighbourhood of $\Phi^{-1}(P_0)$ where $P_0 \in \mathcal{P}$ is the point polytope. The dual polytope of $P_0$ is

$$\Delta_{\mathbb{P}^2} \simeq P_0^\vee. \tag{2.39}$$

For a broken map in $\mathbb{P}^2$, the Mikhalkin graph is the purely tropical part, that is, it consists of the part of the subgraph of the tropical graph lying in $P_0^\vee$. For these components the map part is fully determined by the graph since $P_0$ is 0-dimensional. For each of the one-dimensional polytopes $P_i$, $i = 1, 2, 3$, in the multiple cut, the cut space $X_{P_i}$ is a toric divisor of $\mathbb{P}^2$ minus a neighborhood of fixed points. The thickening $\mathfrak{X}_{P_i}$ is a trivial $\mathbb{P}^1$-fibration over $X_{P_i}$, and corresponding to a leaf of the Mikhalkin graph normal to the $i$-th facet, the broken map has a map component mapping to a fiber of $\mathfrak{X}_{P_i}$. The point constraints for the broken map lie in the neck piece $\mathfrak{X}_{P_0}$; these points in the broken manifold are written using the following notation:

**Definition 2.25.** (An $\mathfrak{X}$-point) Given a broken manifold $\mathfrak{X}$, an $\mathfrak{X}$-*point* is a tuple $\underline{x} := (P_x, \mathcal{T}_x, x)$ consisting of a polytope $P_x \in \mathcal{P}$, a tropical position $\mathcal{T}_x \in P_x^\vee$, and a point $x \in \mathfrak{X}_{P_x}$. We say that a broken map $u : (C, z) \to \mathfrak{X}$ satisfies the point constraint $u(z) = \underline{x} = (P_x, \mathcal{T}_x, x)$ if the component $C_v \subset C$ containing the marking $z$ satisfies

$$P(v) = P_x, \quad \mathcal{T}(v) = \mathcal{T}_x,$$

and $u(z) = x$.

**Proposition 2.26.** *(Tropical curves as broken maps) Let $\mathfrak{X}$ be the broken manifold obtained by applying the multiple cut in Figure 2.21 to $\mathbb{P}^2$. Consider a set of $\mathfrak{X}$-points*

$\underline{x}_1, \ldots, \underline{x}_{3d-1} \in \mathfrak{X}$ *lying in the piece* $\mathfrak{X}_{P_0} \subset \mathfrak{X}$. *That is,* $\underline{x}_i = (P_0, \mathcal{T}_{x_i}, x_i)$. *There is a bijective correspondence between the set of genus zero broken maps passing through the* $\mathfrak{X}$*-points* $\{\underline{x}_i\}_i$ *and the set of genus zero Mikhalkin graphs in* $\mathbb{R}^2$ *passing through the points* $\{\mathcal{T}_{x_i} \in \mathbb{R}^2\}_i$.

*Proof.* Given a Mikhalkin graph $h : \Gamma \to \mathbb{R}^2$ adapted to $\Delta_{\mathbb{P}^2}$, we first construct a tropical graph (as in Definition 4.6) for the polyhedral decomposition $\mathcal{P}$ of $\mathbb{P}^2$ from Figure 2.21. The tropical graph is an augmentation of $\Gamma$ and is denoted by

$$\Gamma_{\text{aug}} \supset \Gamma. \tag{2.40}$$

For all vertices $v \in \text{Vert}(\Gamma)$, we assign $P(v) := P_0$. For every leaf $e \in \text{Edge}(\Gamma)$, the augmented graph $\Gamma_{\text{aug}}$ contains an extra vertex $v_e$ on which $e$ is incident. If $e$ in $\Gamma$ intersects the facet $F_i \subset \Delta_{\mathbb{P}^2}$, then the polytope $P(v_e)$ is $P_i$. The set of univalent vertices thus added is denoted by

$$\text{Vert}_1(\Gamma_{\text{aug}}).$$

Lastly, if an edge $e \in \text{Edge}(\Gamma)$ of the Mikhalkin graph passes through a point constraint $x \in \mathbb{R}^2$ then in $\Gamma_{\text{aug}}$ we subdivide $e$ into two edges $e_1$ and $e_2$ by inserting a new vertex $v_x$ with $P(v_x) := P_0$. The set of vertices $v \in \text{Vert}(\Gamma_{\text{aug}})$ with markings is denoted by

$$\text{Vert}_{\to}(\Gamma_{\text{aug}}).$$

Thus,

$$\text{Vert}(\Gamma_{\text{aug}}) = \text{Vert}(\Gamma) \cup \text{Vert}_{\to}(\Gamma_{\text{aug}}) \cup \text{Vert}_1(\Gamma_{\text{aug}}).$$

The slopes of edges in $\Gamma_{\text{aug}}$ are the same as their slopes in $\Gamma$. The tropical positions for the vertices in $\Gamma_{\text{aug}}$ are given by the map $h : \Gamma \to \mathbb{R}^2$ on the Mikhalkin graph. See Figure 2.22 for an example.

We describe a way of orienting the edges in $\Gamma_{\text{aug}}$ which is useful in the rest of the proof, called the *marking orientation*, that satisfies the following conditions:

- For a vertex $v \in \text{Vert}_{\to}(\Gamma_{\text{aug}})$ containing a marking, both incident edges (corresponding to nodes) are outgoing.
- For a trivalent vertex $v \in \text{Vert}(\Gamma)$ occurring in Mikhalkin's graph, there are two incoming and one outgoing edge.
- For a vertex $v \in \text{Vert}_1(\Gamma_{\text{aug}})$ the only incident edge is incoming.

It is easy to verify that marking orientations can be assigned to all the edges of $\Gamma_{\text{aug}}$, see Figure 2.22 for an example.

The marking orientation has the following interpretation: Consider an ordering of the vertices of $\Gamma_{\text{aug}}$ that respects the orientation, that is, an edge $(v_+, v_-)$ points towards the vertex $v_-$ that occurs later in the ordering. For a vertex $v$, if the tropical positions of vertices prior to $v$ are fixed, then there is a unique possible tropical position for $v$.

Next, we describe the map at each vertex. Choose an ordering of the vertices $v \in \mathrm{Vert}(\Gamma_{\mathrm{aug}})$ that respects the marking orientation as described in the previous paragraph, and define the maps $(u_v)_v$ in that order. For a vertex $v \in \mathrm{Vert}(\Gamma)$ whose incident edges have slopes $\mu_1, \mu_2, \mu_3$ the map is

$$u_v : \mathbb{P}^1 \backslash \{0, 1, 2\} \to \mathfrak{X}_{P_0} \simeq (\mathbb{C}^\times)^2, \quad z \mapsto cz^{\mu_1}(z-1)^{\mu_2}(z-2)^{\mu_3},$$

where the domain is parametrized so that $0, 1, 2 \in \mathbb{P}^1$ are lifts of nodal points. The constant $c \in (\mathbb{C}^\times)^2$ is chosen so that the map satisfies the matching constraint at the nodal points corresponding to the two incoming edges. The constant $c$ is not unique; but, by Lemma 4.43, two distinct values produce broken maps that are related to each other by an element of the tropical symmetry group. For a vertex $v \in \mathrm{Vert}_{\to}(\Gamma)$ with a marking, and whose incident edges have slope $\mu, -\mu$ the map is a trivial cylinder of slope $\mu$

$$u_v : \mathbb{P}^1 \backslash \{0, \infty\} \to \mathfrak{X}_{P_0} \simeq (\mathbb{C}^\times)^2, \quad z \mapsto cz^{\mu},$$

and the constant $c$ is determined by the tropical point constraint at the marking. To define the map $u_v$ for a vertex $v \in \mathrm{Vert}_1(\Gamma_{\mathrm{aug}})$ corresponding to a leaf of the Mikhalkin graph, we first observe that the manifold $\mathfrak{X}_{P(v)}$ is a $\mathbb{P}^1$-fibration

$$\mathbb{P}^1 \to \mathfrak{X}_{P(v)} \xrightarrow{\pi_{P(v)}} X_{P(v)} \tag{2.41}$$

where the manifold $X_{P(v)}$ is a subset of a torus-invariant divisor of $X$. The map $u_{v_e}$ corresponding to the vertex $v_e$ is an injective map to the fiber of (2.41).

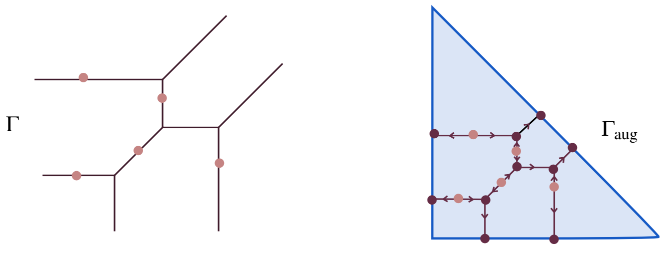



**Figure 2.22.** A Mikhalkin graph $\Gamma$ of degree 2 in $\mathbb{R}^2$ through 5 generic points and the corresponding tropical graph $\Gamma_{\mathrm{aug}}$ of a broken map in the dual complex $B^\vee$ from Figure 2.21. In $\Gamma_{\mathrm{aug}}$ the lighter vertices contain marked points mapping to the point constraints, and the arrows on edges indicate the marking orientation. The lighter vertices are stable since they have two incident edges and a marking.

Conversely, consider a broken map with tropical graph $\Gamma_u$ passing through the tropical point constraints. To prove that the broken map arises from a Mikhalkin graph, it is enough to show that

(a) there are no components mapping to $\mathfrak{X}_P$ with $\dim(P) = 2$; and

(b) if a component $u_v$ maps to $\mathfrak{X}_P$ with $\dim(P) = 1$, then it is a simple map to a $\mathbb{P}^1$-fiber of (2.41), that is, in the trivial $\mathbb{P}^1$-fibration $\overline{\mathfrak{X}}_P \simeq \overline{X}_P \times F$, the compactified map $u_v : \mathbb{P}^1 \to \overline{\mathfrak{X}}_P$ is homologous to $\overline{X}_P \times \{\text{point}\}$.

Both claims follow from considering dimensions of moduli spaces as follows: Markings lie on components mapping to $\mathfrak{X}_{P_0}$. Let $\Gamma_0 \subset \Gamma_u$ be the subgraph spanned by vertices $v$ with $P(v) = P_0$. Each connected component of $\Gamma_0$ is a Mikhalkin graph (where vertices in $\mathrm{Vert}_{\rightarrow}$ are treated as point constraints), and therefore, $\Gamma_0$ is necessarily connected. Consider a vertex $v \notin \Gamma_0$ that has an edge $e$ incident on $v' \in \mathrm{Vert}(\Gamma_0)$. Then, the map $u_v$ has a point constraint at the node $w_e$ and no other point constraints. Furthermore, the intersections with toric divisors are simple, so the only possibility is that $v$ does not have any incident edge besides $e$, $\dim(P(v)) = 1$ and $v$ is of the form (b) above. Thus, we have shown that $\Gamma_0$ is the Mikhalkin graph corresponding to the map $u$, vertices of form (b) lie in $\mathrm{Vert}_1((\Gamma_0)_{\mathrm{aug}})$, and $\Gamma_u = (\Gamma_0)_{\mathrm{aug}}$.

Finally, by Lemma 4.43, the multiplicity [58] of the Mikhalkin graph $\Gamma$ is equal to the size of the tropical symmetry group of the broken map. ■

*Remark* 2.27. The multiple cut of $\mathbb{P}^2$ used in Proposition 2.26 is not an "allowed" cut because at the shaded corner in Figure 2.23, the $S^1$-action corresponding to the cut $P_1$ is not free. However, since none of the broken maps have a node at this orbifold point, our results are still valid. That is, the moduli space of all the broken maps we encountered are transversally cut out, and the set of broken maps (since they are index zero) correspond bijectively to unbroken maps. To avoid this issue altogether, one may alter the multiple cut to $\mathcal{P}_1$ shown in Figure 2.23. The broken maps corresponding to Mikhalkin graphs in $\mathfrak{X}_{\mathcal{P}}$ naturally correspond to broken maps $\mathfrak{X}_{\mathcal{P}_1}$; an example is shown in Figure 2.24.

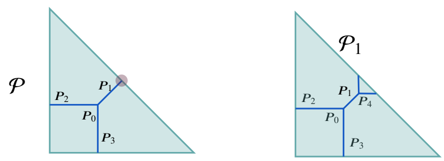



**Figure 2.23.** The shaded orbifold point in the multiple cut $\mathcal{P}$ is avoided in the multiple cut $\mathcal{P}_1$.

*Remark* 2.28. (Origin of the word "tropical") The connection between broken maps and Mikhalkin graphs explains why the objects of study in this paper are called "tropical", as we elaborate in this remark. The *tropical semi-ring* $\mathbb{R}_{\mathrm{trop}}$ is the set $\mathbb{R}$ with the addition and multiplication operations defined as max and + respectively. For example, a tropical quadratic polynomial in $\mathbb{R}_{\mathrm{trop}}$ in variables $x$ and $y$ has the form

$$f(x, y) = \max\{a_{00}, a_{10} + x, a_{01} + y, a_{11} + x + y, a_{20} + 2x, a_{02} + 2y\} \tag{2.42}$$

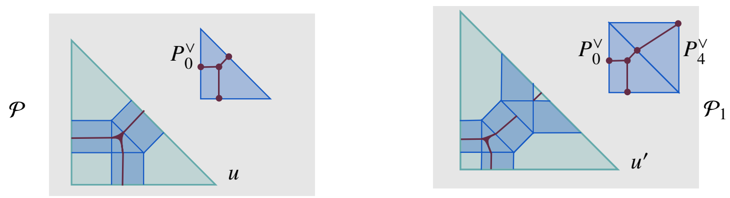



**Figure 2.24.** A broken map $u$ in $\mathfrak{X}_{\mathcal{P}}$ corresponds to a broken map $u'$ in $\mathfrak{X}_{\mathcal{P}_1}$. In both cases, the Figure shows a representation of the map and the tropical graph.

for some real constants $a_{00}, a_{10}, a_{01}, a_{11}, a_{20}, a_{02}$. A *tropical hypersurface* is the zero set of a tropical polynomial in $\mathbb{R}^n$, and is defined to be the set of points where the function is not linear. Thus tropical hypersurfaces are complexes of polytopes. Mikhalkin's

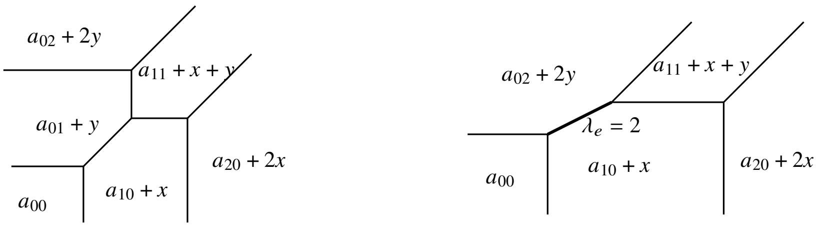



**Figure 2.25.** Tropical curves of degree 2 that arise as zero sets of $f$ in (2.42), and the maximum monomial in regions of $\mathbb{R}^2$.

tropical curve in $\mathbb{R}^2$ is the parametrized version of a tropical hypersurface in $\mathbb{R}^2$.

## Chapter 3

# Broken manifolds

In this Chapter, we review the multiple cut construction and the associated degenerations of almost complex structures. Our approach is much less general than, for example, Parker [67], but we wish to be completely explicit.

In the first half of the chapter we describe cut spaces and broken manifolds as symplectic manifolds. The multiple cut is a generalization of the symplectic cut operation of Lerman [48], which we call "single cut". A multiple cut is defined on a symplectic manifold equipped with a tropical moment map and a polyhedral decomposition $\mathcal{P}$, and yields a symplectic cut space $X_P^\omega$ [1] corresponding to every polytope $P$ in the decomposition $\mathcal{P}$. The broken manifold consists of top-dimensional cut spaces and thickenings of the lower-dimensional cut spaces, denoted by $\mathfrak{X}_P^\omega$ (see Section 3.3). The definition of the thickenings $\mathfrak{X}_P^\omega$ requires the additional datum of a dual complex, Definition 3.18 below.

In the second half of the chapter (starting from Section 3.4), we describe broken manifolds and cut spaces as almost complex manifolds with cylindrical almost complex structures. The manifold $X$ is equipped with a family of neck-stretched almost complex structures $J^\nu$. In the infinite neck length limit $\nu \to \infty$, the almost complex manifolds $X^\nu := (X, J^\nu)$ degenerate into broken almost complex manifolds, denoted by $\mathfrak{X}_P$. The broken almost complex manifold $\mathfrak{X}_P$ is diffeomorphic to the symplectic broken manifold $\mathfrak{X}_P^\omega$, but there is no canonical embedding where the symplectic form tames the almost complex structure.

The definition of the broken manifold $\mathfrak{X}_P$ as the degenerate limit gives a natural family of identifications preserving the almost complex structures and fibered structures between subsets of the neck-stretched manifold $X^\nu$ with a $P$-cylindrical structure almost complex structure and the broken manifold $\mathfrak{X}_P$. These identifications, called *translations*, are defined in Section 3.6, and are analogous to 'target rescalings' of Ionel [44].

Finally, in Section 3.7 we prove the existence of a cylindrical structure on the symplectic form in the neighborhood of cut loci in $(X, \omega_X)$. The cylindrical structure underlying neck-stretched almost complex structure is chosen to be the same as the cylindrical structure on the symplectic form. Choosing the cylindrical structures in this manner will allow us (later in Chapter 7) to construct families of diffeomorphisms from $X$ to itself for which the pullback of $\omega_X$ tames $J^\nu$.

---

[1] We use a superscript $\omega$ to denote symplectic cut spaces and symplectic broken manifold to distinguish them from the corresponding almost complex cut spaces and broken manifolds defined in the second half of the chapter.

## 3.1 Symplectic cut

A multiple cut is the generalization of the *symplectic cut* construction of Lerman [48] which we now review. We call this construction a *single cut*, to distinguish it from a multiple cut. The construction of symplectic cuts uses Hamiltonian circle actions on symplectic manifolds.

**Definition 3.1.** (Lerman's symplectic cut construction)

(a) (Hamiltonian circle actions) Let $(X, \omega_X)$ be a compact symplectic manifold. Let

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

denote the circle group; we identify its Lie algebra $\mathrm{Lie}(S^1)$ with $\mathbb{R}$ by division by $i$. A *Hamiltonian action* of the circle group $S^1$ on $X$ is an action $S^1 \times X \to X$ generated by the Hamiltonian flow of a *moment map*

$$\Phi : X \to \mathbb{R}, \quad \omega_X(\xi_X, \cdot) = -\mathrm{d}\Phi$$

where the generating vector field of an element $\xi \in \mathbb{R}$

$$\xi_X \in \mathrm{Vect}(X), \quad \xi_X(x) = \frac{d}{dt}|_{t=0} \exp(it\xi)x. \tag{3.1}$$

In particular, the affine line $\mathbb{C}$ has symplectic form

$$\omega_{\mathbb{C}} = \frac{-i}{2} \mathrm{d}z \wedge \mathrm{d}\overline{z} \in \Omega^2(\mathbb{C}).$$

The Hamiltonian action of $S^1$ is given by scalar multiplication and has moment map

$$\Phi_{\mathbb{C}} : \mathbb{C} \to \mathbb{R}, z \mapsto \frac{-|z|^2}{2}.$$

(b) (Global symplectic cuts) Let $X$ be a symplectic manifold with symplectic form $\omega_X$ and a free Hamiltonian $S^1$-action with moment map $\Phi$. The product $\hat{X} = X \times \mathbb{C}$ has product symplectic form $\hat{\omega} = \pi_1^*\omega_X + \pi_2^*\omega_{\mathbb{C}}$. The diagonal action of $S^1$ has moment map

$$\hat{\Phi} : \hat{X} \to \mathbb{R}, \quad (x, z) \mapsto \Phi(x) - \frac{|z|^2}{2}.$$

The zero level set is the union

$$\hat{\Phi}^{-1}(0) = (\Phi^{-1}(0) \times \{0\}) \sqcup \left\{ (x, z) \;\middle|\; \Phi(x) = \frac{|z|^2}{2} > 0 \right\}$$

of two pieces where both $\Phi$ and $z$ are zero and the piece where both $\Phi$ and $z$ are non-zero. The action on $z \neq 0$ has a natural slice given by $z \in \mathbb{R}_{>0}$ so that

$$\{(x, z) | \Phi(x) = |z|^2/2 > 0\} \cong \Phi^{-1}(\mathbb{R}_{>0}).$$

The symplectic quotient $\overline{X}_+ := \hat{\Phi}^{-1}(0)/S^1$ is called the *symplectic cut space*. Alternatively, the symplectic cut space is viewed as the compactification of

$$X_+ := \{\Phi > 0\} \subset (X, \omega_X),$$

given by

$$\overline{X}_+ := \hat{\Phi}^{-1}(0)/S^1 \simeq \{\Phi \geq 0\}/\sim,$$

where $\sim$ is the equivalence relation on the boundary $\Phi^{-1}(0)$ given by the $S^1$-action. The cut space is the union of $\{\Phi > 0\}$ and the symplectic quotient $\Phi^{-1}(0)/S^1$. One has a similar construction of a cut space

$$\overline{X}_- := \{\Phi \leq 0\}/\sim,$$

which is the union of $\{\Phi < 0\}$ and the symplectic quotient $\Phi^{-1}(0)/S^1$. The symplectic manifolds $\overline{X}_-, \overline{X}_+$ both contain a copy of $\Phi^{-1}(0)/S^1$ via the embeddings

$$i_- : \Phi^{-1}(0)/S^1 \to \overline{X}_-, \quad i_+ : \Phi^{-1}(0)/S^1 \to \overline{X}_+$$

with opposite normal bundles $N_\pm \to \Phi^{-1}(0)/S^1$.

(c) (Local symplectic cuts) Given an open subset $U \subset X$ with a free Hamiltonian $S^1$-action with moment map $\Phi : U \to \mathbb{R}$, such that $X \backslash U$ is disconnected, gluing together the cut $U_+ \cup U_-$ with $X - \Phi^{-1}(0)$ produces cut spaces $\overline{X}_+, \overline{X}_-$.

## 3.2 Multiple cuts in a symplectic manifold

We recall that the input datum for a single cut consists of a hypersurface and a Hamiltonian $S^1$-action in the neighborhood of the hypersurface. The input datum for a multiple cut consists of a collection of intersecting hypersurfaces with Hamiltonian $S^1$-actions in their neighborhoods. Neighborhoods of intersections of hypersurfaces have Hamiltonian torus actions, whose restrictions coincide with the $S^1$-actions corresponding to individual hypersurfaces. The various Hamiltonian actions are recorded by a *tropical moment map* with target space $\mathfrak{t}^\vee$, and a *polyhedral decomposition* of $\mathfrak{t}^\vee$. These polyhedral decompositions appeared in, for example, Meinrenken [57].

**Definition 3.2.** (Simple resp. Delzant polytopes) Let $T$ be a torus with Lie algebra $\mathfrak{t}$. Let $\mathfrak{t}_\mathbb{Z} \subset \mathfrak{t}$ denote the coweight lattice of points that map to the identity under the exponential map, so that $T \cong \mathfrak{t}/\mathfrak{t}_\mathbb{Z}$. A convex polytope $P$ in $\mathfrak{t}^\vee$ is described by a collection of linear inequalities determined by constants $c_F \in \mathbb{R}$ and normal vectors $\nu_F \in \mathfrak{t}$:

$$P = \{\lambda \in \mathfrak{t}^\vee \mid \langle \lambda, \nu_F \rangle \geq c_F, \quad \forall F \subset P \text{ facets}\}.$$

We allow polytopes to be unbounded. By the *interior* of a polytope, we mean the complement of its proper faces, so that in particular the interior of a zero-dimensional polytope (a point) is itself. The polytope $P$ is *simple* if for each point $v \in P$, the normal primitive vectors $\nu_F \in \mathfrak{t}_\mathbb{Z}$ to the facets $F \subset P$ containing $v$ form a basis for the span of the vectors $\nu_F$ in $\mathfrak{t}$. For a simple polytope $P$, a face $F_0 \subset P$ which is the intersection of facets $F_1, \ldots, F_m$ is *smooth* if the primitive normal vectors form a lattice basis:

$$\operatorname{span}(\nu_{F_i}, i = 1, \ldots, m) \cap \mathfrak{t}_\mathbb{Z} = \operatorname{span}_\mathbb{Z}(\nu_{F_i}, i = 1, \ldots, m). \tag{3.2}$$

A simple polytope is *Delzant* if all of its faces are smooth.

By Delzant [28], there is a bijection between compact, connected Delzant polytopes with trivial generic stabilizer and compact symplectic toric manifolds. A Delzant polytope $P$ corresponds to a symplectic manifold $V_P$ with an effective Hamiltonian action of a torus $T \simeq (S^1)^{(\dim(V_P)/2}$ and moment map and polytope

$$\Psi : V_P \to \mathfrak{t}^\vee, \quad \Psi(V_P) = P.$$

We remark that if in a simple polytope $P$ collection of facets $F_1, \ldots, F_m$ are not smooth then, $\mathfrak{t}_\mathbb{Z} \subset \operatorname{span}_\mathbb{Q}(\nu_{F_i})_i$. In this case $V_P$ is an *orbifold*, that is, it is covered by charts that are finite quotients of $\mathbb{R}^n$.

For later use, we define cones at faces of polytopes.

**Definition 3.3.** (Cones at faces of polytopes) Let $P$ be a simple polytope in a vector space $V$. For a face $Q \subset P$, the *cone* of $P$ at $Q$ is

$$\operatorname{Cone}_Q(P) := \{\lambda(p - q) : p \in P, q \in Q, \lambda \in \mathbb{R}_{\geq 0}\} \subset V. \tag{3.3}$$

For any interior, non-singular point $q$ in $Q$, let $V_Q := T_q Q$ be the tangent space at $q$, independent up to isomorphism of the choice of $q$ using the affine structure. The *normal cone* of $P$ at $Q$ is

$$\operatorname{NCone}_Q(P) := \operatorname{Cone}_Q(P)/V_Q \subset V/V_Q, \tag{3.4}$$

which is the image of $\operatorname{Cone}_Q(P)$ under the projection $V \to V/V_Q$. See Figure 3.1.

**Definition 3.4.** (Tropical Hamiltonian action) A *tropical Hamiltonian action* of a torus $T$ with Lie algebra $\mathfrak{t}$ is a triple $(X, \Phi, \mathcal{P})$ consisting of a

(a) compact symplectic manifold $X$

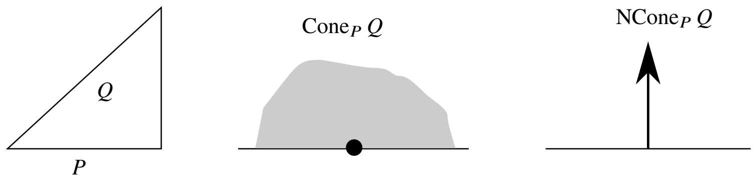



**Figure 3.1.** The cone and normal cone of $Q$ at a face $P$.

(b) (Polyhedral decomposition) a decomposition

$$\mathfrak{t}^\vee = \bigcup_{P \in \mathcal{P}} P^\circ, \quad \mathcal{P} = \{P \subset \mathfrak{t}^\vee\}$$

of $\mathfrak{t}^\vee$ into the disjoint union of the interiors $P^\circ$ of simple polytopes $P \in \mathcal{P}$ such that

- if $P_0, P_1 \in \mathcal{P}$ have non-empty intersection, then $P_0 \cap P_1 \in \mathcal{P}$ and $P_0 \cap P_1$ is a face of both $P_0$ and $P_1$,
- any polytope $P$ has at least one vertex $v \in \mathcal{P}$; and

(c) (Tropical moment map) a *tropical moment map* compatible with $\mathcal{P}$

$$\Phi : X \to \mathfrak{t}^\vee$$

in the following sense. For any $P \in \mathcal{P}$, we denote by

$$\mathfrak{t}_P := \operatorname{ann}(TP) \subset \mathfrak{t}$$

the annihilator of the tangent space of $P$ at any point $p \in P$, and by

$$T_P = \exp(\mathfrak{t}_P)$$

the torus whose Lie algebra is $\mathfrak{t}_P$. Let $\mathfrak{t}_{P,\mathbb{Z}}$ be the coweight lattice in $\mathfrak{t}_P$ so that

$$T_P \cong \mathfrak{t}_P/\mathfrak{t}_{P,\mathbb{Z}}.$$

For any $P \in \mathcal{P}$, there exists an open neighbourhood $U_P$ of $\Phi^{-1}(P)$ such that the composition

$$\pi_{\mathfrak{t}_P^\vee} \circ \Phi : U_P \to \mathfrak{t}_P^\vee$$

is a moment map for a free action of $T_P$ on $U_P$, where $\pi_{\mathfrak{t}_P^\vee} : \mathfrak{t}^\vee \to \mathfrak{t}_P^\vee$ is the projection dual to the inclusion $\mathfrak{t}_P \hookrightarrow \mathfrak{t}$.

*Notation* 3.5. For a polyhedral decomposition $\mathcal{P}$ and any $k \in \mathbb{Z}_{\geq 0}$, we denote by

$$\mathcal{P}_{(k)} \subset \mathcal{P} \quad \text{resp.} \quad \mathcal{P}^{(k)} \subset \mathcal{P} \tag{3.5}$$

the set of polytopes of dimension resp. codimension $k$.

*Remark* 3.6. The definition of the tropical moment map implies that for any zero-dimensional polytope $R \in \mathcal{P}$, $\mathfrak{t}_R = \mathfrak{t}$, and for any pair $Q \subset P$ in $\mathcal{P}$, there is a canonical inclusion $\mathfrak{t}_P \subset \mathfrak{t}_Q$.

*Remark* 3.7. (Tropical manifold for a single cut) In the single breaking case, a tropical Hamiltonian action consists of a map $\Phi : X \to \mathbb{R}$ and a decomposition $\mathbb{R} := (-\infty, c] \cup [c, \infty)$, such that $\Phi$ generates a free $S^1$-action in the neighborhood of $\Phi^{-1}(c)$. Thus the set of polytopes is

$$\mathcal{P} = \{(-\infty, c], \{c\}, [c, \infty)\}.$$

In the case of a single cut, we denote the tropical Hamiltonian action by the triple $(X, \Phi, c)$.

**Definition 3.8.** (Cut space for a multiple cut) Given a tropical symplectic manifold $(X, \Phi, \mathcal{P})$, for any polytope $P \in \mathcal{P}$, the *cut space* is a symplectic manifold (or orbifold, if $P$ is not Delzant) that is a compactification of

$$X_P^\omega := \Phi^{-1}(P^\circ)/T_P$$

given by

$$\overline{X}_P^\omega := \Phi^{-1}(P)/\sim, \tag{3.6}$$

where the equivalence $\sim$ mods out by the following torus actions:

$$x \sim tx, \quad \forall x \in \Phi^{-1}(Q^\circ), t \in T_Q \tag{3.7}$$

for all polytopes $Q \subseteq P$ contained in $\mathcal{P}$. The space $\overline{X}_P^\omega$ is a manifold (or orbifold if $P$ is not Delzant) by an iterative application of Lerman's cut. Cut spaces have natural inclusions

$$\overline{X}_Q^\omega \subset \overline{X}_P^\omega, \quad Q \subset P.$$

For a face $Q \subset P$ with $\operatorname{codim}_P(Q) = 1$, the corresponding subset $\overline{X}_Q^\omega$ is called a *relative divisor of* $\overline{X}_P^\omega$. The relative divisors $\overline{X}_Q^\omega$ intersect $\omega$-orthogonally, and the intersections correspond to a cut space $\overline{X}_R^\omega$ for some polytope $R \in \mathcal{P}$. This ends the Definition.

**An important point about notation:** We use the superscript $\omega$ in the notation for symplectic cut spaces and symplectic broken manifolds in order to to distinguish them from the corresponding almost complex manifolds defined in Section 3.5. Almost complex cut spaces (and broken manifolds) are diffeomorphic to the symplectic cut spaces (and symplectic broken manifolds), but the diffeomorphisms are not canonical. The almost complex versions occur much more frequently in the text and they do not have any superscript. See Lemma 3.52 and Remark 3.55 for related discussion.

*Example* 3.9. For a tropical Hamiltonian action $(X, \Phi, c)$ with a single cut (using notation as in Remark 3.7), the cut spaces are

$$\{\Phi \geq c\}/\sim, \quad \Phi^{-1}(c)/\sim, \quad \{\Phi \leq c\}/\sim,$$

and in all three spaces, the relation $\sim$ quotients $\Phi^{-1}(c)$ by the Hamiltonian $S^1$-action.

*Example* 3.10. The multiple cut in Figure 3.2 is made up of two simultaneous single cuts along hypersurfaces that intersect $\omega$-orthogonally. The set of polytopes is

$$\mathcal{P} = \{P_i, 0 \leq i \leq 3, P_{ij}, j = (i+1) \mod 4, P_\cap\}.$$

The manifolds $\overline{X}_{P_{i(i+1)}}, \overline{X}_{P_{(i-1)i}}$ are relative divisors of $\overline{X}_{P_i}$.

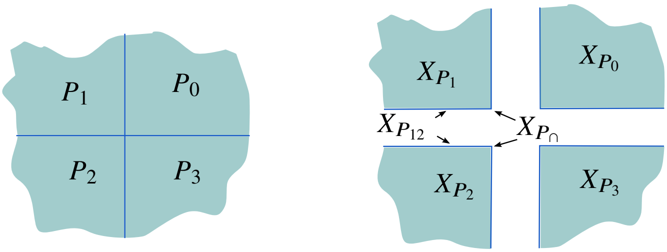



**Figure 3.2.** A multiple cut of $\mathbb{R}^2$.

### 3.2.1 Some generalizations

More generally, we allow the moment map to take values in affine manifolds as follows, continuing the discussion in Section 1.6.

**Definition 3.11.** Let $B$ be a topological manifold of dimension $n$. An *affine structure* on $B$ is an atlas for which the transition maps take values in $\mathbb{R}^n \ltimes GL(n,\mathbb{R})$. A *tropical affine structure* in the language of Gross [39] is one for which the coordinate change maps between coordinate charts take values in $\mathbb{R}^n \ltimes GL(n,\mathbb{Z})$; a manifold with such a structure is a *tropical affine manifold* and the functions provided by the local coordinates are *affine coordinates*. A map $B_1 \longrightarrow B_2$ between tropical affine manifolds is a *tropical affine morphism* if the map is given in local charts by affine linear combinations of the affine local coordinates.

The transition maps preserve the trivial flat connection in affine local coordinates, and these glue together to a flat connection on the smooth locus in $B$, and more generally on $B_i - \cup_{j<i} B_j$ for any $i$.

We also introduce a definition which allows certain kinds of singularities in the affine structure. Let $B$ be a topological manifold.

**Definition 3.12.** A *tropical affine structure with singularities* is a decomposition $B = \cup B_i$ into topological submanifolds $B_i$ of dimension $i$ and a tropical affine structure on each $B_i$ satisfying the following compatibility condition: Each $B_i$ has a topological tubular neighborhood $U_i$, homeomorphic to a topological vector bundle over $B_i$, equipped with a projection $\pi_i : U_i \to B_i$ so that if $\pi_{i,j}$ denotes the restriction of $\pi_i$ to $U_i \cap B_j$ then each $\pi_{i,j}$ is a morphism of tropical affine manifolds, and the projections $\pi_{i,j}$ are compatible in the sense that projection to $B_i$ composed with projection to $B_j$ is projection to $B_i$ whenever both are defined. A manifold with the structure of a tropical affine structure with singularities is a *tropical affine manifold with singularities*.

*Example* 3.13. Almost toric manifolds in the sense of Symington [81] and Leung-Symington [49] have fibrations over tropical affine manifolds with singularities in codimension two.

**Definition 3.14.** Given a tropical affine manifold with singularities $B$, a subset $P \subset B$ is a *polytope* if it is contractible and defined locally by some finite collection of inequalities satisfying the following property: Locally on each tubular neighborhood $U_i$ the set $P$ is defined by a finite collection of linear inequalities in pull-backs of affine coordinates on $B_i$.

The following generalizes the notion of a Hamiltonian torus action:

**Definition 3.15.** A tropical Hamiltonian action with moment map valued in $B$ is a tuple $(X, \Phi, \mathcal{P})$ where $\mathcal{P}$ is a polyhedral decomposition of $B$ and $\Phi : X \to B$ is a continuous map with the following property: For each polytope $P$, the collection of local functions $\langle \Phi, \nu \rangle$ defining $P$ generate a locally free torus action.

That is, for each polytope $P \subset B$, the pull-backs of the functions $\langle \Phi, \nu \rangle$ defining $P$ generate a locally free torus action.

*Example* 3.16. Any Hamiltonian action with a circle-valued moment map $(X, \omega, \Phi : X \to S^1)$ is a tropical Hamiltonian action for the polyhedral decomposition

$$\mathcal{P} = \{\{\theta_1\}, [\theta_1, \theta_2], \{\theta_2\}, [\theta_2, \theta_1]\}$$

for which $\theta_1, \theta_2$ are regular values. For example, one can take $X = S^1 \times S^1$ with moment map given by projection and the decomposition corresponding to any two choices $\theta_1, \theta_2$ of regular value.

*Example* 3.17. We given a example of an action mapping to an affine manifold with singularities. Let $B = \mathbb{R}$ be equipped with the stratification into $B_0 = \{0\}$ and $B_1 = \mathbb{R} - \{0\}$. Let $X = T^*S^2$ equipped with the function $\Phi : X \to B, v \mapsto \|v\|$ using the standard metric on $S^2$. Let $\mathcal{P} = \{(-\infty, c], \{c\}, [c, \infty)\}$ where $c \neq 0$. The map $\Phi$ is continuous and smooth away from 0, and the triple $(X, \Phi, \mathcal{P})$ is a tropical Hamiltonian action.

With these Definitions, Theorem1.6 holds as in the case that the codomain of the moment map is a vector space.

## 3.3 Symplectic broken manifolds

In this Section, we describe the broken manifold as a symplectic space. A component $\mathfrak{X}_P^\omega$ of the symplectic broken manifold corresponding to a polytope $P$ is a thickening of the corresponding cut space $X_P^\omega$ into a toric fibrations. The dimension of the toric fiber is complementary to the dimension of $P$, therefore for a top-dimensional polytope $P$, $X_P^\omega = \mathfrak{X}_P^\omega$. The toric fibrations are defined by considering neighborhoods of cut loci in $X$ and modding out the boundaries as in Lerman's construction. This construction requires the additional datum of a dual complex associated to the polyhedral decomposition $\mathcal{P}$. The *dual complex* $B^\vee$ (Definition 3.18) is a union of dual polytopes $\cup_{P \in \mathcal{P}} P^\vee$, and $P^\vee$ is the moment polytope of the fibers of the toric fibration $\mathfrak{X}_P^\omega$ over the cut space $X_P^\omega$.

**Definition 3.18.** (Dual complex) For a polyhedral decomposition $\mathcal{P}$ of $\mathfrak{t}^\vee$, the *dual complex* is a topological space

$$B^\vee = \left(\cup_{P \in \mathcal{P}} P^\vee\right) / \sim \tag{3.8}$$

defined by a dual polytope $P^\vee \subset \mathfrak{t}_P^\vee$ corresponding to every $P \in \mathcal{P}$ satisfying

(a) $P^\vee$ is top-dimensional in $\mathfrak{t}_P^\vee$, that is, $\dim(P^\vee) = \dim(\mathfrak{t}_P) = \dim(\mathfrak{t}^\vee) - \dim(P)$;

(b) for any $Q \in \mathcal{P}$, there is a bijection

$$\{P \in \mathcal{P} : P \supset Q\} \to \text{Proper faces of } Q^\vee, \quad P \mapsto Q_P^\vee,$$

with $\dim(Q_P^\vee) = \dim(P^\vee)$;

(c) for any pair $Q, P \in \mathcal{P}$ with $Q \subset P$, the projection $\pi_{\mathfrak{t}_P^\vee} : \mathfrak{t}_Q^\vee \to \mathfrak{t}_P^\vee$ maps the face $Q_P^\vee \subset Q^\vee$ isomorphically to a translate of $P^\vee$, where $\pi_{\mathfrak{t}_P^\vee} : \mathfrak{t}_Q^\vee \to \mathfrak{t}_P^\vee$ is the projection dual to the inclusion $\mathfrak{t}_P \hookrightarrow \mathfrak{t}_Q$; and

(d) for any pair $Q, P \in \mathcal{P}$ as in (c), the equivalence relation $\sim$ in (3.8) identifies $Q_P^\vee$ to $P^\vee$ via the map $\pi_{\mathfrak{t}_P^\vee}$.

*Example* 3.19. In the case of a single cut $(X, \Phi, c)$ (see Remark 3.7 for notation), the set of polytopes is

$$\mathcal{P} = \{P_- := (-\infty, c], P_0 := \{c\}, P_+ := [c, \infty)\},$$

and the dual polytopes are

$$P_-^\vee = \{-\epsilon\}, \quad P_0^\vee = [-\epsilon, \epsilon], \quad P_+^\vee = \{\epsilon\},$$

for any $\epsilon > 0$, with identifications $P_\pm^\vee \hookrightarrow P_0^\vee$ given by inclusion of the endpoints. Thus, the dual complex is an interval: $B^\vee \cong [-\epsilon, \epsilon]$.

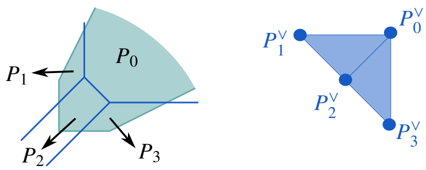



**Figure 3.3.** A polyhedral decomposition (left) and its dual complex (right).

*Example* 3.20. Figure 3.3 shows the dual complex corresponding to a multiple cut on a toric surface. We recall that we used three such multiple cuts to study disks in a cubic surface in Section 2.2.

We prove some properties of the dual complex:

**Lemma 3.21.** *For a polyhedral decomposition $\mathcal{P}$ of $\mathfrak{t}^\vee$, the dual polytope $P^\vee$ corresponding to any $P \in \mathcal{P}$ is a compact simple polytope.*

*Proof.* Consider a top-dimensional dual polytope $Q^\vee$, which, we recall, corresponds to a point $Q \in \mathcal{P}$. Since the polytopes in $\mathcal{P}$ cover $\mathfrak{t}^\vee$,

$$\bigcup_{P \in \mathcal{P}^{(0)} : P \ni Q} \operatorname{Cone}_Q P = \mathfrak{t}^\vee. \tag{3.9}$$

We observe that $\operatorname{Cone}_Q P$ is the normal cone of $\operatorname{Cone}_{P^\vee} Q^\vee$, that is, $\operatorname{Cone}_Q P$ is spanned by all the outward normal vectors to the facets of $\operatorname{Cone}_{P^\vee} Q^\vee$. By (3.9), we conclude that $Q^\vee$ is compact. Any dual polytope $P^\vee$ is a face of a top-dimensional dual polytope, and is therefore compact. To show that a top-dimensional dual polytope $Q^\vee$ is simple, consider a vertex $P_0^\vee$, which corresponds to a top-dimensional $P_0 \in \mathcal{P}$. The simplicity of $P_0$ implies the set of normal vectors $\nu_{F_i} \in \mathfrak{t}$ to the facets $F_i \in \mathcal{P}$ of $P_0$ span $\mathfrak{t}$. The set of one-dimensional edges emanating from $P_0^\vee$ in the polytope $Q^\vee$ are precisely $F_i^\vee$, which are parallel to $\nu_{F_i}$, and hence span $\mathfrak{t}$. By a similar argument applied to every vertex $P_0^\vee$ of $Q^\vee$, we conclude that $Q^\vee$ is simple. ∎

**Definition 3.22.** An *$X$-inner product* is a collection of inner products

$$g_P : \mathfrak{t}_P \times \mathfrak{t}_P \to \mathbb{R} \tag{3.10}$$

for all $P \in \mathcal{P}$ such that for any pair $Q \subset P$, $g_Q|\mathfrak{t}_P = g_P$. The $X$-inner product $(g_P)_P$ is *polytope-independent* if there is a inner product $g$ on $\mathfrak{t}$ and for all $P \in \mathcal{P}$, $g_P$ is the restriction of $g$ to $\mathfrak{t}_P$.

The broken manifold is a collection of manifolds $\overline{\mathfrak{X}}_P$ corresponding to polytopes $P \in \mathcal{P}$, each of which is a toric fibration over the cut space $\overline{X}_P$. The manifold $\overline{\mathfrak{X}}_P$ is modelled on a fibered polytope defined next.

**Definition 3.23.** (Fibered polytope) Let $P \subset \mathfrak{t}^\vee$ be a polytope in $\mathcal{P}$, and let $P^\vee \subset \mathfrak{t}_P^\vee$ be a complementary dimensional polytope. A polytope

$$\tilde{P} \subset \mathfrak{t}^\vee \tag{3.11}$$

is *fibered over $P$ with fiber $P^\vee$* if it is equipped with a diffeomorphism

$$\tilde{P} \xrightarrow{(\pi_P, \pi_{P^\vee})} P \times P^\vee,$$

where, viewing the fibers of $\pi_P$, $\pi_{P^\vee}$ and the polytope $P$ as subsets of $\mathfrak{t}^\vee$, we have

- $\pi_{P^\vee}$ is the restriction of the projection $\mathfrak{t}^\vee \to \mathfrak{t}_P^\vee$,
- for any $x \in P$, the fiber $\pi_P^{-1}(x)$ is a polytope, and
- $P = \pi_{P^\vee}^{-1}(c_{P^\vee})$ for some interior point $c_{P^\vee} \in P^\vee$.

The facets of the fibered polytope $\tilde{P}$ are

$$\text{Facets}(\tilde{P}) = \{\pi_P^{-1}(Q) : Q \in \text{Facets}(P), Q \in \mathcal{P}\} \cup \{\pi_{P^\vee}^{-1}(Q) : Q \in \text{Facets}(P^\vee)\}. \tag{3.12}$$

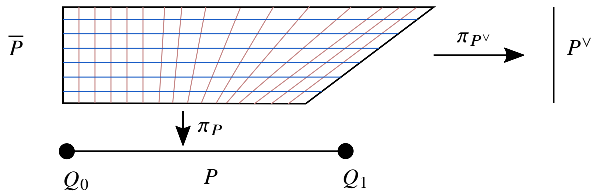



**Figure 3.4.** A fibered polytope $\tilde{P}$. The fibers of $\pi_P$ are in red, and those of $\pi_{P^\vee}$ are in blue.

**Definition 3.24.** (Cutting datum) Given a polyhedral decomposition $\mathcal{P}$ of $X$, and an $X$-inner product $g = (g_P)_{P \in \mathcal{P}}$ on the spaces $\{\mathfrak{t}_P\}_{P \in \mathcal{P}}$ (Definition 3.22), a *cutting datum* consists of a dual complex $B^\vee$, and inclusions of fibered polytopes (as in (3.11))

$$i_{\tilde{P}} : \tilde{P} \hookrightarrow \mathfrak{t}^\vee \tag{3.13}$$

for all $P \in \mathcal{P}$, such that for any $x \in P$ in a neighborhood of a face $Q \subset P$ the fiber $\pi_P^{-1}(x)$ is contained in $\pi_Q^{-1}(\pi_Q(x))$ and is orthogonal to $P \cap \pi_Q^{-1}(\pi_Q(x))$ with respect to the inner product $g_Q$ on $\mathfrak{t}_Q^\vee$. Here, the fibers of $\pi_P$, $\pi_Q$ are viewed as subsets of $\mathfrak{t}^\vee$.

*Remark* 3.25. (The case of a polytope-independent $X$-inner product) For many examples of multiple cuts, a dual complex exists when the $X$-inner product $g = (g_P)_P$ is polytope independent as in Definition 3.22, that is, $g_P$ is the restriction of a $P$-independent inner product $g$ on $\mathfrak{t}$. In this case, there is an embedding $i : B^\vee \hookrightarrow \mathfrak{t}^\vee$ such that any $P \in \mathcal{P}$ is orthogonal to $P^\vee$. For example, the dual complex in Figure 3.3 respects a

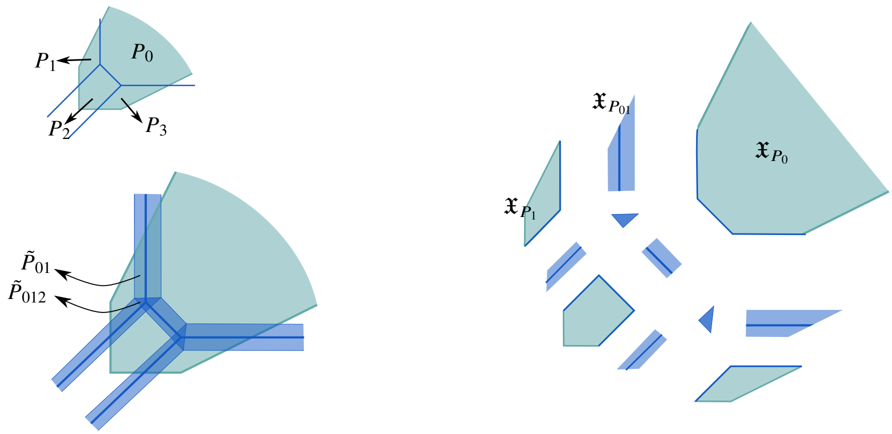



**Figure 3.5.** Top left: Polyhedral decomposition from Figure 3.3. Bottom left: Cutting datum. Right: The symplectic broken manifold is isomorphic to the inverse image of these polytopes under the tropical moment map.

polytope-independent inner product. Furthermore, in this case, the fibered polytopes are rectangles, that is, $\tilde{P} = P \times P^\vee$, where $P$ and $P^\vee$ are orthogonal in $\mathfrak{t}^\vee$. For the cutting datum in Figure 3.5, the $X$-inner product is polytope-independent and is equal to the standard inner product on $\mathbb{R}^2$. The fibered polytopes $\tilde{P}_{01}, \tilde{P}_{12}, \tilde{P}_{23}, \tilde{P}_{30}$ are infinite rectangles.

*Example* 3.26. In Figure 3.4, suppose the vertices $Q_0, Q_1$ of $P$ are elements of $\mathcal{P}$, and suppose that, in the $X$-inner product $g = (g_P)_P$, $g_{Q_0}$ is the standard inner product and $g_{Q_1}$ is such that $\pi_P^{-1}(Q_1)$ is orthogonal to $P$. Then, the fibered polytope in the figure respects the inner product. Indeed, in a neighborhood of $Q_i$, $i = 0, 1$, the fibers of $\pi_P$ are orthogonal to $P$ with respect to the inner product $g_{Q_i}$.

We expect that cutting data exist for a wide class of polyhedral decompositions for suitable choices of $X$-inner products. Here, we list some cases where the cutting data can be explicitly constructed.

**Proposition 3.27.** *(Examples where a cutting datum exists) Let $\mathcal{P}$ be a polyhedral decomposition of* $\mathfrak{t}^\vee$.

(a) *Suppose the* 1*-skeleton* $\mathcal{P}_{(\le 1)} := \cup_{P \in \mathcal{P}, \dim(P) \le 1} P$ *of* $\mathcal{P}$ *does not have a cycle. A cutting datum exists for the polytope-independent $X$-inner product. (A polytope-independent $X$-inner product is the restriction of an inner product on* $\mathfrak{t}$ *to* $\mathfrak{t}_P$ *for all* $P \in \mathcal{P}$ *as in Definition* 3.22*.)*

(b) *Suppose* $\dim(\mathfrak{t}) = 2$. *Suppose the edges in the* 1*-skeleton can be oriented so that any vertex has at most two incoming edges and these two edges are not parallel. Then, there is an $X$-inner product for which a cutting datum exists.*

*Proof.* First, consider the case that there is a single vertex $v$ in $\mathcal{P}$. Fix any inner product on $\mathfrak{t}$. For each edge $e \in \mathcal{P}_{(1)}$, choose any plane perpendicular to $e$ as the dual polytope $e^\vee$. Since each of the polytopes in $\mathcal{P}$ are simple, for any top dimensional polytope $P$ spanned by edges $\{e_i\}_i$, the planes $e_i^\vee$ intersect at a single point $P^\vee$. The other vertices of the polytope $v^\vee$ are similarly determined.

Next to prove (a), orient the edges and order the vertices in $\mathcal{P}^{(0)}$ as $v_1, \ldots, v_n$ so that for any $v_i$, there is at most one edge $(v_j, v_i)$ with $j < i$ It is enough to fix $\tilde{v}_i \subset \mathfrak{t}^\vee$ so that the projections to faces match up, that is, if $v_i, v_j \in P$ then $\pi_{\mathfrak{t}_P^\vee}(v_{i,P}^\vee) = \pi_{\mathfrak{t}_P^\vee}(v_{j,P}^\vee)$. Assume that the fibered polytope $\tilde{v}_i \subset \mathfrak{t}^\vee$ is fixed for every $i < k$. Suppose for $v_k$, $e$ is an incoming edge. The facet $v_{k,e}^\vee$ is fixed up to translation in the direction $e$, we fix $v_{k,e}^\vee$ close enough to $v_k$; this automatically fixes the facets of $\tilde{v}_k$ that intersect $v_{k,e}^\vee$, but we are free to choose the others, and fix $\tilde{v}_k$.

To prove (b), consider a vertex $v$ with incoming edges $e_1 = (v_1, v)$, $e_2 = (v_2, v)$. Assume that the fibered polytopes $\tilde{v}_1, \tilde{v}_2 \subset \mathfrak{t}^\vee$ are fixed, and $\tilde{w}$ is not fixed for any other neighbor $w \in \mathcal{P}^{(0)}$ of $v$. We will choose a inner product $g_v$ for which $e_1$ and $e_2$ are orthogonal, thus the facets $v_{e_1}^\vee$, $v_{e_2}^\vee$ are fixed, in particular, $v_{e_1}^\vee$ resp. $v_{e_2}^\vee$ is parallel to $e_2$ resp. $e_1$. If there is only one other incident edge $e_3$ on $v$, then the face $v_{e_3}^\vee$ is also determined, and there is a unique choice of inner product $g_v$ for which $v_{e_3}^\vee$ is orthogonal to $e_3$. In case, there are four or more edges incident on $v$, then, we may fix any inner product for which $e_1$ and $e_2$ are orthogonal, and the polygon $\tilde{v} \subset \mathfrak{t}$ can be completed with the edges $v_e^\vee$ perpendicular to $e$ for all edges $e$ incident on $v$. ∎

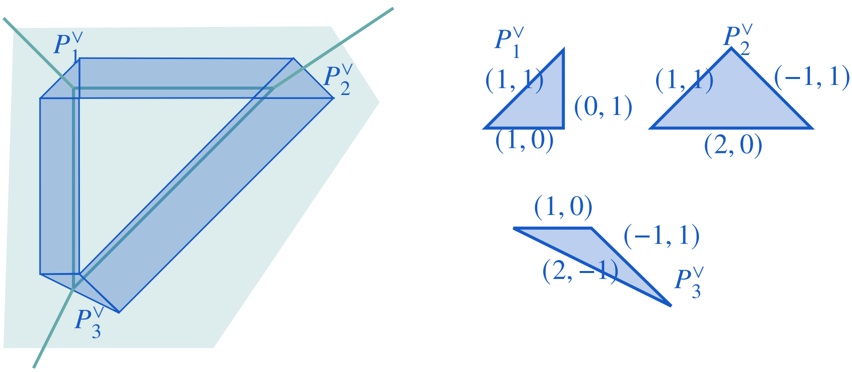



**Figure 3.6.** Dual complex with varying inner products.

*Example* 3.28. The cutting datum in Figure 3.6 is given by applying the algorithm in the proof of Proposition 3.27 (b). For vertices $P_1, P_2 \in \mathcal{P}$, we take the inner products $g_{P_0}$, $g_{P_1}$ to be the standard ones, and for the vertex $P_3 \in \mathcal{P}$, $g_{P_3}$ is the inner product for which $\{(1,0), (1,1)\}$ is an orthonormal basis. Note that in this example, it is not possible to construct a dual complex if the $X$-inner product $(g_P)_P$ is polytope-invariant (as in Definition 3.22); see Figure 3.7.

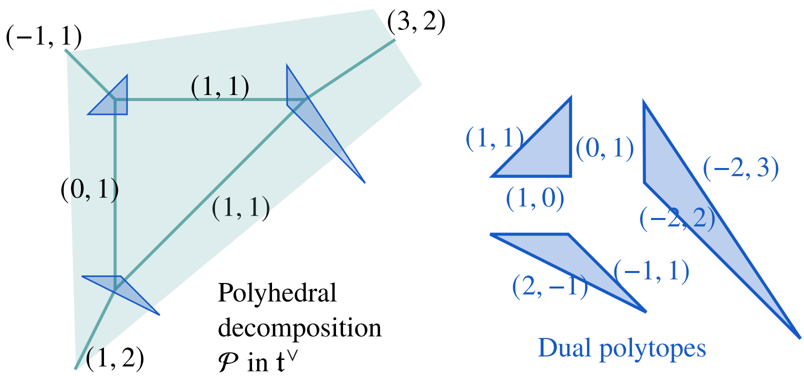



**Figure 3.7.** A dual complex does not exist if each of $g_{P_1}, g_{P_2}, g_{P_3}$ is the standard inner product.

**Definition 3.29.** (Symplectic broken manifold) Suppose $(X, \mathcal{P}, \Phi)$ is a tropical Hamiltonian manifold, and the polyhedral decomposition $\mathcal{P}$ has a cutting datum (Definition 3.24) given by inclusions of fibered polytopes $\tilde{P} \subset \mathfrak{t}^\vee$ for all $P \in \mathcal{P}$.

(a) The *symplectic broken manifold* corresponding to the tropical Hamiltonian action $(X, \Phi, \mathcal{P})$ is a disjoint union

$$\mathfrak{X}_{\mathcal{P}} \quad \text{or} \quad \mathfrak{X} := \sqcup_{P \in \mathcal{P}} \mathfrak{X}_P^\omega,$$

where we use the notation $\mathfrak{X}$ when $\mathcal{P}$ is clear from the context. Here,

$$\mathfrak{X}_P^\omega := \Phi^{-1}(\tilde{P}) \tag{3.14}$$

is a symplectic manifold with corners. In the case when the inner product (3.10) for $\mathcal{P}$ is rational, compactifications of the components of the broken manifold may be defined as

$$\overline{\mathfrak{X}}_P^\omega := \mathfrak{X}_P^\omega / \sim, \tag{3.15}$$

where the equivalence relation $\sim$ quotients any boundary component $\Phi^{-1}(Q)$, $Q \in \text{Facets}(\tilde{P})$ by the $S^1$-action generated by the vector $v_Q \in \mathfrak{t}_{\mathbb{Z}}$ normal to the hyperplane $Q \in \mathfrak{t}^\vee$. The space $\overline{\mathfrak{X}}_P^\omega$ is a smooth symplectic manifold if all faces $F \subset \tilde{P}$ are smooth (as in Definition 3.2). Otherwise $\overline{\mathfrak{X}}_P^\omega$ is a symplectic orbifold. See the “important point about notation” on page 66 regarding the superscript $\omega$.

(b) (Relative divisors of a symplectic broken manifold) Suppose the inner product (3.10) for $\mathcal{P}$ is rational, and compactifications $\overline{\mathfrak{X}}_P^\omega$ of the components of the broken manifold $\mathfrak{X}$ are orbifolds (see (3.15)). Then, for any $P \in \mathcal{P}$ and any facet $Q$ of the thickening $\tilde{P}$, the quotient

$$Y_Q := \Phi^{-1}(Q)/\exp(\mathbb{R}\nu_Q) \subset \overline{\mathfrak{X}}_P^\omega$$

is a *relative divisor* of $\overline{\mathfrak{X}}_P^\omega$. Here, the normal $\nu_Q \in \mathfrak{t}_\mathbb{Z}$ to the facet $Q$ generates a subgroup $\exp(\mathbb{R}\nu_Q) \simeq S^1$. The relative divisor $Y_Q$ is *horizontal* resp. *vertical* if the facet $Q \subset \tilde{P}$ corresponds to a facet of $P$ resp. $P^\vee$ (see (3.12)). Thus

$$\mathfrak{X}_P^\omega = \overline{\mathfrak{X}}_P^\omega - \bigcup_{Q \subset \tilde{P}} Y_Q$$

is the complement of all the relative divisors $Y_Q, Q \subset \tilde{P}$ of $\overline{\mathfrak{X}}_P^\omega$.

*Example* 3.30. In the case of a single cut $(X, \Phi, c)$ the symplectic broken manifold consists of the components

$$\{\Phi > c\}/\sim, \quad \{\Phi < c\}/\sim, \quad \Phi^{-1}([c-\epsilon, c+\epsilon])/\sim,$$

where $\sim$ mods out the boundaries $\Phi^{-1}(c-\epsilon)$ and $\Phi^{-1}(c+\epsilon)$ by $S^1$-actions.

*Remark* 3.31. (Components of the symplectic broken manifold as thickenings) Components of the symplectic broken manifold are fibrations over cut spaces whose fibers are compactifications of the complex tori. That is, there is a fibration

$$V_{P^\vee} \to \mathfrak{X}_P^\omega \xrightarrow{\pi_P} X_P^\omega, \tag{3.16}$$

whose fiber $V_{P^\vee}$ is a symplectic manifold with corners and a $T_P$-action whose moment map has image $P^\vee$. Therefore, the symplectic broken manifold $\mathfrak{X}_P^\omega$ they can be regarded as the thickening of the cut space $X_P^\omega$ for any $P \in \mathcal{P}$. In the particular case when the $X$-inner product (Definition 3.22) is rational, the fiber $V_{P^\vee}$ has a compactification $\overline{V}_{P^\vee}$ obtained by modding out boundaries by $S^1$-actions. The compactification $\overline{V}_{P^\vee}$ is a $T_P$-orbifold whose moment map is $P^\vee$.

*Remark* 3.32. (Torus bundles on cut spaces) Torus bundles over cut spaces are used in the following sections to define neck-stretched almost complex structures. In the symplectic setting, for any $P \in \mathcal{P}$,

$$\Phi^{-1}(P^\circ) \to X_P^\omega,$$

is a principal $T_P$-bundle. It extends to a $T_P$-orbifold bundle

$$\overline{Z}_P^\omega \to \overline{X}_P^\omega \tag{3.17}$$

for which $\overline{\mathfrak{X}}^{\omega}_{P} \to \overline{X}^{\omega}_{P}$ is the associated $V_{P^\vee}$-bundle, that is there is a $T_P$-diffeomorphism

$$\overline{\mathfrak{X}}^{\omega}_{P} = \overline{Z}^{\omega}_{P} \times_{T_P} V_{P^\vee},$$

where $V_{P^\vee}$ is a toric variety whose moment polytope is $P^\vee$ (Remark 3.31). We also observe that the complement of the vertical divisors of $\overline{\mathfrak{X}}^{\omega}_{P}$ is $T_P$-diffeomorphic to $\overline{Z}^{\omega}_{P} \times P^\vee$.

*Example* 3.33. For the multiple cut in Figure 3.2, the dual complex is a rectangle and the symplectic broken manifold $\mathfrak{X}$ is as in Figure 3.8. Relative submanifolds $\overline{X}^{\omega}_{P_{ij}}$ and $\overline{X}^{\omega}_{P_\cap}$ are thickened into toric fibrations $\overline{\mathfrak{X}}^{\omega}_{P_{ij}}$ and $\overline{\mathfrak{X}}^{\omega}_{P_\cap}$

$$\mathbb{P}^1 \to \overline{\mathfrak{X}}^{\omega}_{P_{ij}} \to \overline{X}^{\omega}_{P_{ij}}, \quad (\mathbb{P}^1)^2 \to \overline{\mathfrak{X}}^{\omega}_{P_\cap} \to \overline{X}^{\omega}_{P_\cap}$$

in the symplectic broken manifold.

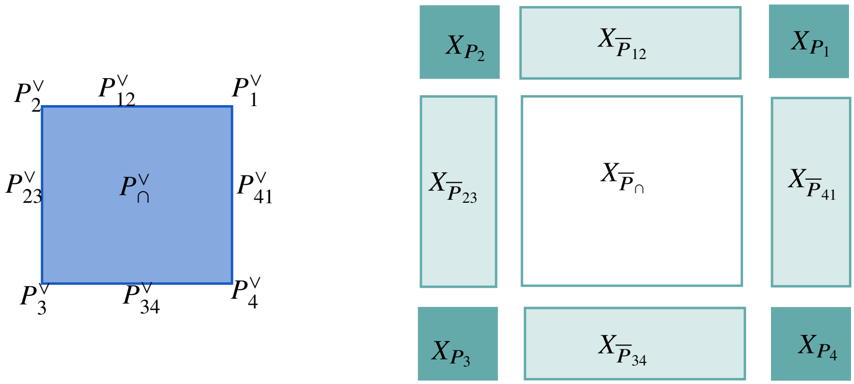



**Figure 3.8.** Dual complex and broken manifold for the multiple cut in Figure 3.2
.

## 3.4 Neck-stretched almost complex structures

We define a family of almost complex structures on the manifold $X$, called neck-stretched almost complex structures. These almost complex structures are "cylindrical" in the sense of the following Definition:

**Definition 3.34.** (Cylindrical almost complex structure)

(a) ($P$-cylinder) Given a polytope $P \in \mathcal{P}$, a $T_{P,\mathbb{C}}$-principal bundle

$$Z_{\mathbb{C}} \to M$$

on a manifold $M$ is called a *P-cylinder*. By choosing a reduction of structure group to the maximal compact subgroup $T_P \subset T_{P,\mathbb{C}}$ we obtain a $T_{P,\mathbb{C}}$-equivariant diffeomorphism

$$Z_{\mathbb{C}} \simeq Z \times i\mathfrak{t}_P$$

where $Z \to M$ is a principal $T_P$-bundle and the $T_{P,\mathbb{C}}$-action on $Z \times i\mathfrak{t}_P$ is

$$te^{is}(z, s_0) = (tz, s_0 + s), \quad t \in T_P, s \in \mathfrak{t}_P.$$

(b) Let $Z_{\mathbb{C}} \to M$ be a $P$-cylinder. An almost complex structure $J$ is *P-cylindrical* if and only if

(i) there exists an almost complex structure $J_M$ on the base manifold $M$ such that the projection

$$\pi : Z_{\mathbb{C}} \to M$$

is $(J, J_M)$ holomorphic almost complex, that is, $d\pi \circ J = J_M \circ d\pi$, and we denote

$$D\pi(J) := J_M; \tag{3.18}$$

(ii) there exists a connection one-form $\alpha \in \Omega^1(Z, \mathfrak{t}_P)$ on the $T_P$-bundle $Z \to M$ such that the horizontal sub-bundle

$$H := \ker(\alpha) \subset TZ \cong TZ \times \{0\} \subset TZ \times \mathbb{R} \cong TZ_{\mathbb{C}} \tag{3.19}$$

is $J$-invariant; and

(iii) on any fiber $\pi^{-1}(m) \subset Z_{\mathbb{C}}$, $J$ is standard in the following sense: For any point $z \in \pi^{-1}(m)$ the map

$$T_{P,\mathbb{C}} \to \pi^{-1}(m), t \mapsto tz$$

is a biholomorphism.

As a result, $J$ is invariant under the $T_{P,\mathbb{C}}$-action on $Z_{\mathbb{C}}$. Denote by

$$\mathcal{J}^{\text{cyl}}(Z_{\mathbb{C}}) := \{ J \in \mathcal{J}(Z_{\mathbb{C}}) \mid (i) - (iii) \}$$

the space of $P$-cylindrical almost complex structures on $Z_{\mathbb{C}}$. Note that a $P$-cylindrical almost complex structure $J$ is determined by its projection $d\pi(J)$ to the base $M$ called the *base almost complex structure* and its *associated connection one-form* $\alpha(J)$.

Given a tropical Hamiltonian action $(X, \mathcal{P}, \Phi)$, we aim to define "neck-stretched manifolds" where the neck region associated to $P \in \mathcal{P}$ has a $P$-cylindrical almost complex structure. To achieve this end, a subset of the manifold $X$ with the action of the torus $T_P$ has to be identified with a subset of a $P$-cylinder. This identification is made via a symplectic cylindrical structure defined below, where a sufficiently small $T_P$-invariant neighborhood of $\Phi^{-1}(P)$ in $X$ is identified to a product $\Phi^{-1}(P) \times P^\vee$. In what follows the identification between $\mathfrak{t}_P$ and $\mathfrak{t}_P^\vee$ from (3.10) is crucial. For the symplectic structure, we view $P^\vee$ as a subset of $\mathfrak{t}_P^\vee$ and the $T_P$-moment map on $\Phi^{-1}(P) \times P^\vee$ is given by the projection to $P^\vee$. For the $P$-cylindrical complex structure, we view $P^\vee$ as a subset of $\mathfrak{t}_P$ so that the fibers of the projection

$$\Phi^{-1}(P) \times P^\vee \to \Phi^{-1}(P)/T_P$$

are subsets of $T_{P,\mathbb{C}}$ via the composition of maps

$$T_P \times P^\vee \subset T_P \times \mathfrak{t}_P^\vee \xrightarrow{\simeq} T_P \times \mathfrak{t}_P \xrightarrow{(t,\xi)\mapsto t\exp(i\xi)} T_{P,\mathbb{C}}, \tag{3.20}$$

where the identification $\mathfrak{t}_P^\vee \simeq \mathfrak{t}_P$ in the middle arrow is via the product on $\mathfrak{t}$ from (3.10). Consequently $\Phi^{-1}(P) \times P^\vee$ has a partial $T_{P,\mathbb{C}}$-action. That is, there is an infinitesimal action

$$\mathfrak{t}_{P,\mathbb{C}} \to \mathrm{Vect}(\Phi^{-1}(P) \times P^\vee) \tag{3.21}$$

whose flows satisfy the axioms for an action of $T_{P,\mathbb{C}}$ wherever they are defined.

**Definition 3.35.** (Symplectic cylindrical structure on tropical Hamiltonian actions) Let $(X, \mathcal{P}, \Phi)$ be a symplectic manifold with a tropical Hamiltonian action. A *symplectic cylindrical structure* $\underline{\phi} = (\phi_P)_{P\in\mathcal{P}}$ consists of a $T_P$-equivariant symplectomorphism $\phi_P$

$$\Phi^{-1}(\tilde{P}) \xrightarrow{\phi_P} (\Phi^{-1}(P) \times P^\vee, \overline{\omega}), \quad \overline{\omega} := (\omega_X|\Phi^{-1}(P)) + d\langle \alpha_P, \Pi_{P^\vee} - c_{P^\vee}\rangle \tag{3.22}$$

for each polytope $P \in \mathcal{P}$. Here

(a) $\alpha_P \in \Omega^1(\Phi^{-1}(P), \mathfrak{t}_P)$ is a $T_P$-connection one-form;

(b) $\Pi_{P^\vee} : \Phi^{-1}(P) \times P^\vee \to P^\vee \subset \mathfrak{t}_P^\vee$ is the projection to the second factor, $c_{P^\vee} \in \mathfrak{t}_P^\vee$ is the constant that is the image of the polytope $P \subset \mathfrak{t}^\vee$ under the projection $\pi_{P^\vee} : \tilde{P} \to P^\vee$ from Definition 3.23, and $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{t}_P$, $\mathfrak{t}_P^\vee$, and so, $\langle \alpha_P, \pi_{P^\vee} - c_{P^\vee}\rangle$ is a one-form on $\Phi^{-1}(P) \times P^\vee$;

(c) the dual polytopes $P^\vee$ are assumed to be small enough that the forms in the right hand side of (3.22) are symplectic;

(d) $\phi_P$ satisfies $\Phi \circ \phi_P = \pi_P \circ \Phi$, where in the left hand side, $\Phi : \Phi^{-1}(P) \times P^\vee \to P$ is independent of the second domain component;

and the maps $\underline{\phi}$ satisfy the following (Patching) condition:

(Patching) For any pair $Q \subset P$, in the overlap region $\tilde{Q} \cap \tilde{P}$ the $T_{P,\mathbb{C}}$-action induced by $\phi_P$ (see the explanation preceding this definition) is the restriction of the $T_{Q,\mathbb{C}}$-action induced by $\phi_Q$.

This ends the Definition.

*Remark* 3.36. For any $P \in \mathcal{P}$, the second component of the symplectic cylindrical structure map, namely $\Pi_P \circ \phi_P$, satisfies

$$\Pi_{P^\vee} \circ \phi_P = \pi_{P^\vee} \circ \Phi,$$

and therefore, it is a $T_P$-moment map. Here, the projection $\pi_{P^\vee} : \tilde{P} \to P^\vee$ is from Definition 3.23.

In the above definition of the symplectic cylindrical structure maps, the (Patching) condition is equivalent to the following consistency condition on connection one-forms: $(\alpha_P)_{P\in\mathcal{P}}$.

**Lemma 3.37.** *(Consistency for connection one-forms) The collection of symplectic cylindrical structure maps $(\phi_P)_P$ satisfy the (Patching) condition if and only if for any $x \in \Phi^{-1}(\tilde{Q} \cap \tilde{P})$,*

$$\ker(\alpha_P(x)) = \ker(\alpha_Q(x)) \oplus (\mathfrak{t}_Q/\mathfrak{t}_P)x, \tag{3.23}$$

*where we view the quotient $\mathfrak{t}_Q/\mathfrak{t}_P$ as a subspace of $\mathfrak{t}_Q$ by the pairing (3.10) on $\mathfrak{t}_Q$.*

*Proof.* For any $x \in \Phi^{-1}(\tilde{Q} \cap \tilde{P})$,

$$T_xX = \mathfrak{t}_{Q,\mathbb{C}}x \oplus \ker(\alpha_Q(x)) = \mathfrak{t}_{P,\mathbb{C}}x \oplus \ker(\alpha_P(x)),$$

and in both decompositions, the summands are $\omega$-complements. The (Patching) condition implies that $\mathfrak{t}_{P,\mathbb{C}}x \subset \mathfrak{t}_{Q,\mathbb{C}}x$, which is equivalent to (3.23). ∎

*Remark* 3.38. In Definition 3.35, condition (d) exists in order to simplify exposition. This condition allows us to state the properties of fibered polytopes purely at the level of polytopes without using the map $\Phi$. In the construction of the symplectic cylindrical structure, we modify the non-essential components of the tropical moment map to achieve this condition, see (3.62).

For the moment, we assume the existence of symplectic cylindrical structures and use them to define a family of neck-stretched almost complex structures. The proof of the existence of symplectic cylindrical structures on tropical Hamiltonian actions is deferred to Proposition 3.65 at the end of Chapter 3. The cylindrical structure maps $\{\phi_P\}_{P\in\mathcal{P}}$ in (3.22) are fixed throughout the paper.

*Remark* 3.39. (A decomposition of polytopes) We describe a polyhedral decomposition that is used to define neck-stretched manifolds. Let $(X, \mathcal{P}, \Phi)$ be a tropical Hamiltonian action and let $B^\vee$ be a dual complex. For any $P \in \mathcal{P}$, let

$$P^{\blacksquare} := P \backslash (\cup_{Q\subset P} \tilde{Q}), \tag{3.24}$$

be the complement of fibered neighborhoods of proper faces of $P$. Here, we recall from (3.13) that the cutting datum gives an embedding $\tilde{Q} \hookrightarrow \mathfrak{t}^\vee$ for any fibered polytope $\tilde{Q}$ corresponding to $Q \in \mathcal{P}$. Corresponding to any facet $Q \subset P$, $P^\blacksquare$ has a facet $Q^\blacksquare$. Let

$$\tilde{P}^\blacksquare := \pi_P^{-1}(P^\blacksquare) \subset \tilde{P} \tag{3.25}$$

be the thickening of $P^\blacksquare$. For a pair $Q \subset P$ with $\operatorname{codim}_P(Q) = 1$, the fibered polytopes $\tilde{P}^\blacksquare, \tilde{Q}^\blacksquare \subset \mathfrak{t}^\vee$ share a facet, which is isomorphic to $Q^\blacksquare \times P^\vee$. The image of $\Phi$ is covered by the union of thickenings

$$\operatorname{im}(\Phi) = \cup_{P \in \mathcal{P}} i_{\tilde{P}}(\tilde{P}^\blacksquare); \tag{3.26}$$

see Figure 3.9. The partition of $\operatorname{im}(\Phi)$ pulls back to a partition of the symplectic manifold $(X, \omega_X)$

$$X := \left( \bigsqcup_{P \in \mathcal{P}} \Phi^{-1}(\tilde{P}^\blacksquare) \right) / \sim \tag{3.27}$$

into manifolds with corners, where the identifications are by the equivalence relation $\sim$ along the boundaries and are induced by the inclusions $\Phi^{-1}(\tilde{P}^\blacksquare) \to X$. The symplectic cylindrical structure map $\underline{\phi} = (\phi_P)_P$ may be used to rewrite the decomposition in (3.27) as

$$X := \left( \bigsqcup_{P \in \mathcal{P}} \Phi^{-1}(P^\blacksquare) \times P^\vee \right) / \sim \tag{3.28}$$

as in Figure 3.9. In (3.28), the equivalence $\sim$ identifies the boundary components

$$\Phi^{-1}(Q^\blacksquare) \times Q^\vee \supset \Phi^{-1}(Q^\blacksquare) \times P^\vee \xrightarrow{\sim} \Phi^{-1}(Q^\blacksquare) \times P^\vee \subset \Phi^{-1}(P^\blacksquare) \times P^\vee \tag{3.29}$$

for all pairs $Q \subset P$, $\operatorname{codim}_P(Q) = 1$.

**Definition 3.40.** (Neck-stretched manifolds) Let $(X, \mathcal{P}, \Phi)$ be a tropical Hamiltonian action with a symplectic cylindrical structure. For any $\nu \in \mathbb{R}_{\geq 1}$, define a *neck-stretched manifold* $X^\nu$ as

$$X^\nu = \left( \bigsqcup_{P \in \mathcal{P}} (\Phi^{-1}(P^\blacksquare) \times \nu P^\vee) \right) / \sim, \tag{3.30}$$

where the equivalence relation $\sim$ is exactly as in (3.29), that is, for all pairs $Q \subset P$, $\operatorname{codim}_P(Q) = 1$, the boundary component $\Phi^{-1}(Q^\blacksquare) \times \nu P^\vee$ in $\Phi^{-1}(P^\blacksquare) \times \nu P^\vee$ is identified with the corresponding boundary component in $\Phi^{-1}(Q^\blacksquare) \times \nu Q^\vee$ by the identity map. This ends the Definition.

*Remark* 3.41. In order to define a manifold structure on $X^\nu$, we need to give identifications between the tubular neighborhoods of boundaries and corners that are glued in (3.30). These identifications are the same as those between the corresponding pieces of the decomposition (3.28) of $(X, \omega_X)$.

*Remark* 3.42. (Projection to dual complex) There is a natural projection map

$$\pi_{\nu B^\vee} : X^\nu \to \nu B^\vee \tag{3.31}$$

which is defined on the subset $\Phi^{-1}(P^\blacksquare) \times \nu P^\vee$ as projection to $\nu P^\vee$ for any $P \in \mathcal{P}$. The map $\pi_{\nu B^\vee}$ is continuous.

*Remark* 3.43. By Definition 3.40, the neck-stretched manifold $X^\nu$ is equipped with

(a) for each $P$, a $P$-cylindrical structure on the subset $\Phi^{-1}(P^\blacksquare) \times \nu P^\vee \subset X^\nu$; that is, there is a projection

$$\Phi^{-1}(P^\blacksquare) \times \nu P^\vee \to \Phi^{-1}(P^\blacksquare)/T_P \tag{3.32}$$

whose fibers are $T_P \times \nu P^\vee \subset T_{P,\mathbb{C}}$; and

(b) a symplectic form on the base manifold $\Phi^{-1}(P^\blacksquare)/T_P$ that is a $T_P$-reduction of the symplectic form $\omega_X$ on $X$.

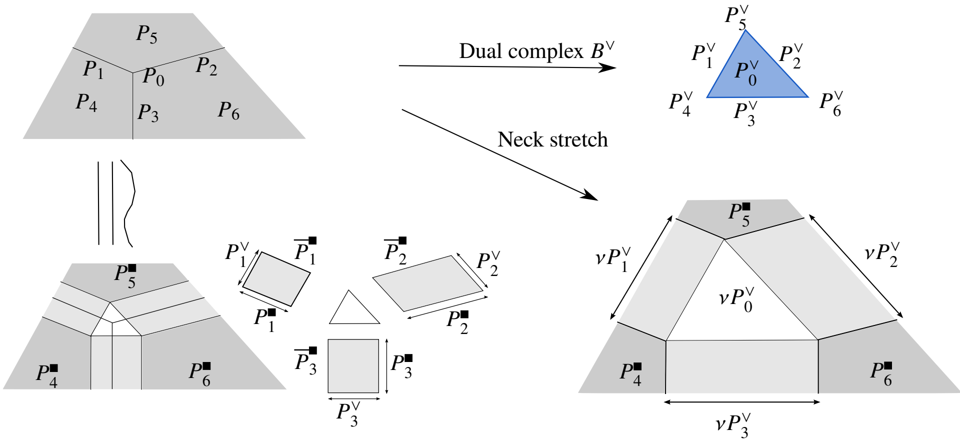



**Figure 3.9.** Stretching

*Remark* 3.44. (A warning about Figure 3.9) Figure 3.9 is a schematic representation of neck-stretched manifolds. We recall symplectic broken manifolds are represented by the images of their tropical moment map, for example in Figure 3.5. In contrast, we do not have a moment map on neck-stretched manifolds. A larger polytope is just used to indicate a larger cylinder.

*Remark* 3.45. The neck-stretched manifold $X^\nu$ is diffeomorphic to $X$, but the diffeomorphism is not canonical. As a result, there is no canonical symplectic form on $X^\nu$.

*Example* 3.46. (Neck-stretched manifolds for a single cut) We describe neck-stretched manifolds in case of a single cut. Let $(X, \Phi, c)$ be a tropical Hamiltonian action with a single cut along $\Phi^{-1}(c)$, and whose polytopes are as in Example 3.19. The symplectic cylindrical structure consists of an $S^1$-equivariant symplectomorphism defined on a neighborhood

$$\Phi^{-1}([c-\epsilon, c+\epsilon]) \xrightarrow{\phi_{P_0}} (\Phi^{-1}(c) \times [-\epsilon, \epsilon], \omega_{/} + d(t\alpha)), \tag{3.33}$$

where $\omega_{/}$ is the reduced symplectic form on the quotient $\Phi^{-1}(c)/S^1$, $t \in (-\epsilon, \epsilon)$ is the coordinate function, and $\alpha \in \Omega^1(\Phi^{-1}(c))$ is a connection one-form on the $S^1$-bundle $\Phi^{-1}(c) \to \Phi^{-1}(c)/S^1$. The other symplectic cylindrical structure maps $\phi_{P_-}, \phi_{P_+}$ are trivial since $P_+, P_-$ are top-dimensional. The neck-stretched manifold is then

$$X^\nu := \{\Phi \le c-\epsilon\} \cup_{\Phi^{-1}(c)} \times [-\epsilon\nu, \epsilon\nu]) \cup_{\Phi^{-1}(c)} \{\Phi \ge c+\epsilon\}, \tag{3.34}$$

where the attachment maps are given by identifying $\{\Phi = c \pm \epsilon\}$ to $\{\Phi = c\}$ via the symplectic cylindrical structure $\phi_{P_\pm}$, and in the middle piece, $\{\Phi = c\}$ is identified to the ends $\{\Phi = \pm\epsilon\nu\}$. [2]

**Definition 3.47.** (Cylindrical almost complex structures on neck-stretched manifolds) Let $(X, \mathcal{P})$ be a tropical Hamiltonian action and let $\{X^\nu\}_\nu$ be a sequence of neck-stretched manifolds. Recall from (3.30) that a neck-stretched manifold has a decomposition

$$X^\nu = \left( \bigsqcup_{P \in \mathcal{P}} \Phi^{-1}(P^\blacksquare) \times \nu P^\vee \right) / \sim,$$

where $\sim$ identifies boundaries of the different components.

(a) An almost complex structure $J^\nu$ on the neck-stretched manifold $X^\nu$ is *cylindrical* if $J^\nu$ is $P$-cylindrical in the sense of Definition 3.34 in the subset

$$\Phi^{-1}(P^\blacksquare) \times \nu P^{\vee,\circ} \subset X^\nu.$$

The space of cylindrical almost complex structures on $X^\nu$ is denoted by

$$\mathcal{J}^{\mathrm{cyl}}(X^\nu).$$

We say that $(X^\nu, J^\nu)$ is a *family of neck-stretched almost complex structures* if there are $\nu$-independent cylindrical almost complex structures $J_P$ on $\Phi^{-1}(P^\blacksquare) \times \mathfrak{t}_P^\vee$ for all $P$, such that on the $P$-cylindrical subset $\Phi^{-1}(P^\blacksquare) \times \nu P^{\vee,\circ}$ of $X^\nu$, $J^\nu$ is the restriction of $J_P$.

[2]The interval $[-\epsilon\nu, \epsilon\nu]$ in (3.34) is chosen to be consistent with definitions, there is no canonical choice of end-points, as long as the length of the segment is $\nu$ times the neck length in $X$.

(b) (Local tamedness and compatibility) Let $(X^\nu, J^\nu)$ be a family of neck-stretched almost complex structures. We say that each element $J^\nu$ is *locally tamed* resp. *locally compatible* if $J^1$ is $\omega_X$-tamed resp. $\omega_X$-compatible. (The definition is based on the observation that the neck-stretched manifold $X^1$ has a canonical diffeomorphism to $(X, \omega_X)$, and this is not so for $X^\nu$, $\nu \neq 1$.)

(c) (Local strong tamedness) A cylindrical locally tamed almost complex structure $J$ on $X^\nu$ is *locally strongly tamed* if for all $P \in \mathcal{P}$ the connection one-form $\alpha_{P,J}$ underlying the $P$-cylindrical almost complex structure on $\Phi^{-1}(P^\blacksquare) \times \nu P^{\vee,\circ}$ in Definition 3.34 is equal to the connection one-form underlying the symplectic cylindrical structure in Definition 3.35.

*Remark* 3.48. The following are some remarks on cylindrical almost complex structures on neck-stretched manifolds.

(a) The cylindrical almost complex structures defined above in Definition 3.47 is sometimes referred to as *X-cylindrical* to emphasize its dependence on the $X$-inner product (3.10) on $\{\mathfrak{t}_P\}_P$. Recall that the dependence on the inner product arises because of the identification (3.20) between the symplectic cylinder $\Phi^{-1}(P) \times \mathfrak{t}_P^\vee$ and the almost complex cylinder $\Phi^{-1}(P) \times \sqrt{-1}\mathfrak{t}_P$

(b) The definition of local (strong) tameness depends not just on the symplectic form $\omega_X$, but also on the symplectic cylindrical structure in Definition 3.35.

(c) In the space of cylindrical almost complex structures, the cylindrical coordinate maps, taking values in $\Phi^{-1}(P^\blacksquare) \times \nu P^\vee$, are held fixed, but the connection one-forms $\alpha_{P,J}$, $P \in \mathcal{P}$ underlying a cylindrical almost complex structure $J$ are allowed to vary.

(d) We give an alternate definition of local tamedness. A cylindrical almost complex structure $J^\nu$ is *locally tamed* on $X^\nu$ if for each $P$, the base almost complex structure (see Definition 3.34) on $X_P$ induced by $J^\nu$ on the $P$-cylinder $\Phi^{-1}(P^\blacksquare) \times \nu P^\vee$ is tamed by $\omega_{X_P} + \langle d\alpha_P, c\rangle$ for all $c \in P^\vee - c_{P^\vee} \subset \mathfrak{t}_P^\vee$. Here $c_{P^\vee} := \pi_{\mathfrak{t}_P^\vee}(P) \in \mathfrak{t}_P^\vee$ is a constant and $\alpha_P$ is the $T_P$-connection one-form underlying the symplectic cylindrical structure in Definition 3.35. Note that the definition of local tameness does not involve $\nu$, since the "same" almost complex structure can be defined for any $\nu$.

The following Lemma is a quantitative version of the statement that local tamedness is a $C^0$-open property in the space of cylindrical almost complex structures. We use the alternate definition of local tamedness from Remark 3.48 (d).

**Lemma 3.49.** *Let $J_P^0$ be an almost complex structure on $\Phi^{-1}(P^\blacksquare)/T_P$ that is tamed by the form $\omega_{X_P} + \langle d\alpha_P, \tau\rangle$ for all $\tau \in P^\vee \subset \mathfrak{t}_P^\vee$. Then there exist constants $\epsilon$, $c$ such* *that if* $J_P \in \mathcal{J}(\Phi^{-1}(P^{\blacksquare})/T_P)$ *is such that* $\|J_P - J_P^0\|_{C^0} < \epsilon$ *then*

$$c^{-1}\omega_{X_P}(v, J_P v) \leq (\omega_{X_P} + \langle d\alpha_P, c\rangle)(v, J_P v) \leq c\omega_{X_P}(v, J_P v) \quad \forall v \in T(\Phi^{-1}(P^{\blacksquare})/T_P). \tag{3.35}$$

*Proof.* By the compactness of the spaces $P^{\blacksquare}/T_P$ and $P^\vee$, and the tamedness of $J_P^0$ for $\omega_{X_P} + \langle d\alpha_P, \tau\rangle$ for all $\tau \in P^\vee$, there are constants $c_0, c_1 > 0$ such that

$$c_0|v|^2 \leq (\omega_{X_P} + \langle d\alpha_P, \tau\rangle)(v, J_P^0 v) \leq c_1|v|^2$$

for all $v \in T(\Phi^{-1}(P^{\blacksquare})/T_P, \tau \in P^\vee$. For any $\epsilon > 0$ and $J_P \in B_\epsilon(J_P^0)$, there are constants $c_0', c_1' > 0$ such that

$$(c_0 - c_0'\epsilon)|v|^2 \leq (\omega_{X_P} + \langle d\alpha_P, \tau\rangle)(v, J_P^0 v) \leq (c_1 + c_1'\epsilon)|v|^2.$$

The Lemma follows by choosing $\epsilon > 0$ such that $c_0 - c_0'\epsilon > 0$ and $c := \frac{c_1 + c_1'\epsilon}{c_0 - c_0'\epsilon}$. ■

## 3.5 Broken manifold as a degenerate limit

In this Section, we define cut spaces and broken manifolds as almost complex manifolds with cylindrical ends. For any $P$, the cut space $X_P$ is defined as the direct limit of the $P$-cylindrical subsets of the neck-stretched manifolds quotiented by the torus $T_{P,\mathbb{C}}$. The broken manifold $\mathfrak{X}_P$ is a $T_{P,\mathbb{C}}$-fibration over $X_P$. For any $P$, the almost complex cut space $X_P$ resp. broken manifold $\mathfrak{X}_P$ is diffeomorphic to the corresponding symplectic object $X_P^\omega$ resp. $\mathfrak{X}_P^\omega$, but there is no canonical diffeomorphism. See Lemma 3.52 and Remark 3.55 for related discussion.

### 3.5.1 Defining almost complex broken manifolds

The broken manifold is the degenerate limit of neck-stretched almost complex manifolds as we now explain. For a polytope $P$ and $\nu \geq 1$, let

$$X_{\tilde{P}}^\nu := \left(\bigsqcup_{Q \in \mathcal{P}: Q \subseteq P} (\Phi^{-1}(Q^{\blacksquare}) \times \nu Q^\vee)/\sim\right) \subset X^\nu \tag{3.36}$$

be the subset of $X^\nu$ that has a $P$-cylindrical structure. Thus $X_{\tilde{P}}^\nu$ has a free partial $T_{P,\mathbb{C}}$-action (see (3.21)). (Here the equivalence relation $\sim$ is the same as the one in the definition 3.30 of $X^\nu$.) Let

$$X_P^\nu := X_{\tilde{P}}^\nu / T_{P,\mathbb{C}} \tag{3.37}$$

be the quotient of $X_{\tilde{P}}^\nu$ under the partial $T_{P,\mathbb{C}}$-action. Consequently, there are projections

$$X_{\tilde{P}}^\nu \xrightarrow{\pi_P'} Z_P^\nu \xrightarrow{\pi_P''} X_P^\nu, \tag{3.38}$$

where $Z_P^\nu \xrightarrow{\pi_P''} X_P^\nu$ is a $T_P$-bundle, and the fibers of $\pi_P'$ are $P^\vee \subset \mathfrak{t}_P^\vee$.

**Lemma 3.50.** *Let $(X^\nu, J^\nu)$ be a family of neck-stretched almost complex structures (as in Definition 3.47). For any $P \in \mathcal{P}$ and $\nu_0 < \nu_1$ there is a natural embedding*

$$i_{P,\nu_0,\nu_1} : (X_P^{\nu_0}, J^{\nu_0}) \to (X_P^{\nu_1}, J^{\nu_1}), \quad i_{Z_P,\nu_0,\nu_1} : Z_P^{\nu_0} \to Z_P^{\nu_1},$$

*and for any $\nu_0 < \nu_1 < \nu_2$,*

$$i_{P,\nu_0,\nu_2} := i_{P,\nu_1,\nu_2} \circ i_{P,\nu_0,\nu_1}, \quad i_{Z_P,\nu_0,\nu_2} := i_{Z_P,\nu_1,\nu_2} \circ i_{Z_P,\nu_0,\nu_1}$$

*Proof.* Let $P \in \mathcal{P}$ be a top-dimensional polytope. Then, $X_P^\nu = X_{\tilde{P}}^\nu$ has a decomposition as in (3.36). Define embeddings

$$i_{P,\nu_0,\nu_1} := (\mathrm{Id}, \tau) : \Phi^{-1}(Q^\blacksquare) \times \nu_0 Q^\vee \to \Phi^{-1}(Q^\blacksquare) \times \nu_1 Q^\vee \tag{3.39}$$

where $\tau$ is the restriction of a translation on $\mathfrak{t}_Q^\vee$ (recall that $\nu Q^\vee \subset \mathfrak{t}_Q^\vee$ for all $\nu$) that maps the point $\nu_0 P^\vee \in \nu_0 Q^\vee$ to the point $\nu_1 P^\vee \in \nu_1 Q^\vee$. The maps (3.39) glue to yield an embedding $i_{P,\nu_0,\nu_1} : X_P^{\nu_0} \to X_P^{\nu_1}$.

Next, consider a face $Q \in \mathcal{P}$ of the top-dimensional polytope $P$. The embedding $i_{P,\nu_0,\nu_1} : X_{\tilde{Q}}^{\nu_0} \to X_{\tilde{Q}}^{\nu_1}$ descends to $i_{Z_Q,\nu_0,\nu_1} : Z_Q^{\nu_0} \subset Z_Q^{\nu_1}$ and $i_{Q,\nu_0,\nu_1} : X_Q^{\nu_0} \subset X_Q^{\nu_1}$. We leave it to the reader to check that the embedding preserves the almost complex structure. ∎

Next, we define cut spaces and the broken manifold as manifolds with cylindrical structures. Cut spaces resp. broken manifolds were already defined as symplectic manifolds in 3.8 resp. 3.29. We now define these spaces as almost complex manifolds. See Lemma 3.52 and Remark 3.55 for a reconciliation of the two viewpoints. The symplectic spaces have a superscript $\omega$ to distinguish them from the corresponding almost complex spaces, see the “important point about notation” in page 66.

**Definition 3.51.** (Almost complex cut spaces and broken manifold) Let $(X, \mathcal{P}, \Phi)$ be a tropical Hamiltonian action and let $\{X^\nu\}_\nu$ be the corresponding family of neck-stretched manifolds.

(a) (Cut space) For a polytope $P \in \mathcal{P}$, the *cut space* $X_P$ is defined as the direct limit [3]

$$X_P := (\cup_\nu X_P^\nu)/\sim, \tag{3.40}$$

where $X_P^\nu$ is the $T_P$-quotient of the $P$-cylindrical subset of $X^\nu$ (see (3.37)), and $\sim$ is given by the embeddings $i_{P,\nu_0,\nu_1} : X_P^{\nu_0} \to X_P^{\nu_1}$ from Lemma 3.50. For any $\nu$, there is a natural embedding

$$i_{P,\nu} : X_P^\nu \to X_P.$$

[3] We do not just define the cut space as $\Phi^{-1}(P)/T_P$ as in done in the case of a symplectic cut space. The definition by direct limits allows us to realize the cut space as the direct limit of almost complex manifolds $X_P^\nu$.

(b) ($T_P$-bundle on the cut space $X_P$) Define the $T_P$-bundle $Z_P \to X_P$ as the direct limit

$$Z_P := (\cup_\nu Z_P^\nu)/\sim \tag{3.41}$$

of $T_P$-bundles $Z_P^\nu$ (defined in (3.38)), where $\sim$ is given by the embeddings $i_{Z_P,\nu_0,\nu_1} : Z_P^{\nu_0} \to Z_P^{\nu_1}$ from Lemma 3.50. There is a natural embedding

$$i_{Z_P,\nu} : Z_P^\nu \to Z_P.$$

(c) (Broken manifold) The broken manifold $\mathfrak{X}$ is the disjoint union

$$\mathfrak{X}_{\mathcal{P}} \quad \text{or} \quad \mathfrak{X} := \bigsqcup_{P \in \mathcal{P}} \mathfrak{X}_P, \tag{3.42}$$

where we use the notation $\mathfrak{X}$ when the polyhedral decomposition $\mathcal{P}$ is clear from the context. Here,

$$\mathfrak{X}_P := Z_P \times \mathfrak{t}_P^\vee$$

and is a $T_{P,\mathbb{C}}$-bundle over $X_P$ with projection

$$T_{P,\mathbb{C}} \to \mathfrak{X}_P \xrightarrow{\pi_{X_P}} X_P. \tag{3.43}$$

(d) (Cylindrical almost complex structures) An almost complex structure $\mathfrak{J} = (\mathfrak{J}_P)_{P \in \mathcal{P}}$ on $\mathfrak{X}$ consists of an almost complex structure $\mathfrak{J}_P$ on each component $\mathfrak{X}_P \subset \mathfrak{X}$. Such a $\mathfrak{J}$ is *cylindrical* (or *$X$-cylindrical* to emphasize the dependence on the $X$-inner product, see Remark 3.48 (a)) if it is the limit of a family $(J^\nu)_\nu$ of neck-stretched cylindrical almost complex structures on $X^\nu$. The space of cylindrical almost complex structures on $\mathfrak{X}$ is denoted by

$$\mathcal{J}^{\text{cyl}}(\mathfrak{X}) = \{\mathfrak{J} \text{ cylindrical}\}.$$

The almost complex structure $\mathfrak{J}$ is *locally tame* or *locally compatible* or *locally strongly tamed* if the corresponding property holds for $J^\nu$ (see Definition 3.47).

(e) (Cylindrical ends) For a pair of polytopes $Q \subset P$, the *$Q$-cylindrical end of* $X_P$ is a subset $U_Q(X_P) \subset X_P$ defined as the exhaustion

$$U_Q(X_P) := \cup_\nu U_Q(X_P^\nu) \tag{3.44}$$

of $Q$-cylindrical subsets $U_Q(X_P^\nu) \subset X_P^\nu$ given by

$$U_Q(X_P^\nu) := \cup_{R \subseteq Q} \Phi^{-1}(R^\blacksquare) \times \nu R^\vee \subset X_P^\nu.$$

The $Q$-cylindrical end in $\mathfrak{X}_P$ resp. $Z_P$ is the lift of $U_Q(X_P)$ by the projection map (3.38) $\pi_P'' : Z_P \to X_P$ resp. $\pi_{X_P} : \mathfrak{X}_P \to X_P$, and is denoted by

$$U_Q(\mathfrak{X}_P) \subset \mathfrak{X}_P \quad \text{resp.} \quad U_Q(Z_P) \subset Z_P.$$

The following is immediate from the definitions:

**Lemma 3.52.** *Given a broken almost complex manifold $\mathfrak{X}$, any piece $\mathfrak{X}_P$ of the broken manifold is diffeomorphic to the symplectic manifold $\mathfrak{X}_P^\omega$.*

The cut space $X_P$ has an orbifold compactification

$$\overline{X}_P \tag{3.45}$$

that is diffeomorphic to the compactification $\overline{X}_P^\omega$ of the symplectic cut space. The boundary $\overline{X}_P \backslash X_P$ is a collection of sub-orbifolds of codimension at least 2, each corresponding to either a relative divisor, or the intersection of a collection of relative divisors in $\overline{X}_P^\omega$ from Definition 3.29 (b). The compactification may be constructed formally (without reference to $X_P^\omega$) by adding to $X_P$ limit points of torus orbits; for example, a divisor $X_Q \subset \overline{X}_P$ corresponding to a facet $Q \subset P$, $Q \in \mathcal{P}$ with outward normal $\nu_Q \in \mathfrak{t}_Q/\mathfrak{t}_P$ is the set of orbits

$$\{[x] := T_{\nu_Q,\mathbb{C}}x \ \mid \ x \in X_P\},$$

where $T_{\nu_Q,\mathbb{C}} \simeq \mathbb{C}^\times$ is the torus generated by $\nu_Q$, and each $[x]$ is the limit point of a $T_{\nu_Q,\mathbb{C}}$-orbit, that is, $\lim_{s\to\infty} e^{(s+it)\nu_Q}x = [x]$. The cylindrical almost complex structure on $X_P$ does not extend to $\overline{X}_P$. Similarly, a component $\mathfrak{X}_P$ of the broken manifold has a compactification $\overline{\mathfrak{X}}_P$ if the $X$-inner product on $\{\mathfrak{t}_P\}_P$ (from (3.10)) is rational.

*Remark* 3.53. The neck-stretched manifolds $X^\nu$ can be recovered from the broken almost complex manifold equipped with cylindrical coordinates on its ends as

$$X^\nu = \left( \bigcup_{P\in\mathcal{P},\operatorname{codim}(P)=0} X_P \right) /\!\sim_\nu$$

where the equivalence relation $\sim_\nu$ is as follows. For top-dimensional polytopes $P_0, P_1$, and $Q := P_0 \cap P_1$,

$$X_{P_0} \ni x_0 \sim_\nu x_1 \in X_{P_1} \iff \\ x_0 \in U_{P_0}Q,\ \ x_1 \in U_{P_1}Q,\ \ i_Q^{P_0}(x_0) = e^{\pi(\nu P_0^\vee - \nu P_1^\vee)} i_Q^{P_1}(x_1). \tag{3.46}$$

Here, $P_0^\vee - P_1^\vee \in \mathfrak{t}_Q^\vee$ and $\pi$ is the identification $\mathfrak{t}_Q^\vee \simeq \sqrt{-1}\mathfrak{t}_Q$ from (3.10). This ends the Remark.

*Example* 3.54. (Cylindrical ends in a single cut) We continue Example 3.46 where we described neck-stretched manifolds in case of a single cut. Let $(X, \Phi, c)$ be a tropical Hamiltonian action with a single cut. First, we describe the $P$-cylindrical part in the neck-stretched manifold $X^\nu$. We have,

$$X^\nu_{P_+} = \{\Phi \geq c + \epsilon\} \cup_{\Phi^{-1}(c)} (\Phi^{-1}(c) \times [-\epsilon\nu, \epsilon\nu]),$$

with $X^{\nu}_{P_-}$ defined analogously, and

$$X^{\nu}_{P_0} = (\Phi^{-1}(c) \times [-\epsilon\nu, \epsilon\nu]).$$

Next, we describe the $P_0$-cylindrical end in $X_{P_+}$. Since $\operatorname{codim}(P_+) = 0$, the spaces $X_{P_+}$, $Z_{P_+}$ and $\mathfrak{X}_{P_+}$ are all the same. Furthermore,

$$X_{P_+} = \{\Phi \geq c + \epsilon\} \cup_{\Phi^{-1}(c)} \Big((-\infty, 0] \times \Phi^{-1}(c)\Big), \tag{3.47}$$

where $\Phi^{-1}(c+\epsilon)$ is mapped to $\Phi^{-1}(c)$ via the symplectic cylindrical structure map, see (3.33). The latter may be identified with $\{0\} \times \Phi^{-1}(c)$. Note that $Z_{P_0} = \Phi^{-1}(c)$, the normal cone $\operatorname{NCone}_{P_+^\vee} P_0^\vee$ is $(-\infty, 0]$. The subset $(-\infty, 0] \times Z$ in $X_{P_+}$ is the $P_0$-cylindrical end.

*Remark* 3.55. (Symplectic versus almost complex broken manifolds) We have defined the broken manifold as a collection of almost complex manifolds $(\mathfrak{X}_P, J_P)$ in (3.42), and as symplectic manifolds $(\mathfrak{X}^{\omega}_P, \omega_{\mathfrak{X}_P})$ in (3.14). In this Remark, we see how to relate the two spaces, and also why they can not be viewed as the same space. Since a component $\mathfrak{X}_P$ of a broken manifold is a $T_{P,\mathbb{C}}$-bundle over a cut space $X_P$, we focus on cut spaces in this discussion.

(a) (Maps respecting cylindrical structures) A surjective map $\phi : X_P \to X^{\omega}_P$ *respects the cylindrical structure* if

- $\phi$ is the natural embedding on the complement of cylindrical ends, that is,

$$\phi(X_P \backslash (\cup_{Q \subset P} U_Q(X_P))) = \Phi^{-1}(P^{\blacksquare})/T_P \subset (\overline{X}^{\omega}_P, \omega_{X_P}), \tag{3.48}$$

  arising from the definition (3.36), (3.37), (3.40) of $X_P$, and the definition (3.6) of the symplectic cut space $X^{\omega}_P$;
- $\phi$ is $T_Q$-equivariant on the $Q$-cylindrical end; and
- $\phi$ commutes with the projection to $X_Q$ on the $Q$-cylindrical end, that is,

$$\pi^P_Q = \pi^{P,\omega}_Q \circ \phi \quad \text{on } U_Q(X_P) \backslash \cup_{R \subset Q} U_R(X_P),$$

  where the projection $\pi^P_Q$ to $X_Q$ on the $Q$-cylindrical end is defined in (3.56) and the corresponding projection $\pi^{P,\omega}_Q$ to $X^{\omega}_Q$ for symplectic manifolds is defined in (5.4). In the above equation, both sides map to the complement of the cylindrical ends in $X_Q$: The left hand side maps to $X_Q \backslash (\cup_{R \subset Q} U_R(X_Q))$, and the right hand side maps to $\Phi^{-1}(Q^{\blacksquare})/T_Q$, and the two spaces are canonically identified as in (3.48).

For a broken manifold $\mathfrak{X}$, there is a large collection of maps that respect the cylindrical structure. In Chapter 7, we construct squashing maps that are continuous and piecewise smooth. Those maps can be approximated by smooth maps that satisfy the above properties. Furthermore, any two maps respecting the cylindrical structure are homotopic via a family of maps respecting the cylindrical structure. These maps are used to define the symplectic area of pseudoholomorphic maps to the broken manifold (Definition 4.21), and to define Donaldson divisors on broken manifolds (Definition 5.12).

(b) (No canonical diffeomorphisms) The manifolds $X_P$ and $X_P^\omega$ are diffeomorphic, but there is no canonical diffeomorphism $\phi : X_P \to X_P^\omega$ that respects the cylindrical structure. Assuming $X_P$ has a cylindrical locally tamed almost complex structure $J_P$, we do not have a method of constructing diffeomorphisms $\phi : X_P \to X_P^\omega$ that respect the cylindrical structure and for which $J_P$ is tamed by $\phi^* \omega_{X_P}$.

(c) (Canonical diffeomorphisms without gluability) The lack of taming embeddings of broken almost complex manifolds into compact symplectic broken manifolds can be remedied by introducing a slight weakening in the definition of cylindrical almost complex structures on broken manifolds. In particular, if one allows the connection one-forms $(\alpha_P)_{P \in \mathcal{P}}$ and the inner products $\{g_P\}_{P \in \mathcal{P}}$ to differ across the set of cut spaces $\{X_P\}_{P \in \mathcal{P}^{(0)}}$, one can construct taming diffeomorphisms of $X_P$ into $X_P^\omega$, see Lemma 5.11. The cost of this choice is that the new kind of almost complex structures on pieces of $\mathfrak{X}$ do not glue to give a neck-stretched almost complex structure in $X$. That is, the almost complex structures are not *gluable*. Gluability of almost complex structures on broken manifolds is necessary for the proof of homotopy equivalence of the Fukaya algebras defined on the unbroken manifold $(X, L)$ and the broken manifold $(\mathfrak{X}, L)$. Once this result is proved, one has greater flexibility in choosing almost complex structures on broken manifolds while keeping the compactness and regularity results. In particular, the requirement of gluability can be dropped and one can choose almost complex structures so that there are taming embeddings into compact symplectic manifolds. Domain-dependent almost complex structures that are not gluable are described in Section 11.1. This finishes the Remark.

### 3.5.2 Coordinates on cylindrical ends

Our next task is to produce identifications between $Q$-cylindrical ends of different components of a broken manifold. These identifications are used in writing down the matching conditions at nodes of a broken map: The matching condition compares the evaluation of maps lying in different manifolds (say $\mathfrak{X}_P, \mathfrak{X}_{P'}$), although neighborhoods of both lifts of the node map to the $Q$-cylindrical end. The identifications between the cylindrical ends are via certain natural cylindrical "coordinates" that take values in cones of torus Lie algebras such as $\mathfrak{t}_P^\vee$.

*Remark* 3.56. We give some relations between dual polytopes, cones and normal cones (Definition 3.3). Later in Remark 3.58, these relations will be shown to be the polytope analogues of the fact that cut spaces are degenerate limits of neck-stretched manifolds. For a pair $Q \subset P$ of polytopes with $\mathrm{codim}(P) = 0$, there are inclusions

$$\tilde{i}_{P,Q,\nu_0,\nu_1} : \nu_0 Q^\vee \to \nu_1 Q^\vee, \quad \forall \nu_0 < \nu_1$$

that are translations and which map the vertex $\nu_0 P^\vee \in \nu_0 Q^\vee$ to $\nu_1 P^\vee \in \nu_1 Q^\vee$. Then $\mathrm{Cone}_{P^\vee} Q^\vee$ is the exhaustion

$$\mathrm{Cone}_{P^\vee} Q^\vee = \sqcup_\nu \nu Q^\vee / \sim, \tag{3.49}$$

where, for any $\nu_0 < \nu_1$, $\sim$ identifies $\nu_0 Q^\vee$ to its image under $\tilde{i}_{P,Q,\nu_0,\nu_1}$. In general for any pair $Q \subset P$ (with $\mathrm{codim}(P) > 0$ possibly) there are inclusions

$$i_{P,Q,\nu_0,\nu_1} : \nu_0 Q^\vee / \mathfrak{t}_P^\vee \to \nu_1 Q^\vee / \mathfrak{t}_P^\vee, \quad \forall \nu_0 < \nu_1$$

which take the point $\nu_0 P^\vee / \mathfrak{t}_P^\vee$ in the domain to the point $\nu_1 P^\vee / \mathfrak{t}_P^\vee$ in the target space. The resulting exhaustion is the normal cone

$$\mathrm{NCone}_{P^\vee} Q^\vee = (\sqcup_\nu \nu Q^\vee / \mathfrak{t}_P^\vee)/\sim, \tag{3.50}$$

where, for any $\nu_0 < \nu_1$, $\sim$ identifies $\nu_0 Q^\vee / \mathfrak{t}_P^\vee$ to its image under $i_{P,Q,\nu_0,\nu_1}$.

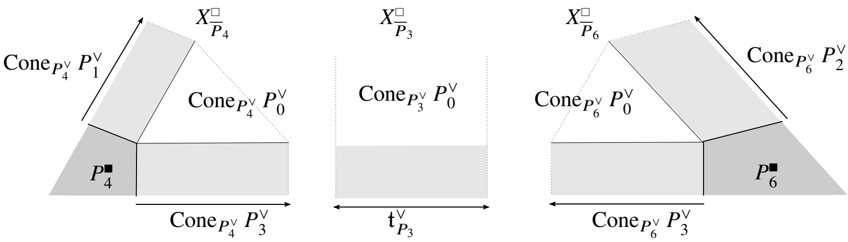



**Figure 3.10.** Some components of the broken manifold in the limit $\nu \to \infty$ of the neck-stretching in Figure 3.9.

The following lemma gives an identification between the $Q$-cylindrical end in a component $\mathfrak{X}_P$ of the broken manifold $\mathfrak{X}$ to the "standard" $Q$-cylinder $Z_Q \times \mathfrak{t}_Q^\vee$ (which is equal to $\mathfrak{X}_Q$ by definition), see Figure 3.10.

**Lemma 3.57.** *(Cylindrical ends on broken manifolds) For any pair of polytopes $Q \subset P$ in $\mathcal{P}$, there are natural embeddings*

$$\begin{aligned} i_Q^P &: U_Q(X_P) \to (Z_Q/T_P) \times \mathrm{NCone}_{P^\vee} Q^\vee, \\ i_Q^{Z,P} &: U_Q(Z_P) \to Z_Q \times \mathrm{NCone}_{P^\vee} Q^\vee, \\ i_Q^{\tilde{P}} &: U_Q(\mathfrak{X}_P) \to Z_Q \times (\mathrm{NCone}_{P^\vee} Q^\vee \times \mathfrak{t}_P^\vee) \subset Z_Q \times \mathfrak{t}_Q^\vee \simeq \mathfrak{X}_Q. \end{aligned} \tag{3.51}$$

*Proof.* We prove the Lemma for the case $\mathrm{codim}(P) = 0$. In this case, the torus $T_P$ is trivial, $X_P = Z_P = \mathfrak{X}_P$, and $\mathrm{NCone}_{P^\vee} Q^\vee = \mathrm{Cone}_{P^\vee} Q^\vee$. The domain of $i_Q^P$ has a decomposition (into sets whose interiors are disjoint)

$$U_Q(X_P) = \cup_{R \subseteq Q}(\Phi^{-1}(R^{\blacksquare}) \times \mathrm{Cone}_{P^\vee} R^\vee). \tag{3.52}$$

Here, we use the fact that $\mathrm{Cone}_{P^\vee} R^\vee$ is the limit of the polytopes $\nu R^\vee$, see (3.49). The target space of $i_Q^P$ has a decomposition (into sets whose interiors are disjoint)

$$Z_Q = \Phi^{-1}(Q^{\blacksquare}) \cup (\cup_{R \subset Q} \Phi^{-1}(R^{\blacksquare}) \times \mathrm{NCone}_{Q^\vee} R^\vee) \tag{3.53}$$

which is a limit of the decompositions

$$Z_Q^\nu = \Phi^{-1}(Q^{\blacksquare}) \cup (\cup_{R \subset Q} \Phi^{-1}(R^{\blacksquare}) \times \nu R^\vee / \mathfrak{t}_Q^\vee),$$

since $\mathrm{NCone}_{Q^\vee} R^\vee$ is the limit of $\nu R^\vee/\mathfrak{t}_Q^\vee$ as $\nu \to \infty$ (see (3.50)). The embedding $i_Q^P$ maps a subset in the domain decomposition (3.52) to the corresponding subset in the target decomposition (3.53). Here $Z_Q^\nu$ is a $T_Q$-bundle defined in (3.38). For any $R \subseteq Q$, the map

$$\mathrm{i}_Q^P : \Phi^{-1}(R^{\blacksquare}) \times \mathrm{Cone}_{P^\vee} R^\vee \to (\Phi^{-1}(R^{\blacksquare}) \times \mathrm{NCone}_{Q^\vee} R^\vee) \times \mathrm{Cone}_{P^\vee} Q^\vee$$

is defined in an following way: It is the identity on $\Phi^{-1}(R^{\blacksquare})$ and

$$\mathrm{Cone}_{P^\vee} R^\vee \to \mathrm{NCone}_{Q^\vee} R^\vee \times \mathrm{Cone}_{P^\vee} Q^\vee$$

is an orthogonal splitting. We leave to the reader the construction of the maps $i_Q^P, i_Q^{Z,P}$ when $\mathrm{codim}(P) > 0$. Finally, we point out that the domain resp. target space of the map $i_Q^{\tilde{P}}$ is the product of $\mathfrak{t}_P^\vee$ and the domain resp. target space of $i_Q^{Z,P}$. Therefore the map $i_Q^{\tilde{P}}$ is defined as such a product once $i_Q^{Z,P}$ is known. ∎

*Remark* 3.58. We can now relate the direct limit definition of cut spaces to the corresponding relations between cones from Remark 3.56. For any pair $Q \subset P$, Lemma 3.57 gives a coordinate map $\pi_{\mathrm{NCone}_{P^\vee} Q^\vee} : U_Q(X_P) \to \mathrm{NCone}_{P^\vee} Q^\vee$ on the $Q$-cylindrical end $U_Q(X_P)$ of $X_P$. The maps $\pi_{\mathrm{NCone}_{P^\vee} Q^\vee}$, for all $Q \subset P$, assemble to give a map

$$\pi_{\mathrm{NCone}_{P^\vee} B^\vee} : X_P \to \mathrm{NCone}_{P^\vee} B^\vee, \tag{3.54}$$

where the complement $X_P \backslash (\cup_{Q \subset P} U_Q(X_P))$ is mapped to the point $\mathrm{NCone}_{P^\vee} P^\vee$. The continuity of the map is a consequence of the construction in Lemma 3.57. There is a commutative diagram

$$\begin{array}{ccc} X_P^\nu & \xrightarrow{i_{P,\nu}} & X_P \\ \downarrow \pi_{\nu P^\vee} & & \downarrow \pi_{\mathrm{NCone}_{P^\vee} B^\vee} \\ \nu B^\vee / \mathfrak{t}_P^\vee & \longrightarrow & \mathrm{NCone}_{P^\vee} B^\vee \end{array} \tag{3.55}$$

where the left downward arrow is the map (3.31) composed with the $\mathfrak{t}_P^\vee$-quotient map, the top arrow is from Definition 3.51 (a), the bottom arrow is from (3.50), and the right downward arrow is from (3.54).

*Remark* 3.59. (Projections on cylindrical ends) The coordinates on the cylindrical ends of broken manifolds naturally yield projection maps. For any pair of polytopes $Q \subset P$, the projection on the $Q$-cylindrical end of $X_P$

$$\pi_Q^P : U_Q(X_P) \to X_Q \tag{3.56}$$

is defined by the first component of $\mathrm{i}_Q^P$ from (3.51) composed with the $T_Q$-quotient.

Neck-stretched manifolds and components of a broken manifold are equipped with a cylindrical metric. We recall that the $X$-inner product (3.10) defines a metric $|\cdot|_{\mathfrak{t}_P}$ resp. $|\cdot|_{\mathfrak{t}_P^\vee}$ on $\mathfrak{t}_P^\vee$ resp. $\mathfrak{t}_P^\vee$ for all $P \in \mathcal{P}$.

*Remark* 3.60. (Compactifications of cut spaces) Since the cylindrical ends of a cut space have a standard form as in (3.51), any cut space $X_P$ with a cylindrical almost complex structure $J_P$ has a compactification

$$\overline{X}_P$$

that is an almost complex orbifold, with $\overline{X}_P \backslash X_P$ being a union of divisors, each corresponding to a facet $Q \subset P, Q \in \mathcal{P}$. In a similar way, if the $\mathfrak{t}$-inner product from (3.10) is rational, any component $\mathfrak{X}_P$ of the broken manifold $\mathfrak{X}$ has a compactification

$$\overline{\mathfrak{X}}_P$$

that is an almost complex orbifold, and $\overline{\mathfrak{X}}_P \backslash \mathfrak{X}_P$ consists of divisors corresponding to the facets of $\tilde{P}$, see (3.12). The same construction defines a $T_P$ orbifold bundle

$$\overline{Z}_P \to \overline{X}_P \tag{3.57}$$

that is an extension of the $T_P$-bundle $Z_P \to X_P$, and is such that $\overline{Z}_P \times \mathfrak{t}_P^\vee$ is the complement of the vertical divisors (Definition 3.29 (b)) of $\overline{\mathfrak{X}}_P$. Note that the spaces $Z_P$ and $\overline{\mathfrak{X}}_P$ depend on the $\mathfrak{t}$-inner product.

**Definition 3.61.** (Cylindrical metric) A metric $g_P$ on $\mathfrak{X}_P \simeq Z_P \times \mathfrak{t}_P^\vee$ is *$P$-cylindrical* if $g_P$ is a product metric, that is, the product of the linear metric $|\cdot|_{\mathfrak{t}_P^\vee}$ on $\mathfrak{t}_P^\vee$ and a $T_P$-invariant metric $g_{Z_P}$ on $Z_P$ that satisfies

$$|\xi_{Z_P}|_{g_{Z_P}} = |\xi|_{\mathfrak{t}_P} \quad \xi \in \mathfrak{t}_P$$

where $\xi_{Z_P}$ is the generating vector field as in (3.1). On the multiply-stretched manifolds $X^\nu$, a metric $g_\nu$ is *cylindrical* if for any $P \in \mathcal{P}$, $g_\nu$ is $P$-cylindrical in the region $\Phi^{-1}(P^\blacksquare) \times \nu P^\vee$.

## 3.6 Translations: Relating neck-stretched and broken manifolds

To examine the convergence behavior of maps in neck-stretched manifolds to a limit map in the broken manifold, we need to embed $P$-cylindrical regions of the neck-stretched manifold into the $P$-cylindrical component of the broken manifold. The embedding maps, called *translations* are parametrized by elements in the scaled dual complex $\nu B^\vee$. Given an element $t \in \nu B^\vee$ lying in the dual polytope $\nu P^{\vee,\circ}$, the translation $\mathrm{e}^{-t}$ is a map from the $P$-cylindrical subset $X^\nu_{\tilde{P}} \subset X^\nu$ to the component of the broken manifold $\mathfrak{X}_P$.

We start by recalling some facts about neck-stretched and broken manifolds:

(a) For any polytope $P \in \mathcal{P}$ and any $\nu$, the $P$-cylindrical subset of $X^\nu$ is $X^\nu_{\tilde{P}}$ which has a projection

$$X^\nu_{\tilde{P}} \xrightarrow{\pi'_{P,\nu}} Z^\nu_P$$

where $Z^\nu_P$ is a $T_P$-bundle over $X^\nu_P$ and the fibers of $\pi'_{P,\nu}$ are subsets of $\nu P^\vee$, see (3.38). In particular, there is a map

$$\pi_{\nu P^\vee} : X^\nu_{\tilde{P}} \to \nu P^\vee, \tag{3.58}$$

which is $\pi_{\nu B^\vee} : X^\nu \to \nu B^\vee$ composed with the orthogonal projection $Q^\vee \to P^\vee$ (from Definition 3.18 (c)).

(b) The $P$-cylindrical component $\mathfrak{X}_P \subset \mathfrak{X}$ is a product

$$\mathfrak{X}_P = Z_P \times \mathfrak{t}^\vee_P$$

where $Z_P \to X_P$ is a $T_P$-bundle over $X_P$.

(c) There is a natural embedding $Z^\nu_P \to Z_P$, $X^\nu_{\tilde{P}} \to X_P$ from Definition 3.51.

**Definition 3.62.** ($P$-translation) For any $P \in \mathcal{P}$, $t \in \nu P^\vee$, denote by

$$\mathrm{e}^{-t}_P : X^\nu_{\tilde{P}} \to \mathfrak{X}_P. \tag{3.59}$$

the lift of the inclusion $Z^\nu_P \to Z_P$ that maps a level set $\{\pi_{\nu P^\vee} = c\} \subset X^\nu_{\tilde{P}}$ to $Z_P \times \{c - t\} \subset \mathfrak{X}_P$, where $\pi_{\nu P^\vee}$ is from (3.58), $c \in \nu P^\vee$, and so $c - t \in \mathfrak{t}^\vee_P$.

The notion of $P$-translation may be expressed via the following commutative diagram for any $t \in \nu P^\vee$:

$$\begin{array}{ccc} X^\nu_{\tilde{P}} & \xrightarrow{\mathrm{e}^{-t}_P} & \mathfrak{X}_P \\ \downarrow{\scriptstyle \pi_{\nu B^\vee}} & & \downarrow{\scriptstyle \pi_{\mathrm{Cone}_{P^\vee} B^\vee}} \\ B^\vee \supseteq \cup_{Q \subseteq P} \nu Q^\vee & \xrightarrow{\mathrm{e}^{-t}_P} & \mathrm{Cone}_{P^\vee} B^\vee \end{array} \tag{3.60}$$

Here, the left downward arrow $\pi_{\nu B^\vee}$ is from (3.31), and for any $Q \subseteq P$, the arrow $\mathfrak{e}_P^{-t} : \nu Q^\vee \to \mathrm{Cone}_{P^\vee} Q^\vee$ is a translation by $t \in \mathfrak{t}_P$.

For any $t \in \nu P^\vee$, the translation $\mathfrak{e}_Q^{-t}$ is the 'same' for all $Q \subseteq P$. Indeed,

$$\mathfrak{e}_P^{-t}|X^\nu_{\tilde{Q}} = \mathfrak{e}_Q^{-t}$$

since the restriction of $\mathfrak{e}_P^{-t}$ to $X^\nu_{\tilde{Q}} \subset X^\nu_{\tilde{P}}$ maps to the $Q$-cylindrical end $U_Q(\mathfrak{X}_P) \subset \mathfrak{X}_P$; the latter may be viewed as a subset of $\mathfrak{X}_Q$. This leads us to view the parameter $t$ in the translation $\mathfrak{e}^{-t}$ as an element in the dual complex $B^\vee$ as in the following definition:

**Definition 3.63.** (Generalized translation) For any $t \in \nu B^\vee$,

$$\mathfrak{e}^{-t} := \mathfrak{e}_P^{-t} \quad \text{if } t \in P^{\vee,\circ}.$$

For an element $t \in \nu P^{\vee,\circ}$, the inverse of the translation $\mathfrak{e}^{-t}$ is well-defined on a subset $\mathfrak{X}_{P,\nu} \subset \mathfrak{X}_P$:

$$\mathfrak{e}^t := (\mathfrak{e}^{-t})^{-1} : \mathfrak{X}_P \supset \mathfrak{X}_{P,\nu} \to X^\nu_{\tilde{P}}. \tag{3.61}$$

The sequence of subsets $\mathfrak{X}_{P,\nu}$ exhaust $\mathfrak{X}_P$ as $\nu \to \infty$.

*Example* 3.64. We illustrate translations in a multiple cut using an example with two non-intersecting single cuts. Consider the tropical Hamiltonian action $(X, \Phi, \mathcal{P})$ where the torus is $T = S^1$, and the polytopes in $\mathcal{P}$ are $P_0, P_1, P_2, P_{01}, P_{12} \subset \mathbb{R}$ shown in Figure 3.11. The dual complex $B^\vee$ is a subset of $\mathbb{R}$. Let the point $P_i^\vee$ be $g_i \in \mathbb{R}$ for $i = 0, 1, 2$.

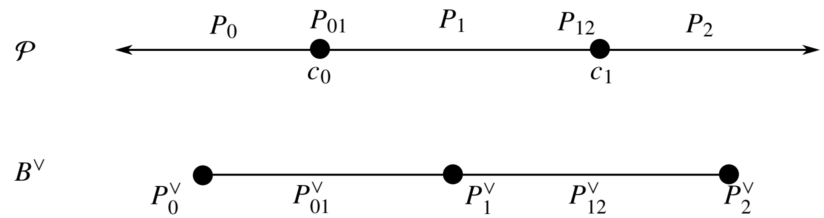



**Figure 3.11.** A polyhedral decomposition $\mathcal{P}$ and its dual complex $B^\vee$.

For $\nu > 0$, the neck-stretched manifold is

$$X^\nu := (X^\blacksquare_{P_0} \cup ([\nu g_0, \nu g_1] \times Z_0) \cup X^\blacksquare_{P_1} \cup ([\nu g_1, \nu g_2] \times Z_1) \cup X^\blacksquare_{P_2})/\sim,$$

where $Z_i := \Phi^{-1}(c_i)$, $X^\blacksquare_{P_i}$ is $\overline{X}_{P_i}$ minus a tubular neighbourhood of relative divisors, and $\sim$ identifies the copies of $Z_0$ and $Z_1$ on the boundaries. The translations are described as follows.

- Suppose $t \in \nu P_1^\vee$. Then $t = \nu g_1$ and $\mathfrak{e}^{-t}$ is the embedding

$$\begin{aligned}([\nu g_0, \nu g_1] \times Z_0) \cup_{Z_0} X^\blacksquare_{P_1} \cup_{Z_1} ([\nu g_1, \nu g_2] \times Z_1)& \\ \to X_{P_1} \simeq ((-\infty, 0] \times Z_0) \cup_{Z_0} X^\blacksquare_{P_1} \cup_{Z_1} ([0, \infty) \times Z_1),&\end{aligned}$$

that is identity on $X^{\blacksquare}_{P_1}$, and on the cylindrical ends it is a translation by $\nu g_1$, namely

$$([\nu g_0, \nu g_1] \times Z_0) \xrightarrow{((\cdot - \nu g_1), \mathrm{Id})} (-\infty, 0] \times Z_0, \quad ([\nu g_1, \nu g_2] \times Z_1) \xrightarrow{((\cdot - \nu g_1), \mathrm{Id})} ([0, \infty) \times Z_1).$$

The translation $\mathbb{e}^{-t}$ is defined similarly when $t \in \nu P_0^\vee$ or $\nu P_2^\vee$.

- Suppose $t \in P^\vee_{i(i+1)}$. Then $t \in [\nu g_i, \nu g_{i+1}]$ and

$$\mathbb{e}^{-t} : [\nu g_i, \nu g_{i+1}] \times Z_i \to \mathbb{R} \times Z_i$$

maps $\{c\} \times Z_i$ to $\{c - t\} \times Z_i$.

This ends the Example.

## 3.7 Existence of symplectic cylindrical structures

We prove that manifolds with tropical Hamiltonian actions possess symplectic cylindrical structures, which were used in the definition of neck-stretched almost complex manifolds. We recall from Definition 3.35 that in a tropical Hamiltonian manifold $(X, \mathcal{P}, \Phi)$, for a polytope $P \in \mathcal{P}$, a *symplectic cylindrical structure* in a neighborhood of $\Phi^{-1}(P) \subset X$ is a diffeomorphism

$$\phi_P : \Phi^{-1}(\tilde{P}) \to \Phi^{-1}(P) \times P^\vee,$$

where the polytope $\tilde{P}$ is a neighborhood of $P$, and $(\phi_P^{-1})^* \omega_X$ has a certain standard cylindrical form on $\Phi^{-1}(P) \times P^\vee$. The space $\Phi^{-1}(P) \times P^\vee$ is a fibration over $\Phi^{-1}(P)/T_P$ whose fibers are subsets of $T_{P,\mathbb{C}}$; via the identification by $\phi_P$, neck-stretched $T_P$-cylindrical almost complex structures were defined on $\Phi^{-1}(\tilde{P})$ in Section 3.4.

However, the result of this Section is not used in the proof of the main result of the book, namely the homotopy equivalence between unbroken and broken Fukaya algebras. For that purpose, neck-stretched manifolds are constructed by gluing a broken manifold equipped with a stabilizing divisor as in Section 5.4.

To construct the symplectic cylindrical structures, we may need to modify the tropical moment map $\Phi$ in the following sense: Given a tropical Hamiltonian manifold $(X, \mathcal{P}, \Phi)$, a tropical moment map $\Phi'$ is a *modification* of $\Phi$ if for every $P \in \mathcal{P}$, there is a sufficiently small $T_P$-invariant neighborhood $U_P \subset X$ of $\Phi^{-1}(P)$, and for which

$$\pi_{P^\vee} \circ \Phi = \pi_{P^\vee} \circ \Phi' : U_P \to \mathfrak{t}_P^\vee. \tag{3.62}$$

In other words, the $\mathfrak{t}_P^\vee$-component of $\Phi$ agrees with that of $\Phi'$, wherever it is an honest $T_P$-moment map.

**Proposition 3.65.** *(Existence of symplectic cylindrical structures) There exists a symplectic cylindrical structure (see Definition 3.35) for a tropical Hamiltonian manifold $(X, \mathcal{P}, \Phi)$, after replacing $\Phi$ by a modification of $\Phi$ (see previous paragraph).*

*Proof of Proposition* 3.65. We construct the maps $\{\phi_P\}_P$ and connection one-forms $\{\alpha_P\}_P$ by induction on the dimension of $P$. Consider a polytope $P \in \mathcal{P}$. We assume that $\phi_Q$ and $\alpha_Q$ are determined for every proper face $Q \subset P$ that is an element of $\mathcal{P}$. We first construct a $T_P$-equivariant diffeomorphism

$$\psi_P : \Phi^{-1}(\tilde{P}) \to \Phi^{-1}(P) \times P^\vee$$

that satisfies the properties expected of $\phi_P$ in $\Phi^{-1}(\tilde{P} \cap \tilde{Q})$ for all proper faces $Q \subset P$ as follows: The inner product on $\mathfrak{t}$ in (3.10) gives the following orthogonal projection

$$Q^\vee \to P^\vee \times \mathfrak{t}_Q^\vee / \mathfrak{t}_P^\vee, \tag{3.63}$$

and the image lies in $P^\vee \times \mathrm{NCone}_Q\, P$ if we restrict the map to an appropriate neighborhood in $Q^\vee$. Viewing $\mathrm{NCone}_Q\, P$ as a subset of $Q^\vee$ via the cylindrical structure map $\phi_Q$, we obtain an embedding

$$\Phi^{-1}(Q) \times \mathrm{NCone}_Q\, P \to \Phi^{-1}(P) \tag{3.64}$$

defined on a neighborhood of the origin in $\mathrm{NCone}_Q\, P$. The map $\psi_P$ is defined on $\Phi^{-1}(\tilde{Q})$ by (3.63) and (3.64). Extend $\psi_P$ to a $T_P$-equivariant diffeomorphism on $\Phi^{-1}(\tilde{P})$ that respects the $T_P$-moment map, that is, satisfies

$$\Pi_{P^\vee} \circ \psi_P = \pi_{P^\vee} \circ \Phi,$$

where $\Pi_{P^\vee}$ is from Definition 3.35 (b) and $\pi_{P^\vee}$ is from Definition 3.23. Next, we define a symplectic form on $\Phi^{-1}(P) \times P^\vee$ by choosing a connection one-form $\alpha_P$: For any proper face $Q \subset P$, on $\Phi^{-1}(\tilde{Q} \cap P)/T_P$, the connection one-form $\alpha_P$ is given by the consistency condition (3.23). We may choose any extension of $\alpha_P$ in the rest of $\Phi^{-1}(P)/T_P$. By our choice of connection one-form $\alpha_P$ on $\Phi^{-1}(\tilde{Q} \cap P)/T_P$, $\psi_P$ is a symplectomorphism on $\Phi^{-1}(\tilde{P} \cap \tilde{Q})$ for all $Q \subset P$. Next, we apply Lemma 3.66 to isotope $\psi_P$ to a sympectomorphism $\phi_P$, so that $\phi_P \equiv \psi_P$ on $\Phi^{-1}(\tilde{P} \cap \tilde{Q})$ for all $Q \subset P$. As a last step, we replace $\Phi$ by a tropical moment map $\Phi'$, such that $\Phi \equiv \Phi'$ in $\Phi^{-1}(\tilde{Q})$, $Q \subset P$, $\Phi'$ satisfies (3.62), and $\Phi' \circ \phi_P = \pi_P \circ \Phi'$. ∎

The proof of Proposition 3.65 uses the following result, which is a version of the coisotropic neighborhood theorem.

**Lemma 3.66.** *Let $Z \subset (X, \omega)$ be a submanifold with boundary. Suppose a neighborhood $U_Z \subset X$ of $Z$ has moment map $\Phi_P : U_Z \to \mathfrak{t}_P^\vee$ that is smooth and generates a free Hamiltonian $T_P$-action, for which $Z = \Phi_P^{-1}(0)$. Then, for any connection one-form $\alpha_P \in \Omega^1(Z, \mathfrak{t}_P)$ on the bundle $Z \to Z/T_P$, there is a neighborhood $\Delta \subset \mathfrak{t}_P^\vee$ of the origin, and a $T_P$-equivariant symplectomorphism*

$$\phi_P : (U_Z, \omega) \to (Z \times \Delta, \omega_{Z/T_P} + d\langle \alpha_P, i \rangle)$$

*on* $U_Z := \Phi_P^{-1}(\Delta)$*, where* $i : \Delta \to \mathfrak{t}_P^\vee$ *is the inclusion map. Furthermore, if* $\psi_P : U_Z \to Z \times \Delta$ *is a diffeomorphism that preserves the symplectic form on* $N := U(\partial Z) \times \Delta$*, where* $U(\partial Z) \subset Z$ *is a neighborhood of* $\partial Z$*, then* $\psi_P$ *can be isotoped to a symplectomorphism* $\phi_P$ *relative to* $N$*.*

The proof of the ordinary symplectic neighborhood ([56, Lemma 3.14]) can be used to prove the slightly stronger statement of Lemma 3.66.

## Chapter 4

# Broken disks

The goal of this chapter is to define *broken treed holomorphic disks*. These are analogues of what Parker [67] calls *exploded* holomorphic maps. These structures combine the features of treed holomorphic disks and tropical (or broken) maps.

- Treed holomorphic disks consist of surface components that are nodal holomorphic disks or spheres in a symplectic manifold whose boundary lies in a Lagrangian submanifold, with the additional feature that disk nodes may be replaced by tree segments mapping to the Lagrangian submanifold. Treed disks are generalizations of pearly trajectories in Biran-Cornea [9].
- On the other hand, a broken map is defined on the normalization of a nodal curve, and each of the domain components maps to a different component of the target broken manifold, and the maps satisfy a matching condition at the nodal lifts. The broken map is equipped with the additional data of a tropical graph in the dual complex associated with the degeneration of the symplectic manifold.

In a broken treed holomorphic disk, certain nodes in the domain are *tropical nodes* which means that the curve components incident on the node map to different target components; and the other nodes, called *internal nodes*, are sphere nodes as seen in Gromov-Witten theory or disk nodes with treed segments as in treed holomorphic disks. Since we assume that the Lagrangian submanifold is disjoint from the relative divisors of the broken manifold, all disk nodes are internal nodes.

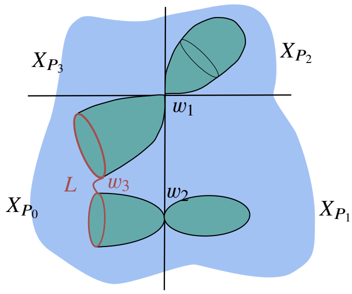



**Figure 4.1.** A broken treed holomorphic disk. The disk node $w_3$ has been replaced by a treed segment in the Lagrangian $L$, and $w_1$, $w_2$ are tropical nodes.

## 4.1 Treed disks

*Treed disks* are the domains of broken treed holomorphic disks, which are the main objects defined in this Chapter. Treed disks are analogues of pearly trajectories of Biran-Cornea [9], Cornea-Lalonde [27] and Seidel [77], and they are combinations of trees, nodal disks and nodal spheres.

**Definition 4.1.** (Treed disks)

(a) (Nodal disks) A nodal disk $S$ is a union

$$S = \left( \bigcup_{\alpha=1,\dots,d(\circ)} S_{\alpha,\circ} \right) \cup \left( \bigcup_{\beta=1,\dots,d(\bullet)} S_{\beta,\bullet} \right) / \sim \tag{4.1}$$

where the index $\alpha$, ranging over $\{1, \dots, d(\circ)\}$, indexes disk components, that is, for each $\alpha$, $S_\alpha$ is biholomorphic to a unit disk $\mathbb{D}^2 \subset \mathbb{C}$, and the index $\beta$, ranging over $\{1, \dots, d(\bullet)\}$, indexes sphere components, that is, for each $\beta$, $S_\beta$ is biholomorphic to the projective line $\mathbb{P}^1$, glued together by an equivalence relation $\sim$. The equivalence relation $\sim$ is generated by pairs of interior or boundary *nodal points*

$$w_e = (w_e^+, w_e^-) \in (\cup_\alpha \partial S_\alpha)^2 \cup (\cup_\alpha (S_\alpha^\circ) \cup \cup_\beta S_\beta)^2 \tag{4.2}$$

of pairs of boundary points of disks, interior points of disks or spheres, with the property that the boundary $\partial S$ is connected and there are no cycles of components $S_{\alpha_1}, \dots, S_{\alpha_k} = S_{\alpha_1}$ connected by nodes (so that in particular, each node connects two different disk or sphere components of $S$). A *marking* of a nodal disk is a collection of boundary and interior points

$$\underline{z}_\circ = (z_{\circ,i} \in \partial S, i = 0, \dots, d(\circ)), \quad \underline{z}_\bullet = (z_{\bullet,i} \in S \backslash \partial S, i = 1, \dots, d(\bullet)) \tag{4.3}$$

distinct from the nodes. A marked nodal disk is *stable* if it admits no automorphisms, or equivalently, if for each disk component $S_{\circ,i}$ the sum of the number of special (nodal or marked) boundary points and twice the number of interior special points is at least three, and each sphere component $S_{\bullet,i}$ has at least three special points.

(b) (Combinatorial type of a nodal disk) The *combinatorial type* of a nodal disk $S$ is the tree $\Gamma$ whose vertices $\mathrm{Vert}(\Gamma)$ correspond to disk or sphere components of $S$, and whose edges $\mathrm{Edge}(\Gamma)$ correspond to markings or nodes, together with

(i) (Sphere and disk vertices) a partition of the vertex set

$$\mathrm{Vert}(\Gamma) = \mathrm{Vert}_\bullet(\Gamma) \cup \mathrm{Vert}_\circ(\Gamma)$$

into disk vertices $\mathrm{Vert}_\circ(\Gamma)$ and sphere vertices $\mathrm{Vert}_\bullet(\Gamma)$;

(ii) (Interior and boundary edges) a partition of the edge set

$$\mathrm{Edge}(\Gamma) = \mathrm{Edge}_{\bullet}(\Gamma) \cup \mathrm{Edge}_{\circ}(\Gamma)$$

into edges $\mathrm{Edge}_{\circ}(\Gamma)$ of boundary type and edges $\mathrm{Edge}_{\bullet}(\Gamma)$ of interior type;

(iii) (Leaf and non-leaf edges) a partition of the edge set $\mathrm{Edge}(\Gamma)$

$$\mathrm{Edge}(\Gamma) = \mathrm{Edge}_{-}(\Gamma) \cup \mathrm{Edge}_{\rightarrow}(\Gamma)$$

into leaf edges $\mathrm{Edge}_{\rightarrow}(\Gamma)$ which correspond to markings, and non-leaf edges $\mathrm{Edge}_{-}(\Gamma)$ which correspond to nodes (and so, leaf edges are incident on a single vertex and non-leaf edges are each incident on two vertices);

(iv) (Ordering of leaves) an ordering of the boundary leaf edges $\mathrm{Edge}_{\circ,\rightarrow}(\Gamma)$ and the interior leaf edges $\mathrm{Edge}_{\bullet,\rightarrow}(\Gamma)$ given by the corresponding ordering of markings (see (4.3));

(v) (Ribbon structure) a *ribbon structure* on $\Gamma$, which is a cyclic ordering $<_v$ on the set of boundary edges (both leaf and non-leaf)

$$\mathrm{Edge}_{\circ}^{v}(\Gamma) := \{e \in \mathrm{Edge}_{\circ}(\Gamma) : v \in e\}$$

incident on each boundary vertex $v \in \mathrm{Vert}_{\circ}(\Gamma)$ such that the induced cyclic ordering on the set $\mathrm{Edge}_{\circ,\rightarrow}(\Gamma)$ of boundary leaves corresponds to the cyclic ordering $z_{\circ,0}, \ldots, z_{\circ,d(\circ)}, z_{\circ,0}$ of boundary markings.

(vi) (Root edge and edge orientations) The edge $e_0 \in \mathrm{Edge}_{\circ}(\Gamma)$ corresponding to the first boundary marking $z_0$ is an outgoing edge and is called the *root edge*, all the other boundary markings are incoming edges, and all edges corresponding to nodes are oriented to point towards the root.

(c) (Treed segments) A *treed segment* consists of a collection of closed intervals $I_1, \ldots, I_k$,

$$I_j \cong [0, \ell(I_j)], \quad (-\infty, 0], \quad \text{or} \quad I_j \cong [a_j, \infty) \quad \text{or} \quad (-\infty, \infty)$$

glued along infinite end-points, to produce a topological space isomorphic to some subset of $\mathbb{R}$. For example,

$$[0, \infty) \cup_{\infty} (-\infty, \infty) \cup_{\infty} (-\infty, 0] \tag{4.4}$$

is a treed segment with three components and finite end-points. Each treed segment $T$ has a length $\ell(T) \in [0, \infty]$ and a number of breakings $b(T) \in \mathbb{Z}_{\geq 0}$.

A treed segment with $b(T) > 0$ is called a *broken segment* and has $\ell(T) = \infty$. We also consider treed segments with one infinite end such as

$$[0, \infty) \cup_\infty (-\infty, \infty)$$

or both infinite ends such as $(-\infty, \infty) \cup_\infty (-\infty, \infty)$.

(d) (Treed disk) A *treed nodal disk* $C = S \cup T$ is

(i) either obtained from a nodal disk $S_0$

- by assigning a length $\ell(e) \in [0, \infty]$ to each boundary node $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$, and replacing any boundary node $w_e$, $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ with $\ell(e) > 0$ by a treed segment $T_e$ with finite end-points, and the treed segment $T_e$ has length $\ell(e)$ if $\ell(e) < \infty$ or it is a broken segment as in (4.4) if $\ell(e) = \infty$; and
- each boundary marking $w_e$, $e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ is replaced by a (possibly broken) treed segment, one of whose end-points is infinite, and thus $\ell(e) = \infty$ for all boundary markings; or

(ii) $C$ has no surface component and consists only of a treed segment with two infinite ends. The $-\infty$ resp. $\infty$ end of the segment is regarded as the input resp. output, and therefore there is one output and one input, so $d(\circ) = 1$.

A treed segment $T_e$, $e \in \mathrm{Edge}_\circ(\Gamma)$, containing a breaking is called a *broken edge*. A non-leaf edge $e \in \mathrm{Edge}_-(\Gamma)$ is broken if and only if $\ell(e) = \infty$.

(e) (Isomorphism of treed disks) An isomorphism of treed disks is a homeomorphism $\phi : C \to C'$ that is a biholomorphism on each sphere or disk component, length-preserving on edges, and preserves the labelling of leaves.

(f) (Combinatorial type of a treed disk) The combinatorial type of a treed disk consists of the combinatorial type $\Gamma$ of the underlying nodal disk which includes the vertex and edge partitions, ordering of markings, ribbon structure and edge orientations; and in addition a partition

$$\mathrm{Edge}_{\circ,-}(\Gamma) = \mathrm{Edge}^0_{\circ,-}(\Gamma) \cup \mathrm{Edge}^{(0,\infty)}_{\circ,-} \cup \mathrm{Edge}^\infty_{\circ,-}(\Gamma)$$

of boundary edges corresponding to boundary nodes with zero, finite, and infinite length edges. Note that for a boundary edge $e$ with edge length $\ell(e) \in (0, \infty)$, the length of the treed segment $T_e$ is not part of the combinatorial type.

(g) (Stable treed disk) A treed nodal disk $C$ is *stable* if the underlying disk $S$ is stable, the treed segment at any node $T_e, e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ has at most one breaking, and treed segments at markings $T_e, e \in \mathrm{Edge}_{\to,\circ}(\Gamma)$ are unbroken. A treed disk with no surface component (that is, consisting of a single possibly broken sequence of segments) is not stable.

(h) (Disconnected types) Let $\Gamma$ be a disjoint union of treed disk types $\Gamma_1, \ldots, \Gamma_k$. A treed curve of type $\Gamma$ is a collection $u_1, \ldots, u_k$ of treed disks of types $\Gamma_1, \ldots, \Gamma_k$.

For integers $d(\bullet), d(\circ) \geq 0$, denote by $\mathcal{M}_{d(\bullet),d(\circ)}$ the (possibly empty) moduli space of isomorphism classes of stable treed disks with $d(\circ)$ incoming boundary markings, one outgoing boundary marking and $d(\bullet)$ interior markings. For each combinatorial type $\Gamma$, denote by $\mathcal{M}_\Gamma \subset \mathcal{M}_{d(\bullet),d(\circ)}$ the set of isomorphism classes of stable treed disks of type $\Gamma$. The moduli space $\mathcal{M}_{d(\bullet),d(\circ)}$ then decomposes as

$$\mathcal{M}_{d(\bullet),d(\circ)} = \bigcup_{\Gamma} \mathcal{M}_\Gamma.$$

For any type $\Gamma$, $\mathcal{M}_\Gamma$ has a compactification $\overline{\mathcal{M}}_\Gamma$ where the boundary is

$$\overline{\mathcal{M}}_\Gamma \backslash \mathcal{M}_\Gamma = \bigcup_{\Gamma'} \mathcal{M}_{\Gamma'},$$

and $\Gamma'$ ranges over all strata such that $\Gamma$ is obtained from $\Gamma'$ by either collapsing an interior edge or a zero length boundary edge, or making a zero length boundary edge non-zero, or making an infinite boundary edge finite, or performing a combination of these operations.

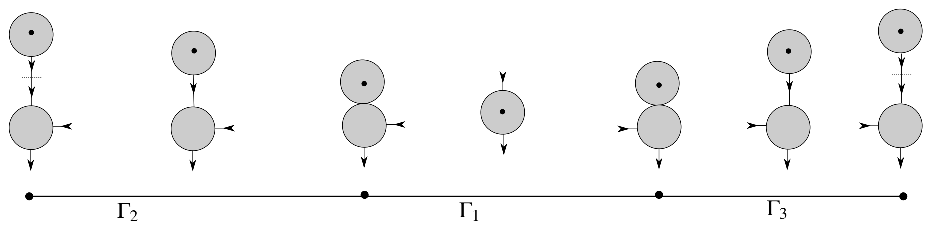



**Figure 4.2.** The moduli space $\mathcal{M}_{1,1}$ has three top-dimensional strata corresponding to the types $\Gamma_1, \Gamma_2, \Gamma_3$.

The top-dimensional cells in $\mathcal{M}_{d(\bullet),d(\circ)}$ correspond to strata where all boundary edges $e$ have finite non-zero length, that is, $\ell(e) \in (0, \infty)$. The dimension of each of the top-dimensional cells is $d(\circ) + 2d(\bullet) - 2$. For the stratum with no finite edges, so containing a single disk, this follows immediately from the fact that each boundary leaf edge resp. interior leaf edge contributes 1 resp. 2 to the dimension, and the automorphism group of the disk is $PSL(2, \mathbb{R})$ which has dimension 3. The dimensions of the other top-dimensional cells can be computed by first computing the corresponding stratum of the moduli spaces of disks without trees, and then adding one for each boundary edge with finite non-zero length. The moduli space $\mathcal{M}_{d(\bullet),d(\circ)}$ is a manifold with corners where the codimension $k$ corner stratum consists of curves containing $k$ broken treed edges. We do not give a proof, since for our purposes it is enough to view $\mathcal{M}_{d(\bullet),d(\circ)}$ as a cell-complex, with a manifold structure on each of the strata $\mathcal{M}_\Gamma$.

**Definition 4.2.** (Orientation of moduli of treed disks) We fix an orientation for moduli spaces of treed disks of type $\Gamma$, where $\Gamma$ does not contain boundary edges $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ of length 0 or $\infty$.

(a) (Single disk) Let $\Gamma$ be a treed disk type with a single disk component with $\geq 2$ incoming boundary leaves and no other surface component. The moduli space $\mathcal{M}_\Gamma$ is the quotient

$$\mathrm{Conf}_{d(\bullet),d(\circ)}/PSL(2,\mathbb{R}),$$

where

$$\mathrm{Conf}_{d(\bullet),d(\circ)} \subset \{\underline{z} = (\underline{z}^\circ, \underline{z}^\bullet) \in (\partial D)^{d(\circ)+1} \times (D^\circ)^{d(\bullet)}\}$$

is the subset of configurations where $\underline{z}$ consists of distinct points and the boundary markings are ordered counter-clockwise. Fixing $z_0^\circ = -1$, $z_1^\circ = 1$, $z_2^\circ = i$ gives a global slice $\Sigma$ of the $PSL(2,\mathbb{R})$-action. Furthermore, $\mathcal{M}_\Gamma$ is oriented via its identification to the slice $\Sigma$ which is an open subset of $\mathbb{R}^{d(\circ)-2} \times \mathbb{C}^{d(\bullet)}$. In the case that $\Gamma$ has less than 2 incoming boundary leaves, we fix the global slice $\Sigma$ by setting $z_0^\circ = -1$, $z_0^\bullet = 0$.

(b) (Multiple disks) Let $\Gamma$ be a treed disk type with no sphere components and all of whose boundary edges have finite non-zero length. The orientation on $\mathcal{M}_\Gamma$ is chosen by inducting on the number of boundary edges $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$. Suppose the treed disk types $\Gamma, \Gamma_0, \Gamma'$ are related by the following morphisms:

$$\Gamma \xleftarrow{\text{Make the edge length } \ell(e) \text{ non-zero}} \Gamma_0 \xrightarrow{\text{Collapse } e} \Gamma'.$$

Here, we point out that the edge length $\ell(e)$ in $\Gamma_0$ is zero. The moduli space $\mathcal{M}_{\Gamma_0}$ is a codimension one boundary stratum of both $\overline{\mathcal{M}}_\Gamma$ and $\overline{\mathcal{M}}_{\Gamma'}$ via inclusions

$$i_{\Gamma,\Gamma_0} : \mathcal{M}_{\Gamma_0} \to \overline{\mathcal{M}}_\Gamma, \quad i_{\Gamma',\Gamma_0} : \mathcal{M}_{\Gamma_0} \to \overline{\mathcal{M}}_{\Gamma'}.$$

Assuming an orientation on $\mathcal{M}'_\Gamma$, the stratum $\mathcal{M}_\Gamma$ is oriented so that the orientations on $\mathcal{M}_{\Gamma_0}$ induced by $i_{\Gamma,\Gamma_0}$, $i_{\Gamma',\Gamma_0}$ are the opposite of each other.

(c) (Disks and spheres) Let $\Gamma$ be a treed disk type, and let $\Gamma'$ be the type obtained by collapsing interior edges $e \in \mathrm{Edge}_{\bullet,-}(\Gamma)$. Then the orientation on $\mathcal{M}_{\Gamma'}$ and the standard orientation on the moduli space of spheres induces an orientation on $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{\Gamma'}$ which is equal to the product orientation $\mathcal{M}_{\Gamma_\circ} \times$

$\prod_{v \in \mathrm{Vert}_\bullet(\Gamma)} \mathcal{M}_{\Gamma_v}$. Here, $\Gamma_\circ \subset \Gamma$ is the subgraph consisting of all the disk vertices and $\Gamma_v$ is a graph with a single vertex $\{v\}$ and markings corresponding to all edges $e \in \mathrm{Edge}_\bullet(\Gamma)$ incident on $v$.

*Remark* 4.3. Let $\Gamma$ be a treed disk type containing a single edge $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ of infinite length, and all of whose other boundary edges $e' \in \mathrm{Edge}_{\circ,-}(\Gamma)$ have $\ell(e') \in (0,\infty)$. Let $\Gamma'$ be a treed disk type obtained from $\Gamma$ by making the edge length of $e$ finite. Then,

$$i_{\Gamma,\Gamma'} : \mathcal{M}_\Gamma \hookrightarrow \overline{\mathcal{M}}_{\Gamma'}$$

is a boundary stratum of codimension one. Suppose $\Gamma_1, \Gamma_2$ are treed disk types obtained by cutting the edge $e$ in $\Gamma$ (see Definition 6.3), and suppose the root of $\Gamma$ is contained in $\Gamma_1$. Then the boundary orientation on $\mathcal{M}_\Gamma$ induced by $i_{\Gamma,\Gamma'}$ differs from the product orientation $\mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2}$ by

$$\epsilon(\Gamma_1, \Gamma_2) := (-1)^\circ \tag{4.5}$$

where $\circ$ depends only on the combinatorial types $\Gamma_1, \Gamma_2$, see [76, (12.22)].

The moduli spaces admit universal curves, which admit partitions into one and two-dimensional parts. For any combinatorial type $\Gamma$, let $\mathcal{U}_\Gamma$ denote the universal treed disk consisting of isomorphism classes of pairs $(C, z)$ where $C$ is a treed disk of type $\Gamma$ and $z$ is a point in $C$, possibly on a disk component, sphere component, or one of the edges of the tree. The map

$$\mathcal{U}_\Gamma \to \mathcal{M}_\Gamma, \quad [C, z] \mapsto [C] \tag{4.6}$$

is the universal projection, whose fiber over $[C]$ is a copy of $C$. The union over types $\Gamma'$ with $\mathcal{M}_{\Gamma'} \subset \overline{\mathcal{M}}_\Gamma$ is denoted $\overline{\mathcal{U}}_\Gamma$. Denote by

$$\mathcal{S}_\Gamma, \quad \text{resp.} \quad \mathcal{T}_\Gamma$$

the locus of points $[C, z] \in \mathcal{U}_\Gamma$ where $z$ lies on a disk or sphere component resp. an edge of $C$. Hence $\mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma$, and $\mathcal{S}_\Gamma \cap \mathcal{T}_\Gamma$ is the set of points on the boundary of the disks meeting the edges of the tree. Denote by

$$\overline{\mathcal{S}}_\Gamma, \quad \text{resp.} \quad \overline{\mathcal{T}}_\Gamma$$

the compactification of $\mathcal{S}_\Gamma$ resp. $\mathcal{T}_\Gamma$ in $\overline{\mathcal{U}}_\Gamma$.

On each stratum, the universal curve admits local trivializations. For a stratum $\Gamma$, the spaces

$$\mathcal{S}_\Gamma \to \mathcal{M}_\Gamma, \quad \text{resp.} \quad \mathcal{T}_\Gamma \to \mathcal{M}_\Gamma$$

are smooth fibrations with 2 resp. 1-dimensional fibers and markings are sections

$$\underline{z} = (z_{i,\circ}, z_{j,\bullet}) : \mathcal{M}_\Gamma \to \mathcal{S}_\Gamma, \quad 0 \le i \le d(\circ), 1 \le j \le d(\bullet).$$

We view breaking points on broken treed segments as sections $\mathcal{M}_\Gamma \to \mathcal{T}_\Gamma$. On a type $\Gamma$, the treed segment $T_e$ corresponding to any edge $e \in \mathrm{Edge}_\circ(\Gamma)$ has a fixed number of breakings, each of which gives rise to a section. The union $\mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma$ has local trivializations: For a curve $[C] \in \mathcal{M}_\Gamma$, $C = S \cup T$, and a small enough neighborhood $U_C \subset \mathcal{M}_\Gamma$ of $[C]$, there is a homeomorphism

$$\mathcal{U}_\Gamma | U_C \simeq U_C \times C. \tag{4.7}$$

When restricted to each stratum of $\mathcal{S}_\Gamma|U_C$ resp. $\mathcal{T}_\Gamma|U_C$, the homeomorphism above is a diffeomorphism onto its image. The markings

$$z_{i,\circ}, z_{j,\bullet}|U_C : U_C \to S$$

are constant functions, whose values we denote by $\underline{z_{U_C}} = (z_{i,\circ,U(C)}, z_{j,\bullet,U(C)})$, where $z_{i,\circ,U(C)}, z_{j,\bullet,U(C)} \in S$. The fibers of $\mathcal{S}_\Gamma \to U_C$ possess a conformal structure, and the trivialization in (4.7) induces a map

$$j : U_C \to \mathcal{J}(S) \tag{4.8}$$

to the space $\mathcal{J}(S)$ of conformal structures on $S$ such that for $[C_1] \in U_C$, $(C, j([C_1], \underline{z_{U_C}})$ is biholomorphic to $(\mathcal{S}_\Gamma, \underline{z})|_{[C_1]}$.

*Remark* 4.4. The structure of a "smooth fibration with sections" on $\mathcal{S}_\Gamma \to \mathcal{M}_\Gamma$ and $\mathcal{T}_\Gamma \to \mathcal{M}_\Gamma$ breaks down on the extension to the compactification $\overline{\mathcal{M}}_\Gamma$. For example, a finite treed segment $T_e$ in the fibers of $\mathcal{T}_\Gamma$ may be transformed to a segment of "zero length" in the compactification $\overline{\mathcal{T}}_\Gamma$; or two disconnected components in the fibers of $\mathcal{S}_\Gamma$ may connect at a disk node in the compactification $\overline{\mathcal{S}}_\Gamma$.

## 4.2 Treed pseudoholomorphic disks

Treed pseudoholomorphic disks are maps from treed disks to a symplectic manifold equipped with a Lagrangian submanifold. The symplectic manifold has a tamed almost complex structure and the Lagrangian submanifold has a Morse function on it. On the two-dimensional part of the treed disk the map is pseudoholomorphic, and the boundaries of the disks map to the Lagrangian submanifold. On the one-dimensional part of the domain, the map is a gradient flow line of the Morse function on the Lagrangian submanifold, whose length is the same as the length of the tree edge. Later in this chapter, we will adapt the definition of treed holomorphic disks in a symplectic manifold to define broken treed holomorphic disks in a broken manifold. Even later in the text, the almost complex structure, the Morse function, and the metric on the Lagrangian will be given domain-dependent perturbations in order to regularize the moduli spaces of treed holomorphic (broken) disks.

We introduce the necessary notation for defining treed holomorphic disks. Let $(X, \omega_X)$ be a symplectic manifold and $L \subset X$ be a Lagrangian submanifold. Let $J$ be an $\omega_X$-tame almost complex structure. Let $G_L$ be a Riemannian metric on $L$ and let $F \in C^\infty(L, \mathbb{R})$ be a Morse function such that the pair $(F, G_L)$ is Morse-Smale. The *gradient vector field* is defined by the condition

$$\operatorname{grad}_F \in \operatorname{Vect}(L), \quad \mathrm{d}F(\cdot) = G_L(\operatorname{grad}_F, \cdot).$$

**Definition 4.5.** A *treed holomorphic disk* with boundary in $L \subset X$ consists of a treed disk $C = S \cup T$ and a continuous map

$$u : C \to X$$

satisfying the following conditions:

(a) The tree components $T$ and the boundary $\partial S$ of the surface components are mapped to the Lagrangian submanifold $L$; that is, $u(T \cup \partial S) \subset L$;

(b) the map $u|_S$ is a pseudoholomorphic map on the surface part: $Jd(u|_S) = d(u|_S) \circ j$; and

(c) the map $u|_T$ is a union of gradient trajectories of $F$:

$$\frac{d}{ds} u|_T = -\operatorname{grad}_F(u|_T)$$

where $s$ is a unit velocity coordinate on $T$.

A treed holomorphic disk $u : C = S \cup T \to X$ is *stable* if it has finitely many automorphisms, $\#\operatorname{Aut}(u) < \infty$, or equivalently

(a) each surface component $S_v \subset S$ on which the map $u$ is constant is stable as a component of a nodal disk $S$ (see Definition 4.1); and

(b) each treed segment $T_e$ on which the map $u$ is constant has at most one infinite end, that is, one of the ends of $T_e$ is an attaching point to a sphere or disk $S_v \subset S$.

Note that the case $C \cong \mathbb{R}$ equipped with a non-constant Morse trajectory $u : C \to L$ is allowed under this stability condition and corresponds to the case of a single incoming edge, that is, $d(\circ) = 1$. The area of a sphere or disk $u : S \to X$ is the symplectic area

$$\operatorname{Area}(u) = \int_S (u|_S)^* \omega_X.$$

## 4.3 Multiply-broken disks

A broken map is a map from a nodal curve to a multiply cut manifold that is discontinuous at tropical nodes. Different components of the nodal curve map to different pieces of the multiply cut manifold, and the lifts of the tropical nodal points satisfy an edge-matching condition. The nodal points carry an additional data of intersection multiplicity with relative divisors. This data is packaged into a *tropical structure*, which is part of the combinatorial type of the broken map.

**Definition 4.6.** (Tropical graph) Let $B^\vee$ be the dual complex for a set of polytopes $\mathcal{P} = \{P \subset \mathfrak{t}^\vee\}$ as in Definition 3.18. A *pre-tropical graph* is a triple $(\Gamma, P, \mathcal{T})$ consisting of

(a) a graph $\Gamma$ with vertex set $\mathrm{Vert}(\Gamma)$ and edge set $\mathrm{Edge}(\Gamma) \subset \mathrm{Vert}(\Gamma) \times \mathrm{Vert}(\Gamma)$,

(b) polytope assignments

$$P : \mathrm{Vert}(\Gamma) \cup \mathrm{Edge}(\Gamma) \to \mathcal{P} \tag{4.9}$$

such that for any edge $e = (v_+, v_-)$, $P(e) = P(v_+) \cap P(v_-)$, and

(c) edge *slopes*

$$\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}} \backslash \{0\}, \quad \forall \ e \in \mathrm{Edge}(\Gamma).$$

A *tropical graph* is a pre-tropical graph $(\Gamma, P, \mathcal{T})$ that is *realizable* in sense that there is a collection of *tropical vertex positions*

$$\mathcal{T}(v) \in P(v)^{\vee,\circ} \subset B^\vee, \quad v \in \mathrm{Vert}(\Gamma) \tag{4.10}$$

satisfying the following *slope condition*

$$\text{(Slope condition)} \quad \mathcal{T}(v_+) - \mathcal{T}(v_-) \in \mathbb{R}_{>0}\mathcal{T}(e). \tag{4.11}$$

Here, we view $\mathcal{T}(e)$ as an element in $\mathfrak{t}^\vee_{P(e)}$ via the identification $\mathfrak{t} \simeq \mathfrak{t}^\vee$ in (3.10). The set of tropical vertex position maps on a tropical graph $\Gamma$ is denoted

$$\mathcal{W}(\Gamma) = \{(\mathcal{T}(v))_{v \in \mathrm{Vert}(\Gamma)} : \mathcal{T} \text{ is a tropical vertex position map on } \Gamma\}. \tag{4.12}$$

A tropical graph $\Gamma$ is *rigid* if $\mathcal{W}(\Gamma)$ has exactly one element.

*Remark* 4.7. A deformation of vertex positions for a tropical graph $\Gamma$ is shown in Figure 4.3 in the dual complex $B^\vee$ corresponding to two orthogonal cuts (see Figure 1.2 and Figure 1.6). By the definition of tropical graphs, moving the vertices $v_3$ and $v_4$ to the dotted positions produces a one-parameter space of tropical vertex positions. Thus, the space of vertex positions is

$$\mathcal{W}(\Gamma) \simeq (0, 1).$$

This remark is continued in Remark 4.22.

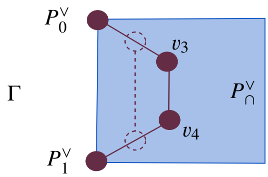



**Figure 4.3.** Moving the vertices $v_3$ and $v_4$ to the dotted positions gives a different vertex position map for the same tropical graph $\Gamma$.

*Remark* 4.8. (The set of vertex positions is the interior of a polytope) For a tropical graph $\Gamma$, the closure $\overline{\mathcal{W}}(\Gamma)$ of the space $\mathcal{W}(\Gamma)$ of vertex positions is given by

$$\overline{\mathcal{W}}(\Gamma) := \{(\mathcal{T}(v) \in P(v)^\vee)_{v\in \mathrm{Vert}(\Gamma)} : \\ \mathcal{T}(v_+) - \mathcal{T}(v_-) \in \mathbb{R}_{\geq 0}\mathcal{T}(e), \ \ \forall e = (v_+, v_-) \in \mathrm{Edge}(\Gamma)\}.$$

Thus $\overline{\mathcal{W}}(\Gamma)$ is obtained by weakening the conditions (4.10) and (4.11) in the definition of $\mathcal{W}(\Gamma)$. The closure $\overline{\mathcal{W}}(\Gamma)$ is a compact polytope in the affine space (of the same dimension)

$$A_\Gamma := \{(\mathcal{T}(v) \in \mathfrak{t}^\vee_{P(v)})_{v\in \mathrm{Vert}(\Gamma)} : \\ \mathcal{T}(v_+) - \mathcal{T}(v_-) \in \mathbb{R}\mathcal{T}(e), \ \ \forall e = (v_+, v_-) \in \mathrm{Edge}(\Gamma)\}.$$

The facets of the polytope $\overline{\mathcal{W}}(\Gamma)$ are given by the conditions

$$\mathcal{T}(v) \in P(v)^\vee, \quad \mathcal{T}(v_+) - \mathcal{T}(v_-) \in \mathbb{R}_{\geq 0}\mathcal{T}(e),$$

for each vertex $v$ and each edge $e = (v_+, v_-)$ of $\Gamma$.

**Definition 4.9.** (Tropical structure on a treed disk type) Let $\Gamma$ be the combinatorial type of a treed disk (see Definition 4.1(f)). Let $\mathfrak{X}$ be a broken manifold with an underlying polyhedral decomposition $\mathcal{P}$, and a Lagrangian submanifold $L$ contained in the component $X_{P_0} \subset \mathfrak{X}$, $P_0 \in \mathcal{P}$.

A *tropical structure* on $\Gamma$ consists of a tropical graph $(\Gamma_{\mathrm{tr}}, P, \mathcal{T})$ and an edge collapse morphism

$$\mathrm{tr} : \Gamma \to \Gamma_{\mathrm{tr}} \tag{4.13}$$

called the *tropicalization*. Here $P : \mathrm{Vert}(\Gamma_{\mathrm{tr}}) \cup \mathrm{Edge}(\Gamma_{\mathrm{tr}}) \to \mathcal{P}$ is the polytope assignment on the tropical graph (see (4.9)), and $\mathcal{T} = (\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}})_{e\in \mathrm{Edge}(\Gamma_{\mathrm{tr}})}$ is the collection of edge slopes. The edges collapsed by tr are called *internal edges* and uncollapsed edges are called *tropical edges*. The subset of internal resp. tropical edges of $\Gamma$ is denoted by

$$\mathrm{Edge}_{\mathrm{int}}(\Gamma) \quad \text{resp.} \quad \mathrm{Edge}_{\mathrm{trop}}(\Gamma) \subset \mathrm{Edge}_-(\Gamma).$$

All boundary edges are collapsed by assumption. Therefore,

$$\mathrm{Edge}_{-,\circ}(\Gamma) \subset \mathrm{Edge}_{\mathrm{int}}(\Gamma), \quad \mathrm{Edge}_{\mathrm{trop}}(\Gamma) \subset \mathrm{Edge}_{-,\bullet}(\Gamma).$$

The map tr is often suppressed in the notation. The polygon assignment $P \circ \mathrm{tr}$ and edge slope $\mathcal{T} \circ \mathrm{tr}$ maps on $\Gamma$ are often denoted by $P, \mathcal{T}$.

*Remark* 4.10. In a treed disk type $\Gamma$ equipped with a tropical structure, since the tropicalization map collapses all the boundary edges $e \in \mathrm{Edge}_{\circ}(\Gamma)$, all the boundary vertices $v \in \mathrm{Vert}_{\circ}(\Gamma)$ are mapped to a single vertex in $\Gamma_{\mathrm{tr}}$.

We recall some notation required to define broken maps. The domain of a broken map is a treed disk $C = T \cup S$ (Definition 4.1) where $S$ is the surface part and $T$ is the tree part. For a treed disk $C$ of type $\Gamma$ equipped with a tropical structure, for every vertex $v \in \mathrm{Vert}(\Gamma)$ we denote by

$$S_v^\circ = S_v \backslash \{w_e : e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)\} \tag{4.14}$$

the surface part with cylindrical ends obtained by deleting nodal points corresponding to tropical edges. The target space of a broken map is a broken manifold $\mathfrak{X}$. We recall from (3.42) that a broken manifold with an underlying polyhedral decomposition $\mathcal{P}$ is a disjoint union

$$\mathfrak{X} = \bigsqcup_{P \in \mathcal{P}} \mathfrak{X}_P,$$

and there is a top-dimensional polytope $P_0 \in \mathcal{P}$ such that $X_{P_0}$ contains a Lagrangian submanifold $L$. Note that $X_{P_0} = \mathfrak{X}_{P_0}$ since the torus $T_{P_0}$ is trivial.

**Definition 4.11.** (A surface component of a broken map) Let $\Gamma$ be the combinatorial type of a treed disk equipped with a *tropical structure* and let $v$ be a vertex of $\Gamma$. Define the domain $S_v$ to be a disk if $v \in \mathrm{Vert}_{\circ}(\Gamma)$ and a sphere if $v \in \mathrm{Vert}_{\bullet}(\Gamma)$. The *surface component of a broken map* corresponding to $v$ consists of a punctured domain curve $S_v^\circ := S_v \backslash \{w_e : v \in e, e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)\}$ and a holomorphic map

$$u_v : S_v^\circ \to X_{P(v)}$$

with the following behavior at punctures: For any tropical edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, $e \ni v$, and any holomorphic coordinate in a punctured neighborhood $U_{w_e}$ of $w_e$,

$$z_e : (U_{w_e} \backslash \{w_e\}) \to (\mathbb{C}, 0),$$

the map $z_e^{-\mathcal{T}(e)} u_v$ has a removable singularity at $w_e$. The quantity

$$\mathrm{ev}_{w_e}^{\mathcal{T}(e)} u_v := \lim_{z_e \to 0} z_e^{-\mathcal{T}(e)} u_v(z_e) \in \mathfrak{X}_{P(e)}. \tag{4.15}$$

is called the *tropical evaluation* at the puncture $w_e$ for the choice of holomorphic coordinate $z_e$. Here, $\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}}$ is the slope of the edge $e$, [1] and for any $z \in \mathbb{C}^\times$, $z^{\mathcal{T}(e)}$ is an element of the one-dimensional torus $T_{\mathcal{T}(e),\mathbb{C}} \subset T_{P(e),\mathbb{C}}$. Via the cylindrical coordinate map (3.51) which identifies the $P(e)$-cylindrical end to $\mathfrak{X}_{P(e)}$, we may assume that

$$u_v(U_{w_e} \backslash \{w_e\}) \subset \mathfrak{X}_{P(e)}.$$

Recall that $\mathfrak{X}_{P(e)}$ has a holomorphic $T_{P(e),\mathbb{C}}$-action.

It is often useful to have a projected version of the tropical evaluation map that does not involve the choice of a holomorphic coordinate in the neighborhood of the puncture.

**Definition 4.12.** (Projected tropical evaluation map) In the notation of Definition 4.11 of a component of a broken map, for a puncture $w_e$ and coordinates $z_e$ in the neighborhood of $w_e$, the *projected tropical evaluation* at a puncture $w_e$ corresponding to an edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ is

$$\pi^\perp_{\mathcal{T}(e)}(\mathrm{ev}^{\mathcal{T}(e)}_{w_e} u_v) \in \mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}, \tag{4.16}$$

which is equal to $\lim_{w_e}(\pi^\perp_{\mathcal{T}(e)} \circ u_v)$, and therefore, is independent of the coordinate $z_e$. Here, $\pi^\perp_{\mathcal{T}(e)} : \mathfrak{X}_{P(e)} \to \mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}$ is the quotient map by the complex subtorus $T_{\mathcal{T}(e),\mathbb{C}} \subset T_{P(e),\mathbb{C}}$.

**Definition 4.13.** (Unframed broken map) Let $\mathfrak{X}_{\mathcal{P}}$ be a broken manifold and $L \subset X_{P_0}$ be a Lagrangian submanifold as above. An *unframed broken map* $u$ to $\mathfrak{X}$ is a datum consisting of

(a) (Domain type and tropical structure) a *domain type* $\Gamma$, which is the combinatorial type of a treed disk equipped with a *tropical structure* (Definition 4.9) for which disk vertices map to $P_0$ :

$$v \in \mathrm{Vert}_\circ(\Gamma) \implies P(v) = P_0;$$

(b) (Domain curve) a treed nodal disk $C = S \cup T$ of type $\Gamma$ consisting of a surface component $S_v$ (sphere or disk) for every vertex $v$ of $\Gamma$, and a treed segment $T_e$ for every boundary edge $e \in \mathrm{Edge}_\circ(\Gamma)$;

[1] It is enough to just consider broken maps whose edges have integral slopes $\mathcal{T}(e)$, and not rational slopes, because the free torus actions in the neighborhood of cut loci implies that in the neighborhood of a puncture, a map is asymptotic to a torus orbit generated by an integral element. Also, see the related Remark 7.31 following the proof of the removal of singularities result.

(c) (Map) a collection of holomorphic maps on punctured surface components

$$u_v : S_v^\circ \to \mathfrak{X}_{P(v)}, \quad v \in \mathrm{Vert}(\Gamma)$$

as in Definition 4.11, and continuous maps on the treed segments

$$u_e : T_e \to L, \quad e \in \mathrm{Edge}_\circ(\Gamma)$$

collectively denoted $u : C \to \mathfrak{X}$, such that

- (Behavior on the boundary) the restriction to the components with boundary, namely

$$u|(\cup_{v\in \mathrm{Vert}_\circ(\Gamma)} S_v^\circ \cup \cup_{e\in \mathrm{Edge}_\circ(\Gamma)} T_e),$$

  of $C$ is a treed holomorphic map (in the sense of Definition 4.5) to the target space $(X_{P_0}, L)$;
- (Matching at internal nodes) for an interior internal edge

$$e = (v_+, v_-) \in \mathrm{Edge}_\bullet(\Gamma) \cap \mathrm{Edge}_{\mathrm{int}}(\Gamma),$$

  the corresponding nodal points $w_e^\pm \in S_{v_\pm}$ map to $\mathfrak{X}_{P(v_\pm)}$ and the map $u$ is continuous at the node, that is,

$$u(w_e^+) = u(w_e^-) \in \mathfrak{X}_{P(v_\pm)}, \tag{4.17}$$

- (Matching at tropical nodes) for a tropical edge $e = (v_+, v_-) \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, the projected tropical evaluations (4.16) on the nodal lifts are equal, that is,

$$(\pi^\perp_{\mathcal{T}(e)} \circ u_{v_+})(w_e^+) = (\pi^\perp_{\mathcal{T}(e)} \circ u_{v_-})(w_e^-). \tag{4.18}$$

This ends the Definition.

**Definition 4.14.** (Broken map) A broken map $u : C \to \mathfrak{X}$ is an unframed broken map (Definition 4.13) with the additional data of a *framing* at tropical nodes $w_e$, $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$. A framing is a linear isomorphism

$$\mathrm{fr}_e : T_{w_e^+} S_{v_+} \otimes T_{w_e^-} S_{v_-} \to \mathbb{C}, \tag{4.19}$$

satisfying the following: For any holomorphic coordinate

$$z_\pm : (U_{w_e^\pm}, w_e^\pm) \to (\mathbb{C}, 0) \tag{4.20}$$

in the neighborhood of nodal lifts $w_e^\pm$ that respect the framing, that is,

$$dz_+(w_e^+) \otimes dz_-(w_e^-) = \mathrm{fr}_e, \tag{4.21}$$

the following matching condition is satisfied at tropical nodes:

$$\lim_{z_- \to 0} z_-^{-\mathcal{T}(e)} u_{v_-}(z_-) = \lim_{z_+ \to 0} z_+^{\mathcal{T}(e)} u_{v_+}(z_+). \tag{4.22}$$

The coordinates $z_+, z_-$ are called *matching coordinates* at the node $w_e$.

**Definition 4.15.** (Isomorphisms) An *isomorphism* between two unframed broken maps $u : C \to \mathfrak{X}, u' : C' \to \mathfrak{X}$ is an isomorphism $\phi : C \to C'$ of treed disks (see 4.1(e)) such that $u = u' \circ \phi$. Two framed broken maps $(u, \mathrm{fr}), (u', \mathrm{fr}')$ are isomorphic if the preceding relation holds, and in addition for any node $w$ in $C$ corresponding to an edge $e$ in the underlying graph, and lifts $w^\pm$, $\mathrm{fr}_e \circ (\mathrm{d}\Phi(w^+) \otimes \mathrm{d}\Phi(w^-)) = \mathrm{fr}'_e$.

*Remark* 4.16. In this paper, we only use the zero-dimensional components of the moduli space, and the above notion of isomorphism is sufficient. To construct higher-dimensional moduli spaces of isomorphism classes of broken maps, one also needs to take into account the symmetries of the target manifolds. In this case, one would define an isomorphism between two broken maps $u : C \to \mathfrak{X}, u' : C' \to \mathfrak{X}$ to be an isomorphism of domains $\phi : C \to C'$ and a tropical symmetry

$$(g_v \in T_{P(v),\mathbb{C}}, v \in \mathrm{Vert}(\Gamma), z_e \in \mathbb{C}^\times, e \in \mathrm{Edge}(\Gamma))$$

intertwining the maps $u, u'$ in the sense that

$$u'_v = g_v u_v \circ (\phi|_{C_v}), \quad \forall v \in \mathrm{Vert}(\Gamma). \tag{4.23}$$

See Theorem 8.2 below.

**Definition 4.17.** In a broken map $u$, a surface component $u|_{S_v^\circ} : S_v^\circ \to X_{\tilde{P}(v)}$ is *horizontally constant* if its projection to $X_{P(v)}$ is constant.

**Definition 4.18.** A broken map $u : C \to \mathfrak{X}$ is *stable* if any surface component $S_v \subset C$ on which the map $u_v : S_v^\circ \to X_{P(v)}$ is horizontally constant is stable as a marked curve, and any tree component $T_e \subset C$ on which $u|_{T_e}$ is constant does not contain an infinite segment $\mathbb{R} \subset T_e$.

*Remark* 4.19. A broken map cannot have an *extraneous* sphere component $u_v : C_v \to \overline{X}_{P(v)}, v \in \mathrm{Vert}_\bullet(\Gamma)$ with no markings, two nodal points $e_1, e_2 \ni v$, and for which the projection

$$\pi_{P(v)} \circ u_v : C_v^\circ \to X_{P(v)}$$

is a constant map. An extraneous component in a broken map $u$ can be deleted to yield a broken map with one less component. Indeed, an extraneous component is isomorphic to a trivial cylinder

$$u_v : C_v^\circ \simeq \mathbb{P}^1 \backslash \{0, \infty\} \to \mathfrak{X}_{P(v)}, \quad z \mapsto z^\mu x,$$

where $\mu \in \mathfrak{t}_{P,\mathbb{Z}}$ is the slope of both the edges $e_1$, $e_2$ incident on $v$; the vertex $v$ and edges $e_1, e_2$ can be deleted and replaced by a single edge $e$ of slope $\mu$. Conversely, if extraneous components were allowed in broken maps, any tropical edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ can be subdivided to insert an extraneous vertex, and such vertices can be inserted an arbitrary number of times.

*Remark* 4.20. In the absence of extraneous components, the stability condition is equivalent to requiring that the the automorphism group $\{\phi : C \to C, \phi^* u = u\}$ is finite. In the presence of extraneous components, the stability condition is equivalent to requiring that there are only finitely many isomorphisms in the sense of Remark 4.16.

**Definition 4.21.** (Area of a broken map) The *area* of a surface component $u_v : S_v^\circ \to \mathfrak{X}_{P(v)}$ of a broken map is the symplectic area of its horizontal projection $\pi_{P(v)} \circ u_v : S_v^\circ \to X_{P(v)}$. That is,

$$\mathrm{Area}(u_v) = \langle (\pi_{P(v)} \circ u_v)_* [S_v], \omega_{X_{P(v)}} \rangle. \tag{4.24}$$

Note that the pairing is defined via a map $\phi : X_{P(v)} \to X_{P(v)}^{\omega}$ that respects the cylindrical structure (as made precise in Remark 3.55 (a)). Since at any puncture in $S_v^\circ$, $\pi_{P(v)} \circ u_v$ is asymptotic to a trivial cylinder (see (4.15)), $\phi(\pi_{P(v)} \circ u_v)$ extends to a continuous map $\overline{u}_v : S_v \to \overline{X}_{P(v)}^{\omega}$. A different choice of $\phi$ changes $\overline{u}_v$ by a homotopy.

*Remark* 4.22. (Tropical graph as a combinatorial invariant) The tropical graph of a broken map is a combinatorial invariant consisting of the data of edge slopes $\mathcal{T}(e)$, $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ and vertex polytopes $P(v)$, $v \in \mathrm{Vert}(\Gamma)$, since varying the tropical positions of vertices does not produce a new broken map, see Remark 4.7.

The following remarks are easy conclusions of the definition of broken maps, and are intended to help the reader process the definition.

*Remark* 4.23. (Continuity away from tropical nodes) Suppose the tropical structure on a broken map is given by $\Gamma \xrightarrow{\mathrm{tr}} \Gamma_{\mathrm{tr}}$ where tr is the tropicalization map and $\Gamma_{\mathrm{tr}}$ is a tropical graph. The domain of a broken map breaks up into connected components

$$C \backslash \{w_e : e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)\} = \cup_{v \in \mathrm{Vert}(\Gamma_{\mathrm{tr}})} C_v^\circ.$$

For a vertex $v \in \mathrm{Vert}(\Gamma_{\mathrm{tr}})$ in the tropical graph, if $\mathrm{tr}^{-1}(v)$ does not have disk vertices, then $C_v$ is a nodal sphere with punctures, with nodes corresponding to internal edges in $\mathrm{tr}^{-1}(v)$, and punctures corresponding to tropical edges incident on $\mathrm{tr}^{-1}(v)$. If $\mathrm{tr}^{-1}(v)$ has disk components then $C_v^\circ$ is a treed disk with punctures. For any vertex $v \in \mathrm{Vert}(\Gamma_{\mathrm{tr}})$ in the tropical graph, the restriction

$$u | C_v^\circ : C_v^\circ \to \mathfrak{X}_{P(v)}$$

is continuous.

*Remark* 4.24. (Node matching for a single cut) In the case of a single cut, the familiar form of the node matching condition from symplectic field theory corresponds to the matching condition (4.18) for unframed maps. Indeed, for any edge $e$ in a tropical graph, $\operatorname{codim}(P(e)) = \dim T_{P(e)} = 1$, and so, $T_{\mathcal{T}(e),\mathbb{C}} = T_{P(e),\mathbb{C}}$; the unframed matching condition (4.18) then translates to the matching of evaluations on $X_{P(e)}$, namely, $(\pi_{P(e)} \circ u_{v_+})(w_e^+) = (\pi_{P(e)} \circ u_{v_-})(w_e^-)$.

**Definition 4.25.** (Primitive slope, multiplicity of a tropical edge) In a tropical graph $\Gamma$, the slope $\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}} \backslash \{0\}$ of a tropical edge $e$ can be written as a product

$$\mathcal{T}(e) = \mu_e \mathcal{T}(e)_{\text{prim}}$$

of a primitive integer vector $\mathcal{T}(e)_{\text{prim}} \in \mathfrak{t}_{P(e),\mathbb{Z}}$, called the *primitive slope*, and a positive integer $\mu_e \in \mathbb{Z}_{\geq 1}$ which is called the *multiplicity* of the edge $e$.

*Remark* 4.26. (Number of framings) Any unframed broken map possesses framings. For an unframed broken map $u$ of type $\Gamma$, the number of framings is equal to $\Pi_{e \in \text{Edge}_{\text{trop}}(\Gamma)} \mu_e$, where $\mu_e \in \mathbb{Z}_{\geq 1}$ is the multiplicity of the edge $e$ (Definition 4.25). Indeed, the trivial cylinder

$$\mathbb{C}^\times \to T_{P(e)}, \quad z \mapsto z^{\mathcal{T}(e)}$$

which is the one-parameter subgroup induced by the Lie algebra element $\mathcal{T}(e)$, is a $\mu_e$-cover. As a consequence, if $\text{fr}_e$ is a framing for the node $w_e$, then $e^{2\pi i k/\mu_e} \text{fr}_e$ is also a framing for $k = 0, 1, \ldots, \mu_e - 1$.

*Remark* 4.27. (Extending broken map components over punctures) In the special case that the compactifications of the components of the broken manifold are manifolds, a broken map extends over lifts of nodal points to yield a map

$$u_v : S_v \to \overline{\mathfrak{X}}_{P(v)}, \quad v \in \text{Vert}(\Gamma). \tag{4.25}$$

Indeed, the existence of the tropical evaluation (4.15) implies that, in $\mathfrak{X}_{P(v)}$, $u_v$ is asymptotically close to trivial cylinders at punctures. If $\overline{\mathfrak{X}}_{P(v)}$ has orbifold singularities, that is, if the polytope $P(v)$ is simple and not Delzant, the extension (4.25) is defined on a domain curve with orbifold singularities. Since compactifications of cut spaces (3.45) are orbifolds, the projections extend over punctures to yield $\pi_{P(v)} \circ u_v : S_v \to \overline{X}_{P(v)}$. For any nodal lift $w \in S_v$ corresponding to an edge $e = (v, v')$, we have

$$(\pi_{P(v)} \circ u)(w) \in X_{P(e)} \subset \overline{X}_{P(v)}. \tag{4.26}$$

This can be seen as follows: For each relative divisor $X_Q \subset \overline{X}_{P(v)}$, the normal vector $\mu_Q$ in the moment polytope lies in $\mathfrak{t}_Q/\mathfrak{t}_{P(v)}$, and a punctured neighborhood of $X_Q$ in $X_{P(v)}$ is a subset of a $T_{\mu_Q,\mathbb{C}}$-fibration. If the edge slope $\mathcal{T}(e) \in \mathfrak{t}_{P(e)}$ is spanned by normal vectors $\mu_{Q_1}, \ldots, \mu_{Q_k}$, where each $Q_i \in \mathcal{P}$ is a facet of $P(v)$, then $X_{P(e)} = \cap_i X_{Q_i}$, and $(\pi_{P(v)} \circ u)(w) \in X_{P(e)}$ since the map is asymptotically close to a $T_{\mathcal{T}(e),\mathbb{C}}$-orbit near the puncture $w$.

*Remark* 4.28. (Relationship of slope with intersection multiplicities) Consider the special case when the compactifications of the broken manifold are smooth manifolds. Consider a component $u_v : S_v \to \overline{\mathfrak{X}}_{P(v)}$ of a broken map. Suppose for a nodal lift $w \in S_v$ corresponding to an edge $e = (v, v')$, $u_v(w)$ lies on the intersection of relative divisors $Y_1, \dots, Y_k \subset \overline{\mathfrak{X}}_{P(v)}$. Furthermore, suppose $\nu_1, \dots, \nu_k \in \mathfrak{t}$ are the primitive outward pointing normal vectors to the facets of $\tilde{P}(v)$ corresponding to the divisors $Y_1, \dots, Y_k$, and that the map $u_v$ intersects the divisor $Y_i$ with multiplicity $\mu_i \in \mathbb{Z}_+$. Then we have the following relation to the slope of the edge:

$$\sum \mu_i \nu_i = \mathcal{T}(e). \tag{4.27}$$

In the case when $\overline{\mathfrak{X}}_{P(v)}$ is an orbifold, the intersection multiplicities $\mu_i$ may be fractional, though the relation (4.27) still holds. However, we do not use orbifold intersection numbers in any of the proofs. This ends the Remark.

*Remark* 4.29. (Exponential convergence to a trivial cylinder at tropical nodes) The removal of singularity of punctures from Remark 4.28 can alternately be stated as that a broken map is asymptotically close to a trivial cylinder at the punctured neighborhood of any nodal lift. For a tropical node $w_e$ corresponding to $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ and holomorphic coordinates

$$(s,t) : U(w_e^\pm) \backslash \{w_e^\pm\} \to \mathbb{R}_{\geq 0} \times S^1 \tag{4.28}$$

in punctured neighborhoods of the nodal lifts, the limit

$$x_e^\pm := \lim_{s \to \infty} e^{\pm \mathcal{T}(e)(s+it)} u_{v_\pm}(s,t)$$

exists and is a point in the $P(e)$-cylindrical end $U_{P(e)}(\mathfrak{X}_{P(v_\pm)})$. Furthermore, the map $u_{v_\pm}$ exponentially converges to a trivial cylinder with slope $\mathcal{T}(e)$ in the punctured neighborhood $U(w_e^\pm) \backslash \{w_e^\pm\}$, that is, there is a constant $c > 0$ such that

$$d(u_{v_\pm}(s,t), e^{\mp \mathcal{T}(e)(s+it)} x_e^\pm) \leq c e^{-s}, \quad s \geq s_0,$$

using the cylindrical metric in $\mathfrak{X}_{P(e)}$. The proof is left to the reader.

*Remark* 4.30. (A different view of node matching) In the special case that the multiple cut is a collection of orthogonally intersecting single cuts, and each cut space is a manifold, the tropical node matching condition (4.22) splits into the following conditions:

- a *horizontal matching* condition on the intersection of relative divisors, and
- a *vertical matching* condition involving leading order Taylor coefficients in the directions normal to each of the relative divisors.

We consider a multiple cut given by a tropical moment map $\Phi : X \to \mathbb{R}^k$ with cuts along the hypersurfaces $\Phi_i^{-1}(0), i = 1, \ldots, k$. Denote the relative divisors by $Y_i := \Phi_i^{-1}(0)/S^1$ (though $Y_i$ is a broken manifold). Consider a broken map $u$ with components $u_+$, $u_-$ in the cut spaces for the polytopes

$$P_+ := \cap_{i=1}^k \{\Phi_i \geq 0\}, \quad P_- := \cap_{i=1}^k \{\Phi_i \leq 0\}$$

respectively, sharing a node $w_e$. As discussed in Remark 4.28, the map $u_\pm$ extends over the puncture at the nodal point $w_e^\pm$. The intersection multiplicities with the divisors $Y_1, \ldots, Y_k$ are the same for $u_+, u_-$, and are denoted by $\mu_1, \ldots, \mu_k \in \mathbb{Z}_{>0}$. The matching condition consists of

- (Horizontal matching) a horizontal condition which says that

$$u_+(w_e^+) = u_-(w_e^-) \in X_{P(e)},$$

  where $X_{P(e)} = \Phi^{-1}(0)/(S^1)^k \simeq Y := \cap_{i=1}^k Y_i$;
- (Vertical matching) and a vertical condition which says that for holomorphic coordinates

$$z_\pm : (U(w_e^\pm), w_e^\pm) \to (\mathbb{C}, 0)$$

  in a neighborhood of the nodal lifts that respect the framing (as in (4.21)), for all $i$, the $(\mu_i + 1)$-th derivative normal to $Y_i$ is equal for $u_{v_+}$ and $u_{v_-}$. In the direction normal to $Y_i$, since derivatives up to order $\mu_i$ vanish, the $(\mu_i + 1)$-th normal derivative, or the $(\mu_i + 1)$-jet normal to $Y_i$, denoted by

$$j_{Y_i}^{\mu_i} u_\pm(w_e^\pm) \in N_\pm Y_i \backslash Y_i$$

  is well-defined (see [26, Section 6]), where $N_\pm Y_i$ is the normal bundle of $Y_i$ in $\overline{X}_{P_\pm}$. The vertical matching condition says that

$$j_{Y_i}^{\mu_i} u_+(w_e^+) = j_{Y_i}^{\mu_i} u_-(w_e^-).$$

This view of the matching condition is relevant only in case of a collection of orthogonal single cuts. In general cases, given a node, both nodal lifts do not lie on the same set of relative divisors. For example, in the broken map in Figure 2.22 corresponding to a Mikhalkin graph, there are edges of the form

- $e = (v_+, v_-)$, $\mathcal{T}(e) = (1, 1)$
- with $P(v_+) = P(v_-) = P_0$ which is the zero-dimensional polytope; and
- one of the lifts lies on a divisor of $\overline{\mathfrak{X}}_{P_0}$ corresponding to a facet with normal vector $(1, 1)$ and another lift lies on the intersection of divisors corresponding to facets with normal vectors $(0, -1)$ and $(-1, 0)$.

This ends the Remark.

*Example* 4.31. We unpack the horizontal and vertical matching conditions from Remark 4.30 in the first broken map in Figure 4.4. The multiple cut consists of two orthogonal cuts shown in Figure 1.2, and the node $w$ lies between maps $u_0 : C_0 \to \overline{X}_{P_2}$, $u_1 : C_1 \to \overline{X}_{P_0}$. The nodes $u_0(w), u_1(w)$ lies in $\overline{X}_{P_\cap}$. Suppose the leading order Taylor term of $u_0$ in the direction normal to $X_{P_{12}}$ resp. $X_{P_{23}}$ is $a_+ z_+^{\mu_h^+}$ resp. $b_+ z_+^{\mu_v^+}$, and suppose the leading order Taylor term of $u_1$ in the direction normal to $X_{P_{30}}$ resp. $X_{P_{01}}$ is $a_- z_-^{\mu_h^-}$ resp. $b_- z_-^{\mu_v^-}$. Then the matching condition says that

- (Matching of intersection multiplicities) $\mu_h^+ = \mu_h^-$, $\mu_v^+ = \mu_v^-$,
- (Horizontal matching) the projections of the nodal evaluation maps to $X_{P_\cap}$ are equal: $\pi_{P_\cap}(u(w^+)) = \pi_{P_\cap}(u(w^-)) \in X_{P_\cap}$,
- (Vertical matching) and the leading order Taylor coefficients are equal: $a_+ = a_-$, $b_+ = b_-$.

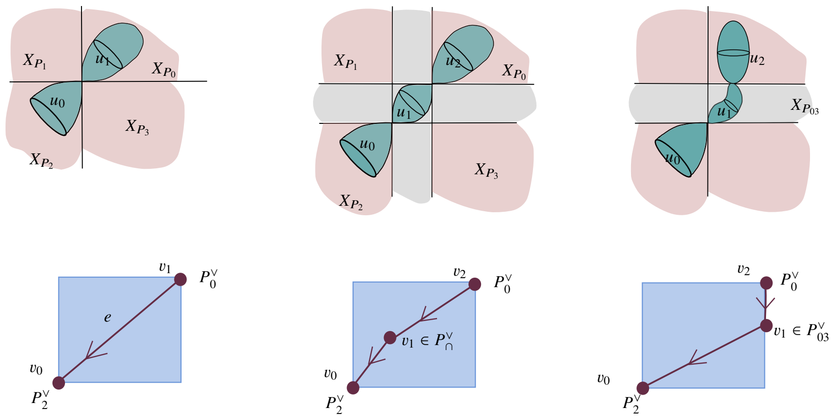



**Figure 4.4.** Broken maps and their dual graphs in the broken manifold of Figure 3.8.

*Remark* 4.32. (Comparison to Ionel's refined matching) In [44, p14], Ionel states the edge-matching condition using a "refined evaluation map", which is analogous to the projected tropical evaluation map defined in (4.18). Suppose a node $w_e$ maps to the intersection of relative divisors $Y_1, \dots, Y_n$ and the intersection multiplicity with each of the divisors is $\mu_1, \dots, \mu_n$. (Here we work with the extension of the broken map $u$ over nodal lifts as in Remark 4.28.) The refined evaluation map $\mathrm{ev}_{\mathrm{ref}}(u, 0)$ for the map $u$ and the nodal lift $0 \in C$ is a point in the weighted projective space

$$\mathrm{ev}_{\mathrm{ref}}(u, 0) = (x_1, \dots, x_n) \in \mathbb{P}_{(\mu_1, \dots, \mu_n)}(\oplus_i NY_i).$$

The point $\mathrm{ev}_{\mathrm{ref}}(u, 0)$ is well-defined since replacing the domain coordinate $z$ by $az$ for some $a \in \mathbb{C}^\times$ has the effect of changing $(x_1, \dots, x_n)$ to $(a^{\mu_1} x_1, \dots, a^{\mu_n} x_n)$. The projected tropical evaluation in (4.18) and (4.16)

$$\pi^{\perp}_{\mathcal{T}(e)}(\mathrm{ev}^{\mu}(0)) \in (Z \times \mathbb{R}^n)/T_{\mathcal{T}(e),\mathbb{C}}$$

is Ionel's evaluation map written using cylindrical coordinates on the target space, and the slope $\mathcal{T}(e)$ of the edge $e$ is the vector $(\mu_1, \dots, \mu_n)$. The cylindrical viewpoint appears more natural because in the image of the refined evaluation map none of the coordinates $x_i$ vanish. On the other hand, all points in the cylinder $(Z \times \mathbb{R}^n)/T_{\mathcal{T}(e),\mathbb{C}}$ can possibly be in the image of the evaluation map.

*Remark* 4.33. (Balancing property) We discuss the generalization of the balancing condition (2.38) for Mikhalkin graphs to broken maps. First, consider the special case that the projection $(\pi_{P(v)} \circ u_v) : S_v^\circ \to X_{P(v)}$ of a map component $u_v : S_v^\circ \to \mathfrak{X}_{P(v)}$ is constant. Then, the edges $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ emanating from the vertex $v$ satisfy

$$\sum_{e \ni v} \mathcal{T}(e)_{\mathrm{vert}} = 0, \tag{4.29}$$

where $\mathcal{T}(e)_{\mathrm{vert}} \in \mathfrak{t}_{P(v)}$ is the orthogonal projection of $\mathcal{T}(e) \in \mathfrak{t}$ to $\mathfrak{t}_{P(v)}$ with respect to the inner product (3.10) on $\mathfrak{t}$.

We can generalize this property to non-constant projections, assuming that the inner product (3.10) on $\mathfrak{t}$ is rational, in which case the compactification $\overline{\mathfrak{X}}_P$ of the broken manifold is an orbifold, and $\overline{Z}_P \to \overline{X}_P$ is an orbifold torus bundle (see Remark 3.60) for all $P \in \mathcal{P}$. Then we have the equality

$$\sum_{e \ni v} \mathcal{T}(e)_{\mathrm{vert}} = c_1((\pi_{P(v)} \circ u_v)^* \overline{Z}_{P(v)} \to \overline{X}_{P(v)})), \tag{4.30}$$

where $c_1((\pi_{P(v)} \circ u_v)^* \overline{Z}_{P(v)} \to \overline{X}_{P(v)}))$ is the first Chern class of the pull-back bundle, viewed as a vector in $\mathfrak{t}_{P(v)} \simeq H^2(S_v, \mathfrak{t}_{P(v)})$, and $S_v$ is the possibly orbifold compactification of $S_v^\circ$ over which the map $u_v$ extends. Indeed, the map $u_v$ gives a section of the $T_{P(v)}$-principal bundle $(\pi_{P(v)} \circ u_v)^* \overline{Z}_{P(v)} \to \overline{X}_{P(v)}$ on the complement of nodal points $w_e$, $e \ni v$, and the monodromy of the section around each such intersection is determined by $\mathcal{T}(e)_{\mathrm{vert}}$. This ends the Remark.

*Remark* 4.34. (A comparison with symplectic field theory) Holomorphic buildings in symplectic field theory [11] are defined slightly differently to broken maps on manifolds with a single cut. The target space of a holomorphic building consists of $X^+$, $X^-$ and $k-1$ copies of the neck piece $Z(\mathbb{P}^1)$ for some $k \geq 1$:

$$\mathfrak{X}[k] := \overline{X}_+ \cup_Y Z(\mathbb{P}^1) \cup_Y \cdots \cup_Y Z(\mathbb{P}^1) \cup_Y \overline{X}_-,$$

and any pair of consecutive pieces are identified along a divisor $Y$. A holomorphic building $u : C \to \mathfrak{X}[k]$ is a continuous map, where nodes map to the divisor $Y$, and intersection multiplicities $m_{w_e^\pm}(u_{v_\pm}, Y)$ are equal on both sides.

A holomorphic building differs from a broken map in two ways: A holomorphic building is a continuous map, and the data for a holomorphic building includes an ordering for the neck piece components. In the broken map view, this ordering is not important: With suitable regularity assumptions a broken $u$ map with $m$ components in neck pieces can be glued to give a $2m$-dimensional family of unbroken maps in $X^\nu$ for any $\nu$. Any sequence of maps $u_\nu : C_\nu \to X^\nu$ lying in the glued family converges to a broken map $u' : C' \to \mathfrak{X}$ that is related to $u : C \to \mathfrak{X}$ by a tropical symmetry (as in Definition 4.35). In contrast, the holomorphic building limits of different sequences in the glued family may not all be the same, since the choice of the sequence $(u_\nu)_\nu$ determines the component of $\mathfrak{X}[k]$ to which a curve component $C_v \subset C$ maps. For a broken map $u = (u_v)$, the ordering of the pieces $u_v$ is not part of the data of the the map. One effect of the differing definitions is the following: Unlike holomorphic buildings, broken maps do not have components $u_v$ that are trivial cylinders in the sense that they map into a fiber of $\overline{\mathfrak{X}}_{P(v)}$ with only two marked points and so are unstable. In holomorphic buildings, trivial cylinders have to be inserted whenever there is a node $w_e$ between components $C_{v_+}, C_{v_-}$ that are not in adjacent levels in order to achieve continuity. This ends the Remark.

## 4.4 Symmetries of broken maps

The moduli space of broken maps of a fixed type $\Gamma$ has the action of a group, called the *tropical symmetry group*, arising from torus actions on the "neck pieces" in the degeneration. The tropical symmetry group $T_{\text{trop}}(\Gamma)$ can be read off from the tropical graph $\mathcal{T}$ of $\Gamma$. The identity component of $T_{\text{trop}}(\Gamma)$, is generated by the degrees of freedom of vertex positions in the tropical graph $\mathcal{T}$. For example, the graph $\Gamma_1$ in Figure 4.5 has one degree of freedom, and therefore, the identity component of $T_{\text{trop}}(\Gamma_1)$ is a one-dimensional complex torus. A tropical graph is *rigid* if there is no way of moving the vertices while keeping the edge slopes fixed. Such graphs have a finite tropical symmetry group. For example, in Figure 4.5, $|T_{\text{trop}}(\Gamma_2)| = 2$. The Mikhalkin graphs from Section 2.4 give a class of examples of rigid tropical graphs. We prove later in this section that the size of the symmetry group for Mikhalkin graphs is equal to the multiplicity of the graph.

The tropical symmetry group also incorporates a group action on the framings of the broken map. Even one of the simplest kind of broken maps occuring in a single cut, namely one whose tropical graph consists of two vertices each lying in $P_+^\vee$ and $P_-^\vee$ (in Figure 1.5) connected by an edge of multiplicity $k$ has a tropical group of size $k$ arising from the action on framings.

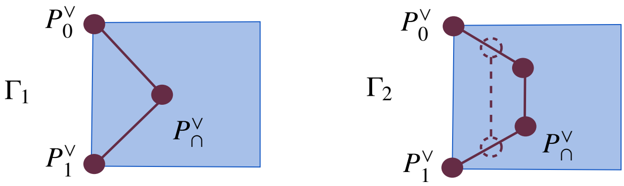



**Figure 4.5.** The dual complex $B^\vee$ for the multiple cut in Figure 1.2 is a rectangle. The tropical graph $\Gamma_1$ (left figure) is rigid, but $\Gamma_2$ (right figure) is not rigid since the vertices inside the square can be moved to the dotted positions.

**Definition 4.35.** (Tropical symmetry) A *tropical symmetry* for a tropical graph $\Gamma$ is a tuple

$$(\underline{g}, \underline{z}) = ((g_v)_{v \in \mathrm{Vert}(\Gamma)}, (z_e)_{e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)}), \quad g_v \in T_{P(v),\mathbb{C}}, \quad z_e \in \mathbb{C}^\times$$

consisting of a translation $g_v$ for each vertex and a change of local coordinate $z_e$ for each tropical edge that satisfies

$$g_{v_+} g_{v_-}^{-1} = z_e^{\mathcal{T}(e)} \quad \forall e = (v_+, v_-) \in \mathrm{Edge}_\bullet(\Gamma), \tag{4.31}$$

where we assume $z_e = 1$ for $e \in \mathrm{Edge}_\bullet(\Gamma) \setminus \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$. A tropical symmetry $(g, z)$ acts on a broken map $(u, \mathrm{fr})$ as

$$u_v \mapsto g_v u_v, \quad \mathrm{fr}_e \mapsto z_e \mathrm{fr}_e.$$

The group of tropical symmetries is denoted by

$$T_{\mathrm{trop}}(\Gamma) := \{((g_v)_{v \in \mathrm{Vert}(\Gamma)}, (z_e)_{e \in \mathrm{Edge}_\bullet(\Gamma)}) | (4.31)\}. \tag{4.32}$$

The condition (4.31) is a necessary and sufficient condition for the translations $(g_v)_v$ to preserve the matching condition at nodes.

*Remark* 4.36. (Framing symmetry) There are finitely many tropical symmetries $(\underline{g}, \underline{z}) \in T_{\mathrm{trop}}(\Gamma)$, called *framing symmetries*, for which the action on the unframed map $u$ of type $\Gamma$ is trivial, that is $g_v = \mathrm{Id}$ for all vertices $v \in \mathrm{Vert}(\Gamma)$. The group of such framing symmetries $(\underline{g}, \underline{z})$ is the product

$$\prod_{e \in \mathrm{Edge}_\bullet(\Gamma)} \mathbb{Z}_{\mu_e} \subset T_{\mathrm{trop}}(\Gamma), \tag{4.33}$$

where $\mu_e$ is the multiplicity of the edge $e$ (Definition 4.25), and also the order of ramification of the broken map at the nodal point $w_e$.

**Lemma 4.37.** *Let $u_0$, $u_1$ be broken maps of type $\Gamma$ whose underlying unframed broken maps are the same. Then, $u_0$ and $u_1$ are related by an element of the framing symmetry group (from* (4.33)*).*

*Proof.* By Remark 4.26, the number of possible framings of an unframed broken map is equal to $\prod_{e \in \mathrm{Edge}_\bullet(\Gamma)} \mu_e$. This number is equal to the size of the framing symmetry group from (4.33) which has a free action on the set of broken maps. ■

*Remark* 4.38. (Tropical symmetry for unframed broken maps) A tropical symmetry on an unframed broken map is a tuple

$$\underline{g} = (g_v)_{v \in \mathrm{Vert}(\Gamma)}$$

that satisfies

$$g_{v_+} g_{v_-}^{-1} \in \{ z^{\mathcal{T}(e)} : z \in \mathbb{C}^\times \} \quad \forall e = (v_+, v_-) \in \mathrm{Edge}_\bullet(\Gamma). \tag{4.34}$$

The tropical symmetry group for unframed maps is equal to the quotient of the tropical symmetry group by the group of framing symmetries (4.33).

The next result shows that the identity component of the tropical symmetry group is generated by tropical positions of vertices of the tropical graph.

**Lemma 4.39.** *(Tropical vertex positions generate tropical symmetries)*

(a) *For a tropical graph $\Gamma$, the set of tropical vertex positions $\mathcal{W}(\Gamma)$ (defined in* (4.12)*) is convex.*

(b) *If a tropical graph $\Gamma$ has two distinct tropical vertex position maps*

$$(\mathcal{T}_0(v), v \in \mathrm{Vert}(\Gamma)), \quad (\mathcal{T}_1(v), v \in \mathrm{Vert}(\Gamma))$$

*then the difference $\mathcal{T}_1 - \mathcal{T}_0$ generates a real-two-dimensional subgroup* $\exp((\mathcal{T}_1 - \mathcal{T}_0)(\cdot))$ *of the tropical symmetry group $T_{\mathrm{trop}}(\Gamma)$ (defined in* (4.32)*).*

(c) *The subgroup*

$$T_{\mathrm{trop},\mathcal{W}}(\Gamma) := \langle \exp((\mathcal{T}_1 - \mathcal{T}_0)z) | \mathcal{T}_0, \mathcal{T}_1 \in \mathcal{W}(\Gamma), z \in \mathbb{C} \rangle \tag{4.35}$$

*generated by tropical vertex position maps is the identity component of $T_{\mathrm{trop}}(\Gamma)$.*

*Proof.* For the first statement, if $\mathcal{T}_0, \mathcal{T}_1 \in \mathcal{W}(\Gamma)$ are vertex position maps for a tropical graph $\Gamma$, then for any $t \in [0, 1]$

$$(1 - t)\mathcal{T}_0 + t\mathcal{T}_1 \in \mathcal{W}(\Gamma)$$

is also a vertex position map on $\Gamma$. Assume that $\mathcal{T}_0$, $\mathcal{T}_1$ are distinct, and that $l_{e,i} \in \mathbb{R}$ describe the difference between the tropical vertex positions in the sense that

$$\mathcal{T}_i(v_+) - \mathcal{T}_i(v_-) = l_{i,e}\mathcal{T}(e), \quad i = 0, 1, e \in \mathrm{Edge}_\bullet(\Gamma).$$

Then the set of elements

$$g : \mathbb{C} \to T_{\text{trop}}(\Gamma), \quad z \mapsto \begin{cases} e^{(\mathcal{T}_1(v) - \mathcal{T}_0(v))z}, & v \in \text{Vert}(\Gamma) \\ e^{(l_{1,e} - l_{0,e})z}, & e \in \text{Edge}_\bullet(\Gamma) \end{cases}$$

is a non-trivial group of symmetries for broken maps modelled on $\Gamma$. In a similar way, one sees that the Lie algebra $\mathfrak{t}_{\text{trop}}(\Gamma)$ is equal to the vector space generated by differences of tropical vertex position maps $\mathcal{T}_1 - \mathcal{T}_0$, $\mathcal{T}_0, \mathcal{T}_1 \in \mathcal{W}(\Gamma)$, from which (c) follows. ∎

The following is a corollary of Lemma 4.39.

**Corollary 4.40.** *(Rigid tropical graphs) A tropical graph is rigid if and only if its tropical symmetry group $T_{\text{trop}}(\Gamma)$ is finite.*

**Lemma 4.41.** *For any tropical graph $\Gamma$, the quotient $T_{\text{trop}}(\Gamma)/T_{\text{trop},\mathcal{W}}(\Gamma)$ is finite and the group $T_{\text{trop}}(\Gamma)$ has a finite number of connected components.*

*Proof.* The quotient $T_{\text{trop}}(\Gamma)/T_{\text{trop},\mathcal{W}}(\Gamma)$ is discrete, because by Lemma 4.39, $T_{\text{trop},\mathcal{W}}(\Gamma)$ is the identity component of $T_{\text{trop}}(\Gamma)$. Furthermore, every connected component of $T_{\text{trop}}(\Gamma)$ deformation retracts to a component of the maximal compact subtorus

$$T_{\text{trop}}(\Gamma)^{im} := \{(\underline{g}, \underline{z}) \in T_{\text{trop}}(\Gamma) : g_v \in T_{P(v)}, |z_e| = 1\}.$$

Indeed, an element $(g, z) \in T_{\text{trop}}(\Gamma)$ can be written as

$$g_v = k_v e^{is_v}, \quad k_v \in T_{P(v)}, s_v \in \mathfrak{t}_{P(v)}, \quad z_e = \theta_e e^{\alpha_e}, \quad \theta_e \in S^1, \alpha_e \in \mathbb{R}$$

The tuple $(k, \theta) := ((k_v)_v, (\theta_e)_e)$ is a tropical symmetry element in $T_{\text{trop}}(\Gamma)^{im}$ which is connected to $(g, z)$ via the path

$$[0, 1] \ni \tau \mapsto ((k_v e^{i\tau s_v})_v, (\theta_e e^{\tau \alpha_e})_e) \in T_{\text{trop}}(\Gamma).$$

The quotient $T_{\text{trop}}(\Gamma)/T_{\text{trop},\mathcal{W}}(\Gamma)$ is finite because it is in bijection with the connected components of the compact subgroup $T_{\text{trop}}(\Gamma)^{im}$. ∎

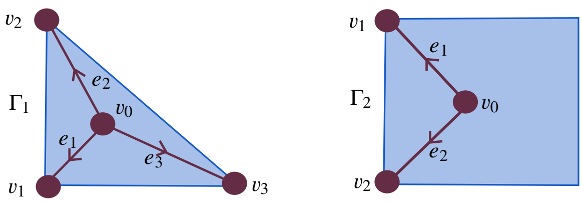



**Figure 4.6.** Rigid tropical graphs in Example 4.42.

*Example* 4.42. The tropical graphs in Figure 4.6 are rigid, but have non-trivial tropical symmetry groups. Suppose the tropical graph $\Gamma_1$ has edge slopes

$$\mathcal{T}(e_1) = (-1,-1), \quad \mathcal{T}(e_2) = (-1,2), \quad \mathcal{T}(e_3) = (2,-1).$$

A tropical symmetry $(\underline{g}, \underline{z})$ on $\Gamma_1$ satisfies the equations

$$g_1 = g_2 = g_3 = \mathrm{Id}, \quad g_1 g_0^{-1} = z_{e_1}^{(-1,-1)}, \quad g_2 g_0^{-1} = z_{e_2}^{(-1,2)}, \quad g_3 g_0^{-1} = z_{e_3}^{(2,-1)},$$

and is therefore given by

$$g_0 = (\omega, \omega), \quad z_{e_1} = z_{e_2} = z_{e_3} = \omega,$$

where $\omega \in e^{2\pi i k/3}$ is a cube root of unity. Thus

$$|T_{\mathrm{trop}}(\Gamma_1)| = 3. \tag{4.36}$$

This tropical graph is similar to the example studied by Abramovich-Chen-Gross-Siebert [2, p51] and Tehrani [33, Section 6]. The tropical graph $\Gamma_2$ has edge slopes

$$\mathcal{T}(e_1) = (-1,1), \quad \mathcal{T}(e_2) = (-1,-1).$$

By a similar calculation $|T_{\mathrm{trop}}(\Gamma_2)| = 2$.

Recall from Section 2.4 that the multiplicity of a Mikhalkin graph $\Gamma$ is the product of multiplicities of the vertices of $\Gamma$, that is,

$$\mathrm{mult}(\Gamma) := \prod_{v \in \mathrm{Vert}(\Gamma)} \mathrm{mult}(v).$$

Assuming that all vertices are trivalent, the multiplicity of a vertex $v$ of $\Gamma$ is the area of the parallelogram spanned by two of the three edges incident on $v$. The next result, which was used in Section 2.4, shows that he multiplicity of the Mikhalkin graph is equal to the size of the tropical symmetry group of the augmented graph $\Gamma_{\mathrm{aug}}$. We recall from the proof of Proposition 2.26, that $\Gamma_{\mathrm{aug}}$ is a tropical graph for the polyhedral decomposition of $\mathbb{P}^2$ from Figure 2.21. We also recall that

$$\mathrm{Vert}(\Gamma_{\mathrm{aug}}) = \mathrm{Vert}(\Gamma) \cup \mathrm{Vert}_{\to}(\Gamma_{\mathrm{aug}}) \cup \mathrm{Vert}_1(\Gamma_{\mathrm{aug}}),$$

where a vertex $v_e \in \mathrm{Vert}_1(\Gamma_{\mathrm{aug}})$ corresponds to a leaf $e$ of the Mikhalkin graph, and each vertex in $\mathrm{Vert}_{\to}(\Gamma_{\mathrm{aug}})$ has an interior marking, that is constrained to pass through a fixed point in the broken manifold.

**Lemma 4.43.** *(Mikhalkin multiplicity and tropical symmetry) Let $\Gamma$ be a Mikhalkin graph all whose vertices are trivalent, and whose leaf edges have multiplicity* 1. *Let $\Gamma_{\mathrm{aug}}$ be the augmentation of $\Gamma$. Then,*

$$|T_{\mathrm{trop}}(\Gamma_{\mathrm{aug}})| = \mathrm{mult}(\Gamma).$$

*Any two broken maps $u_1$, $u_2$ modelled on $\Gamma_{\text{aug}}$ are related by a tropical symmetry $g \in T_{\text{trop}}(\Gamma_{\text{aug}})$.*

*Proof.* A tropical symmetry of a broken map with tropical graph $\Gamma_{\text{aug}}$ is a tuple

$$(\underline{g}, \underline{z}) = ((g_v)_{v \in \text{Vert}(\Gamma_{\text{aug}})}, (z_e)_{e \in \text{Edge}(\Gamma_{\text{aug}})}), \quad g_v \in T_{P(v),\mathbb{C}}, \quad z_e \in \mathbb{C}^\times$$

satisfying

$$g_{v_+} g_{v_-}^{-1} = z_e^{\mathcal{T}(e)} \quad \forall e = (v_+, v_-) \in \text{Edge}(\Gamma_{\text{aug}}), \tag{4.37}$$

see Definition 4.35. We count the number of solutions of (4.37). For vertices $v \in \text{Vert}_{\rightarrow}(\Gamma_{\text{aug}})$ containing markings, we must have $g_v = \text{Id}$ in order to ensure that the evaluation of the marking satisfies the point constraint. We solve for $(\underline{g}, \underline{z})$, one vertex at a time, with vertex ordering respecting the marking orientation defined in Page 57. Consider a vertex $v \in \text{Vert}(\Gamma)$ with incoming edges $e_1 = (v_1, v)$ and $e_2 = (v_2, v)$, and outgoing edge $e_3 = (v, v_3)$. Assuming the values of $g_{v_1}, g_{v_2}$ to be given, we solve the equations

$$g_v g_{v_1}^{-1} = z_{e_1}^{\mathcal{T}(e_1)}, \quad g_v g_{v_2}^{-1} = z_{e_2}^{\mathcal{T}(e_2)}$$

for $g_v \in T_{P_0,\mathbb{C}} \simeq (\mathbb{C}^\times)^2$, $z_{e_1}, z_{e_2} \in \mathbb{C}^\times$ and show that the number of solutions is $\text{mult}(v)$. Use an integral basis of $\mathfrak{t}_{P_0,\mathbb{Z}} \simeq \mathbb{Z}^2$ so that

$$\mathcal{T}(e_1) = (p_1, 0), \quad \mathcal{T}(e_2) = (q, p_2)$$

for some integers $p_1, p_2, q$. In this notation, the multiplicity $\text{mult}(v)$ is $|p_1 p_2|$. We may write $g_v = (g_v^0, g_v^1)$. We solve in the order $g_v^1, z_{e_2}, g_v^0, z_{e_1}$. For a fixed value of $g_{v_1}, g_{v_2}$, there are $p_1 p_2$ solutions: There are $p_1$ solutions for $z_{e_2}$, and for each of these solutions, there are $p_2$ solutions of $z_{e_1}$. Thus we have shown that for a vertex $v \in \text{Vert}(\Gamma) \subset \text{Vert}(\Gamma_{\text{aug}})$, the number of solutions for $(g_v, z_{e_1}, z_{e_2})$ is equal to the multiplicity of $v$. Finally, given a solution of (4.37) for vertices in the set $\text{Vert}(\Gamma) \cup \text{Vert}_{\rightarrow}(\Gamma)$ and connecting edges, it remains to solve (4.37) for $g_v, z_{e(v)}$ where $v \in \text{Vert}_1(\Gamma_{\text{aug}})$ is a univalent vertex corresponding to a leaf of the Mikhalkin graph $\Gamma$ and $e(v) \in \text{Edge}(\Gamma_{\text{aug}})$ is the edge incident on $v$. There is a unique solution of (4.37) for $g_v \in T_{P(v),\mathbb{C}}$ and $z_{e(v)} \in \mathbb{C}^\times$ because the slope $\mathcal{T}(e(v))$ of the edge $e(v)$ is primitive, and $T_{P(v),\mathbb{C}}$ is orthogonal to $T_{\mathcal{T}(e(v)),\mathbb{C}}$ which contains the element $z_{e(v)}^{\mathcal{T}(e(v))}$. This shows that

$$|T_{\text{trop}}(\Gamma_{\text{aug}})| = \prod_{v \in \text{Vert}(\Gamma)} \text{mult}(v).$$

To prove the second statement of the Lemma, consider two unframed broken maps $u$, $u'$ modelled on $\Gamma_{\text{aug}}$. For vertices $v \in \text{Vert}_{\rightarrow}(\Gamma_{\text{aug}})$ containing markings, $u_v = u'_v$. Indeed, $u_v(z) = cz^\mu$ for some $\mu \in \mathbb{Z}^2$, and the point constraint determines the constant $c \in (\mathbb{C}^\times)^2$ uniquely. For all other vertices $v$, there exist elements $g_v \in (\mathbb{C}^\times)^2$ satisfying $g_v u_v = u'_v$ as follows: For any trivalent vertex $v$ we assume the domain is parametrized so that $z_1, z_2, z_3 \in C_v$ are the nodal points, and the edge incident at $z_i$ has slope $\mu_i \in \mathbb{Z}^2$. Then,

$$u_v = c \prod_{1 \le i \le 3} (z - z_i)^{\mu_i}, \quad u'_v = c' \prod_{1 \le i \le 3} (z - z_i)^{\mu_i},$$

for some $c, c' \in (\mathbb{C}^\times)^2$, and therefore $g_v u_v = u'_v$ for some $g_v \in (\mathbb{C}^\times)^2$. For a univalent vertex $v \in \mathrm{Vert}_1(\Gamma_{\mathrm{aug}})$ corresponding to an end of $\Gamma$, we have $u_v(z) = c(z - z_0)^\mu$, $u'_v(z) = c'(z - z_0)^\mu$ for some $c, c' \in (\mathbb{C}^\times)^2$, where $z_0$ is the nodal point and $\mu \in \mathbb{Z}^2$ is the slope of the edge $e(v)$. The matching condition at nodes for unframed broken maps implies that for any edge $e = (v_+, v_-)$, the condition (4.34) for a tropical symmetry element for unframed maps, namely $g_{v_+} g_{v_-}^{-1} \in T_{\mathcal{T}(e),\mathbb{C}}$, is satisfied. Finally consider (framed) broken maps $u, u'$ modelled on $\Gamma_{\mathrm{aug}}$. The preceding discussion for unframed maps implies that, after applying a tropical symmetry element, the unframed maps underlying $u$, $u'$ are the same, and therefore, by Lemma 4.37, $u$, $u'$ are related by a framing symmetry, which is an element of the tropical symmetry group $T_{\mathrm{trop}}(\Gamma)$. This finishes the proof of the Lemma. ∎

## Chapter 5

# Stabilizing divisors

We use domain-dependent perturbations of the almost complex structure to regularize the moduli space of pseudoholomorphic disks or spheres. Moduli spaces of maps are defined modulo automorphisms of the domain curve, and in order to define domain-dependent perturbations on every component of the map, we would like the domain components to be stable. As in Cieliebak-Mohnke [26], we consider domains with interior markings corresponding to points at which the pseudoholomorphic map intersects a Donaldson divisor. As a consequence, the almost complex structure can be perturbed to attain regularity on all domain components. Of course, one could envision using any of the current perturbation schemes to achieve virtual fundamental chains; the stabilizing divisor approach is merely a convenience.

## 5.1 Stabilizing divisors in symplectic manifolds

We recall results about stabilizing divisors for unbroken manifolds. We consider divisors of Donaldson type, meaning Poincaré dual to $k[\omega]$ where $k \gg 0$ is a large integer. To define and construct Donaldson divisors we assume that all the symplectic forms are rational.

**Definition 5.1.** A symplectic orbifold $(X, \omega)$ is *rational* if $[\omega] \in H^2(X, \mathbb{Q})$, and *integral* if $[\omega] \in H^2(X, \mathbb{Z})$. Given an integral form $\omega$, a *prequantum line bundle* is a line-bundle-with-connection $\mathcal{L}_X \to X$ whose curvature is

$$\operatorname{curv}(\mathcal{L}_X \to X) = (2\pi/i)\omega \in \Omega^2(X, i\mathbb{R}).$$

A Lagrangian $L \subset X$ disjoint from the singular locus of $X$ is *rational* if there exists a prequantum bundle $\mathcal{L}_X$, an integer $k$ and a flat section $s : L \to \mathcal{L}_X^{\otimes k}$ of the restriction $\mathcal{L}_X^{\otimes k} | L$.

**Definition 5.2.** (a) A *divisor* in $X$ is a symplectic suborbifold $D \subset X$ of real codimension $\operatorname{codim}(D) = 2$ such that locally if $X$ is the quotient of a symplectic manifold $X$ by a finite group $\Gamma$, then the divisor $D$ is the quotient of a $\Gamma$-invariant divisor in $X$. A *Donaldson divisor* is a divisor $D$ that is Poincaré dual to $k[\omega]$ for some large integer $k > 0$, and $k$ is called the *degree* of the divisor $D$.

(b) An almost complex structure $J$ on $X$ is *adapted* to $D$ if $D$ is a $J$-almost complex submanifold, that is, $J(TD) = TD$. The space of $D$-adapted almost complex structures is denoted by

$$\mathcal{J}(X, D) = \{J \in \mathcal{J}(X) | J(TD) = TD\}.$$

(c) Let $D \subset X - L$ be a divisor disjoint from the Lagrangian $L$. For $E > 0$, a $D$-adapted almost complex structure $J$ is *$E$-stabilizing* for the divisor $D$ if and only if

(i) (No non-constant spheres) $D$ does not contain any $J$-holomorphic *orbifold sphere* $u : C \to X$ with symplectic area $< E$ (by an orbifold sphere we mean that $C$ is an orbifold Riemann surface of genus zero), and

(ii) (Sufficient intersections) any $J$-holomorphic orbifold sphere in $X$ with symplectic area $< E$ has at least 3 distinct points of intersection with $D$, and any $J$-holomorphic disk with symplectic area less than $E$ has at least one intersection with $D$.

A pair $(J, D)$ consisting of a divisor $D$ and an $\omega$-tamed $J \in \mathcal{J}(X, D)$ is *stabilizing* if $J$ is $E$-stabilizing for all $E > 0$. We call the divisor $D$ in a stabilizing pair a *stabilizing divisor*.

**Proposition 5.3.** *(Existence of a stabilizing pair,* [16, Section 4], [26, Section 8]*) Suppose $\omega \in \Omega^2(X)$ is a rational symplectic form on an orbifold $X$ and $L \subset X$ is a compact rational Lagrangian submanifold disjoint from the singular locus of $X$. Then, there exists*

(a) *a divisor $D \subset X - L$ such that $L$ is exact in $X - D$; and*

(b) *a tamed almost complex structure $J_0 \in \mathcal{J}(X, D)$ such that $(J_0, D)$ is a stabilizing pair and for every $E > 0$ there is an open neighbourhood*

$$\mathcal{J}(X, D; J_0, E) \subset \{J \in \mathcal{J}(X, D) : J|TD = J_0|TD\}$$

*of $J_0$ consisting of $E$-stabilizing almost complex structures adapted to $D$.*

The proof of the Proposition is outlined below. The construction of a stabilizing pair uses the following notion of a degree bound.

**Definition 5.4.** A constant $k_* > 0$ is a *degree bound* for a tamed almost complex structure $J$ on $(X, \omega)$ if for any $J$-holomorphic orbifold sphere $u : C \to X$,

$$c_1(u) := \int_C u^* c_1(TX) \leq k_* \omega(u).$$

**Lemma 5.5.** *(Existence of uniform degree bounds) (*[26, Lemma 8.11]*) Let $J_0$ be a compatible almost complex structure on the compact symplectic orbifold $(X, \omega)$. For any $0 < \epsilon < 1$, there exists a constant $k_*$ which is a degree bound for all tamed almost complex structures $J$ satisfying $\|J - J_0\|_{C^0(X)} < \epsilon$.*

The proof in [26], carried out for the manifold case, extends verbatim to the case of orbifolds. We reproduce the proof since an adaptation of the technique is used in the case of broken manifolds.

*Proof of Lemma 5.5.* Let $\gamma \in \Omega^2(X)$ be a closed two-form in the class of the first Chern class $c_1(TX) \in H^2(X)$. Using the norm $|v|^2 := \omega(v, J_0 v)$ on $TX$, and a $C^0$ norm on forms, for any $J \in B_\epsilon(J_0)$ and $v \in TX$, we have the bounds

$$\gamma(v, Jv) \le \|\gamma\|(1 + \|J - J_0\|)|v|^2 \le \|\gamma\|(1+\epsilon)|v|^2,$$
$$\omega(v, Jv) \ge (1 - \|J - J_0\|)|v|^2 \ge (1-\epsilon)|v|^2.$$

Define $k_* := \frac{1+\epsilon}{1-\epsilon}$. We have

$$\gamma(v, Jv) \le k_* \omega(v, Jv).$$

For a $J$-holomorphic orbifold sphere $u : C \to X$ this implies $c_1(u) \le k_* \omega(u)$. ■

*Remark* 5.6. (Using the degree bound for stabilizing) The degree bound on almost complex structures is related to the stabilizing property in Definition 5.2 as follows. In the manifold case, by [26, Lemma 8.13], if a divisor $D$ is Poincaré dual to $k[\omega]$, and a $D$-adapted tamed almost complex structure $J$ has a degree bound $k_*$ satisfying

$$k > 2\max\{k_*, k_* + \tfrac{1}{2}(\dim(X)) - 2\}, \tag{5.1}$$

then the expected dimension of moduli space of $J$-holomorphic spheres that are not stabilized by $D$ is negative. The same holds in the orbifold case also. Indeed, for an orbifold sphere $u : C \to X$, in the Hirzebruch-Riemann-Roch formula for the expected dimension of the moduli space of maps, the Chern number $c_1(u^*TX)$ is replaced by the first Chern number of the "de-singularization" of $u^*TX$, and the latter is lesser than the former, see [19, p24], [20, Proposition 4.2.1]. Therefore if $J$ were chosen generically, $(J, D)$ would be a stabilizing pair.

*Outline of proof of Proposition 5.3.* In the case when $X$ is a manifold, the adaptation of Donaldson's construction in Auroux-Gayet-Mohsen [5] produces an approximately holomorphic divisor in $X - L$. By [16, Theorem 3.6], if the Lagrangian $L$ is rational, then for any divisor $D$ in $X - L$ produced by Auroux-Gayet-Mohsen [5], $L$ is exact in $X - D$. (We refer the reader to [16, Example 3.2] for the definition of "exactness".) Exactness allows the following relation of area to intersection numbers: If $L$ is exact in $X - D$, then, the intersection number of a disk $u : (C, \partial C) \to (X, L)$ with the divisor $D$ is proportional to the area of the disk: If $[D]^\vee = k[\omega]$ for some $k \in \mathbb{Z}$, then,

$$\# u^{-1}(D) = k \int_C u^* \omega. \tag{5.2}$$

See [16, Lemma 3.4]. As a consequence, for any $J$ that is adapted to $D$, any non-constant $J$-holomorphic disk is automatically stabilized since it has at least one marked point.

Donaldson's divisor construction is extended to the orbifold case by Gironella-Muñoz-Zhou [37]. Since the Lagrangian is smooth the modifications of Auroux-Gayet-Mohsen [5] do not interfere with the arguments in Gironella-Muñoz-Zhou [37]. The relation between symplectic area and the number of divisor intersections extends to the orbifold case if the Lagrangian does not contain orbifold singularities, the proof from [16] carries over verbatim.

To construct a stabilizing pair one starts with a preliminary almost complex structure $J^{\mathrm{pre}}$ that is $\omega$-compatible. For any $\epsilon > 0$ there exists a constant $k_* := k_*(\epsilon, J^{\mathrm{pre}})$ that is a degree bound for any $\omega$-tamed almost complex structure $J$ on $X$ satisfying $\|J - J^{\mathrm{pre}}\|_{C^0} \leq \epsilon$. This fact gives enough wiggle room to find a stabilizing pair as follows: For any $\epsilon' > 0$ there is a constant $k_d(\epsilon')$ such that if the degree of the Donaldson divisor $D$ is $\geq k_d(\epsilon')$, then there is a tamed almost complex structure $J_1 \in \mathcal{B}_{\epsilon'}(J^{\mathrm{pre}})$ that is $D$-adapted. We choose the degree $k$ of the Donaldson divisor to be high enough that it satisfies the bound (5.1), that is,

$$k > 2\max\{k_*, k_* + \tfrac{1}{2}(\dim(X)) - 2\}$$

and $k \geq k_d(\epsilon/2)$. Then, there is an open subset of $\mathcal{J}(X, D)$ contained in $B_\epsilon(J^{\mathrm{pre}})$, and for any $J \in \mathcal{J}(X, D) \cap B_\epsilon(J^{\mathrm{pre}})$, $k_*$ is a degree bound in the sense of Definition 5.4. Therefore by Remark 5.6, for a generic $J_0$ in $\mathcal{J}(X, D) \cap B_\epsilon(J^{\mathrm{pre}})$, the pair $(J_0, D)$ is stabilizing for all orbifold spheres. See [26, Section 8] for details. ■

## 5.2 Cylindrical almost complex structures, without gluability

Donaldson's construction of approximately holomorphic divisors requires a rational symplectic manifold with a compatible almost complex structure. Broken manifolds with cylindrical almost complex structures, as defined in Chapter 3, do not meet this requirement because they do not have natural embeddings into symplectic broken manifolds. We define a modified version of cylindricity, called $\mathfrak{X}$-*cylindricity* for almost complex structures, so that symplectic broken manifolds possess $\mathfrak{X}$-cylindrical almost complex structures. We point out that an $\mathfrak{X}$-cylindrical almost complex structure is not *gluable*, that is, it can not be glued on the ends to yield neck-stretched almost complex structures on $X^\nu$. Indeed, the $\mathfrak{t}$-inner product underlying an $\mathfrak{X}$-cylindrical almost complex structure is not the same for all the components $\overline{X}_P$ of the broken manifold $\mathfrak{X}$, and has the following form.

**Definition 5.7.** ($\mathfrak{X}$-inner product) Let $g^P_Q : \mathfrak{t}_Q \times \mathfrak{t}_Q \to \mathbb{Q}$ be a collection of inner products for all $Q \subset P$, where $P \in \mathcal{P}^0$ ranges over top-dimensional polytopes, that satisfy

(a) (Restriction) $R \subset Q \implies g^P_R|\mathfrak{t}_Q = g^P_Q$; and
(b) (Orthogonality) for $Q \subset P$, $\dim(Q) = 0$, which is the intersection of facets $Q = \cap_{i=1}^n Q_i$ of $P$, the inner product $g^P_Q$ satisfies the condition that the outward normals to $Q_i$ are orthogonal.

We define symplectic cylindrical structure maps on symplectic cut spaces $\{\overline{X}^\omega_P\}_{P\in\mathcal{P}}$ using $\mathfrak{X}$-inner products for the consistency condition.

**Definition 5.8.** ($\mathfrak{X}$-symplectic cylindrical structure maps) For any top-dimensional polytope $P$ let $(\phi^P_Q)_{Q\subset P}$ be a collection of symplectic cylindrical structure maps

$$U_{\overline{X}^\omega_P}(\overline{X}^\omega_Q) \xrightarrow{\phi^P_Q} ((\mathrm{Cone}_Q P \times \overline{Z}^\omega_Q)/\sim, \omega_{\tilde{Q}}), \quad \omega_{\tilde{Q}} := \omega_{X_Q} + d\langle \alpha^P_Q, \pi_{\mathfrak{t}^\vee_Q}\rangle \tag{5.3}$$

on neighborhoods $U_{\overline{X}^\omega_Q}(\overline{X}^\omega_P) \subset \overline{X}^\omega_P$ of $\overline{X}^\omega_Q$, where

- $\overline{Z}^\omega_Q \to \overline{X}^\omega_Q$ is the $T_Q$-bundle from (3.17),
- $\alpha^P_Q \in \Omega^1(\overline{Z}^\omega_Q, \mathfrak{t}_Q)$ is a connection one-form, where the collection $(\alpha^P_Q)_{Q\subset P}$ satisfies the consistency condition (3.23) with respect to the $\mathfrak{X}$-inner product; and
- $\pi_{\mathfrak{t}^\vee_Q} : \mathfrak{t}^\vee \to \mathfrak{t}^\vee_Q$ is the projection on $\mathrm{Cone}_Q P \subset \mathfrak{t}^\vee$, and so, $\langle \alpha_Q, \pi_{\mathfrak{t}^\vee_Q}\rangle$ is a one-form on $(\overline{Z}^\omega_Q \times \mathrm{Cone}_P Q)$;
- for any facet $R \subset P$ with $Q \subseteq R$, the equivalence $\sim$ mods out the boundary $\mathrm{Cone}_Q R \subset \mathrm{Cone}_Q P$ by the action of $T_R \simeq S^1$; and [1]
- $\overline{X}^\omega_Q$ is identically mapped by $\phi^P_Q$ to $Q \times \overline{Z}^\omega_Q/T_Q$.

The map $\phi^P_Q$ induces a projection map

$$\pi^P_Q : U_{\overline{X}^\omega_Q}(\overline{X}^\omega_P) \to \overline{X}^\omega_Q. \tag{5.4}$$

Finally, we define $\omega_{\mathfrak{X}}$-compatible almost complex structures. In addition to being $\omega$-compatible on each piece, in neighborhoods $U_{\overline{X}^\omega_Q}(\overline{X}^\omega_P)$ of boundary submanifolds these almost complex structures are integrable on the fibers of the map $\pi^P_Q$.

**Definition 5.9.** ($\mathfrak{X}$-cylindrical almost complex structures) Let $\mathfrak{X}$ be a broken symplectic manifold equipped with $\mathfrak{X}$-symplectic cylindrical structure maps as in (5.3)

[1]By Lerman's construction $(\mathrm{Cone}_Q P \times \overline{Z}^\omega_Q, \omega_{\tilde{Q}})/\sim$ is a manifold resp. orbifold if $\mathrm{Cone}_Q P$ is Delzant resp. simple.

and resulting projection maps

$$\pi_Q^P : U_{\overline{X}_Q^\omega}(\overline{X}_P^\omega) \to \overline{X}_Q^\omega$$

on neighborhoods $U_{\overline{X}_Q^\omega}(\overline{X}_P^\omega) \subset \overline{X}_P^\omega$ of $\overline{X}_Q^\omega$ for $Q \subset P$.

(a) ($\mathfrak{X}$-cylindrical) An $\mathfrak{X}$-*cylindrical* almost complex structure $\mathfrak{J} = (J_P)_{P \in \mathcal{P}}$ on $\mathfrak{X}$ consists of an almost complex structure $J_P$ on each (compactified) cut space $\overline{X}_P^\omega$, $P \in \mathcal{P}$ satisfying

  (i) (Restriction) for any pair $Q \subset P$ of polytopes, $\overline{X}_Q^\omega$ is a $J_P$-holomorphic submanifold of $\overline{X}_P^\omega$, and $J_P|\overline{X}_Q^\omega = J_Q$;

  (ii) (Cylindrical structure) for any pair $Q \subset P$ of polytopes, the fibers of $\pi_Q^P$ are $\mathfrak{J}$-holomorphic and $\mathfrak{J}$ is integrable on each of the fibers.

(b) ($\omega_{\mathfrak{X}}$-compatibility) An $\mathfrak{X}$-cylindrical almost complex structure $\mathfrak{J} = (J_P)_{P \in \mathcal{P}}$ is $\omega_{\mathfrak{X}}$-*compatible* if for any $P \in \mathcal{P}$, $J_P$ is $\omega_{X_P}$-compatible.

*Remark* 5.10. A collection of $\mathfrak{X}$-cylindrical almost complex structures can not be glued along the necks because the underlying $\mathfrak{t}$-inner products and connection one-forms vary across cut spaces.

**Lemma 5.11.** *An $\omega_{\mathfrak{X}}$-compatible almost complex structure exists.*

*Proof.* Starting from a broken symplectic manifold $\mathfrak{X}$, we construct an $\mathfrak{X}$-inner product, $\mathfrak{X}$-symplectic cylindrical structure maps, and an $\omega_{\mathfrak{X}}$-compatible almost complex structure $\mathfrak{J}$. We first construct these structures in the neighborhoods of vertices $Q \in \mathcal{P}_{(0)}$. Consider a top-dimensional polytope $P \in \mathcal{P}^{(0)}$ and a vertex $Q \in P$ which is the intersection of facets $Q_1, \dots Q_q \subset P$. There is a $T_Q$-equivariant symplectic embedding

$$U_{X_Q^\omega}\overline{X}_P^\omega \xrightarrow{\phi_Q^P} (\mathbb{C}^q/\Gamma, \omega_{\mathrm{std}}),$$

where $\Gamma$ is a finite group acting on $\mathbb{C}^q$ ($\Gamma$ is trivial if $Q$ is a smooth corner of $P$ as in (3.2)), $T_Q$ is the torus acting on $U_{X_Q^\omega}\overline{X}_P^\omega$ with an identification $T_Q = (S^1)^q/\Gamma$, and $(S^1)^q$ has a standard action on $\mathbb{C}^q$ that descends to a $T_Q$-action on $\mathbb{C}^q/\Gamma$. On $U_{X_Q^\omega}\overline{X}_P^\omega$, we define the almost complex structure $\mathfrak{J}$ by pulling back via $\phi_Q^P$; the inner product $g_Q^P$ is automatically defined. We may similarly define $\mathfrak{J}$ for all vertex neighborhoods $U_{X_Q^\omega}\overline{X}_P^\omega$, $P \in \mathcal{P}^{(0)}$, $Q \in \mathcal{P}_{(0)}$ so that the $\mathfrak{X}$-inner product satisfies the (Restriction) condition.

With the $\mathfrak{X}$-inner product fixed, we proceed to extend the $\omega_{\mathfrak{X}}$-compatible almost complex structure $\mathfrak{J}$ to all of the broken manifold by induction on the face structure of the polytopes in the polyhedral decomposition. Consider a polytope $Q \in \mathcal{P} \backslash \mathcal{P}^{(0)}$ that is not top-dimensional, and assume that an $\omega_{\mathfrak{X}}$-compatible collection of almost complex structures

$$\mathfrak{J}|\left(\cup_{R\subset Q, P\in\mathcal{P}^{(0)}:Q\subset P} U_{\overline{X}_R^\omega}(\overline{X}_P^\omega)\right) \tag{5.5}$$

is already defined. First, we extend $\mathfrak{J}| \cup_{R\subset Q} U_{\overline{X}_R^\omega}(\overline{X}_Q^\omega)$ to an $\omega_{X_Q}$-compatible almost complex structure over $\overline{X}_Q^\omega$. Next, we extend the definition of $\mathfrak{J}$ to neighborhoods $U_{\overline{X}_Q^\omega}(\overline{X}_P^\omega)$ for any $P \supset Q$, $P \in \mathcal{P}^{(0)}$ as follows. Suppose $Q_1, \ldots, Q_q \in \mathcal{P}$ are facets of $P$ such that $\cap_i Q_i = Q$. Identify $T_Q \simeq \prod_{i=1}^q T_{Q_i} \simeq (S^1)^q$. Choose a symplectic cylindrical structure map $\phi_Q^P$ (as in (5.3)) which is consistent with the existing maps, and let $\pi_Q^P$ be the resulting projection map as in (5.4). From $\phi_Q^P$ we obtain a family of $T_Q$-equivariant symplectic embeddings for $x \in \overline{X}_Q^\omega$

$$(\pi_Q^P)^{-1}(x) \xrightarrow{\phi_x} (\mathbb{C}^q/\Gamma, \omega_{\mathrm{std}})$$

where $\Gamma$ is a finite group and there is an identification $T_Q \simeq (S^1)^q/\Gamma$ so that $T_Q$ has a standard action on $\mathbb{C}^q/\Gamma$. The embeddings $\phi_x$ vary smoothly with $x \in Q$, and $(\pi_Q^P)^{-1}(x) \cap \overline{X}_{Q_i}^\omega$ is mapped to the subspace $\{z_i = 0\} \subset \mathbb{C}^q$. The fibers of the projection $\pi_Q^P : U_{\overline{X}_Q^\omega}(\overline{X}_P^\omega) \to \overline{X}_Q^\omega$ are equipped with the complex structure pulled back by $\phi_x$. For any $y \in U_{\overline{X}_Q^\omega}(\overline{X}_P^\omega)$, on the $\omega$-complement of $\ker(d\pi_Q^P)_y$ in the tangent space $T_y\overline{X}_P^\omega$, define $\mathfrak{J}$ to be equal to $\mathfrak{J}_{\overline{X}_Q^\omega}(\pi_Q^P(y))$. The definition of $\mathfrak{J}$ in neighborhoods of $\overline{X}_Q^\omega$ agrees with the existing definition in neighborhoods of $\overline{X}_R^\omega$ for any $R \subset Q$. Indeed, for any $R \subset Q$, and top-dimensional polytope $P \supset Q$, the definition of the $\mathfrak{X}$-inner product implies that $\mathfrak{J}|U_{\overline{X}_Q^\omega}(\overline{X}_P^\omega)$ agrees with $\mathfrak{J}|U_{\overline{X}_R^\omega}(\overline{X}_P^\omega)$ from (5.5) in the overlaps. In the concluding step of the induction, for each top-dimensional polytope $P \in \mathcal{P}^{(0)}$, we extend $\mathfrak{J}| \cup_{Q\subset P} U_{\overline{X}_Q^\omega}(\overline{X}_P^\omega)$ to an $\omega_{X_P}$-compatible almost complex structure on $\overline{X}_P^\omega$. ∎

## 5.3 Stabilizing divisors in broken manifolds

In this Section, we construct stabilizing divisors in a broken manifold by a modification of Donaldson's construction. A broken divisor $\mathfrak{D}$ in a broken manifold $\mathfrak{X}$ is a Donaldson divisor $D_P$ in each cut space $\overline{X}_P$ of $\mathfrak{X}$ that is cylindrical in the ends of $\overline{X}_P$. Such submanifolds are constructed as approximately holomorphic submanifolds with respect to a compatible almost complex structure.

**Definition 5.12.** (Broken Donaldson divisor) Let $(X, \mathcal{P})$ be a tropical Hamiltonian manifold, where on any symplectic cut space $\overline{X}_P^\omega$, $P \in \mathcal{P}$, the symplectic form $\omega_{X_P}$ is rational. Let $\mathfrak{X} := \mathfrak{X}_{\mathcal{P}}$ be the broken manifold in the sense of Definition 3.51. For

$k \gg 0$, a *broken Donaldson divisor* of degree $k$ on $\mathfrak{X}$ is a collection of codimension two submanifolds

$$\mathfrak{D} = \{\mathfrak{D}_P \subset \mathfrak{X}_P, \quad P \in \mathcal{P}\}$$

such that

(a) (Cylindricity on neck pieces) for any $P \in \mathcal{P}$, there is a codimension two submanifold

$$D_P \subset X_P \quad \text{such that} \quad \mathfrak{D}_P = \pi_P^{-1}(D_P),$$

where $\pi_P : \mathfrak{X}_P \to X_P$ is the projection map;

(b) (Cylindricity on ends) for any pair of polytopes $Q \subset P$, the divisor $D_P \subset X_P$ is $Q$-cylindrical in the $Q$-cylindrical end $U_Q(X_P)$, that is, $D_P \cap U_Q(X_P) = (\pi_Q^P)^{-1}(D_Q)$, where the projection $\pi_Q^P : U_Q(X_P) \to X_Q$ is from (3.56); and

(c) (Poincaré dual) for any map $F : X_P \to \overline{X}_P^\omega$ that respects the cylindrical structure at the ends of $X_P$ (in the sense of Remark 3.55 (a)), the closure $\overline{F(D_P)} \subset \overline{X}_P^\omega$ is a suborbifold that is Poincaré dual to $k[\omega_{X_P}]$.

In the rest of this section, we construct a stabilizing divisor in the cut spaces of a broken manifold. The construction of the divisors is via a slight modification of Donaldson's technique [29]. We give an outline of Donaldson's construction following Auroux [4]. Let $(X, \omega)$ be a symplectic manifold with a compatible almost complex structure $J$. Let $\mathcal{L}_X \to X$ be a Hermitian line-bundle with connection $\alpha$ over $X$ whose curvature two-form $\text{curv}(\alpha)$ satisfies $\text{curv}(\alpha) = (2\pi/i)\omega$. Since our symplectic manifolds are rational we may always assume this to be the case after taking a suitable integer multiple of the symplectic form. We will construct approximately holomorphic sections $s_k$ of the line bundles $\mathcal{L}_X^k$ for large $k$ that are transverse in the sense that $|\overline{\partial} s_k| \ll |\partial s_k|$ on the zero set $s_k^{-1}(0)$. Then for large enough $k$ the zero set $s_k^{-1}(0)$ is transversally cut out, and is a divisor of $X$ with degree $k$. To study sections on the bundle $\mathcal{L}_X^k$, we use the metric

$$g_k := k\omega(\cdot, J\cdot)$$

on $X$. Under this metric, the effect of the non-integrability of $J$ becomes negligible as $k$ increases. We define the notions of *approximate holomorphicity* and *transversality*:

**Definition 5.13.** (Asymptotically holomorphic sequences of sections) Let $(X, \omega)$ be a symplectic manifold with $\omega$-compatible almost complex structure $J$ and a prequantum line bundle $\mathcal{L}_X \to X$. Let $(s_k)_{k \geq 0}$ be a sequence of sections of $\mathcal{L}_X^k \to X$.

(a) The sequence $(s_k)_{k \geq 0}$ is *asymptotically holomorphic* if there exists a constant $C$ and integer $k_0$ such that for $k \geq k_0$,

$$|s_k| + |\nabla s_k| + |\nabla^2 s_k| \leq C, \quad |\overline{\partial} s_k| + |\nabla \overline{\partial} s_k| \leq C k^{-1/2}. \tag{5.6}$$

(b) The sequence $(s_k)_{k\geq 0}$ is *uniformly transverse* to 0 if there exists a constant $\eta$ independent of $k$, and $k_0 \in \mathbb{Z}$ such that for any $x \in X$ and $k \geq k_0$ with $|s_k(x)| < \eta$, the derivative $\nabla s_k$ of $s_k$ is surjective at any point and satisfies $|\nabla s_k(x)| \geq \eta$.

In both definitions, the norms of the derivatives $\nabla s_k$ are evaluated using the metric $g_k = k\omega(\cdot, J\cdot)$.

Donaldson's construction uses a family of asymptotically holomorphic sections on $\mathcal{L}_X^k$, that are concentrated at a given point in $X$ : The Gaussian section centered at $x \in X$ is constructed by choosing Darboux coordinates $(z_1, \ldots, z_n) : (U_x, x) \to (\mathbb{C}^n, 0)$ that are approximately holomorphic in the following sense: Assuming that the Darboux coordinates map to a neighborhood $B \subset \mathbb{C}^n$ of the origin, and $\psi : \mathbb{C}^n \supset B \to X$ is the inverse map, then the norm $|\psi^* J - i|$ of the difference between the complex structures $\psi^* J$ and $i$ is at most $c|z|$, where $c$ is a uniform constant independent of $x$, and the derivative $|\nabla(\psi^* J - i)|$ is uniformly bounded. We choose a trivialization of the bundle $\mathcal{L}_X$ so that the Hermitian connection is $\sum_i (z_i d\overline{z}_i - \overline{z}_i dz_i)$ plus terms of higher order in $z, \overline{z}$. The Gaussian section is then defined as

$$\sigma_{k,x} := \beta_{k,x} \cdot e^{-k|z|^2} \tag{5.7}$$

on $U_x$, where $\beta_{k,x}$ is a cut-off function vanishing at a $g_k$-distance of $k^{1/6}$ from $x$, and extended by zero outside $U_x$.

The globalization process in Donaldson's construction uses the following result:

**Lemma 5.14.** *(Quantitative Sard's theorem,* [29, Theorem 20]*) Suppose* $0 < \delta < \frac{1}{4}$*, and* $f : B_+ \to \mathbb{C}$ *is a function defined on a ball* $B_+ \subset \mathbb{C}^n$ *of radius* $\frac{11}{10}$ *that satisfies* $\|f\|_{C^1} \leq \eta$*, where* $\eta := \delta \log(\delta^{-1})^{-p}$*. Then, there exists* $w \in \mathbb{C}$, $|w| \leq \delta$ *such that* $f - w$ *is* $\eta$*-transverse to* 0 *over the interior ball* $B$ *of radius* 1.

Lemma 5.14 is used to modify an approximately holomorphic section in order to achieve uniform transversality in the neighborhood of a given point: Given an asymptotically holomorphic section $s_k : X \to \mathcal{L}_X^k$ and a point $x \in X$, by applying Lemma 5.14 to the section $f_k := s_k/\sigma_{k,x} : B_{g_k}(x, \frac{11}{10}) \to \mathbb{C}$, we obtain a constant $w_k \in \mathbb{C}$ such that $f_k - w_k$ is uniformly transverse to the zero section. As a consequence, the difference $s_k - w_k\sigma_{k,x}$ is also uniformly transverse to the zero section in a neighborhood of $x$.

Uniform transversality on the entire manifold is obtained by iteratively adding successively smaller contributions, so that the transversality over previous neighborhoods is not disturbed. In particular, Donaldson's construction chooses a lattice of points $\Lambda_k \subset X$ such that $X$ is covered by the unit balls $B_{g_k}(x, 1)$, $x \in \Lambda_k$; and partitions the lattice into $N$ sets (where $N$ is $k$-independent)

$$\Lambda_k = I_1^k \cup \cdots \cup I_N^k$$

such that any two points in a set $I_j^k$ are separated by a uniform $g_k$-distance guaranteeing the following: For any $I_j^k$, the constants $w_{k,x}$ for the balls centered at points $x \in I_j$ can be chosen simultaneously and independently of each other. If $j_1 < j_2$, the constants $w_{k,x} \in \mathbb{C}$ for $x \in I_{j_2}^k$ are chosen to be small enough to not break the transversality in the balls belonging to $I_k^{j_1}$. We start with the zero section, and run the iteration described above. The iteration terminates in $N$ steps, which is $k$-independent, and the resulting section is a sum

$$\sigma_k := \sum_{x \in \Lambda_k} w_{k,x} \sigma_{k,x}.$$

that is uniformly transverse on $X$ for large $k$.

*Remark* 5.15. In case of a symplectic orbifold $(X, \omega)$, we consider the pre-quantum orbibundle $\mathcal{L}_X$, and a sequence of sections $s_k : X \to \mathcal{L}_X^k$ for $k$ such that $\mathcal{L}_X^k$ is a line bundle on $X$. Notions of asymptotic holomorphicity and transversality extend naturally to the orbifold case and are defined in Gironella-Muñoz-Zhou [37].

Next, we give a preliminary result constructing a gluable cylindrical almost complex structure on the broken manifold $\mathfrak{X}$, that will be used to construct Donaldson divisors. We start with compatible almost complex structures on the symplectic cut spaces (as in Definition 5.9 and constructed in Lemma 5.11), and produce gluable almost complex structures by twisting the inner product on $\mathfrak{t}$ in a truncation of the cylindrical end.

**Lemma 5.16.** *(Taming map) Let $\mathfrak{X}$ be a broken manifold with cylindrical end $U_Q(X_P) \subset X_P$ and projections $\pi_Q : U_Q(X_P) \to X_Q$ for each pair $Q \subset P$, and an $\omega_{\mathfrak{X}}$-compatible almost complex structure $\mathfrak{J} = (J_P)_P$ as in Definition 5.9. Then,*

(1) *there are are truncated cylindrical ends*

$$U''_Q(X_P) \subset U'_Q(X_P) \subset U_Q(X_P),$$

(2) *an almost complex structure $\mathfrak{J}' = (J'_P)_P$ on $\mathfrak{X}$ that is strongly compatible (as in Definitions 3.51 (d), 3.47 (b)) and $X$-cylindrical (Definition 3.51 (d)) on the ends $U''_Q(X_P)$ for all $Q \subset P$, and*

(3) *maps $\phi_P : X_P \to \overline{X}_P^{\omega}$*

*such that*

(a) *on $X_P \backslash U'_Q(X_P)$, $\phi_P$ is a diffeomorphism onto $X_P^{\omega}$, and $\phi_P^* J_P = J'_P$, and*

(b) *on $U'_Q(X_P)$, $\phi_P$ is projection to $X_Q$, that is, $\phi_P | U'_Q(X_P) = \phi_Q \circ \pi_Q$.*

*Proof.* Choose truncated cylindrical ends $U'_Q(X_P)$, $U''_Q(X_P)$ for all pairs $Q \subset P$, and choose domain-dependent inner products

$$g_Q^P : U_Q(X_P) \to (t_Q \otimes t_Q)^{\vee}$$

that is

- constant and equal to the $X$-inner product $g_Q$ from (3.10) on $U''_Q(X_P)$; and
- is constant and equal to an $\mathfrak{X}$-inner product (Definition 5.7) $g^P_Q$ on $U_Q(X_P)\backslash U'_Q(X_P)$.

On cylindrical ends, the maps $\phi_P$ are determined by maps on cones. For example, for a pair $Q \subset P$, on the domain of $\phi_P$, we have

$$U_Q(X_P) = (0,\infty)^k \times (Z_Q/T_P), \quad U'_Q(X_P) = \Pi_{i=1}^k(\delta_i,\infty) \times (Z_Q/T_P), \tag{5.8}$$

where $k = \operatorname{codim}_P(Q)$, and on the target space of $\phi_P$,

$$U_{\overline{X}^\omega_Q}\overline{X}^\omega_P = (\Pi_{i=1}^k(-\epsilon_i, 0] \times (Z_Q/T_P))/\sim, \tag{5.9}$$

where $\sim$ mods out boundaries by circle actions as in (5.3). The map $\phi_P$ is coordinate-wise given by maps $(0,\infty) \to (-\epsilon_i, 0]$ that are equal to 0 on $[\delta_i,\infty)$, and strictly increasing on $(0,\delta_i)$. On the complement of the cylindrical ends, the map $\phi_P$ is standard as in (3.48).

The almost complex structure $J'_P$ is defined to be the pullback $\phi_P^* J_P$ on the complement of $\cup_Q U'_Q(X_P)$. The pullback can be extended to the ends $\cup_Q U_Q(X_P)$ by keeping the base almost complex structures $d\pi_Q(J_P)$ fixed on $U_Q(X_P)$, ensuring $X$-cylindricity on $U''_Q(X_P)$, and interpolating the inner products and connection one-forms in the intervening subset $U''_Q(X_P)\backslash U'_Q(X_P)$. ■

The following is the main result of the section.

**Proposition 5.17.** *(Construction of a broken divisor) Let $\mathfrak{J}_0$ be a cylindrical almost complex structure as constructed in Lemma 5.16. For any $\theta > 0$, and a large enough $k \in \mathbb{N}$, there is a cylindrical Donaldson-type divisor $\mathfrak{D} \subset \mathfrak{X}$ that is $\theta$-approximately holomorphic, and such that on any cut space $\overline{X}_P$, the induced divisor $D_P \subset \overline{X}_P$ is Poincaré dual to $k[\omega_{X_P}]$.*

We refer the reader to [29] for the definition of $\theta$-approximate holomorphicity.

*Proof of Proposition 5.17.* The sections are constructed by running Donaldson's procedure simultaneously for all the manifolds in the set $\{\overline{X}_P\}_{P\in\mathcal{P}}$. Our modification of Donaldson's algorithm is limited to choosing appropriate Gaussian sections in order to ensure (Cylindricity on ends) is satisfied. The step of achieving global transversality by applying Lemma 5.14 is the same as the original algorithm, and therefore not discussed.

The prequantum bundle on a cut space $\overline{X}_P$ is defined as the pullback by a map to the the symplectic cut space $\overline{X}^\omega_P$. Lemma 5.16 constructs a map $\phi_P : \overline{X}_P \to \overline{X}^\omega_P$ and produces an almost complex structure $J_P$ such that

- a truncated cylindrical end $U'_Q(X_P) \subset U_Q(X_P)$ is mapped by $\phi_P$ to $X_Q$ via a projection map,

- on the complement of the truncated cylindrical end, $\phi_P$ is a diffeomorphism onto its image and $J_P$ is $\phi_P^* \omega_{X_P}$-compatible, and
- $\phi_P | X_Q = \phi_Q$.

Assuming that $\mathcal{L}_P \to \overline{X}_P^{\omega}$ is a prequantum line bundle with connection, we will construct asymptotically holomorphic sections on the pullback bundle $\phi_P^* \mathcal{L}_P \to X_P$. On $U'_Q(X_P)$, the connection on the pullback bundle is flat on the fibers of $\pi_Q$, and we will define sections that are constant in the fiber direction.

The notions of asymptotically holomorphicity and transversality are defined with respect to a dilated metric on the cut spaces. On any $X_P$, the metric $g$ is equal to $\omega_{X_P}(\cdot, J_P \cdot)$ in the complement of the cylindrical ends $U'_Q(X_P)$, $Q \subset P$, and on the cylindrical end $g_P$ is a product metric on the fibration $U_Q(X_P) \to X_Q$ that extends the metric $g_Q$ on $X_Q$. For any $k$, define a dilated metric

$$g_k := kg.$$

We describe a set of lattice points at which the Gaussian section for each tensor power of the given line bundle are centered. Given $k \gg 0$, a set of lattice points

$$\Lambda_{k,P} \subset X_P \backslash \cup_{Q \subset P} U'_Q(X_P)$$

is defined so that it contains $Ck^{\dim(\overline{X}_P)}$ number of points, and there is a covering

$$X_P \backslash \left( \cup_{Q \subset P} B_{g_k}(U'_Q(X_P), 1) \right) = \left( \cup_{x \in \Lambda_{k,P}} B_{g_k}(x, 1) \right).$$

(The set $\Lambda_{k,P}$ is called a lattice because it resembles a lattice locally in Darboux coordinates.) For any $p \in \Lambda_{k,Q}$, we will write down Gaussian sections

$$\sigma_{k,p,P} : \overline{X}_P \to \phi_P^* \mathcal{L}_P$$

centered at $p$ for all $P \supseteq Q$, and determine coefficients $w_{p,k} \in \mathbb{C}$ so that

$$\sum_{p \in \Lambda_Q, Q \subseteq P} w_{p,k} \sigma_{k,p,P} : \overline{X}_P \to \phi_P^* \mathcal{L}_P$$

is asymptotically holomorphic and uniformly transverse on $\overline{X}_P$. The set $\Lambda_k$ is partitioned into subsets $I_1, \ldots, I_N$ where $N$ is independent of $k$ while satisfying the following: For any pair

$$x \in \Lambda_{k,P} \backslash \cup_{Q \subset P} \Lambda_{k,Q}, \quad y \in \cup_{Q \subset P} \Lambda_{k,Q}$$

we have

$$x \in I_\alpha, y \in I_\beta \implies \beta < \alpha,$$

and at the $\alpha$-th iteration the coefficients $w_{k,p}$ are fixed for $p \in I_\alpha$. As a result, coefficients of the Gaussian sections centered at lattice points on $X_Q$ are fixed before those centered at lattice points on $X_P$ if $Q \subset P$.

We define Gaussian sections concentrated at the lattice points. Consider $p \in \Lambda_{k,Q}$. We wish to construct Gaussian sections $\sigma_{k,p,P} : X_P \to \phi_P^* \mathcal{L}_P^k$ for all $P \supseteq Q$ such that

$$\sigma_{k,p,P} | X_Q = \sigma_{k,p,Q}.$$

We will first define $\sigma_{k,p,Q}$ and then define $\sigma_{k,p,P}$ so that it is constant on the fibers of the projection $U_Q(X_P) \to X_Q$, and and decays at an exponential rate along the fiber $U_Q(X_P) \backslash U'_Q(X_P) \to X_Q$. The detailed construction of the sections is as follows: For any $P \supseteq Q$, choose Darboux coordinates $(z_1, \dots, z_{n(P)})$ on a neighborhood $U_p \subset \overline{X}_P^\omega$ centered at $p$, where $n(P) := \frac{1}{2}\dim(X_P)$, and the first $n(Q)$ coordinates are Darboux coordinates on $X_Q^\omega$. Choose a trivialization of $\mathcal{L}_P^k | U_P$ for which the connection is $k \sum_i (\overline{z}_i dz_i - z_i d\overline{z}_i)$. Let

$$s_{k,p,P}(z) := \beta_{k,z} e^{-k|z|^2} : \overline{X}_P^\omega \to \mathcal{L}_P^k,$$

where $\beta_{k,z}$ is a cut-off function vanishing at a $g_k$-distance of $k^{1/6}$ from $x$, and extended by zero outside $U_p$, and let

$$\sigma_{k,p,P} := \phi_P^* s_{k,p,P} : X_P \to \phi_P^* \mathcal{L}_P^k.$$

As a consequence, $\sigma_{k,p,Q}$ is an ordinary Gaussian section. For $P \supset Q$, the section $\sigma_{k,p,P}$ is constant on the fibers decays outside $U'_Q(X_P)$.

The globalization process consists of finding coefficients $\{w_p \in \mathbb{C}\}_{p \in \Lambda_k}$ such that

$$\sigma_{k,P} := \sum_{p \in \Lambda_{k,P}} w_p \sigma_{k,p,P}$$

is a uniformly transverse sequence of sections for each $P$. The proof of globalization carries over from [29]. The only new feature is to determine each coefficient $w_p$ in a $P$-independent way. This can be done by Lemma 5.18 which is a modification of Lemma 5.14.

Finally, for any pair $Q \subset P$, the zero set $\sigma_{k,P}^{-1}(0)$ is $Q$-cylindrical, because a lattice point $p \in \Lambda_{k,P}$ lies in the complement of the $Q$-cylindrical end, and the section $\sigma_{k,p,P}$ is supported in $B_{g_k}(p, k^{1/6})$, which is equal to $B_g(p, k^{-1/3})$. Therefore, there is a truncated cylindrical end $U''_Q(X_P) \subset U'_Q(X_P)$ on which, for large enough $k$, $\sigma_{k,P}^{-1}(0)$ is $Q$-cylindrical, since $U''_Q(X_P)$ is disjoint from $B_g(p, k^{-1/3})$ for all $p \in \Lambda_{k,P}$. ∎

**Lemma 5.18.** *(Quantitative Sard's theorem) Given a tuple of positive integers $(n_1, \dots, n_k) \in \mathbb{Z}^k$ there is an integer $p$ for which the following is satisfied. Suppose $0 < \delta < \frac{1}{4}$, and $f_i : B_+^{n_i} \to \mathbb{C}$ is a set of $k$ functions on balls $B_+^{n_i} \subset \mathbb{C}^{n_i}$ of radius $\frac{11}{10}$ that satisfy $\|f_i\|_{C^1} \leq \eta$, where $\eta := \delta \log(\delta^{-1})^{-p}$. Then, there exists $w \in \mathbb{C}$, $|w| \leq \delta$ such that $f_i - w$ is $\eta$-transverse to $0$ over the interior ball $B^{n_i}$ of radius $1$ for each $i$.*

The case $k = 1$ is Theorem 20 of [29], and is stated as Lemma 5.14. The proof in the case $k = 1$ is by bounding the size of the image $f(B_+)$ in the range. For a finite $k$, the volume is multiplied by a constant, and the proof in [29] can be replicated by altering the constants.

*Remark* 5.19. The construction of a broken divisor in Proposition 5.17 extends to the orbifold case. Indeed, in the orbifold adaptation of the Donaldson construction in Gironella-Muñoz-Zhou [37], the new features are the choice of an appropriate lattice compatible with the stratification of the orbifold; and adjusting the uniform transversality constants. Both of these features are compatible with the modifications we have introduced in the proof of Proposition 5.17.

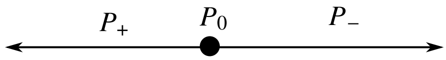



**Figure 5.1.** The polyhedral decomposition of a single cut.

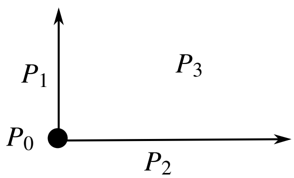



**Figure 5.2.** Multiple cut case.

*Remark* 5.20. (On the construction of broken stabilizing divisors) This is a technical remark justifying the proof technique of Proposition 5.17 wherein we construct asymptotically holomorphic sequences of sections $\sigma_{k,P}$ simultaneously for all polytopes $P \in \mathcal{P}$ rather than construct one sequence $(\sigma_{k,P})_k$ at a time. We point out that the latter approach, which appears cleaner, is taken in [17] for the case of single breaking. Consider a single cut with polyhedral decomposition shown in Figure 5.1. Given a sequence of sections

$$(\sigma_{k,P_0} : \overline{X}_{P_0} \to \mathcal{L}^k_{\overline{X}_{P_0}})_k,$$

[17, Lemma 7.15] constructs an extension of the sequence of sections to

$$(\sigma_{k,P_\pm} : \overline{X}_{P_\pm} \to \mathcal{L}^k_{\overline{X}_{P_\pm}})_k.$$

The first step in the construction is to consider the extension $\sigma_{k,P_0} e^{-k|x|^2}$ and then turn on the contributions from Gaussians centered away from the divisor $\overline{X}_{P_0} \subset \overline{X}_{P_\pm}$ so that the sections become transverse. If we apply this approach to spaces corresponding to the polytopes shown in Figure 5.2, we would construct a sequence in the order $\sigma_{k,P_0}$, then $\sigma_{k,P_1}$, $\sigma_{k,P_2}$ and then finally $\sigma_{k,P_3}$. We would like the sequence $\sigma_{k,P_3}$ on $\overline{X}_{P_3}$ to be an extension of $\sigma_{k,P_0}$ , $\sigma_{k,P_1}$ and $\sigma_{k,P_2}$. Therefore, we would like to start Donaldson's globalization iteration with a sequence of Gaussian sections that are equal to $\sigma_{k,P_0}e^{-k(|z_1|^2+|z_2|^2)}$ in a neighborhood of $\overline{X}_{P_0}$, equal to $\sigma_{k,P_1}e^{-k|z_1|^2}$ in a neighborhood of $\overline{X}_{P_1}$, and equal to $\sigma_{k,P_2}e^{-k|z_2|^2}$ in a neighborhood of $\overline{X}_{P_2}$. The approach fails because the three sections do not agree on overlaps. Indeed, to define the sequence $\sigma_{k,P_1}$, we would have used $\sigma_{k,P_0}e^{-k|z_2|^2}$ as a starting sequence, but then these sections would have been modified when contributions from Gaussians in nearby balls are turned on to achieve transversality.

## 5.4 Stabilizing pairs in neck-stretched manifolds

In this Section, we prove the existence of a stabilizing pair $(\mathfrak{J}_0, \mathfrak{D})$ on a broken manifold such that the family $(J^\nu, D^\nu)$ obtained by gluing the pair $(\mathfrak{J}_0, \mathfrak{D})$ consists of stabilizing pairs on neck-stretched manifolds. We recall from Section 5.1 that in a smooth symplectic manifold $(X, \omega_X)$, a stabilizing pair $(J, D)$ consists of a Donaldson-type divisor $D \subset X$ and a $\omega_X$-tamed almost complex structure $J$ that is adapted to $D$ and for which any non-constant $J$-holomorphic sphere intersects $D$ at least three points and is not contained in $D$. We start by defining the analogous notion for broken manifolds.

**Definition 5.21.** (Adapted broken almost complex structure) Given a broken divisor $\mathfrak{D} \subset \mathfrak{X}$ as in Definition 5.12, we denote by

$$\mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D}) = \{\mathfrak{J} \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}) | \mathfrak{J}(T\mathfrak{D}_P) = T\mathfrak{D}_P \text{ on } \overline{\mathfrak{X}}_P \ \ \forall P \in \mathcal{P}\}$$

the space of cylindrical almost complex structures that are adapted to $\mathfrak{D}$.

**Definition 5.22.** (Stabilizing pair in a broken manifold) Let $\mathfrak{D} \subset \mathfrak{X}$ be a broken divisor as in Definition 5.12 which is disjoint from the Lagrangian $L \subset \mathfrak{X}$. For $E > 0$, an adapted almost complex structure $\mathfrak{J} \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D})$, is $E$*-stabilizing* if for every polytope $P \in \mathcal{P}$, the almost complex structure $\mathfrak{J}|\overline{X}_P$ is $E$-stabilizing in $(\overline{X}_P, \omega_{X_P}, D_P)$. A pair $(\mathfrak{J}, \mathfrak{D})$ is *stabilizing* if $\mathfrak{J}$ is locally strongly tamed and is $E$-stabilizing for all $E > 0$. We call a divisor $\mathfrak{D}$ appearing in a stabilizing pair a *stabilizing divisor.*

The following is the main result of the Section where we construct stabilizing pairs on broken manifolds. We adapt the construction of stabilizing pairs in smooth manifolds outlined in Section 5.1.

**Proposition 5.23.** *(Stabilizing pair in a broken manifold) Suppose $\mathfrak{X} := \mathfrak{X}_{\mathcal{P}}$ is a broken manifold, such that on each cut space $\overline{X}_P$, $P \in \mathcal{P}$ the symplectic form $\omega_{X_P}$ is rational. There exists a stabilizing pair $(\mathfrak{J}_0, \mathfrak{D})$ in $\mathfrak{X}$ such that*

(a) *the family $(J^\nu, D^\nu)$ obtained by gluing consists of stabilizing pairs for all $\nu \in [1, \infty)$, and*

(b) *for every $E > 0$ there is a neighbourhood*

$$\mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D}; \mathfrak{J}_0, E) \subset \{\mathcal{J} \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D}) : \mathcal{J}|T\mathfrak{D} = \mathfrak{J}_0|T\mathfrak{D}\}$$

*of $\mathfrak{J}_0$ consisting of $E$-stabilizing locally tamed cylindrical almost complex structures adapted to $\mathfrak{D}$.*

*Proof of Proposition 5.23.* Step 1: We start with a preliminary almost complex structure, and find a uniform degree bound on its neighborhood, in order to determine the degree of the Donaldson divisor that we require. Suppose $\mathfrak{J}^{\mathrm{pre}}_\omega \in \mathcal{J}(\mathfrak{X})$ is a $\omega_{\mathfrak{X}}$-compatible almost complex structure (Definition 5.9). By Lemma 5.16, there are maps $\phi_P : X_P \to \overline{X}^\omega_P$ from cut spaces to symplectic cut spaces, and a locally compatible (hence gluable, see Definitions 3.51 (d), 3.47 (b)) almost complex structure on $(X_P)_P$, which we denote by $\mathfrak{J}^{\mathrm{pre}}$. The almost complex structure $\mathfrak{J}^{\mathrm{pre}}$ is cylindrical on a truncation of the cylindrical ends, henceforth in this proof, the cylindrical ends are replaced by these truncations, and are denoted by $U_Q(X_P) \subset X_P$ for any pair $Q \subset P$. We recall that cylindrical ends are equipped with a projections $\pi_Q : U_Q(X_P) \to X_Q$. For any $P \in \mathcal{P}^{(0)}$, Lemma 5.16 says that $\phi_P^* \omega_{X_P}$ is a *basic* form on the cylindrical ends of $X_P$, in the sense that for any $Q \subset P$, on $U_Q(X_P)$, $\phi_P^*\omega_{X_P}$ is a pullback of a form by $\pi_Q$. Gluing the forms $\phi_P^*\omega_{X_P}$ along the ends produces $\omega_\nu \in \Omega^2(X^\nu)$ which is cohomologous to $\omega_X \in \Omega^2(X)$, and which is basic in the cylindrical subsets. To derive the bound we also choose representatives of the first Chern classes $[c_1(T\overline{X}_P)]$ which are basic on cylindrical ends. That is, we choose forms

$$\gamma = (\gamma_P)_{P \in \mathcal{P}}, \quad \gamma_P \in \Omega^2(\overline{X}_P), \quad [\gamma_P] = c_1(T\overline{X}_P).$$

We choose forms on the neck-stretched manifolds

$$\gamma_\nu \in \Omega^2(X^\nu), \quad \gamma_\nu \in [c_1(TX)],$$

which are basic in the the sense that on the $P$-cylindrical subset $\pi^\nu_P : X^\nu_{\tilde{P}} \to X_P$, each of the forms $\gamma_\nu$ is the pullback of the same form on $X_P$.

For any $P \in \mathcal{P}$, the supremum of the ratio of the first Chern class form and the basic symplectic form given by

$$k_*(\epsilon, P) := \sup_{\substack{\mathfrak{J} : \|\mathfrak{J}, \mathfrak{J}^{\mathrm{pre}}\|_{C^0} < \epsilon \\ 0 \neq v \in TX_P}} \frac{\gamma_P(v, \mathfrak{J}v)}{\omega_{X_P}(v, \mathfrak{J}v)}$$

is finite. The proof of finiteness is as in the proof of Lemma 5.5, together with the fact that the forms $\gamma_P$ and $\omega_{X_P}$ are basic on $X_P$. Similarly, the supremum

$$k_*(\epsilon, X) := \sup_{\nu} \sup_{\substack{\mathfrak{J}:\|\mathfrak{J},\mathfrak{J}^{\text{pre}}\|_{C^0}<\epsilon \\ 0\neq v\in TX^\nu}} \frac{\gamma_\nu(v, J^\nu v)}{\omega_\nu(v, J^\nu v)},$$

where $J^\nu$ is given by gluing $\mathfrak{J}$ on the cylindrical ends, is finite. We choose $k_*$ satisfying

$$k_* \geq k_*(\epsilon, X), k_*(\epsilon, P) \quad \forall P \in \mathcal{P}^{(0)}. \tag{5.10}$$

Step 2 : We find a stabilizing divisor in the broken manifold. We will find a stabilizing divisor of degree

$$k > 2\max\{k_*, k_* + \tfrac{1}{2}(\dim(X)) - 2\}. \tag{5.11}$$

for reasons explained in Remark 5.6. Let $\theta_0 > 0$ be such that for any $\theta_0$-approximately $\mathfrak{J}^{\text{pre}}$-holomorphic divisor, there is an adapted tamed almost complex structure that is $\epsilon/2$-close to $\mathfrak{J}^{\text{pre}}$. In addition to satisfying (5.11), we may choose $k$ to be large enough that a Donaldson divisor of degree $k$ is $\theta_0$-approximately holomorphic. By Proposition 5.17, there is a divisor $\mathfrak{D}$ cylindrical in $\{U_Q(X_P)\}_{Q\subset P}$. There is an adapted strongly tamed almost complex structure

$$\mathfrak{J}_0 \in B_{\epsilon/2}(\mathfrak{J}^{\text{pre}}),$$

and furthermore, there is a neighborhood of adapted almost complex structures

$$U_{\mathfrak{J}_0} := \{\mathfrak{J} \in \mathcal{J}^{\text{cyl}}(\mathfrak{X}, \mathfrak{D}) : \mathfrak{J} \text{ is locally strongly tamed}, \|\mathfrak{J} - \mathfrak{J}_0\|_{C^0} < \epsilon/2\}$$

that satisfy the degree bound of $k_*$ on the cut spaces $\{x_P\}_P$, and their gluings satisfy the same degree bound on the neck-stretched manifolds $\{X_\nu\}_\nu$.

Step 3: We finish the proof closely following the method in the unbroken case in Cieliebak-Mohnke [26], which is outlined in Proposition 5.3. So far, we have shown that there is a neighborhood $U_{\mathfrak{J}_0}$ in the space of locally strongly tamed $\mathfrak{D}$-adapted cylindrical almost complex structures on $\mathfrak{X}$ such that for any $\mathfrak{J} \in U_{\mathfrak{J}_0}$, $k_*$ is a degree bound for the glued family $J^\nu$. By Remark 5.6 generic almost complex structures in the set $U_{\mathfrak{J}_0}$ are stabilizing.

Next, we claim that for any $E > 0$, an open and dense subset of $U_{\mathfrak{J}_0}$ is $E$-stabilizing for all neck lengths. The subset

$$\mathcal{J}^{\text{reg},E,\infty} = \{\mathfrak{J} \in U_{\mathfrak{J}_0} : \mathfrak{J} \text{ is } E\text{-stabilizing on } \mathfrak{X}\}$$

is open and dense: Openness is a consequence of Gromov convergence applied to each of the cut spaces $\overline{X}_P$, $P \in \mathcal{P}$ (see the proof of [26, Corollary 8.16]), and denseness follows because the $E$-stabilizing condition is generic. The subset

$$\mathcal{J}^{\text{reg},E} = \{\mathfrak{J} \in \mathcal{J}^{\text{reg},E,\infty} : J^\nu \text{ is } E\text{-stabilizing on } X^\nu \text{ for } \nu \in [1, \infty)\}.$$

is also open in $\mathcal{J}^{\mathrm{reg},E,\infty}$ by Gromov compactness. Openness at the infinite neck length parameter is proved by Lemma 5.24. The subset $\mathcal{J}^{\mathrm{reg},E} \subset \mathcal{J}^{\mathrm{reg},E,\infty}$ is comeager by an application of Sard's theorem : The universal moduli space

$$\mathcal{M}_{\mathcal{J}} := \{u_\nu : \mathbb{P}^1 \to X^\nu : u_\nu \text{ is } J^\nu\text{-holomorphic}, \mathfrak{J} \in \mathcal{J}^{\mathrm{reg},E,\infty}, \nu \in [1,\infty)\}$$

is a Banach manifold and the projection $\pi_{\mathcal{J}} : \mathcal{M}_{\mathcal{J}} \to \mathcal{J}^{\mathrm{reg},E,\infty}$ is a Fredholm map. Therefore, the subset $\mathcal{J}^{\mathrm{reg},E}$ of regular values of $\pi_{\mathcal{J}}$ is comeager.

Finally, let $E_k \to \infty$ be any sequence of real numbers with limit infinity. The set of almost complex structures

$$\mathcal{J}^{\mathrm{reg}} = \bigcap_{k=1}^{\infty} \mathcal{J}^{\mathrm{reg},E_k}$$

that is stabilizing for all $\nu \in [1,\infty]$ is the intersection of the set of $E_k$-stabilizing almost complex structures for all $k$. The intersection $\mathcal{J}^{\mathrm{reg}}$ is non-empty because each of the sets in the intersection is open and dense. ∎

The following Lemma, used above in the proof of Proposition 5.23, is an openness statement for stabilizing almost complex structures at $\nu = \infty$.

**Lemma 5.24.** *Suppose $\mathfrak{D} \subset \mathfrak{X}$ is a cylindrical broken divisor, and $\mathfrak{J} \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X},\mathfrak{D})$ is a tamed adapted almost complex structure that is $E$-stabilizing. Suppose the divisors $D^\nu \subset X^\nu$ are obtained by gluing, and the sequence $J^\nu \in \mathcal{J}(X^\nu, D^\nu)$ converges to $\mathfrak{J}$. Then, there exists $\nu_0$ such that $J^\nu$ is $E$-stabilizing for $\nu \geq \nu_0$.*

The proof of the Lemma requires terminology from Chapter 7 and Section 8.2. The proof is by a hard rescaling argument similar to the one used in the proof of Gromov convergence for breaking maps.

*Proof of Lemma 5.24.* Suppose $u_\nu : C \to X^\nu$ is a sequence of non-constant $J^\nu$-holomorphic spheres with area $\leq E$ that are not stabilized. That is, either the images are contained in the divisor $D^\nu$ or they have $\leq 2$ distinct points of intersection with the divisor.

First, consider the situation that the derivatives of $u_\nu$ are uniformly bounded. By Lemma 8.18, after passing to a subsequence, the sequence $u_\nu$ converges horizontally in some polytope $P$. By Lemma 8.17, there is a sequence of translations $t_\nu \in \nu P^\vee$ such that a subsequence of $\mathfrak{e}^{-t_\nu} u_\nu$ uniformly converges to a non-constant $\mathfrak{J}_P$-holomorphic map $u : C \to \mathfrak{X}_P$, that is unstabilized. The projection $\pi_P \circ u : C \to X_P$ is non-constant, because it is not possible for the image of $u$ to be non-constant and lie in a single fiber of $\pi_P$, which is isomorphic to $T_{P,\mathbb{C}}$. By the definition of $P$-Hofer energy, for closed curves in $X_P$,

$$\langle (\pi_P \circ u)_*[C], \omega_{X_P} \rangle \leq E^*_{P,\mathrm{Hof}}(\pi_P \circ u).$$

We have

$$E^*_{P,\mathrm{Hof}}(\pi_P \circ u) \leq \lim_\nu E_{\mathrm{Hof}}(u_\nu) = \langle (u_\nu)_*[C], \omega_X \rangle \leq E,$$

where the first inequality is by Proposition 7.27, and from the fact that Hofer energy is monotonic for maps that are holomorphic with respect to a locally strongly tamed almost complex structure (Lemma 7.22), and the equality is from Remark 7.19. As a consequence,

$$\mathrm{Area}(u) = \langle (\pi_P \circ u)_*[C], \omega_{X_P} \rangle \leq E.$$

If the derivatives on $u_\nu$ are not uniformly bounded, we produce a sphere by hard rescaling. By following the procedure in Step 2 of the proof of Theorem 8.2, we obtain a rescaled sequence of $J^\nu$-holomorphic maps $v_\nu : B_{r_\nu} \to X^\nu$ on balls $B_{r_\nu}$ that exhaust $\mathbb{C}$, a polytope $P$, and a sequence of translations $t_\nu \in \nu P^\vee$ such that $\mathrm{e}^{-t_\nu} u_\nu$ converges in $C^\infty_{\mathrm{loc}}$ to a non-constant map $v : \mathbb{C} \to \mathfrak{X}_P$. In this preceding step, the rescaled maps are found in the same way as in [55, Lemma 4.6.5], the polytope $P$ and translation sequences are given by Lemma 8.17. As in Step 2 in the proof of Theorem 8.2, the map $v$ has finite $P$-Hofer energy, and by Proposition 7.1, the infinite end of $v$ is asymptotic to a trivial cylinder. Therefore, the projection $\pi_P \circ v$ extends to $\pi_P \circ v : C_0 \to \overline{X}_P$, for some weighted projective space $C_0 = \mathbb{P}(1, n)$. The projection $\pi_P \circ v : C_0 \to \overline{X}_P$ is non-constant: Otherwise, the image of $v(\mathbb{C})$ is in a fiber $V_{P^\vee}$ which is a toric variety, and the map extends to $v : C_0 \to V_{P^\vee}$ with only one point $\infty \in C_0$ that maps to toric divisors $V_{Q^\vee}, Q \supset P$ of $V_{P^\vee}$, and therefore, $v$ is constant. By the argument in the previous paragraph, $\mathrm{Area}(v) = \langle (\pi_P \circ v)_*[C_0], \omega_{X_P} \rangle \leq E$.

In both cases, the limit (orbifold) sphere is not stabilized and the Lemma follows.

∎

# Chapter 6

# Coherent perturbations and regularity

In order to obtain the necessary transversality for moduli spaces of treed holomorphic maps, we use the Cieliebak-Mohnke perturbation scheme [26]. This scheme has been adapted to define Fukaya algebras of Lagrangians by Charest-Woodward [16]. The Morse function on the Lagrangian submanifold and the almost complex structure on the broken manifold are allowed to be domain-dependent. Stabilizing divisors from the previous chapter are essential in defining domain-dependent perturbations.

A domain-dependent perturbation is defined as a map from a universal curve to the space of tamed almost complex structures. We recall from Chapter 4 that domain curves are treed disks. For any type $\Gamma$ of treed disks the moduli space $\mathcal{M}_\Gamma$ of treed disks has a universal curve

$$\mathcal{U}_\Gamma \to \mathcal{M}_\Gamma$$

whose fiber over a point $[C] \in \mathcal{M}_\Gamma$ is isomorphic to $C$. The perturbation datum is then a map from $\mathcal{U}_\Gamma$ to the space of cylindrical almost complex structures on the broken manifold and Morse functions on the Lagrangian submanifold. Under this perturbation scheme, marked points on the domain curve are the inverse images of the intersection of the holomorphic map with the stabilizing divisor. The stabilizing divisor is Poincaré dual to a large multiple of the symplectic form. This ensures that generically there are no holomorphic spheres in the divisor, and that any non-constant holomorphic sphere has enough divisor intersections to ensure that its domain is stable.

Domain-dependent perturbations are necessary for solving the multiple cover problem. Indeed, for a fixed almost complex structure the compactification of the moduli space of pseudoholomorphic maps may contain nodal maps with components that are multiple covers of maps with negative Chern class. It is not possible to achieve transversality for the moduli space of such maps. Multiple covers can be perturbed away by domain-dependent perturbations. As a toy example, suppose for a domain-independent almost complex structure $J_0$, there is a simple $J_0$-holomorphic sphere $S \subset X$ with negative Chern number and one intersection point with the stabilizing divisor. Via a domain-dependent perturbation of $J_0$, we can ensure that an $n$-covered sphere ($n > 1$) homologous to $n[S]$ does not occur in the moduli space of perturbed holomorphic maps. Indeed, for such a cover the domain would have $n$ marked points labelled $z_1, \dots, z_n$, and the perturbation is not required to be invariant under automorphisms of the domain curve that permute the marked points. Breaking the permutation symmetry adds a multiplicative factor of $\frac{1}{n!}$ to the curve count (see Remark 6.15).

## 6.1 Domain-dependent perturbations

Domain-dependent perturbations $\mathfrak{p}_\Gamma$ are defined for each domain type $\Gamma$, and later in the section we describe a set of *coherence* conditions on the set of perturbation data $(\mathfrak{p}_\Gamma)_\Gamma$ to ensure that the perturbations on various strata fit in together in an expected manner.

To describe perturbations for a domain type, we first fix subsets of the universal treed disk outside of which perturbations vanish. Let $\Gamma$ be a combinatorial type of treed disk and $\overline{\mathcal{U}}_\Gamma = \overline{\mathcal{S}}_\Gamma \cup \overline{\mathcal{T}}_\Gamma$ its universal curve from (4.6). Fix a compact subset

$$\overline{\mathcal{T}}_\Gamma^{\mathrm{cp}} \subset \overline{\mathcal{T}}_\Gamma$$

in the complement of breaking points and disk nodes (that is, disk nodes $w_e$ where the length $\ell(e)$ of the treed segment is zero), and containing in its interior, for every edge $e \in \mathrm{Edge}_\circ(\Gamma)$ and every curve $C \subset \overline{\mathcal{U}}_\Gamma$, at least one point $z \in T_e \subset C$ on any infinite segment. Also fix a compact subset

$$\overline{\mathcal{S}}_\Gamma^{\mathrm{cp}} \subset \overline{\mathcal{S}}_\Gamma - \{w_e \in \overline{\mathcal{S}}_\Gamma,\ e \in \mathrm{Edge}_-(\Gamma)\}$$

disjoint from the boundary and spherical nodes $w_e \in C, e \in \mathrm{Edge}(\Gamma)$, containing in its interior at least one point $z \in S_v$ on each sphere and disk component $S_v \subset C$ for each fiber $C \subset \overline{\mathcal{U}}_\Gamma$. Furthermore, the complement $\overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}_\Gamma^{\mathrm{cp}} \subset \overline{\mathcal{S}}_\Gamma$ is a neighbourhood of the boundary and nodes; these neighborhoods must be chosen compatibly with those already chosen on the boundary for the inductive construction later.

Following Floer [35], we use a $C^\varepsilon$-topology on the space of almost complex structures. For a section $\xi$ of a vector bundle $E \to X$, the $C^\varepsilon$-norm is

$$\|\xi\|_{C^\varepsilon} := \textstyle\sum_{k=0}^\infty \varepsilon_k \|\xi\|_{C^k(X,E)}.$$

Here $\varepsilon = (\varepsilon_i)_{i\in\mathbb{N}}$ is a fixed sequence of positive numbers that converges fast enough to 0 as $i \to \infty$. If the convergence is sufficiently rapid, then the space of sections with a bounded norm is a Banach space [35, Lemma 5.1] and contains sections supported in arbitrarily small neighbourhoods of $X$.

**Definition 6.1.** (a) (Domain-dependent Morse functions) Suppose that $\Gamma$ is a type of stable treed disk, and $\overline{\mathcal{T}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ is the tree part of the universal treed disk. Let

$$(F_0 : L \to \mathbb{R}, G : T^{\otimes 2}L \to \mathbb{R})$$

be a Morse-Smale pair. For an integer $l \geq 0$ a *domain-dependent perturbation* of $F_0$ of class $C^l$ is a $C^l$ map

$$F_\Gamma : \overline{\mathcal{T}}_\Gamma \times L \to \mathbb{R} \tag{6.1}$$

equal to the given function $F$ away from the compact part:

$$F_\Gamma|(\overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}_\Gamma^{\mathrm{cp}}) = \pi_2^* F_0$$

where $\pi_2$ is the projection on the second factor in (6.1). Here $F_0$ is called the *background Morse function* for the domain-dependent perturbation $F_\Gamma$. The set of critical points of the background Morse function is denoted by

$$\mathcal{I}(L) := \mathrm{crit}(F). \tag{6.2}$$

(b) (Domain-dependent almost complex structure) Let $\mathfrak{J}_0 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ be a locally strongly tamed cylindrical almost complex structure. A *domain-dependent almost complex structure* of class $C^\varepsilon$ for treed disks of type $\Gamma$ with *background almost complex structure* $\mathfrak{J}_0$ is a map from the two-dimensional part $\overline{\mathcal{S}}_\Gamma$ of the universal curve $\overline{\mathcal{U}}_\Gamma$ to $\mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ given by

$$\mathfrak{J}_\Gamma : \overline{\mathcal{S}}_\Gamma \to \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})^{1} \tag{6.3}$$

equal to the given $\mathfrak{J}_0$ away from the compact part:

$$\mathfrak{J}_\Gamma|(\overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}_\Gamma^{\mathrm{cp}}) = \mathfrak{J}_0, \tag{6.4}$$

and on any fiber $S_\Gamma \subset \overline{\mathcal{S}}_\Gamma$, $\mathfrak{J}_\Gamma - \mathfrak{J}_0$ has finite norm in $C^\varepsilon(S_\Gamma \times \mathfrak{X}, \mathrm{End}(T\mathfrak{X}))$.

**Definition 6.2.** (Perturbation data) A perturbation datum for a type $\Gamma$ of stable treed disks is a pair $\mathfrak{p}_\Gamma = (F_\Gamma, \mathfrak{J}_\Gamma)$ consisting of a domain-dependent Morse function $F_\Gamma$ and a domain-dependent almost complex structure $\mathfrak{J}_\Gamma$.

The following morphisms on the set of combinatorial types of stable treed disks are used to define coherence of perturbations.

**Definition 6.3.** (Morphisms on treed disk types)

(a) (Cutting edges) There is a (Cutting edges) morphism $\Gamma \to \Gamma'$ between combinatorial types $\Gamma, \Gamma'$ of stable treed disks if and only if $\Gamma$ is obtained by cutting a boundary edge $e \in \mathrm{Edge}_{\circ,-}$ of $\Gamma'$ that contains a breaking, see Figure 6.1.

(b) (Collapsing edges) A morphism $\Gamma \to \Gamma'$ is a (Collapsing edges) morphism if $\Gamma'$ is obtained from $\Gamma$ by collapsing an interior edge $e \in \mathrm{Edge}_{\bullet,-}(\Gamma)$ or a boundary edge $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ with length zero, $\ell(e) = 0$.

[1]Here we do not say that $\mathfrak{J}_\Gamma$ maps to the locally tamed elements in $\mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$. But later in Definition 6.9 we constrain $\mathfrak{J}_\Gamma$ to take values in a small enough neighborhood of $\mathfrak{J}_0$ that automatically ensures local tamedness.

(c) (Making an edge length finite or non-zero) A morphism $\Gamma \to \Gamma'$ is a (Making an edge length finite or non-zero) morphism if $\Gamma'$ is obtained from $\Gamma$ by changing the edge length of a boundary edge $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ from infinite or zero to finite non-zero.

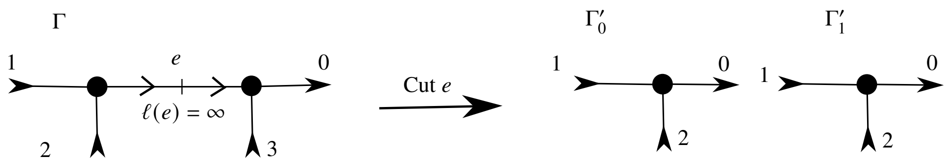



**Figure 6.1.** Cutting an edge $e$ relabels the boundary and interior markings while preserving their ordering on the pieces $\Gamma'_0, \Gamma'_1$.

Perturbation data $\mathfrak{p}_\Gamma$ for various treed disk types $\Gamma$ must be chosen coherently to obtain compactness. For any two treed types $\Gamma, \Gamma'$ related by a morphism from Definition 6.3, there is a coherence condition relating $\mathfrak{p}_\Gamma, \mathfrak{p}_{\Gamma'}$.

**Definition 6.4.** (Morphisms of perturbation data)

(a) (Collapsing edges/Making an edge length finite/non-zero) Let $\Gamma' \to \Gamma$ be a (Collapsing edges/making an edge length finite/non-zero) morphism. For perturbation data $\mathfrak{p}_{\Gamma'}, \mathfrak{p}_\Gamma$ there is a morphism $\mathfrak{p}_{\Gamma'} \to \mathfrak{p}_\Gamma$ if $\mathfrak{p}_{\Gamma'}$ is induced by pullback of $\mathfrak{p}_\Gamma$ under the natural inclusion of the universal curve

$$\iota_\Gamma^{\Gamma'} : \overline{\mathcal{U}}_{\Gamma'} \to \overline{\mathcal{U}}_\Gamma.$$

(b) (Cutting edges) Suppose $\Gamma \to \Gamma'$ is a (Cutting edges) morphism, where an edge $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ is cut to yield leaf edges $e_+, e_- \in \mathrm{Edge}(\Gamma')$. There is a morphism of perturbation data $\mathfrak{p}_\Gamma \to \mathfrak{p}_{\Gamma'}$ if $\mathfrak{p}_{\Gamma'}$ is obtained by pushing forward $\mathfrak{p}_\Gamma$ under the map

$$\pi_\Gamma^{\Gamma'} : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{\Gamma'}$$

defined by gluing at the leaf edges $e_\pm$ to form a single non-leaf edge $e$ with $\ell(e) = \infty$. That is, define

$$\mathfrak{J}_{\Gamma'}(z', x) = \mathfrak{J}_\Gamma(z, x), \quad \forall z \in (\pi_\Gamma^{\Gamma'})^{-1}(z).$$

The definition for the perturbed Morse datum $F_{\Gamma'}$ is similar.

We are now ready to define coherent collections of perturbation data. These are data that behave well with each type of operation in Definition 6.3.

**Definition 6.5.** (Coherent families of perturbation data) A collection of perturbation data $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$ is *coherent* if it is compatible with the morphisms of moduli spaces of different types in the sense that

(a) (Collapsing edges)/(Making an edge length finite/non-zero) if $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making an edge finite/non-zero, then $\mathfrak{p}_{\Gamma'}$ is the pullback of $\mathfrak{p}_\Gamma$;

(b) (Cutting edges) if $\Gamma$ is obtained from $\Gamma'$ by cutting a boundary edge $e \in \mathrm{Edge}^{\infty}_{\circ,-}(\Gamma')$ of infinite length, then $\mathfrak{p}_{\Gamma'}$ is the push-forward of $\mathfrak{p}_\Gamma$. Assuming $\Gamma$ is the union of types $\Gamma_1, \Gamma_2$, $\mathfrak{p}_\Gamma$ is obtained from $\mathfrak{p}_{\Gamma_1}$ and $\mathfrak{p}_{\Gamma_2}$ as follows: For $k = 1, 2$, let
$$\pi_k : \overline{\mathcal{M}}_\Gamma \cong \overline{\mathcal{M}}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2} \to \overline{\mathcal{M}}_{\Gamma_k}$$
denote the projection on the $k$-th factor, and therefore, $\overline{\mathcal{U}}_\Gamma$ is the union of $\pi_1^*\overline{\mathcal{U}}_{\Gamma_1}$ and $\pi_2^*\overline{\mathcal{U}}_{\Gamma_2}$. We require that $\mathfrak{p}_\Gamma$ is equal to the pullback of $\mathfrak{p}_{\Gamma_k}$ on $\pi_k^*\overline{\mathcal{U}}_{\Gamma_k}$:
$$\mathfrak{p}_\Gamma | \overline{\mathcal{U}}_{\Gamma_k} = \pi_k^* \mathfrak{p}_{\Gamma_k}. \tag{6.5}$$

We also require the perturbation data to satisfy the following locality axiom which ensures that the perturbations on any component only depend on special points on that component, and the length of the treed segments on the boundary of the disk. We first set up some notation: For a type $\Gamma$ underlying treed disks, and a vertex $v \in \mathrm{Vert}(\Gamma)$, let $\Gamma(v)$ be a graph with a single vertex $v$ and markings
$$\{e \in \mathrm{Edge}(\Gamma) : v \in e\}.$$
corresponding to each edge incident on $v$. Let $\mathcal{U}_{\Gamma,v} \subset \mathcal{U}_\Gamma$ be a fibration over $\mathcal{M}_\Gamma$ whose fiber over $m \in \mathcal{M}_\Gamma$ consists of the curve component represented by $v$. Define a map
$$\pi_v : \mathcal{U}_{\Gamma,v} \to \mathcal{U}_{\Gamma(v)} \times ([0,\infty])^{|\mathrm{Edge}_{\circ,-}(\Gamma)|}, \tag{6.6}$$
whose first component $\mathcal{U}_{\Gamma,v} \to \mathcal{U}_{\Gamma(v)}$ is the natural projection map, and the second component is the length function on boundary edges $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$.

(Locality Axiom) The restriction of the perturbation datum $\mathfrak{p}_\Gamma$ to $\mathcal{U}_{\Gamma,v}$ is the pullback via $\pi_v$ of some datum on $\mathcal{U}_{\Gamma(v)} \times ([0,\infty])^{|\mathrm{Edge}_{\circ,-}(\Gamma)|}$.

This ends the Definition.

Let $C$ be a possibly unstable treed disk of type $\Gamma$. The *stabilization* of $C$ is the stable treed disk $\mathrm{st}(C)$ of some type $\mathrm{st}(\Gamma)$ obtained by collapsing unstable surface and tree components. Thus the stabilization $\mathrm{st}(C)$ of any treed disk $C$ is the fiber of a universal treed disk $\mathcal{U}_{\mathrm{st}(\Gamma)}$. Given a perturbation datum for the type $\mathrm{st}(\Gamma)$, we obtain a domain-dependent almost complex structure and Morse function for $C$, still denoted $\mathfrak{J}_\Gamma, F_\Gamma$, by pull-back under the map $C \to \mathcal{U}_{\mathrm{st}(\Gamma)}$. If $\Gamma$ does not contain vertices, i.e. if $C$ is a single infinite segment $T_e$, $e \in \mathrm{Edge}(\Gamma)$, then the perturbation $\mathfrak{p}_\Gamma$ vanishes on $C$.

*Remark* 6.6. Coherence conditions for perturbation data are motivated as follows.

(a) If a sequence $C_\nu$ of (domain) curves of type $\Gamma$ converges to a limit curve $C$ of type $\Gamma'$, we would like the perturbation data on $C_\nu$ to converge to the perturbation datum on $C$. If $\Gamma' \neq \Gamma$ then there is a (Collapsing edges)/(Making an edge length finite/non-zero) morphism $\Gamma' \to \Gamma$, and the convergence of the perturbation datum is ensured by coherence under (Collapsing edges)/(Making an edge length finite/non-zero).

(b) Suppose $\Gamma \to \Gamma'$ is a cutting edges morphism, and suppose the disconnected type $\Gamma'$ has components $\Gamma'_0, \Gamma'_1$. Once moduli spaces of perturbed holomorphic maps in a symplectic manifold $X$ are defined, we would like to be able to say that the moduli space $\mathcal{M}_\Gamma(X)$ of maps with domain type $\Gamma$ is a product

$$\mathcal{M}_\Gamma(X) = \mathcal{M}_{\Gamma'_0}(X) \times \mathcal{M}_{\Gamma'_1}(X),$$

The product relation would hold only if the perturbation data is coherent under (Cutting edges).

*Remark* 6.7. (On the locality axiom) The locality axiom ensures that forgetting a marking $z_{e'}$ on a treed curve $C$ affects the perturbation datum only on the component containing $z_{e'}$. This feature is used in Proposition 8.50. The dependence on the boundary edge lengths is useful for the following reason. Suppose $\Gamma$ is a combinatorial type of treed disk depicted in Figure 6.1. By cutting an edge $e \in \mathrm{Edge}(\Gamma)$, we obtain two identical types. The cutting edge axiom requires that a coherent perturbation datum $\mathfrak{p}_\Gamma$ for $\Gamma$ is equal after restriction to the universal curve for the type $\Gamma'$ on both sides of the edge $e$. If in the locality axiom, the perturbation is $\mathfrak{p}_\Gamma$ defined by pulling back by the map $\pi_v : \mathcal{U}_{\Gamma,v} \to \mathcal{U}_{\Gamma(v)}$, then the perturbation datum on both surface components will be required to be equal even when the edge length of $e$ is finite. This creates a problem, because in order to obtain transversality in the case $\ell(e) = 0$, we need the perturbation datum on both surface components to be independent of each other.

*Remark* 6.8. (Perturbations for moduli spaces of spheres) The coherence conditions required to define moduli spaces of spheres are much simpler than those required for treed disks: The only coherence condition is (Collapsing edges), which translates to the statement that the perturbation datum is continuous on the compactified moduli space $\overline{\mathcal{M}}_n$ of $n$-marked spheres. An important difference from the disk case is that the perturbation datum on $\overline{\mathcal{M}}_n$ can be defined independently of $\overline{\mathcal{M}}_k$ for any $k < n$.

We require perturbations to be adapted to the stabilizing pair $(\mathfrak{J}_0, \mathfrak{D})$ in the sense that $\mathfrak{J}_0$ is the background almost complex structure; the domain-dependent almost complex structures $\mathfrak{J}_\Gamma$ are adapted to the stabilizing divisor (as in Definition 5.21) in that the tangent space of the stabilizing divisor is $\mathfrak{J}_\Gamma(z)$-invariant for all points $z \in \overline{\mathcal{U}}_\Gamma$; and finally, the domain-dependent almost complex structures take values in small enough neighborhoods of $\mathfrak{J}_0$, with the size of these neighborhoods being smaller for types $\Gamma$ with more interior markings. The reason for the last requirement is explained later in Remark 6.11 after perturbed holomorphic maps are defined.

**Definition 6.9.** (Perturbations adapted to a stabilizing divisor) Let $k \gg 0$, and $(\mathfrak{J}_0, \mathfrak{D})$ be a stabilizing pair on the broken manifold $\mathfrak{X}$ (as in Definition 5.22), such that $D_P \subset \overline{X}_P$ is dual to $k[\omega_{X_P}]$ for all polytopes $P \in \mathcal{P}$. Furthermore, let $\mathfrak{D}$ be disjoint from the Lagrangian submanifold $L$. Suppose $\Gamma$ is a type of treed disk. A perturbation datum $\mathfrak{p}_\Gamma = (\mathfrak{J}_\Gamma, F_\Gamma)$ on $(\mathfrak{X}, L)$ is adapted to the pair $(\mathfrak{J}_0, \mathfrak{D})$ if

- $\mathfrak{J}_0$ is the background almost complex structure for $\mathfrak{J}_\Gamma$, meaning that each $\mathfrak{J}_\Gamma$ is a domain-dependent perturbation of $\mathfrak{J}_0$, and
- for any treed curve $C = S \cup T$, and a connected component $S' \subset S$ with $d_\bullet(S')$ interior markings,

$$\mathfrak{J}_\Gamma(S') \subset \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D}; \mathfrak{J}_0, \tfrac{1}{k} d_\bullet(S')) \cap \mathcal{U}_{\mathfrak{J}_0}. \tag{6.7}$$

Here

$$\mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D}; \mathfrak{J}_0, \tfrac{1}{k} d_\bullet(S')) \subset \{\mathfrak{J} \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D}) : \mathfrak{J}|T\mathfrak{D} = \mathfrak{J}_0|T\mathfrak{D}\}$$

is the neighbourhood of $\mathfrak{J}_0$ consisting of $\frac{1}{k} d_\bullet(S')$-stabilizing almost complex structures (see Proposition 5.23), and $U_{\mathfrak{J}_0} \subset \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ is a neighborhood of $\mathfrak{J}_0$ on which the results on Hofer energy hold (see Lemma 7.21).

The set of perturbation data adapted to $(\mathfrak{J}_0, \mathfrak{D})$ is denoted by $\mathscr{P}_\Gamma(\mathfrak{X}, \mathfrak{J}_0, \mathfrak{D})$.

## 6.2 Perturbed maps

Given domain-dependent perturbations adapted to a stabilizing divisor as in the previous section, we define perturbed pseudoholomorphic maps and their moduli spaces.

**Definition 6.10.** (Perturbed holomorphic treed disks) Given a rational symplectic manifold $(X, \omega_X)$, a rational Lagrangian $L \subset X$, a coherent perturbation datum $\underline{\mathfrak{p}} = \{\mathfrak{p}_\Gamma = (J_\Gamma, F_\Gamma)\}_\Gamma$ on $X$ adapted to a stabilizing divisor $D$ (as in Definition 6.9), a *perturbed holomorphic treed disk u* of stable domain type $\Gamma$ consists of

(a) a treed disk $C = S \cup T$ of type $\Gamma$, and

(b) a continuous map

$$u : C \to X,$$

such that the following hold:

(a) (Boundary condition) The tree components and the boundary of the surface components are mapped to the Lagrangian submanifold :

$$u(\partial S \cup T) \subset L.$$

(b) (Surface equation) On the surface part $S$ of $C$, the map $u$ is holomorphic with respect to the given domain-dependent almost complex structure: if $j$ denotes the complex structure on $S$ then

$$J_\Gamma(z, u(z))\, \mathrm{d}u_S = \mathrm{d}u_S\, j. \tag{6.8}$$

(c) (Boundary tree equation) On the boundary tree part $T \subset C$ the map $u$ is a collection of gradient trajectories:

$$\frac{d}{ds} u_T = -\operatorname{grad}_{F_\Gamma(s,u(s))} u_T \tag{6.9}$$

where $s$ is a coordinate on the segment so that the segment has the given length. Thus, for each treed edge $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ the length of the trajectory $u|_{T_e}$ is equal to $\ell(e)$.

(d) (Adapted to stabilizing divisor) Each interior marking $z_e$, $e \in \mathrm{Edge}_{\bullet,\to}(\Gamma)$ maps to $D$ under $u$ and each connected component $C'$ of $u^{-1}(D) \subset C$ contains an interior marking.

We say that the map $u$ is $\mathfrak{p}_\Gamma$-*holomorphic*. A perturbed holomorphic disk $u$ with an unstable domain type $\Gamma$ is defined similarly except that $u_v, u_e$ are $\mathcal{P}_{\mathrm{st}\,\Gamma}$-holomorphic, where $\mathrm{st}\,\Gamma$ is the stabilization of $\Gamma$. On surface and treed components that get collapsed in the stabilization, the map $u$ is pseudoholomorphic with respect to the background data $(J_0, F_0)$.

*Remark* 6.11. Our domains below are stable on the surface parts because of the additional markings arising from intersections with the stabilizing divisor. So the content of the last sentence is that on broken Morse trajectories, for segments that are infinite in both directions the gradient flow equation is unperturbed.

**Definition 6.12.** (Perturbed holomorphic broken treed disks) Given a broken manifold $\mathfrak{X} := \mathfrak{X}_{\mathcal{P}}$ with rational symplectic cut spaces $X_P, P \in \mathcal{P}$, a rational Lagrangian $L \subset X_{P_0}$, a coherent perturbation datum $\mathfrak{p} = \{\mathfrak{p}_\Gamma = (\mathfrak{J}_\Gamma, F_\Gamma)\}_\Gamma$ on $\mathfrak{X}$ that is adapted to a broken stabilizing divisor $\mathfrak{D}$ (as in Definition 6.9), a *perturbed holomorphic broken treed disk* $u$ of stable domain type $\Gamma$ consists of

(a) a treed disk $C = S \cup T$ of type $\Gamma$;

(b) a tropical structure $\mathcal{T}$ on $\Gamma$;

(c) a collection of maps on surface components

$$u_v : S_v^\circ \to \mathfrak{X}_{P(v)}, \quad v \in \mathrm{Vert}(\Gamma)$$

that are $\mathfrak{J}_\Gamma$-holomorphic as in (6.8) where $S_v^\circ$ is the punctured domain curve with lifts of tropical nodal points deleted (see (4.14)), satisfy the Lagrangian boundary condition $u_v(\partial S_v) \subset L$ for all $v \in \mathrm{Vert}_\circ(\Gamma)$ and are adapted to the stabilizing divisor $\mathfrak{D}$ (as in Definition 6.10 (d)); and

(d) a collection of $F_\Gamma$-gradient flow lines as in (6.9)

$$u_e : T_e \to L \subset X_{P_0}, \quad e \in \mathrm{Edge}_\circ(\Gamma)$$

on the treed parts, that satisfy all the matching conditions on broken treed holomorphic maps (Definition 4.13).

We say that the map $u$ is $\mathfrak{p}_\Gamma$-*holomorphic*. Broken maps on unstable domain types are defined similarly as the unbroken case in Definition 6.10.

*Remark* 6.13. We explain why we use $\frac{1}{k}d_\bullet(S')$-stabilizing almost complex structures in (6.7). Let $\underline{J} = (J_\Gamma)_\Gamma$ be a collection of coherent domain-dependent almost complex structures satisfying (6.7). Let $S_\nu$ be a sequence of nodal curves of type $\Gamma$ with $d(\bullet)$ markings. Then the limit $u$ of any sequence $u_\nu : S_\nu \to \mathfrak{X}$ of $J_\Gamma$-holomorphic broken maps can not have an unstable domain component. Indeed, the area of the maps $u_\nu$ is $\frac{d_\bullet}{k}$, and so an unstable component $u_v$ of the $u$ has area $\leq \frac{d_\bullet}{k}$. Therefore, $u$ is holomorphic with respect to a domain-independent almost complex structure $\mathfrak{J}_v$ which, by (6.7), is $\frac{d_\bullet}{k}$-stabilizing. This contradicts the existence of a non-constant map $u_v$.

We now define moduli spaces of perturbed broken treed holomorphic disks. The moduli space of such maps is stratified by combinatorial type.

**Definition 6.14.** The *combinatorial type of an perturbed holomorphic broken treed disk* $u : C \to \mathfrak{X}$ adapted to a broken divisor $\mathfrak{D} \subset \mathfrak{X} - L$ consists of

(a) the combinatorial type $\Gamma$ of its domain $C$,

(b) the tropical structure on $\Gamma$, which consists of an assignment of polytopes for vertices, and slopes for tropical edges :

$$\mathrm{Vert}(\Gamma) \ni v \mapsto P(v) \in \mathcal{P}, \quad \mathrm{Edge}_{\mathrm{trop}}(\Gamma) \ni e \mapsto \mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}};$$

(c) a labelling

$$d(v) := (\pi_{P(v)} \circ u_v)_*[(S_v, \partial S_v)] \in \begin{cases} H_2(\overline{X}_{P_0}, L), & v \in \mathrm{Vert}_\circ(\Gamma), \\ H_2(\overline{X}_{P(v)}), & v \in \mathrm{Vert}_\bullet(\Gamma) \end{cases}$$

that assigns to each vertex $v$ of $\Gamma$ the homology class of the disk/sphere obtained by extending $\pi_{P(v)} \circ u_v : S_v^\circ \to X_{P(v)}$ over the punctures corresponding to nodal lifts; [2]

(d) a labelling

$$\mu_{\mathfrak{D}} : \mathrm{Edge}_{\bullet,\to}(\Gamma) \to \mathbb{Z}_{>0}$$

[2]In case the space $\mathfrak{X}_{P(v)}$ has an orbifold compactification $\mathfrak{X}_{P(v)}$ (see Remark 3.60), the homology class of $(u_v)_*[S_v] \in H_2(\overline{\mathfrak{X}}_{P(v)})$ can be read off from $d(v)$ and the tropical graph.

that records the order of tangency of the map $u$ to the divisor $\mathfrak{D}$ at markings that do not lie on horizontally constant components (with the convention that a transverse intersection has order 1).

The type is denoted simply as $\Gamma$, suppressing $\mathcal{T}, d, \mu_{\mathfrak{D}}$ in the notation, or by $\Gamma_X$ if we wish to distinguish a type of map $\Gamma_X$ from a type of treed disk $\Gamma$.

We introduce the following notations for moduli spaces. Given a collection of coherent perturbation data $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$, let $\mathcal{M}^{\mathrm{brok}}(L, \underline{\mathfrak{p}})$ be the moduli space of isomorphism classes of stable treed broken holomorphic disks in $\mathfrak{X}$ with boundary in $L$, where isomorphism is modulo reparametrizations of domains and is defined in Definition 4.15. Let

$$\mathcal{M}^{\mathrm{brok}}_\Gamma(L, \mathfrak{p}_\Gamma) \subset \mathcal{M}^{\mathrm{brok}}(L, \underline{\mathfrak{p}})$$

be the *moduli space of broken maps* of combinatorial type $\Gamma$ modulo the action of domain reparametrizations. We drop the perturbation datum $\underline{\mathfrak{p}}$ from the notation if it is obvious from the context. We recall that domain reparametrizations are isomorphisms of broken maps as in Definition 4.15. The group of tropical symmetries $T_{\mathrm{trop}}(\Gamma)$ acts naturally on $\mathcal{M}^{\mathrm{brok}}_\Gamma(L, \mathfrak{p}_\Gamma)$. The quotient

$$\mathcal{M}^{\mathrm{brok}}_{\Gamma,\mathrm{red}}(L, \mathfrak{p}_\Gamma) := \mathcal{M}^{\mathrm{brok}}_\Gamma(L, \mathfrak{p}_\Gamma)/T_{\mathrm{trop}}(\Gamma)$$

is the *reduced moduli space of broken maps* of combinatorial type $\Gamma$. For $\underline{x} \in \mathcal{I}(L)^{d(\circ)+1}$, let

$$\mathcal{M}^{\mathrm{brok}}_\Gamma(L, \mathfrak{p}_\Gamma, \underline{x}) \subset \mathcal{M}^{\mathrm{brok}}_\Gamma(L, \mathfrak{p}_\Gamma) \tag{6.10}$$

denote the subset of perturbed holomorphic treed disks of type $\Gamma$ with limits $\underline{x} = (x_0, \ldots, x_{d(\circ)}) \in \mathcal{I}(L)$ along the root and leaves. The union over all types with $d(\circ)$ incoming leaves is denoted

$$\mathcal{M}^{\mathrm{brok}}_{d(\circ)}(L, \underline{\mathfrak{p}}) := \bigcup_{\Gamma, \underline{x}} \mathcal{M}^{\mathrm{brok}}_\Gamma(L, \mathfrak{p}_\Gamma, \underline{x}).$$

The space $\mathcal{M}^{\mathrm{brok}}_{d(\circ)}(L, \underline{\mathfrak{p}})$ has a natural topology for which convergence is a version of Gromov convergence defined in Chapter 8. For a type $\Gamma$ and labels $\underline{x}$ on leaves, the expected dimension of the moduli space $\mathcal{M}^{\mathrm{brok}}_\Gamma(L, \mathfrak{p}_\Gamma, \underline{x})$ of broken maps is given by the index $i^{\mathrm{brok}}(\Gamma, \underline{x})$ defined in (6.30) in Section 6.4. For $d = 0, 1$, we denote by

$$\mathcal{M}^{\mathrm{brok}}_{d(\circ)}(L, \underline{\mathfrak{p}})_d := \bigcup_{(\Gamma, \underline{x}): i^{\mathrm{brok}}(\Gamma, \underline{x}) = d} \mathcal{M}^{\mathrm{brok}}_\Gamma(L, \mathfrak{p}_\Gamma, \underline{x})$$

the union of $d$-dimensional strata with $d(\circ)$ incoming boundary edges in the moduli space of broken maps. In this paper, we show that the compactifications of zero and one-dimensional components of the moduli space $\mathcal{M}^{\mathrm{brok}}_{d(\circ)}(L, \underline{\mathfrak{p}})^{<E}$ are manifolds. For one-dimensional components, the boundary consists of configurations with disk

bubbling. For zero and one-dimensional components of $\mathcal{M}^{\text{brok}}_{d(\circ)}(L, \underline{\mathfrak{p}})$ the symmetry tropical group is finite.

*Remark* 6.15. (Perturbed maps and curve counts) A perturbed holomorphic map $u$ in a zero-dimensional moduli space makes a contribution of $\frac{1}{d(\bullet)!}$ to curve counts. Here $d(\bullet)$ is the number of interior leaves of $u$, and hence is the number of intersections of the map $u$ with the stabilizing divisor. To justify this multiplicative factor, let us consider the special case where for a certain type of map, the moduli space can be regularized using a domain-independent perturbation. Then there are $d(\bullet)!$ ways of labelling the divisor intersections as $z_1, \dots, z_{d(\bullet)}$. Each of these labellings is a different map in the moduli space, and therefore a corrective factor of $\frac{1}{d(\bullet)!}$ needs to be inserted.

## 6.3 Fredholm theory for broken maps

We introduce a weighted Sobolev space needed for the transversality result. The norm is defined by viewing the domain as having punctures that map to cylindrical ends in the target. With this Sobolev completion, we can enforce higher order tangencies with relative divisors. We focus on surface components of a broken map, since the transversality and gluing results use a standard norm on tree components. Moduli spaces of broken holomorphic maps are cut out as zero sets of a Cauchy-Riemann operator on the weighted Sobolev space. Later in the section, we define another Cauchy-Riemann operator on the Sobolev completion of the space of maps on compact curves, and show that the linearizations of both Cauchy-Riemann operators have kernels and cokernels of the same dimension as each other. The latter operator is often useful for index computations.

The Sobolev norm is defined component-wise for a broken map. We start by defining a *relative map*, which is a single component of a broken map. Tropical nodes in the broken map appear as markings on a relative map, called *tropical markings*; they correspond to one-sided edges $\text{Edge}_{\text{trop}}(\Gamma)$ on the combinatorial type $\Gamma$ of the relative map.

**Definition 6.16.** (Relative map) A *relative type* is a graph $\Gamma$ with a single vertex $v$, a collection $\text{Edge}_{\text{trop}}(\Gamma)$ of edges each with a single end-point, a polytope $P(v) \in \mathcal{P}$, slopes for the edges

$$\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}}, \quad e \in \text{Edge}_{\text{trop}}(\Gamma),$$

and homology data $d(v) \in H_2(\overline{X}_{P(v)})$. A *relative map* is a map $u : C^\circ \to \mathfrak{X}_{P(v)}$ from a punctured domain $C^\circ := C \backslash \{z_e : e \in \text{Edge}_{\text{trop}}(\Gamma)\}$ where $C$ is a disk or sphere, to the manifold $\mathfrak{X}_{P(v)}$ with cylindrical ends, such that at a puncture $z_e$ $u$ is asymptotic to a $\mathcal{T}(e)$-cylinder, and the extension of $\pi_{P(v)} \circ u$ over the punctures has homology class $d(v) \in H_2(\overline{X}_{P(v)})$.

We introduce the space of smooth relative maps whose Sobolev completion will be given later. Let $(C, j)$ be a connected Riemann surface with a set $\{z_e : e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)\}$ of tropical markings. Let

$$C^\circ := C \backslash \{z_e : e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)\} \tag{6.11}$$

be the curve with the tropical marked points removed. For the purpose of defining weighted Sobolev norms, we introduce cylindrical coordinates

$$(s_e, t_e) : U_{z_e} \backslash \{z_e\} \to [0, \infty) \times S^1 \tag{6.12}$$

on the neighborhood $U_{z_e} \subset C$ of every puncture $z_e$. Let

$$\mathrm{Map}_\Gamma(C^\circ, \mathfrak{X}_P)$$

be the set of relative maps, which at a puncture $z_e \in C \backslash C^\circ$ corresponding to an edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ is asymptotically close to a $\mathcal{T}(e)$-cylinder

$$u_{\mathrm{vert},e} : [0, \infty) \times S^1 \to \mathfrak{X}_{P(e)}, \quad (s_e, t_e) \mapsto e^{\mathcal{T}(e)(s_e + it_e)} x_e$$

for some point $x_e \in \mathfrak{X}_{P(e)}$. Here, we recall that the punctured neighborhood at $z_e$ in $C^\circ$ maps to a $P(e)$-cylindrical end of $\mathfrak{X}_P$ and the $P(e)$-cylindrical end is identified to $\mathfrak{X}_{P(e)}$, see (3.51). An infinitesimal deformation of a $\mathcal{T}(e)$-cylinder is given by a vector $\xi_e \in T_{x_e} \mathfrak{X}_{P(e)}$. It defines a section

$$\xi_{e,\mathcal{T}(e)} := \tfrac{\mathrm{d}}{\mathrm{d}\tau} (e^{\mathcal{T}(e)(s+it)} \exp_{x_e}(\tau \xi_e))|_{\tau=0} \in \Gamma(\mathbb{R}_+ \times S^1, u^*_{\mathrm{vert},e} T\mathfrak{X}_{P(e)}). \tag{6.13}$$

The tangent space of $\mathrm{Map}_\Gamma(C^\circ, \mathfrak{X}_P)$ at $u$ consists of sections

$$\xi \in \Gamma(C^\circ, u^* T\mathfrak{X}_P)$$

which, for any edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, have limits

$$\xi_e := \lim_{z \to z_e} \xi \in T_{x_e} \mathfrak{X}_{P(e)} \tag{6.14}$$

in the sense that a section $\xi$ is asymptotically close to the section $\xi_{e,\mathcal{T}(e)}$ (6.13) as $z \to z_e$.

*Remark* 6.17. The quantity $\xi_e$ defined in (6.14) is the infinitesimal change in the tropical evaluation map $\mathrm{ev}_{z_e}^{\mathcal{T}(e)}(u) \in \mathfrak{X}_{P(e)}$ (see (4.15)) when the broken map $u$ is infinitesimally deformed by $\xi$. Since the tropical evaluation map depends on the choice of cylindrical coordinates (6.12) in the punctured neighborhood of $z_e$ in the domain curve, the quantity $\xi_e$ depends on infinitesimal changes in the domain cylindrical coordinates. On the other hand, the projected evaluation map $\pi^\perp_{\mathcal{T}(e)}(\mathrm{ev}_{z_e}(u)) \in \mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}$ is independent of the domain cylindrical coordinates, and the same is true of its infinitesimal version $\pi^\perp_{\mathcal{T}(e)}(\xi_e) \in T(\mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}})$.

We now define the Sobolev norm. Define a cutoff function

$$\beta \in C^\infty(\mathbb{R}, [0,1]), \qquad \begin{cases} \beta(s) = 0 & s \le 0 \\ \beta(s) = 1 & s \ge 1 \end{cases}. \tag{6.15}$$

Let

$$\kappa : C^\circ \to \mathbb{R} \tag{6.16}$$

be equal to $\beta(s_e)s_e$ near the puncture corresponding to $z_e$, for any edge $e$, and 0 outside all the edge neighbourhoods. For a section $\xi : C^\circ \to E$ of a vector bundle $E$ with connection $\nabla$, integers $k \ge 0$, $p > 1$ and constant $\lambda \in (0,1)$ define the $W^{k,p,\lambda}$-norm of $\xi$ as

$$\|\xi\|^p_{W^{k,p,\lambda}} := \sum_{0 \le i \le k} \int_{C^\circ} |\nabla^i \xi|^p \exp(\lambda \kappa p) \, \mathrm{dvol}_{C^\circ} .$$

The norm on a section $\xi$ is

$$\|\xi\|^\circ_\Gamma := \textstyle\sum_e |\xi_e| + \|\xi - \sum_e \beta(s_e) \mathbb{T}^e_u \xi_{e,\mathcal{T}(e)}\|_{W^{1,p,\lambda}}, \tag{6.17}$$

where

$$\xi_e := \lim_{z \to z_e} \xi \in T_{x_e} \mathfrak{X}_{P(e)} \tag{6.18}$$

near the puncture $z_e$ as in (6.18), $u$ is asymptotic to a vertical cylinder $u_{\mathrm{vert},e}$,

$$\mathbb{T}^e_u : u^*_{\mathrm{vert},e} T\mathfrak{X}_P \to u^* T\mathfrak{X}_P$$

is the parallel transport map along geodesics, and $\xi_{e,\mathcal{T}(e)}$ is a section of $u^*_{\mathrm{vert},e} T\mathfrak{X}_P$ as in (6.13).

There are similar Sobolev spaces of maps and one-forms. Let $\mathrm{Map}^{1,p,\lambda}_\Gamma(C^\circ, \mathfrak{X}_P)$ be the Banach completion of the space of maps $\mathrm{Map}_\Gamma(C^\circ, \mathfrak{X}_P)$ under the norm (6.17). That is, for a smooth map $u \in \mathrm{Map}_\Gamma(C^\circ, \mathfrak{X}_P)$ and a section $\xi \in \Gamma(C^\circ, u^* T\mathfrak{X}_P)$ satisfying $\|\xi\|^\circ_\Gamma < \infty$, the map $\exp_u \xi$ belongs to the completion $\mathrm{Map}^{1,p,\lambda}_\Gamma(C^\circ, \mathfrak{X}_P)$. Let

$$\mathcal{E}^{p,\lambda} \to \mathrm{Map}^{1,p,\lambda}_\Gamma(C^\circ, \mathfrak{X}_P)$$

be the vector bundle whose fiber

$$\mathcal{E}^{p,\lambda}_u = \Omega^{0,1}(C^\circ, u^* T\mathfrak{X}_P)_{L^{p,\lambda}}$$

is the space of $(0,1)$-forms with respect to $(j, J)$, where $J$ is a fixed cylindrical almost complex structure. The vertical projection of the linearization of the Cauchy-Riemann operator $\overline{\partial}_{j,J}$ at $u$

$$D^\circ_u : T_u \mathrm{Map}^{1,p,\lambda}_\Gamma \to \mathcal{E}^{p,\lambda}_u$$

is a Fredholm operator by results of Lockhart-McOwen [51].

Next, we describe a linearized $\overline{\partial}$-operator on the compact curve $C$, denoted by $D_u$. This operator will be useful for index computations. It is not used for transversality and gluing proofs where we stick to $D_u^\circ$. Given a relative map $u : C^\circ \to \mathfrak{X}_P$, we define an extension of the pullback bundle $u^* T\mathfrak{X}_P \to C^\circ$ to a bundle over $C$, as follows. Define a (partial) almost complex orbifold compactification of $\mathfrak{X}_P$, denoted by $\mathfrak{X}_{P,\mathcal{T}}$, as the union

$$\mathfrak{X}_{P,\mathcal{T}} := \mathfrak{X}_P \cup \cup_{e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)} Y_{\mathcal{T}(e)} \tag{6.19}$$

obtained by adding a divisor at infinity $Y_{\mathcal{T}(e)}$ corresponding to every tropical marking $z_e$.[3] The divisor $Y_{\mathcal{T}(e)}$ is defined as

$$Y_{\mathcal{T}(e)} := \{[x] := T_{\mathcal{T}(e),\mathbb{C}} x \mid x \in \mathfrak{X}_P\}, \tag{6.20}$$

where each $[x]$ is the limit point of a $T_{\mathcal{T}(e),\mathbb{C}}$-orbit, that is, $\lim_{s\to\infty} e^{(s+it)\mathcal{T}(e)} x = [x]$, and $Y_{\mathcal{T}(e)} = Y_{\mathcal{T}(e')}$ if $\mathcal{T}(e) = \lambda \mathcal{T}(e')$ for some $\lambda \in \mathbb{R}_{>0}$. The map $u$ extends over all tropical markings $z_e$ to yield $u : C \to \mathfrak{X}_{P,\mathcal{T}}$, with $u(z_e) \in Y_{\mathcal{T}(e)}$. The extension of $u^* T\mathfrak{X}_P \to C^\circ$ is the pullback bundle $u^* T\mathfrak{X}_{P,\mathcal{T}} \to C$. Recall from Definition 4.25 that for any tropical marking $z_e \in C \backslash C^\circ$, the slope $\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}}$ is the product

$$\mathcal{T}(e) = \mu_e \mathcal{T}(e)_{\mathrm{prim}}, \tag{6.21}$$

of the primitive slope $\mathcal{T}(e)_{\mathrm{prim}} \in \mathfrak{t}_{P(e),\mathbb{Z}}$ and a multiplicity $\mu_e \in \mathbb{Z}_{\geq 1}$. Denote by

$$D_u : W^{k,p}(\Omega^0_{\underline{\mu}}(C, u^* T\mathfrak{X}_{P,\mathcal{T}})) \to W^{k-1,p}(\Omega^{0,1}_{\underline{\mu}-1}(C, u^* T\mathfrak{X}_{P,\mathcal{T}})) \tag{6.22}$$

the linearization of the $\overline{\partial}$-operator defined on the space of sections on the compactification $\mathfrak{X}_{P,\mathcal{T}}$. Here, $\underline{\mu} = (\mu_e \in \mathbb{N})_e$, and the subscript in the domain and codomain indicate the order of vanishing of the sections at the tropical markings $z_e$, $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ in the direction normal to the divisor at infinity $Y_{\mathcal{T}(e)}$; and the Sobolev indices are such that

$$k \in \mathbb{Z}_{>0}, \quad p > 1, \quad k > \mu_e + 2/p \ \forall e$$

so that the sections with prescribed orders of vanishing at the divisors are well-defined.

**Proposition 6.18.** *Let $u \in \mathrm{Map}_\Gamma(C^\circ, \mathfrak{X}_P)$ be a relative holomorphic map. There are isomorphisms*

$$\ker(D_u) \simeq \ker(D_u^\circ), \quad \mathrm{coker}(D_u) \simeq \mathrm{coker}(D_u^\circ).$$

*Proof.* Since the proof is by a local computation near each of the special points, it is enough to prove the result for a map whose domain is a single curve component with a

[3] The space $\mathfrak{X}_{P,\mathcal{T}}$ is a different compactification of $\mathfrak{X}_P$ compared to $\overline{\mathfrak{X}}_P$ (when it exists).

single tropical marking $z_e$. By the description of $D_u$ (see (6.22)), the curve $u$ extends over $z_e$ to yield $u : C \to \mathfrak{X}_{P,\mathcal{T}} := \mathfrak{X}_P \cup Y$ (as in (6.19)), with $u(z_e)$ mapping to the divisor at infinity $Y$, and $u$ intersecting the divisor $Y$ at $z_e$ with multiplicity $\mu \in \mathbb{N}$. We abbreviate $\hat{\mathfrak{X}}_P := \mathfrak{X}_{P,\mathcal{T}}$. We assume that the marked point is $0 \in C$.

The proof is based on the fact that in a neighborhood of the divisor the tangent space splits into a vertical and a horizontal subspace. Indeed, because of the cylindrical almost complex structure, there is a neighborhood $U_Y \subset \hat{\mathfrak{X}}_P$ of $Y$ for which there is a projection $\pi_Y : U_Y \to Y$ with holomorphic fibers. The tangent space splits into a $J$-holomorphic horizontal and vertical part:

$$T\hat{\mathfrak{X}}_P|U_Y \simeq V \oplus H, \quad H := \pi_Y^* TY, \quad V := \ker(d\pi_Y). \tag{6.23}$$

Under the splitting (6.23),

$$D_u = \begin{pmatrix} \overline{\partial} & A \\ 0 & D_u^Y \end{pmatrix} \tag{6.24}$$

in a neighbourhood $U_0 \subset C$ of the marked point. Here $\overline{\partial} : \Gamma(U_0, u^*V) \to \Omega^{0,1}(U_0, u^*V)$ is the standard Cauchy-Riemann operator, $A : \Gamma(U_0, u^*H) \to \Omega^{0,1}(U_0, u^*V)$ is multiplication with a tensor and

$$D_u^Y : \Gamma(U_0, u^*H) \to \Omega^{0,1}(U_0, u^*H)$$

is the lift of the linearized operator $D_{u_Y}$, where $u_Y := \pi_Y \circ u$.

The correspondence for the kernels follows by decay estimates. For an element $\xi \in W^{k,p}(C, u^*T\hat{\mathfrak{X}}_P)$, we denote the horizontal part and vertical part by

$$\xi_h \in W^{k,p}(U_0, u^*H), \quad \xi_v \in W^{k,p}(U_0, u^*V).$$

in the neighbourhood $U_0 \subset C$ of 0. Denote by $|\cdot|$ resp. $|\cdot|^\circ$ the ordinary metric on $\hat{\mathfrak{X}}_P$ resp. the cylindrical metric on $\mathfrak{X}_P$. Then, we have

$$|\xi_h(z)|^\circ \sim |\xi_h(z)|, \quad |\xi_v(z)|^\circ \sim |\xi_v(z)|/|u(z)|, \quad z \neq 0, \tag{6.25}$$

where the norm on $u(z)$ is a norm on the fiber of the projection $U_Y \to Y$. Since

$$|\xi_h| \leq c, \quad |\xi_v(z)| \leq c|z|^\mu, \quad |u(z)| \sim c|z|^\mu,$$

we conclude that $|\xi_h|^\circ$ and $|\xi_v(z)|^\circ$ are uniformly bounded and have a limit as $z \to 0$. Since $\xi$ is smooth on $C$, the convergence to $\xi(0)$ is exponential on $C^\circ$:

$$|\xi(z) - \xi(0)|^\circ \sim c|z| \sim ce^{-s},$$

where $(s,t)$ are cylindrical coordinates on $\mathbb{P}^1 \backslash \{0\}$ near the puncture, and

$$|\nabla\xi_h(z)|^\circ, |\nabla\xi_v(z)|^\circ \sim ce^{-s}$$

since the derivative is with respect to the cylindrical coordinates on $C^\circ$. We conclude

$$\xi \in W^{k,p}_{\Gamma}(C, \hat{\mathfrak{X}}_P) \implies \xi - \xi(0) \in W^{1,p,\lambda}(C^\circ, u^*T\mathfrak{X}_P)$$

for any $0 < \lambda < 1$, which implies $\ker D_u \subset \ker D_u^\circ$. The converse $\ker D_u^\circ \subset \ker D_u$ is proved using removal of singularity, elliptic regularity, and the estimate (6.25).

A similar argument holds for the cokernels. We first prove the inclusion $\operatorname{coker}(D_u) \subset \operatorname{coker}(D_u^\circ)$. Consider $\eta \in \operatorname{coker}(D_u)$. We view $\operatorname{coker}(D_u)$ as a subspace in the $(W^{k-1,p})^\vee$-completion of $\Omega^{1,0}(C, u^*(T^*\hat{\mathfrak{X}}_P))$, which is a space of distributions. (Here we use $\Omega^{1,0}(u^*(T^*\hat{\mathfrak{X}}_P))$ instead of $\Omega^{0,1}(u^*T\hat{\mathfrak{X}}_P)$ to avoid making choices of metrics on $C$ and $\hat{\mathfrak{X}}_P$.) By elliptic regularity, the distribution $\eta$ is represented by a smooth section in the complement of marked points. So we focus attention in a neighbourhood $U_0 \subset C$ of 0. For any $W^{k,p}$-section $\xi : C \to u^*T\hat{\mathfrak{X}}_P$ that is supported in $U_0$, and is vertical, we have $z^\mu \xi \in W^{k,p}_\Gamma(C, u^*T\hat{\mathfrak{X}}_P)$. So for any such $\xi$,

$$0 = \int_C (\overline{\partial}(z^\mu \xi), \eta_v).$$

Therefore, $\overline{\partial}(z^\mu \eta_v) = 0$ weakly in $U_0$ and so, $z^\mu \eta_v$ can be represented by a smooth function. Next, using the split form (6.24), we observe that any section $\xi$ that is supported in $U_0$ and is horizontal satisfies

$$(A\xi, \eta_v) + (D_u^Y \xi, \eta_h) = 0,$$

and so, $A^*\eta_v + (D_u^Y)^*\xi_h = 0$ weakly in $U_0$. The tensor $A$ vanishes to order $\mu$ at $0 \in C$ in the $|\cdot|$-norm, as a consequence of the transformation relation (6.25) and the fact that $A$ is bounded in the $|\cdot|^\circ$ norm. So, $A^*\eta_v$ is smooth in $U_0$. By elliptic regularity $\xi_h$ is smooth in $U_0$. Finally, $\eta$ is in $\operatorname{coker} D_u^\circ$ because of the following transformations valid in $U_0$ :

$$|\eta_h(z)|^\circ \sim |\eta_h(z)|, \quad |\eta_v(z)|^\circ \sim |\eta_v(z)| \cdot |u(z)|, \quad z \neq 0. \tag{6.26}$$

Indeed, $|\eta(z)|^\circ$ is bounded, and therefore is in $L^{p^*,-\lambda}$ the dual space of $L^{p,\lambda}$. The reverse inclusion $\operatorname{coker}(D_u^\circ) \subset \operatorname{coker}(D_u)$ follows formally from the inclusion relation

$$W^{k-1,p}_{\mu-1}(\Omega^{0,1}(C, u^*T\hat{\mathfrak{X}}_P)) \to L^{p,\lambda}(\Omega^{0,1}(C^\circ, u^*T\mathfrak{X}_P))$$

between the target spaces of $D_u$ and $D_u^\circ$. ∎

## 6.4 The index of a broken map

The index of a broken map is the expected dimension of the moduli space containing the broken map. In this section, we give an expression for the index in terms of the

Fredholm index of the linearized Cauchy-Riemann operator defined in the last section. We also prove that collapsing tropical edges in the tropical graph of the broken map does not change the expected dimension of the moduli space.

We start by stating the expected dimension of the moduli space of relative maps, which are single components of broken maps. We recall that the *boundary Maslov index* $I(E,F)$ is an integer assigned to a pair $(E,F)$ consisting of a complex vector bundle $E$ on a Riemann surface $C$ with boundary, and a totally real subbundle $F \subset E|_{\partial C}$ on the boundary (see [55, Appendix C]). In case $\partial C = \emptyset$, the Maslov index $I(E,\emptyset)$ is twice the first Chern number $c_1(E)$.

**Lemma 6.19.** *(Expected dimension for relative maps) Let $u$ be a relative map of type $\Gamma_v$ with a collection of tropical markings $z_e$ corresponding to $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma_v)$, each with multiplicity $\mu_e \in \mathbb{N}$, and with simple intersections with the stabilizing divisor $\mathfrak{D}$. Then, the expected dimension of the moduli space $\mathcal{M}_{\Gamma_v}(\mathfrak{X})$ of relative maps of type $\Gamma_v$ is*

$$i(\Gamma_v) = \begin{cases} (d(\circ)+1) + i_{\mathrm{Morse}}(\underline{x}) + I_{\mathrm{adj}}(\Gamma_v) - \dim \mathrm{Aut}(\mathbb{D}^2), & \Gamma_v \text{ is a disk}, \\ \dim(X) + I_{\mathrm{adj}}(\Gamma_v) - \dim \mathrm{Aut}(\mathbb{P}^1), & \Gamma_v \text{ is a sphere} \end{cases} \tag{6.27}$$

*where $I_{\mathrm{adj}}(\Gamma_v)$ is the adjusted Maslov index defined as*

$$\text{(Adjusted Maslov index)} \quad I_{\mathrm{adj}}(\Gamma_v) := I_{\mathrm{adj}}(u) := I((u^*T\mathfrak{X}_{P(v),\mathcal{T}}), (\partial u)^*TL) - \sum_{e\in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)} 2(\mu_e^{\mathrm{trop}} - 1); \tag{6.28}$$

*the Morse index of the tuple is the difference*

$$i_{\mathrm{Morse}}(\underline{x}) := i_{\mathrm{Morse}}(x_0) - \textstyle\sum_{i=1}^{d(\circ)} i_{\mathrm{Morse}}(x_i),$$

*and $i_{\mathrm{Morse}}(x_i)$ is the dimension of the stable manifold of $x_i$ in $L$. Furthermore,*

$$I_{\mathrm{adj}}(u) = \mathrm{ind}(D_u) + 2|\,\mathrm{Edge}_{\mathrm{trop}}(\Gamma)|. \tag{6.29}$$

*Proof.* We recall from the paragraph preceding (6.22) that, assuming the tropical markings are fixed, the moduli space of relative maps of type $\Gamma_v$ is cut out as the zero set of $D_u$ from the Banach space of sections $W^{k,p}_{\underline{\mu}}(C, u^*T\mathfrak{X}_{P,\mathcal{T}})$ that vanish to order $\mu_e$ in the direction normal to the divisor $Y_{\mathcal{T}(e)}$ at the tropical marking $z_e$, and therefore the expected dimension is $\mathrm{ind}(D_u)$. Since the tropical markings are free to move on the domain, the expected dimension of $\mathcal{M}_{\Gamma_v}(\mathfrak{X})$ is given by the first expression in (6.29). To obtain the second expression, we observe that the expected dimension of moduli space of holomorphic maps $u : C \to \mathfrak{X}_{P,\mathcal{T}}$ is the Maslov index $I(u^*T\mathfrak{X}_{P,\mathcal{T}}, (\partial u)^*TL)$, and for any $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma_v)$, the choice of $z_e$ on $C$ adds 2 to the expected dimension, and the requirement that there is a tangency of order $\mu_e$ with the divisor $Y_{\mathcal{T}(e)}$ at $z_e$ subtracts $2\mu_e$. ∎

*Example* 6.20. Suppose $\overline{X}_{P(v)} = \mathbb{P}^2$ with a Hamiltonian $(S^1)^2$-action, $X_{P(v)}$ is the complement of the 3 torus-invariant divisors, $L \subset \overline{X}_{P(v)}$ is a toric Lagrangian, and $u : (D, \partial D) \to (\overline{X}_{P(v)}, L)$ is a disk through the point $[1 : 0 : 0]$ whose boundary Maslov index is 4. In contrast, the adjusted Maslov index $I_{\mathrm{adj}}(u)$ is 2, because the adjusted Maslov index accounts for the constraint that the disk passes through the point $[1 : 0 : 0]$ which cuts down the index by 2. The reason for the discrepancy is that the pullback bundle $u^* T\overline{X}_{P(v)}$ is different from $u^* T\mathfrak{X}_{P(v),\mathcal{T}}$ since a tropical marking maps to the intersection of two divisors in $\overline{X}_{P(v)}$, which is replaced by a single divisor in $\mathfrak{X}_{P(v),\mathcal{T}}$.

For a type of broken maps the expected dimension of the moduli space is the same as that of the glued type of unbroken maps. We define *glued type*:

**Definition 6.21.** (Glued type) Let $\Gamma$ be the combinatorial type of a broken map. Then the *glued type* $\Gamma_{\mathrm{glue}}$ is the type of the unbroken map obtained by collapsing the tropical edges $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ in $\Gamma$.

**Proposition 6.22.** *(Expected dimension) Suppose $\Gamma$ is the combinatorial type of a broken map.*

(a) *The expected dimension of the moduli space $\mathcal{M}_\Gamma^{\mathrm{brok}}(L, \underline{x})$ of broken maps of type $\Gamma$ with limits $\underline{x} \in \mathcal{I}(L)^{d(\circ)+1}$ along the root and leaves is equal to*

$$i^{\mathrm{brok}}(\Gamma, \underline{x}) := \sum_{v \in \mathrm{Vert}(\Gamma)} i(\Gamma_v) - (\dim(X) - 2)\,\mathrm{Edge}_{\mathrm{trop}}(\Gamma) - c(\Gamma), \qquad (6.30)$$

*where $i(\Gamma_v)$ is the expected dimension of the moduli space of relative maps of type $\Gamma_v$ (see (6.27)), and*

$$c(\Gamma) := 2|\mathrm{Edge}_{\mathrm{int},\bullet}(\Gamma)| + |\mathrm{Edge}^0_{\circ,-}| + |\mathrm{Edge}^\infty_{\circ,-}| + 2\textstyle\sum_{e \in \mathrm{Edge}_{\bullet,\to}(\Gamma)} (\mu_{\mathfrak{D}}(e) - 1)$$

*is a factor accounting for internal disk and sphere nodes in $\Gamma$, and the last term accounts for higher order intersections with the stabilizing divisor.*

(b) *Collapsing tropical edges does not affect the index, that is, denoting by $i(\Gamma_{\mathrm{glue}}, \underline{x})$ the expected dimension of the moduli space of unbroken maps of type $\Gamma_{\mathrm{glue}}$ with limits $\underline{x} \in \mathcal{I}(L)^{d(\circ)+1}$ along the root and leaves,*

$$i^{\mathrm{brok}}(\Gamma, \underline{x}) = i(\Gamma_{\mathrm{glue}}, \underline{x}).$$

*Proof of Proposition* 6.22. The formula (6.30) for the expected dimension of the moduli space of broken maps follows in a straightforward way: the expected dimension is the sum of expected dimensions $i(\Gamma_v)$ for each of its components, viewed as relative maps minus the codimension of the matching conditions at nodes, and higher order tangency conditions with the stabilizing divisor. The matching condition at a tropical node $w_e$ is a condition of codimension $(\dim(X) - 2)$, since it requires that the projected tropical evaluations for both nodal lifts agree (see (4.18)), which accounts for the second term in (6.30). The last term incorporates matching conditions at non-tropical nodes and higher intersections with the stabilizing divisor.

To prove (b), we proceed by obtaining a relation between the Maslov indices of the components of the broken map and the Maslov index of the glued type. For simplifying exposition, we first consider the case of a broken map $u$ of type $\Gamma$ with two components $u_{v_+}, u_{v_-}$ connected by a tropical edge $e = (v_+, v_-)$ with nodal lifts $w_e^{\pm} \in C_{v_\pm}$. The map $u_{v_\pm} : C_{v_\pm} \setminus \{w_e^{\pm}\} \to \mathfrak{X}_{P(v_\pm)}$ extends to a map on the compactified curve, denoted by

$$\overline{u}_{v_\pm} : C_{v_\pm} \to \mathfrak{X}_{P(v_\pm),\mathcal{T}},$$

mapping to the compactification $\mathfrak{X}_{P(v_\pm),\mathcal{T}}$ of $\mathfrak{X}_{P(v_\pm)}$ (from (6.19)). The evaluations $\overline{u}_{v_\pm}(w_e^{\pm})$ lie in $Y_{\mathcal{T}(e)}$, where $Y_{\mathcal{T}(e)} \subseteq \mathfrak{X}_{P(v_\pm),\mathcal{T}} \setminus \mathfrak{X}_{P(v_\pm)}$ is a divisor at infinity as in (6.20). Recall from definition 4.25 that the slope $\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}}$ is the product

$$\mathcal{T}(e) = \mu_e^{\mathrm{trop}} \mathcal{T}(e)_{\mathrm{prim}}$$

of a primitive integer vector $\mathcal{T}(e)_{\mathrm{prim}} \in \mathfrak{t}_{P(e),\mathbb{Z}}$ and a positive integer $\mu_e^{\mathrm{trop}}$. Because of the cylindrical complex structure on the ends, a neighbourhood $U^{\pm}_{Y_{\mathcal{T}(e)}}$ of $Y_{\mathcal{T}(e)}$ in $\mathfrak{X}_{P(v_\pm),\mathcal{T}(e)}$ has a projection map

$$\pi_\pm : U^{\pm}_{Y_{\mathcal{T}(e)}} \to Y_{\mathcal{T}(e)}$$

with holomorphic fibers, and the tangent space splits as the sum of horizontal and vertical sub-bundles

$$TU^{\pm}_{Y_{\mathcal{T}(e)}} = H \oplus V_\pm, \quad H = \pi_\pm^* TY_{\mathcal{T}(e)}, \quad V_\pm = \ker(d\pi_\pm).$$

Topologically, the maps $\overline{u}_{v_+}, \overline{u}_{v_-}$ are obtained by cutting $u_{\mathrm{glue}}$ to yield $u_{v_+}, u_{v_-}$ and adding a capping disk to either side. The horizontal bundle $(u_{v_\pm})^* H$ extends trivially over the capping disk, but for $(u_\pm)^* V_\pm$, the bundle over the capping disk is glued in with a $\mu_e^{\mathrm{trop}}$ twist. Thus gluing at the node $w_e$ has the effect of adding the Maslov indices on both sides and subtracting $4\mu_e^{\mathrm{trop}}$.

Next, consider a general broken map type $\Gamma$, that is, it may have more than two components. Using the preceding observation, and summing over contributions from all tropical edges $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, we obtain

$$\sum_{v \in \mathrm{Vert}(\Gamma)} I(u_v^* T\mathfrak{X}_{P(v),\mathcal{T}}, (\partial u_v)^* TL) - \sum_{e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)} 4\mu_e^{\mathrm{trop}} = \sum_{v \in \mathrm{Vert}(\Gamma_{\mathrm{glue}})} I(u_v^* TX, (\partial u_v)^* TL). \quad (6.31)$$

Rewriting the left hand side of (6.31) using (6.28), we get

$$\sum_{v\in\mathrm{Vert}(\Gamma)} I_{\mathrm{adj}}(\Gamma_v) - 4\,\mathrm{Edge}_{\mathrm{trop}}(\Gamma) = \sum_{v\in\mathrm{Vert}(\Gamma_{\mathrm{glue}})} I(u_v^*TX, (\partial u_v)^*TL). \tag{6.32}$$

The expected dimension of $\mathcal{M}_\Gamma^{\mathrm{brok}}(L, \underline{\mathfrak{p}}, \underline{x})$ can be related to the index of the glued type by using the relation between the Maslov indices from the previous paragraph. To lighten notation, let us assume that all nodes are tropical, since internal nodes can be accounted for in an obvious way, and that all intersections with the stabilizing divisor $\mathfrak{D}$ are simple, and consequently $c(\Gamma) = 0$. The assumption also implies that there is a single disk component and $\mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ number of sphere components. Invoking (6.30), followed by (6.27), and then (6.32), we get

$$\begin{aligned} i^{\mathrm{brok}}(\Gamma, \underline{x}) &= \sum_{v\in\mathrm{Vert}(\Gamma)} i(\Gamma_v) - (\dim(X) - 2)|\,\mathrm{Edge}_{\mathrm{trop}}(\Gamma)| \\ &= \sum_{v\in\mathrm{Vert}(\Gamma)} I_{\mathrm{adj}}(\Gamma_v) + (d(\circ) - 2) + i_{\mathrm{Morse}}(\underline{x}) - 4|\,\mathrm{Edge}_{\mathrm{trop}}(\Gamma)| = i(\Gamma_{\mathrm{glue}}, \underline{x}). \end{aligned}$$

■

*Remark* 6.23. (On the adjusted Maslov index) Consider a relative map $u_v : S_v \to \overline{\mathfrak{X}}_P$ whose target space $\overline{\mathfrak{X}}_P$ is a manifold. The adjusted Maslov index of $u$ is then the ordinary Maslov index $I(u)$ adjusted by the constraint that for any tropical marking $w_e$ corresponding to an edge $e \ni v$, the map has the appropriate tangency with relative divisors. In particular, if at $w_e$, the sum of orders of tangency with all relative divisors is given by $\mu_{e,v} \in \mathbb{N}$, then, the adjusted Maslov index is

$$I_{\mathrm{adj}}(u) = I(u) - 2\sum_{e:v\in e} (\mu_{e,v} - 1). \tag{6.33}$$

*Example* 6.24. (Adjusted Maslov index computation) In this example, we consider the multiple cut of the cubic surface from Section 2.2. We consider the cubic surface $X$ with a deformed symplectic form so that it is a toric manifold (see (2.11)), $L \subset X$ is a toric Lagrangian, and the multiple cut is as in Figure 2.8. We list all broken maps whose gluing has homology class $2[\delta_{E_2}] + [E_2]$, and write down the adjusted Maslov indices of all the components. Here $\delta_{E_2} \in H_2(X, L)$ is the class of the disk of Maslov index two with a single intersection with the short divisor $E_2$, see the center of Figure 6.2. There are seven types $\Gamma$ of broken disks whose gluing has homology class $2[\delta_{E_2}] + [E_2]$ shown in Figure 6.2. We state the adjusted Maslov indices of the components of some of these types.

(a) In $\Gamma^1$ the adjusted Maslov indices (defined in (6.28)) are

$$I_{\mathrm{adj}}(v_1) = 4, \quad I_{\mathrm{adj}}(v_2) = 4,$$

which are equal to the ordinary Maslov indices of the components.

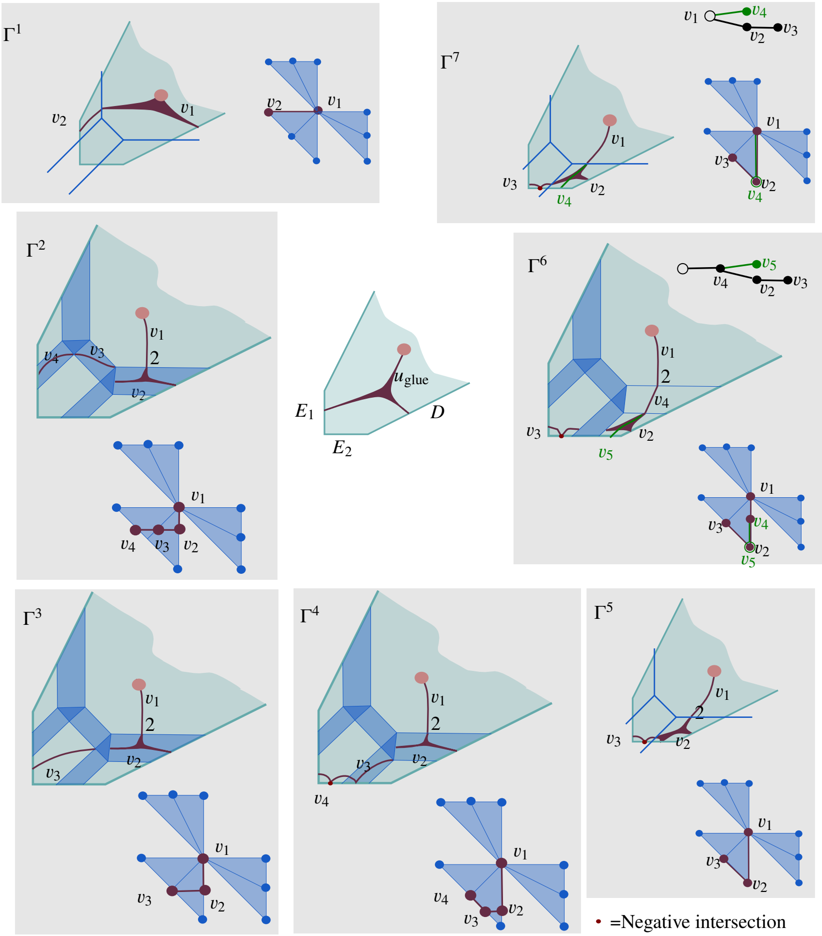



**Figure 6.2.** Broken maps corresponding to the Maslov index four disk class $[u_{\text{glue}}] = 2[\delta_{E_2}] + [E_2] = [\delta_{E_1}] + [\delta_D]$.

(b) The edge slopes in $\Gamma^3$ are

$$\mathcal{T}(v_1, v_2) = (0, -2), \quad \mathcal{T}(v_2, v_3) = (-1, 0).$$

The adjusted Maslov indices of the vertices in $\Gamma^3$ are

$$I_{\text{adj}}(v_1) = 2, \quad I_{\text{adj}}(v_2) = 6, \quad I_{\text{adj}}(v_3) = 4.$$

Indeed $u_{v_2}$ is a curve with self-intersection number 2 in $\overline{\mathfrak{X}}_{P(v_2)}$ which is the second Hirzebruch surface $F_2$, and therefore, $c_1(u_{v_2}^* T\overline{\mathfrak{X}}_{P(v_2)}) = 4$. The ordinary Maslov indices are $I(v_1) = 4$, $I(v_2) = 8$. The adjusted Maslov indices $I_{\text{adj}}(v_1)$, $I_{\text{adj}}(v_2)$ are each 2 lower because the node between $v_1$, $v_2$ has an intersection multiplicity of 2 with the relative divisor. To calculate $I_{\mathrm{adj}}(v_3)$, we need to choose an alternate almost complex compactification of $X_{P(v_3)}$ since $u_{v_3}$ has a nodal point $w_e$ corresponding to the edge $e = (v_2, v_3)$ mapping to an orbifold singularity. We choose the compactification to be a resolution of the $A_1$-singularity as in Figure 6.3 so that the nodal point $w_e$ lies in a single divisor, from where, we conclude $I_{\mathrm{adj}}(v_3) = 4$.

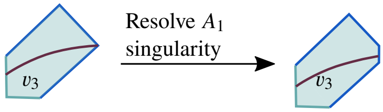



**Figure 6.3.** Orbifold singularities can be eliminated by changing the compactification.

(c) The edge slopes in $\Gamma^2$ are

$$\mathcal{T}(v_1, v_2) = (0, -2), \quad \mathcal{T}(v_2, v_3) = (-1, 0), \quad \mathcal{T}(v_3, v_4) = (-1, 0),$$

and the adjusted Maslov indices are

$$I_{\mathrm{adj}}(v_1) = 2, \quad I_{\mathrm{adj}}(v_2) = 6, \quad I_{\mathrm{adj}}(v_3) = 4, \quad I_{\mathrm{adj}}(v_4) = 4.$$

The adjusted Maslov indices of $v_3$ and $v_4$ are computed by passing to a toric resolution as in as in Figure 6.3.

(d) In $\Gamma^5$ the edge $e = (v_1, v_2)$ has multiplicity 2. The adjusted Maslov indices are

$$I_{\mathrm{adj}}(v_1) = 2, \quad I_{\mathrm{adj}}(v_2) = 8, \quad I_{\mathrm{adj}}(v_3) = 2.$$

Note that the ordinary Maslov indices are different: $I(v_1) = 4$, $I(v_2) = 10$. The value of $I_{\mathrm{adj}}$ is lower for $v_1$, $v_2$ because it accounts for the intersection multiplicity of 2 at the node $w_e$ corresponding to the edge $e = (v_1, v_2)$. We point out that the map $u_{v_3}$ is homologous to the $(-1)$-divisor $E_2'$ which is obtained by cutting the $(-2)$-divisor $E_2$, see Figure 2.10 for notation. Since the almost complex structure is a perturbation of the toric almost complex structure, the image of $u_{v_3}$ does not lie on the toric divisor $E_2'$.

(e) In a broken map of type $\Gamma^7$, the component $u_{v_1}$ has two distinct intersections with the relative divisor, each of multiplicity 1. For each of the vertices $v_i$ in $\Gamma^0$ the Maslov index $I(u_{v_i})$ is equal to the adjusted Maslov index $I_{\mathrm{adj}}(u_{v_i})$, and

$$I_{\mathrm{adj}}(v_1) = 4, \quad I_{\mathrm{adj}}(v_2) = 6, \quad I_{\mathrm{adj}}(v_3) = 2, \quad I_{\mathrm{adj}}(v_4) = 4.$$

For each type $\Gamma^i$, the Maslov index sum formula (6.32) implies that the Maslov index $I(\Gamma^i_{\mathrm{glue}})$ of the glued disk is 4. We discuss the moduli spaces of broken maps of types $\Gamma_i$ in the following Remark 6.25.

*Remark* 6.25. We conjecturally describe the moduli space of of broken disks in the multiply cut cubic surface whose glued homology class is $[\delta_{E_1}] + [\delta_D]$. We refer to Figure 6.2 for notations. We can not make this discussion rigorous because under the Cieliebak-Mohnke perturbation scheme, we can only compactify moduli spaces of dimension at most 1. For $i = 1, \dots, 7$ we denote by

$$\mathcal{M}_i := \mathcal{M}^{\text{brok}}_{\Gamma^i}(\mathfrak{X})/T_{\text{trop}}(\Gamma^i)$$

the reduced moduli space of broken disks whose tropical graph is $\Gamma^i$ and which have a point constraint on the boundary. In other words, the disks do not have any inputs and the output marking is required to map to the maximum point of the Morse function on the Lagrangian $L$. Since in each of the types the glued disk has Maslov index 4, we get $\dim(\mathcal{M}^{\text{brok}}_{\Gamma^i}(\mathfrak{X})) = 2$ for all $i$. For odd $i$, the tropical graph is rigid and $T_{\text{trop}}(\Gamma^i)$ is finite. For even $i$, $\dim_{\mathbb{C}}(T_{\text{trop}}(\Gamma^i)) = 1$, and so $\mathcal{M}_i$ is 0-dimensional. For any even $i$,

$$\mathcal{M}_i \subset \overline{\mathcal{M}}_{i-1} \backslash \mathcal{M}_{i-1} \quad \text{and} \quad \mathcal{M}_i \subset \overline{\mathcal{M}}_{i+1} \backslash \mathcal{M}_{i+1},$$

because there are tropical edge collapse morphisms $\Gamma^{i+1} \to \Gamma^i$, $\Gamma^{i-1} \to \Gamma^i$ (see Definition 8.31 and Example 8.34). Besides having a non-compact end, the moduli spaces $\mathcal{M}_1$, $\mathcal{M}_7$ also have codimension one boundary components consisting of configurations with a broken edge $e \in \text{Edge}_-(\Gamma^i)$, $\ell(e) = \infty$. The moduli space of broken disks of class $[\delta_{E_1}] + [\delta_D]$ is

$$\overline{\mathcal{M}}_1 \cup_{\mathcal{M}_2} \overline{\mathcal{M}}_3 \cup_{\mathcal{M}_4} \overline{\mathcal{M}}_5 \cup_{\mathcal{M}_6} \overline{\mathcal{M}}_7.$$

Here we take Parker's viewpoint [69] that the moduli spaces of broken maps are themselves broken manifolds (which in his papers, are called "exploded manifolds"). The spaces $\overline{\mathcal{M}}_i$ for odd $i$ are the top-dimensional cut spaces, and $\mathcal{M}_i$ for even $i$ are the cut spaces of codimension 2.

## 6.5 Transversality

In this section, we show that for a set of comeager domain-dependent perturbations, moduli spaces of broken maps with index at most one are transversally cut out. The perturbation scheme we use here only achieves transversality for certain combinatorial types, as in Cieliebak-Mohnke [26]. In particular, the Cieliebak-Mohnke [26] perturbation scheme can not achieve transversality on *crowded components*, which are components where the map is constant and which contain more than one marking. The moduli space of such maps cannot be transversally cut out, since the constraint that the first marking in such a component $S$ maps to the stabilizing divisor $D$ together with the fact that $u|S$ is a constant map guarantees that the second marking does as well. In the context of broken maps, the "constant" condition in the definition of crowdedness needs to be replaced by "horizontally constant". Recall that a component $u_v : S_v^\circ \to \mathfrak{X}_{P(v)}$ is horizontally constant if its projection to $X_{P(v)}$ is constant. We define crowdedness for broken maps.

**Definition 6.26.** The combinatorial type $\Gamma$ of a broken map $u : C \to \mathfrak{X}$ is *crowded* if there exists a connected subgraph $\Gamma' \subset \Gamma$ such that $u|S_v$ is horizontally constant (see Definition 4.17) on all vertices $v \in \mathrm{Vert}(\Gamma')$ and $\Gamma'$ contains more than one interior leaf.

**Definition 6.27.** For a type $\Gamma$ of treed disks, a perturbation datum $\mathfrak{p}_\Gamma$ is *regular* if any $\mathfrak{p}_\Gamma$-holomorphic map is regular, and so for any type $\Gamma_X$ of broken maps whose domain type is $\Gamma$, the moduli space $\mathcal{M}^{\mathrm{brok}}_{\Gamma_X}(L, \mathfrak{p}_\Gamma)$ is a manifold of expected dimension.

The following theorem on the existence of regular perturbation data for broken maps is the main result of this section. Perturbation data is defined strata-wise, and at each step we assume that the data on the smaller strata is fixed, where the ordering on the strata is as follows: For types $\Gamma'$, $\Gamma$ of broken maps,

$$\Gamma' < \Gamma \tag{6.34}$$

if $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making the length of a boundary edge finite or non-zero. The ordering relation helps in describing the boundary of the moduli spaces of treed curves as

$$\overline{\mathcal{M}}_\Gamma \backslash \mathcal{M}_\Gamma = \cup_{\Gamma' < \Gamma} \mathcal{M}_{\Gamma'}.$$

The background data $(\mathfrak{J}_0, F_0)$ and the stabilizing divisor $\mathfrak{D}$ are the same for $\mathfrak{p}_\Gamma$ for all domain types $\Gamma$. For any $\Gamma$, we will require that the perturbation datum $\mathfrak{p}_\Gamma$ is adapted to the stabilizing pair $(\mathfrak{J}_0, \mathfrak{D})$ in the sense of Definition 6.9.

**Theorem 6.28.** *(Transversality) Let $\mathfrak{X}$ be a broken manifold. Let $\mathfrak{D} \subset \mathfrak{X}$ be a cylindrical broken divisor and $\mathfrak{J}_0 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D})$ be a locally strongly tamed cylindrical almost complex structure adapted to $\mathfrak{D}$ such that $(\mathfrak{D}, \mathfrak{J}_0)$ is a stabilizing pair (Definition 5.22), and let $F_0 : L \to \mathbb{R}$ be a Morse function. Suppose $\Gamma$ is a type of stable treed disks, and the perturbation data*

$$\{\mathfrak{p}_{\Gamma'}\}_{\Gamma' < \Gamma}$$

*is coherent, regular, has background $(\mathfrak{J}_0, F_0)$, and is adapted to the pair $(\mathfrak{J}_0, \mathfrak{D})$ (as in Definition 6.9). Then, for type $\Gamma$, there exists a regular perturbation datum*

$$\mathfrak{p}_\Gamma$$

*with background $(\mathfrak{J}_0, F_0)$ and adapted to the pair $(\mathfrak{J}_0, \mathfrak{D})$ that is coherent with the data $\{\mathfrak{p}_{\Gamma'}\}_{\Gamma' < \Gamma}$ (as in Definition 6.5). For an uncrowded type $\Gamma_X$ of broken maps whose* *domain type is* $\Gamma$ *(see Definition 6.14), the moduli space* $\mathcal{M}_{\Gamma_X}(L, \mathfrak{p}_\Gamma)$ *is a smooth oriented manifold of expected dimension. Consequently, there is a collection of coherent perturbation data*

$$\underline{\mathfrak{p}} := (\mathfrak{p}_\Gamma)_\Gamma$$

*that has background* $(\mathfrak{J}_0, F_0)$*, and is adapted to the pair* $(\mathfrak{J}_0, \mathfrak{D})$*, where* $\Gamma$ *ranges over all types of stable treed disks.*

*Proof of Theorem* 6.28. Transversality, as in the unbroken case, is an application of Sard-Smale as in Cieliebak-Mohnke [26] and Charest-Woodward [17] on the universal space of maps. The new feature is that in neck pieces of the broken manifold the almost complex structure is fixed in the fiber direction. This does not pose any issues for maps whose horizontal projection is non-constant. Components of the map whose horizontal projection is constant will shown to be automatically transversal.

The moduli space is cut out as a zero set of a section of a Banach bundle which we now describe. We restrict our attention to types of maps for which all intersections with the stabilizing divisor have multiplicity one. Other types are discussed later. We construct the moduli space of broken maps without framing, since the framed version is a finite cover of the unframed one. (See Definition 4.14 of framing and Remark 4.26.) We cover the moduli space of treed disks $\mathcal{M}_\Gamma$ by charts $\cup_i \mathcal{M}_\Gamma^i$, so that on a trivialization of the universal curve $\mathcal{U}_\Gamma^i$, each of the fibers is a fixed treed curve $C = S \cup T$ with fixed special points (see (4.7)). The complex structure on $S$ varies smoothly in the sense that it is given by a map

$$\mathcal{M}_\Gamma^i \to \mathcal{J}(S_\Gamma), \quad m \mapsto j(m).$$

In order to apply Sard-Smale, we pass to maps from the normalized curve of a fixed Sobolev class. The domain of the map is the punctured curve $C^\circ \subset C$ with punctures at tropical nodal points $w_e$ (corresponding to $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$). Let

$$\tilde{C}^\circ := \bigsqcup_{v \in \mathrm{Vert}(\Gamma)} S_v^\circ \sqcup \bigsqcup_{e \in \mathrm{Vert}(\Gamma)} T_e$$

denote the normalized curve in the sense that nodes in $C^\circ$ (corresponding to internal edges $e \in \mathrm{Edge}_{\mathrm{int}}(\Gamma)$) are lifted to double points in $\tilde{C}^\circ$ and the tree components in $C^\circ$ are detached from the surface components. Choose $p > 2$ and $\lambda \in (0, 1)$. Let

$$\mathrm{Map}_\Gamma^{1,p,\lambda}(\tilde{C}^\circ, \mathfrak{X}, L, \mathfrak{D})$$

denote the completion of maps under the weighted Sobolev norm $\|\cdot\|_\Gamma^\circ$ in (6.17) for surface components, and the ordinary $W^{1,p}$-norm for the tree components. That is, an element of this space consists of

(a) a collection of $W^{1,p,\lambda}$-maps

$$u_v : S_v^\circ \to \mathfrak{X}_{P(v)}, \quad u(\partial S_v) \subset L, \quad v \in \mathrm{Vert}(\Gamma)$$

for each vertex $v$ (where $S_v^\circ$ is defined in (4.14)), with the puncture at the node corresponding to any tropical edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ asymptotic to a $\mathcal{T}(e)$-cylinder;

(b) and $W^{1,p}$ maps

$$u_e : T_e \to L, \quad e \in \mathrm{Edge}(\Gamma)$$

from tree components to $L$ asymptotic to the critical points $\underline{x}$ on the leaves $T_e, e \in \mathrm{Edge}_\circ(\Gamma)$. If $\ell(e)$ is finite and non-zero, that is $e \in \mathrm{Edge}_\circ^{(0,\infty)}(\Gamma)$, then $T_e \simeq [0, 1]$ and if $\ell(T_e) = \infty$, each segment in $T_e$ is either $[0, \infty)$, $(-\infty, 0]$ or $\mathbb{R}$.

The metric on $\mathfrak{X}_{P(v)}$ is chosen to be cylindrical on the ends, and so that $L$ is totally geodesic.

The perturbation data on the strata lower than $\Gamma$ glue to give regular perturbation datum in a small neighborhood in the boundary of the moduli space $\overline{\mathcal{M}}_\Gamma$: Any perturbation data for $\Gamma'$ with $\Gamma' < \Gamma$ induces perturbations for $\Gamma$ in a neighborhood of $\mathcal{U}_{\Gamma'}$ in $\overline{\mathcal{U}_\Gamma}$ by the gluing construction in (8.3) and a similar gluing construction for the tree parts. Let

$$\mathscr{P}_\Gamma = \{\mathfrak{p}_\Gamma = (J_\Gamma, F_\Gamma)\}$$

be the space of perturbation data defined on $\overline{\mathcal{U}}_\Gamma$ whose distance from the background data $(\mathfrak{J}_0, F_0)$ is bounded in the $C^\varepsilon$-norm, and that agree with the glued perturbation datum on a fixed open neighbourhood $\mathcal{N}_{\Gamma'}$ of $\mathcal{U}_{\Gamma'}$ for the strata $\Gamma' < \Gamma$. For a sufficiently small neighborhood $\mathcal{N}_{\Gamma'}$, any such perturbation $\mathfrak{p}_\Gamma$ is already regular for maps $u$ with domain in $\mathcal{N}_{\Gamma'}$. Indeed, otherwise one could take a sequence $u_\nu$ of irregular maps of fixed area with domain converging to $\mathcal{U}_{\Gamma'}$. By Gromov compactness in Chapter 8, after passing to a subsequence we may assume that $u_\nu$ converges to a limit $u_\infty$ of type $\Gamma'$. By surjectivity of gluing in Section 9.7 and Lemma 9.6, the linearized operator $D_{u_\nu}$ is surjective for $\nu$ sufficiently large, which is a contradiction. Since the gluing and compactness arguments do not use regularity, circularity of argument is avoided.

We use the $C^\varepsilon$-norm for $J_\Gamma$ and a $C^l$-norm for $F_\Gamma$ where $l > 1$ is a fixed number. For any broken curve $C$ of type $\Gamma$ we obtain perturbation data on $C$ by identifying it conformally with a fiber of the universal tree disk $\mathcal{U}_\Gamma$. Let

$$\mathcal{B}^i_{p,\lambda,l,\Gamma} := \mathcal{M}^i_\Gamma \times \mathrm{Map}^{1,p,\lambda}_\Gamma(\tilde{C}, \mathfrak{X}, L, \mathfrak{D}) \times \mathscr{P}_\Gamma(\mathfrak{X}, \mathfrak{D}).$$

Let $\mathcal{E}^i = \mathcal{E}^i_{p,\lambda,\Gamma}$ be the Banach bundle over $\mathcal{B}^i_{p,\lambda,l,\Gamma}$ given by

$$(\mathcal{E}^i_{p,\lambda,\Gamma})_{j,u,J} \subset L^{p,\lambda}(\Omega^{0,1}_{j,J}(S, (u|S)^*T\mathfrak{X})) \oplus L^p(\Omega^1(T, (u|T)^*TL)).$$

Here the first summand is the space of 0, 1-forms with respect to $(j(m), J)$. The Cauchy-Riemann and shifted gradient operators applied to the restrictions $u|S$ resp.

$u|T$ of $u$ to the two resp. one dimensional parts of $C = S \cup T$ define a $C^{l-1}$ section

$$\overline{\partial}_\Gamma : \mathcal{B}^i_{p,\lambda,l,\Gamma} \to \mathcal{E}^i_{p,\lambda,\Gamma}, \quad (C, u, (J_\Gamma, F_\Gamma)) \mapsto \left( \overline{\partial}_{j(m),J_\Gamma} u|S, \left( \frac{1}{\lambda_e} \frac{d}{ds} + \operatorname{grad}_{F_\Gamma} \right) u|T \right) \tag{6.35}$$

where $s$ is a local coordinate on the tree components with unit speed, and

$$\lambda_e = \begin{cases} \ell(e), & e \in \mathrm{Edge}^{(0,\infty)}_{\circ,-}(\Gamma), \\ 1, & e \in \mathrm{Edge}^{\infty}_{\circ}(\Gamma). \end{cases}$$

is a factor accounting for the length of treed segments in case the edge length is finite and non-zero. The evaluation maps at lifts of nodal points, markings, and lifts of $S \cap T$ give a smooth map

$$\mathrm{ev}_\Gamma : \mathcal{B}^i_{p,\lambda,l,\Gamma} \to \mathfrak{X}(\Gamma) \tag{6.36}$$

where

$$\mathfrak{X}(\Gamma) = \left( \prod_{e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)} (\mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}})^2 \right) \times \left( \prod_{e=(v_+,v_-) \in \mathrm{Edge}_{\mathrm{int},\bullet}(\Gamma)} (\mathfrak{X}_{P(v_\pm)})^2 \right)$$
$$\times \left( \prod_{x \in S \cap T} L^2 \right) \times \left( \prod_{e \in \mathrm{Edge}^0_\circ(\Gamma)} L^2 \right) \times \left( \prod_{e \in \mathrm{Edge}_{\bullet,\to}(\Gamma)} \mathfrak{X}_{P(v(e))} \right). \tag{6.37}$$

The first two factors in (6.37) correspond to lifts of interior nodes, the third term is a lift of boundary nodes $w_e$ with no treed segments (that is, $\ell(e) = 0$), the fourth term is a lift of $S \cap T$, and the last term corresponds to evaluation at an interior marking. All the factors of $\mathrm{ev}_\Gamma$ are standard evaluation maps, except for the first factor which is the projected tropical evaluation map as in (4.16). Let

$$\Delta(\Gamma) \subset \mathfrak{X}(\Gamma)$$

be the submanifold that is the product of diagonals in the first four factors of $\mathfrak{X}(\Gamma)$ in (6.37), and the stabilizing divisor $D_{P(v(e))} \subset \mathfrak{X}_{P(v(e))}$ in the last factor. The *local universal moduli space* is

$$\mathcal{M}^{\mathrm{univ},i}_\Gamma(L, \mathfrak{D}) = (\overline{\partial}, \mathrm{ev}_\Gamma)^{-1}(\mathcal{B}^i_{p,\lambda,l,\Gamma}, \Delta(\Gamma)), \tag{6.38}$$

where $\mathcal{B}^i_{k,p,l,\Gamma}$ is embedded as the zero section in $\mathcal{E}^i_{p,\lambda,\Gamma}$.

We will next show that this subspace is cut out transversally. We first consider two-dimensional components of $\tilde{C}$ on which the map is not horizontally constant, and show that the linearization of $(\overline{\partial}, \mathrm{ev}_\Gamma)$ is surjective. For components whose target space is not a neck piece, the surjectivity of the differential $D(\overline{\partial}, \mathrm{ev}_\Gamma)$ follows from [55, Proposition 3.4.2]. Recall from (3.43) that a neck piece $\mathfrak{X}_P$ is a $T_{P,\mathbb{C}}$-bundle

$$T_{P,\mathbb{C}} \to \mathfrak{X}_P \to X_P,$$

and the almost complex structure on $\mathfrak{X}_P$ is $P$-cylindrical. (For non-neck pieces $T_P$ is trivial.) Consider a component $S_v^\circ \simeq \mathbb{P}^1 \backslash \{\text{special points}\}$ that maps to a neck piece $\mathfrak{X}_P$, and the horizontal projection $\pi_P \circ u : S_v^\circ \to X_P$ is non-constant. The linearized operator

$$D_{u,J}^\circ(\xi, K) = D_u^\circ \xi + \tfrac{1}{2} K D u j.$$

is surjective as follows. Let

$$\eta \in \operatorname{coker}(D_{u,J}^\circ) \subset \Omega^{0,1}(u^* T\mathfrak{X}_P)$$

be a one-form in the cokernel of $D_{u,J}^\circ$. Variations of tamed almost complex structure of cylindrical type are $J$-antilinear maps

$$K : T\mathfrak{X}_P \to T\mathfrak{X}_P$$

that vanish on the vertical sub-bundle and are $T_{P,\mathbb{C}}$-invariant.[4] Since the horizontal part of $D_z u$ is non-zero at some $z \in S_v^\circ$, we may find an infinitesimal variation $K$ of almost complex structure *of cylindrical type* by choosing $K(z)$ so that $K(z) D_z u j(z)$ is an arbitrary $(j(z), J(z))$-antilinear map from $T_z C$ to $T_{u(z)} \mathfrak{X}_P$. Choose $K(z)$ so that $K(z) D_z u j(z)$ pairs non-trivially with $\eta(u(z))$ and extend $K(z)$ to an infinitesimal almost complex structure $K$ by a cutoff function on the domain curve. For two-dimensional components that are horizontally constant, by Proposition 6.30, the linearization $D\overline{\partial}$ is surjective, and additionally the evaluation map at a single marked point is surjective. (Note that, if we consider all the markings together, the linearization $D\,\mathrm{ev}_\Gamma$ may not be surjective on horizontally constant components.) Finally, for tree components, the linearization of the shifted gradient operator and the evaluation map at the finite end is surjective, since we can perturb the Morse function on the Lagrangian.

From the discussion so far, we conclude that the local moduli space (6.38) is cut out transversally except if there is a connected component in the domain on which the map is horizontally constant, and which contains more than one irreducible surface component. In this exceptional case, it remains to prove that the matching conditions at nodes between two horizontally constant components are cut out transversally. Such nodes are necessarily internal interior nodes, corresponding to edges $e \in \mathrm{Edge}_{\bullet,\mathrm{int}}(\Gamma)$. We consider a maximal connected subgraph $\Gamma' \subset \Gamma$ so that the map is horizontally constant on the vertices of $\Gamma'$, and $\Gamma'$ does not have tree components. By uncrowdedness, there is at most one marked point in $\Gamma'$. So, it is possible to choose at most one special point (marked point or a lift of a nodal point) on each component of $\tilde{C}_{\Gamma'}$, so

[4]Perturbing $K$ is equivalent to perturbing the connection one-form $\alpha_P$ of the $P$-cylindrical almost complex structure, see (3.19).

that for every nodal point one of its ends is chosen. By Proposition 6.30, for a horizontally constant component with a single tropical marking $z_e$ corresponding to an edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, the linearized map $D(\overline{\partial}, \pi_{\mathcal{T}(e)}(\mathrm{ev}_{z_e}^{\mathcal{T}(e)}))$ is surjective. Since the evaluation map is surjective at each of the chosen lifts, an inverse of the linearized map $D(\overline{\partial}, \mathrm{ev}_\Gamma)$ can be constructed inductively, see [17, p63].

For types where the map has higher order intersections with the stabilizing divisor, the universal moduli space is cut out inductively as in [26, Lemma 6.5]. Each step of the induction cuts out a moduli space where the tangencies at one of the markings is increased by one. We start out with a moduli space cut out of $W^{k,p,\lambda}$ where $k - \frac{2}{p} > \mu$ and $\mu$ is the largest order of tangency with the stabilizing divisor $\mathfrak{D}$ that occur in the type $\Gamma$.

By the implicit function theorem, $\mathcal{M}_\Gamma^{\mathrm{univ},i}(L, \mathfrak{D})$ is a smooth Banach manifold, and the forgetful morphism

$$\varphi_i : \mathcal{M}_\Gamma^{\mathrm{univ},i}(L, \mathfrak{D})_{k,p,l} \to \mathscr{P}_\Gamma(L, \mathfrak{D})_l$$

is a smooth Fredholm map. By the Sard-Smale theorem, the set of regular values $\mathscr{P}_\Gamma^{i,\mathrm{reg}}(L, D)$ of $\varphi_i$ on $\mathcal{M}_\Gamma^{\mathrm{univ},i}(L, \mathfrak{D})_d$ in $\mathscr{P}_\Gamma(L, \mathfrak{D})$ is comeager. Let

$$\mathscr{P}_\Gamma^{\mathrm{reg}}(L, \mathfrak{D}) = \bigcap_i \mathscr{P}_\Gamma^{i,\mathrm{reg}}(L, \mathfrak{D}).$$

A standard argument shows that the set of smooth domain-dependent $\mathscr{P}_\Gamma^{\mathrm{reg}}(L, \mathfrak{D})$ is also comeager. Fix $(J_\Gamma, F_\Gamma) \in \mathscr{P}_\Gamma^{\mathrm{reg}}(L, \mathfrak{D})$. By elliptic regularity, every element of $\mathcal{M}_\Gamma^i(L, \mathfrak{D})$ is smooth. The transition maps for the local trivializations of the universal bundle define smooth maps

$$\mathcal{M}_\Gamma^i(L, \mathfrak{D})|_{\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j} \to \mathcal{M}_\Gamma^j(L, \mathfrak{D})_{\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j}.$$

This construction equips the space

$$\mathcal{M}_\Gamma(L, \mathfrak{D}) = \cup_i \mathcal{M}_\Gamma^i(L, \mathfrak{D})$$

with a smooth atlas. Since $\mathcal{M}_\Gamma$ is Hausdorff and second-countable, so is $\mathcal{M}_\Gamma(L, \mathfrak{D})$ and it follows that $\mathcal{M}_\Gamma(L, \mathfrak{D})$ has the structure of a smooth manifold. Orientation of the moduli space is discussed in Remark 6.29 below. ■

For regular perturbations, the tangent space to the moduli space can be identified with the kernel of an operator called the *linearized operator*. Using notations from the proof of transversality, we write down the linearized operator for future reference. For a regular perturbation $\mathfrak{p}_\Gamma$, consider the map

$$(\overline{\partial}, \mathrm{ev}_\Gamma) : \mathcal{B}_{k,p,\Gamma}^i \to \mathcal{E}_{k,p,\Gamma}^i \times \mathfrak{X}(\Gamma)$$

where $\mathrm{ev}_\Gamma$ is defined in (6.36). Its linearization is denoted

$$D_u : T_{[C],u}\mathcal{B}^i_{k,p,\Gamma} \to (\mathcal{E}^i_{k,p,\Gamma})_{[C],u} \oplus \mathrm{ev}_\Gamma^* T\mathfrak{X}(\Gamma)/T\Delta(\Gamma). \tag{6.39}$$

For a regular broken map $u$, the operator $D_u$ is surjective.

*Remark* 6.29. (Orientation of moduli spaces of maps) Moduli spaces of broken maps are oriented using a relative spin structure on the Lagrangian as in Fukaya-Oh-Ohta-Ono [36] as follows. An orientation for the moduli space at a given map is by definition a a non-zero determinant line bundle of the Fredholm operator $D_u$ that is the linearization of the Fredholm map cutting out the moduli space, where the determinant line is defined as the tensor product of top exterior powers

$$\det(D_u) := \Lambda^{\mathrm{top}}(\ker(D_u)) \wedge \Lambda^{\mathrm{top}}(\mathrm{coker}(D_u)).$$

Let $L$ be an oriented Lagrangian equipped with a relative spin structure (see [36, Chapter 44] for the definition). Given a holomorphic disk $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (X, L)$ and a complex linear Cauchy-Riemann operator

$$\overline{\partial} : \Gamma(\mathbb{D}^2, \partial\mathbb{D}^2; u^*TX, (\partial u)^*TL) \to \Omega^{0,1}(\mathbb{D}^2, u^*TX),$$

the determinant line bundle can be identified with $\det(T_l L)$ for any point $l \in L$, that is, $\det(\overline{\partial}) \simeq \det(T_l L)$, and the identification is canonical up to multiplication by positive scalars $\lambda \in \mathbb{R}_+$, see [36, Proposition 44.4] or an explanation in [21, Proposition 5.2]. The argument in [36, Proposition 44.4] also gives such an identification in case of a nodal disk with interior nodes, since that involves degenerating the bundle $u^*TX$ into a bundle on the disk and a complex vector bundle on the sphere. The determinant bundle for the linearized operator for broken maps can be identified to $\det(T_l L)$ in a similar way, since the matching conditions at the cylindrical ends are complex-linear. There are similar orientations for moduli spaces of treed disks with contributions to the orientation from the infinite treed segments attached at boundary markings. We choose orientations on Morse unstable and stable manifolds

$$o(x) : W^\pm(x) \to TW^\pm(x), \quad \forall x \in \mathrm{crit}(F)$$

for the Morse function $F$ on the Lagrangian, such that for any $x \in \mathrm{crit}(F)$ the map

$$\det(T_x W^-(x) \oplus T_x W^+(x)) \to \det(T_x L)$$

is orientation preserving. We view a broken treed holomorphic disk as defined by a condition $u(z_i^\circ) \in W^-(x_i)$ on each of the boundary markings $e_i^\circ \in \mathrm{Edge}_{\circ,\to}$. This gives an isomorphism of line bundles

$$\det(D_u) \simeq \det(T\mathcal{M}_\Gamma) \wedge \det(TL) \wedge \det(W^+(x_0)) \wedge (\wedge_{i=1}^{d(\circ)} \det(W^-(x_i))),$$

up to multiplication by positive scalars. The choices made in the previous paragraphs fix a positive section on the right-hand side. The construction induces an orientation on the moduli space $\mathcal{M}_\Gamma(\mathfrak{X}, \underline{x})$ for any type $\Gamma$ of broken maps and a collection of inputs and output $\underline{x}$. The orientation of $\det(T\mathcal{M}_\Gamma)$ is described in Definition 4.2. For a broken map type $\Gamma$ with a broken edge, the difference in orientation induced by (Making an edge length finite) morphism and the product orientation induced by (Cutting an edge) morphism differ by a quantity that depends on the type $\Gamma$ and the Morse indices of the labels on the end-points of the treed segments, see [76, (12.25)]. This ends the Remark.

## 6.6 The toric case

In this section, we consider the question of regularity for horizontally constant maps. These are maps in a toric fibration whose image is contained in a single fiber. Therefore, we wish to show that, in a toric variety with standard almost complex structure, spheres that intersect torus-invariant divisors at isolated points are regular. The maps are allowed to have a single tropical marking, in which case, the evaluation map at the marking is submersive. This result is used in the proof of transversality in the previous section to analyze horizontally constant components.

**Proposition 6.30.** *Suppose $\mathfrak{X}$ has a cylindrical almost complex structure. Let $\Gamma$ be a type of relative map with a single vertex $v$, the vertex $v$ being of sphere type and horizontally constant. Any map $u : S_v^\circ \to \mathfrak{X}_{P(v)}$ of type $\Gamma$ is regular, and therefore, the moduli space of maps $\mathcal{M}_\Gamma(\mathfrak{X})$ is a manifold of expected dimension. For a tropical marking $z_e$ corresponding to any edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, the projected tropical evaluation map*

$$\pi^\perp_{\mathcal{T}(e)} \circ \mathrm{ev}^{\mathcal{T}(e)}_{z_e} : \mathcal{M}_\Gamma(\mathfrak{X}) \to \mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}$$

*is submersive.*

**Lemma 6.31.** *Let $X$ be a toric orbifold with the standard almost complex structure, and let $u : \mathbb{P}^1 \to X$ be a holomorphic sphere whose intersection points with the torus-invariant divisors $Y_1, \dots, Y_N$ are isolated. Then, $u$ is regular in $X$, that is, the linearized operator $D\overline{\partial}$ is surjective at $u$.*

*Proof.* Holomorphic spheres meeting the interior of $X$ are regular by the following argument. As in Delzant [28], $X$ can be viewed as a geometric invariant theory quotient $\mathbb{C}^N /\!\!/ G$ where $N$ is the number of prime torus-invariant relative divisors of $X$, and $G \subset (\mathbb{C}^\times)^N$ is a complex torus whose quotient $(\mathbb{C}^\times)^N/G$ is $T_\mathbb{C}$. Each of the torus-invariant divisors $Y_1, \dots, Y_N$ in $X$ lifts to a coordinate hyperplane $\{z_1 = 0\}, \dots, \{z_N = 0\}$ in $\mathbb{C}^N$. Consider a holomorphic sphere $u : \mathbb{P}^1 \to X$, that is not contained in any toric divisor.

The vector bundle $u^*TX \oplus \underline{\mathfrak{g}}$ on $\mathbb{P}^1$ is a sum of line bundles

$$u^*TX \oplus \underline{\mathfrak{g}} = \bigoplus_{i=1}^{N} u^*\mathcal{O}(Y_i)$$

where $\underline{\mathfrak{g}} := \mathfrak{g} \times \mathbb{P}^1$ is the trivial bundle. The degree $\deg(u^*\mathcal{O}(Y_i))$ of the line bundle $u^*\mathcal{O}(Y_i)$ is given by the intersection of $u$ with $Y_i$. Hence, each of the degrees $\deg(u^*\mathcal{O}(Y_i))$ is non-negative. As a result, the operator

$$\overline{\partial} : \Gamma(\mathbb{P}^1, \oplus_i u^*\mathcal{O}(Y_i)) \to \Omega^{0,1}(\mathbb{P}^1, \oplus_i u^*\mathcal{O}(Y_i))$$

is onto. Consequently, the cohomology group

$$H^{0,1}(\mathbb{P}^1, \oplus_i u^*\mathcal{O}(Y_i)) \simeq H^1(\mathbb{P}^1, \oplus_i u^*\mathcal{O}(Y_i))$$

vanishes. Consider the long exact sequence in Čech cohomology, corresponding to the short exact sequence of sheaves

$$0 \to \underline{\mathfrak{g}} \to \oplus_i u^*\mathcal{O}(Y_i) \to u^*TX \to 0.$$

Vanishing of the zeroth resp. first cohomology of the first resp. second terms implies that the first cohomology of the third term

$$H^{0,1}(\mathbb{P}^1, u^*TX) \simeq H^1(\mathbb{P}^1, u^*TX)$$

also vanishes. Therefore the sphere $u$ is regular in $X$. ■

*Proof of Proposition* 6.30. We recall that we may equivalently use the operators $D_u$ or $D_u^\circ$ since they have isomorphic kernels and cokernels. First, we use $D_u$ to show that $u$ is regular. Recall that

$$\mathfrak{X}_{P(v),\mathcal{T}} := \mathfrak{X}_{P(v)} \cup \cup_{e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)} Y_{\mathcal{T}(e)}.$$

is a compactification of $\mathfrak{X}_{P(v)}$ obtained by adding a divisor $Y_{\mathcal{T}(e)}$ (see (6.20)) to $\mathfrak{X}_{P(v)}$ corresponding to every edge $e$ incident on $v$. A map $u : S_v^\circ \to \mathfrak{X}_{P(v)}$ of type $\Gamma$ extends holomorphically to $u : \mathbb{P}^1 \to \mathfrak{X}_{P(v),\mathcal{T}}$ with $u(z_e) \in Y_{\mathcal{T}(e)}$. The fibers of the projection map $\pi_{P(v)} : \mathfrak{X}_{P(v),\mathcal{T}} \to X_{P(v)}$ are toric orbifolds with the standard almost complex structure, and the image of $u$ is contained in a single fiber. Since the almost complex structure on $\mathfrak{X}_{P(v)}$ is cylindrical, the pullback bundle splits into holomorphic bundles

$$u^*T\mathfrak{X}_{P(v)} = V \oplus H \to \mathbb{P}^1, \quad V = u^* \ker(d\pi_{P(v)}),$$

with the horizontal bundle $H$ being trivial, since $u$ is horizontally constant. The linearization $D\overline{\partial}|V$ surjects onto $\Lambda^{0,1}_{\mathbb{P}^1} \otimes V$ by Lemma 6.31, and since $H$ is trivial, the map $D\overline{\partial} : H \to \Lambda^{0,1}_{\mathbb{P}^1} \otimes H$ is surjective.

Next, we will show that the moduli spaces of spheres with prescribed tangencies at relative marked points are cut out transversally. For a fixed point $p \in X_{P(v)}$, consider holomorphic maps $u : \mathbb{P}^1 \to \pi^{-1}_{P(v)}(p) \subset \mathfrak{X}_{P(v),\mathcal{T}}$ of type $\Gamma$, and hence not contained in the divisors $Y_{\mathcal{T}(e)}$. The first Chern class of the sphere $u$, viewed as a map to the fiber $\pi^{-1}_{P(v)}(p)$, is equal to the sum of its intersection multiplicities $(u.Y_{\mathcal{T}(e)})$ with all divisors at infinity $Y_{\mathcal{T}(e)}$. In fact, the moduli space containing $u$ is parametrized by the set of intersection points $z \in C$ with the toric divisors $Y_{\mathcal{T}(e)}$. For maps with a higher order tangency with a toric divisor $Y_{\mathcal{T}(e)}$, some subsets of intersection points of $u$ with $Y_{\mathcal{T}(e)}$ coincide. Since the zeros of sections of positive line bundles may be chosen arbitrarily in genus zero, the locus of such maps is cut out transversally from the space of all maps. Therefore, the moduli space of relative maps of any type $\Gamma$ is regular. We now switch to working with relative maps $u : S_v^\circ \to \mathfrak{X}_{P(v)}$ defined on punctured curves.

The linearization of the tropical evaluation map (4.15) at tropical markings is surjective because of the torus-equivariance of the evaluation map. In particular, for a tropical marking $z_e$ and a choice of holomorphic coordinate in a neighborhood of $z_e$, the tropical evaluation

$$\mathrm{ev}_{z_e}^{\mathcal{T}(e)} : \mathcal{M}_\Gamma(\mathfrak{X}) \to \mathfrak{X}_{P(e)}$$

satisfies

$$\mathrm{ev}_{z_e}^{\mathcal{T}(e)}(tu) = t\,\mathrm{ev}_{z_e}^{\mathcal{T}(e)}(u), \quad \forall t \in T_{P(v),\mathbb{C}}. \tag{6.40}$$

We choose an embedding of $\mathfrak{X}_{P(e)}$ into the cylindrical end of $\mathfrak{X}_{P(v)}$, and view $\mathrm{ev}_{z_e}^{\mathcal{T}(e)}$ as mapping to $\mathfrak{X}_{P(v)}$. The fiber of the projection map $\pi_{P(v)}$ is a $T_{P(v),\mathbb{C}}$-orbit. By (6.40), the image of $\mathrm{ev}_{z_e}^{\mathcal{T}(e)}$ consists of $T_{P(v),\mathbb{C}}$-orbits, and therefore, $D\,\mathrm{ev}_{z_e}^{\mathcal{T}(e)}$ surjects onto the vertical part of the tangent space $V \subset T\mathfrak{X}_{P(v)}$. The differential $D\,\mathrm{ev}_{z_e}^{\mathcal{T}(e)}$ also surjects onto the horizontal subspace $H$, since constant sections on $H$ are contained in $\ker(D\overline{\partial}) = T\mathcal{M}_\Gamma(\mathfrak{X})$. ∎

Chapter 7

# Hofer energy and exponential decay

Compactness for sequences of broken pseudoholomorphic maps requires bounds on area. Neck-stretched manifolds do not have a canonically-defined taming symplectic form. However, neck-stretched manifolds are non-canonically diffeomorphic to a tropical Hamiltonian manifold and inherit a cohomology class of degree two from the diffeomorphism. The Hofer energy of a pseudoholomorphic curve is defined as a supremum of symplectic areas where the symplectic form ranges over a family of taming forms in this symplectic class. Each symplectic form in the family is given by a map of complexes

$$\aleph : B_J \to B_\omega$$

between the *$J$-complex* $B_J$ underlying the cylindrical almost complex structure and the *$\omega$-complex* $B_\omega$ underlying the symplectic form on the tropical manifold. For example, in a neck-stretched manifold $X^\nu$ the $\omega$-complex is the dual complex $B^\vee$ and the $J$-complex is $\nu B^\vee$. Such a map $\aleph$ between complexes gives a map between manifolds

$$\psi_\aleph : (X^\nu, J^\nu) \to (X, \omega_X)$$

which is taming (that is, $J^\nu$ is $\psi_\aleph^*\omega_X$-tame) if $\aleph$ satisfies certain conditions. In case of a single cut, $\psi_\aleph$ is taming if $\aleph : [\frac{-\nu}{2}, \frac{\nu}{2}] \to [0, 1]$ is an increasing diffeomorphism. This notion of energy as a supremum over a family of symplectic areas was originally defined by Hofer in the context of symplectic field theory [43].

The taming condition is more complicated for a multiple cut and leads us to define a class of maps between complexes, called *squashing maps*. A squashing map is a continuous piecewise smooth map between complexes, which on any piece, is the composition of a translation, dilation and an orthogonal projection. For a pseudoholomorphic map $u : C \to X^\nu$ in a neck-stretched manifold (or a broken manifold), we define the *Hofer energy* as

$$E_{\mathrm{Hof}}(u) := \sup_{\aleph \text{ is a squashing map}} \int_C u^*(\psi_\aleph^* \omega_X).$$

Of course, if $C$ is a closed curve, or if $C$ is a disk and $u$ maps the boundary $\partial C$ to a Lagrangian submanifold, the value of the integral is independent of $\aleph$ since the cohomology class of $\psi_\aleph^*\omega_X$ is independent of $\aleph$.

The main result of this Chapter is that a punctured pseudoholomorphic curve $u$ with finite Hofer energy is asymptotic to a trivial cylinder, which is a map

$$u_{\mathrm{triv}} : \mathbb{C}^\times \to X_P, \quad z \mapsto z^\mu x_0$$

for some slope $\mu \in \mathfrak{t}_{P,\mathbb{Z}}$ and $x_0 \in X_P$. More precisely, we show that for such a map $u$, the tropical evaluation is well-defined at the puncture point, that is, the twisting by $-\mu$, given by

$$v(z) := z^{-\mu} u(z),$$

has a removable singularity at the puncture, where $z$ is a holomorphic coordinate in the neighborhood of the puncture $z = 0$.

For the removal-of-singularities result, we consider maps on a punctured disk $B_1 \backslash \{0\} \subset \mathbb{C}$, which is holomorphically identified with

$$\mathrm{Cyl} := \mathbb{R}_{\geq 0} \times S^1.$$

For any $l \geq 0$, we refer to a truncated semi-infinite cylinder by

$$\mathrm{Cyl}(l) := [l, \infty) \times S^1.$$

We use a product metric on the domain Cyl, and on the target broken manifold $\mathfrak{X}$, we use a fixed cylindrical metric as in Definition 3.61.

The following result holds for maps that are holomorphic with respect to a domain-dependent almost complex structure that takes values in a $C^0$-neighborhood $U_{\mathfrak{J}_0}$ of the space of cylindrical almost complex structures. The neighborhood $U_{\mathfrak{J}_0}$ is determined in Lemma 7.21, and is such that any $J \in U_{\mathfrak{J}_0}$ is weakly tamed (Definition 7.4) by the form $\psi_{\aleph}^* \omega_X$ for any squashing map $\aleph$.

**Proposition 7.1.** *(Removal of singularities) Suppose $u : \mathrm{Cyl} \to \mathfrak{X}_P$ is a map that is holomorphic with respect to the domain-dependent almost complex structure $J : B_1 \to U_{\mathfrak{J}_0}$ (holomorphically identifying $\mathrm{Cyl} \simeq B_1 \backslash \{0\}$) and $U_{\mathfrak{J}_0}$ is as in the preceding paragraph. Suppose*

$$E^*_{P,\mathrm{Hof}}(u) < \infty, \quad \|\mathrm{d}u\|_{L^\infty(\mathrm{Cyl})} < \infty. \tag{7.1}$$

*Then, there exists a polytope $Q \subseteq P$, $\mu \in \mathfrak{t}_{Q,\mathbb{Z}}$, and $L \geq 0$, such that the image $u(\mathrm{Cyl}(L))$ lies in the $Q$-cylindrical end $\mathfrak{X}_P$, and the twisted map*

$$\overline{u} : \mathrm{Cyl}(L) \to \mathfrak{X}_Q,^{1} \quad (s,t) \mapsto e^{-\mu(s+it)} u(s,t) \tag{7.2}$$

*has a removable singularity at $\infty$. The same result holds when the broken manifold $\mathfrak{X}_P$ is replaced by the cut space $X_P$.*

Here, $E^*_{P,\mathrm{Hof}}$ is the unpartitioned $P$-Hofer energy of a map defined below in Definition 7.24.

We describe some other results from this Chapter, and indicate how they are used in the proof of Gromov compactness in Chapter 8. Given a sequence of pseudoholomorphic maps in neck-stretched manifolds which belong to a fixed homology class,

[1]The $Q$-cylindrical end of $\mathfrak{X}_P$ is embedded in $\mathfrak{X}_Q$ by (3.51).

Remark 7.19 says that the Hofer energy of these maps is uniformly bounded, since it is a topological quantity for maps on closed domains and disks with Lagrangian boundary conditions. Next, the monotonicity result (Lemma 7.21) says that the Hofer energy of a map can not increase if we restrict the map to a subdomain. By Proposition 7.27, Hofer energy is well-behaved for a sequence of maps on subdomains (not necessarily closed) under the neck-stretching limit, which allows us to obtain Hofer energy bounds on pieces of the Gromov limit. As a last step, this bound allows us to apply the removal of singularities result (Proposition 7.1) on components of the Gromov limit.

## 7.1 Symplectic forms on neck-stretched manifolds

In this section, we define a family of symplectic forms on neck-stretched manifolds that tame cylindrical almost complex structures that are locally strongly tame (Definition 3.47 (c)). For a pseudoholomorphic disk $u : (C, \partial C) \to (X^\nu, L)$ in a neck stretched manifold $X^\nu$ whose boundary maps to a Lagrangian submanifold $L \subset X^\nu$, the symplectic area

$$\mathrm{Area}(u) := \langle u_*[C], [\omega_X] \rangle, \tag{7.3}$$

is defined by identifying the neck-stretched manifold $X^\nu$ (not canonically) with the symplectic manifold $X$. Our task is to construct a family of symplectic forms in the class $[\omega_X] \in H^2(X^\nu)$. As a first step, we describe how a map

$$\aleph : \nu B^\vee \to B^\vee$$

between complexes determines a map

$$\psi_\aleph : X^\nu \to X$$

from the neck-stretched manifold $X^\nu$ to the corresponding symplectic manifold $X$. In the second half of the Section, we describe an *increasing* condition on the map $\aleph$ that ensures that $\psi_\aleph^* \omega$ tames cylindrical almost complex structures that are locally strongly tamed (Definition 3.47 (c)) and that lie in a small neighborhood. The condition on $\aleph$ may appear fairly restrictive, and the reason is that the symplectic form needs to tame a class of cylindrical almost complex structures whose underlying connection one-form is allowed to vary (see Remark 7.7 for more details). The extra freedom of varying the connection one-forms is needed for defining domain-dependent almost complex structures with which one can attain transversality of the linearized $\overline{\partial}$ operator.

In order to construct the map $\psi_\aleph$ from the map $\aleph$ of complexes, we first define a map $\pi_{B^\vee}$ from the symplectic manifold $(X, \omega_X)$ to its $\omega$-complex $B^\vee$ that is a piecewise projection. We recall from (3.22) that on a tropical manifold $(X, \omega_X, \mathcal{P}, \Phi)$ there is a symplectic cylindrical structure on neck regions, that is, there is a $T_P$-equivariant symplectomorphism

$$(X, \omega_X) \supset \Phi^{-1}(\tilde{P}) \xrightarrow{\phi_P} (\Phi^{-1}(P) \times P^\vee, \tilde{\omega}_P),$$
$$\tilde{\omega}_P := (\omega_X | \Phi^{-1}(P)) + d\langle \alpha_P, \pi_{P^\vee} \rangle, \quad (7.4)$$

for each polytope $P \in \mathcal{P}$, where $\tilde{P} \subset \mathfrak{t}^\vee$ is a tubular neighborhood of the polytope $P \subset \mathfrak{t}^\vee$ with projection (see Figure 3.4)

$$\pi_P : \tilde{P} \to P,$$

and $\alpha_P \in \Omega^1(\Phi^{-1}(P), \mathfrak{t}_P)$ is a $T_P$-connection one-form. The projection $\pi_{P^\vee} : \Phi^{-1}(\tilde{P}) \to P^\vee \subset \mathfrak{t}_P^\vee$ of the moment map $\Phi$ to $\mathfrak{t}_P^\vee$ is a moment map for the $T_P$-action. Let $P^\blacksquare \subset P$ be the complement of a neighborhood of faces of $P$, namely

$$P^\blacksquare := P \backslash (\cup_{Q \subset P} \tilde{Q}). \quad (7.5)$$

Let

$$\tilde{P}^\blacksquare := \pi_P^{-1}(P^\blacksquare) \subset \tilde{P} \quad (7.6)$$

be the thickening of $P^\blacksquare$. For a pair $Q \subset P$ with $\operatorname{codim}_P(Q) = 1$, the fibered polytopes $\tilde{P}^\blacksquare, \tilde{Q}^\blacksquare \subset \mathfrak{t}^\vee$ share a facet, which is isomorphic to $Q^\blacksquare \times P^\vee$. Then the image of $\Phi$ has a cover

$$\operatorname{im}(\Phi) = \cup_{P \in \mathcal{P}} i_{\tilde{P}}(\tilde{P}^\blacksquare)/\sim, \quad (7.7)$$

where $\sim$ identifies shared facets of polytopes, see Figure 7.1. The partition of $\operatorname{im}(\Phi)$

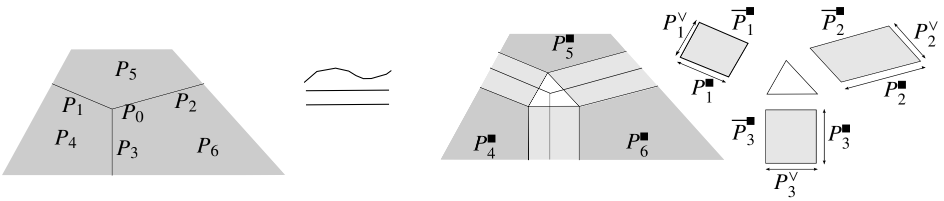



**Figure 7.1.** Decomposition of moment polytope induced by thickenings of polytopes in $\mathcal{P}$.

pulls back to a partition of the manifold $X$

$$(X, \omega_X) \simeq \left( \bigsqcup_{P \in \mathcal{P}} \Phi^{-1}(\tilde{P}^\blacksquare) \right) / \sim \quad (7.8)$$

into manifolds with corners, where the identifications in $\sim$ are along the boundaries and are induced by the inclusions $\Phi^{-1}(\tilde{P}^\blacksquare) \to X$. Furthermore, the symplectic cylindrical structure map $\underline{\phi} = (\phi_P)_P$ may be used to rewrite the decomposition in (7.8) as

$$(X, \omega_X) \simeq \left( \bigsqcup_{P \in \mathcal{P}} \Phi^{-1}(P^\blacksquare) \times P^\vee \right) / \sim . \quad (7.9)$$

(See the derivation of (3.28) for more details.) The decomposition (7.9) implies that there is a continuous projection map

$$\pi_{B^\vee} : (X, \omega_X) \to B^\vee \tag{7.10}$$

defined by projecting $\Phi^{-1}(P^\blacksquare) \times P^\vee \subset X$ to $P^\vee$ for each $P \in \mathcal{P}$. For any polytope $P \in \mathcal{P}$, $\pi_{B^\vee}$ is a $T_P$-moment map for the $T_P$-action on $\pi_{B^\vee}^{-1}(P^{\vee,\circ})$. Thus, $P^\vee$ is the $\omega$-polytope on $\Phi^{-1}(P^\blacksquare) \times P^\vee \subset X$, and the union $B^\vee = \cup_{P\in\mathcal{P}} P^\vee$ is the *$\omega$-complex* for $X$.

Next, we give map relating the neck-stretched manifold $X^\nu$ to the $J$-complex $\nu B^\vee$. We recall that the neck-stretched manifold is

$$X^\nu = \left( \bigsqcup_{P\in\mathcal{P}} \Phi^{-1}(P^\blacksquare) \times \nu P^\vee \right) /\sim \tag{7.11}$$

where the equivalence $\sim$ from (3.30) identifies boundary components. The space $X^\nu$ is a smooth manifold, with the identifications between tubular neighborhoods of boundaries and corners that are glued in (7.11), being the same as the corresponding identifications in the decomposition (7.9) of $(X, \omega_X)$. Analogously to (7.10), neck-stretched manifolds project to the scaled dual complex (see (3.31)): For any $\nu \geq 1$, there is a continuous projection map

$$\pi_{\nu B^\vee} : X^\nu \to \nu B^\vee \tag{7.12}$$

such that for any polytope $P \in \mathcal{P}$, $(\pi_{\nu B^\vee})^{-1}(P^{\vee,\circ})$ is a $P$-cylinder $\Phi^{-1}(P^\blacksquare) \times \nu P^{\vee,\circ}$. Thus the $J$-polytope for the $P$-cylindrical subset of $X^\nu$ is $\nu P^\vee$, and the *$J$-complex* of $X^\nu$ is $\nu B^\vee$.

Neck-stretched almost complex manifolds are mapped to compact symplectic manifolds via maps of complexes between the $J$-complex of the neck-stretched manifold and the $\omega$-complex of the symplectic manifold:

**Definition 7.2.** For any $\nu \geq 1$, a map

$$\aleph : \nu B^\vee \to B^\vee$$

is a *map of complexes* if it is a collection of maps of polytopes, that is, for any $P \in \mathcal{P}$, $\aleph(\nu P^\vee) \subseteq P^\vee$.

A map $\aleph$ of complexes induces a map of manifolds

$$\psi_\aleph^\nu : X^\nu \to (X, \omega_X), \tag{7.13}$$

where, for any $P \in \mathcal{P}$, $\psi_\aleph$ maps $\Phi^{-1}(P^\blacksquare) \times \nu P^\vee$ to $\Phi^{-1}(P^\blacksquare) \times P^\vee$ by $(\mathrm{Id}, \aleph)$. We leave it to the reader to verify that the maps on the subsets patch to yield a continuous map.

The resulting map $\psi_\aleph$ fits into a commutative diagram

$$\begin{array}{ccc} X^\nu & \xrightarrow{\psi_\aleph} & (X, \omega_X) \\ {\scriptstyle \pi_{B_J}}\downarrow & & \downarrow{\scriptstyle \pi_{B^\vee}} \\ \nu B^\vee & \xrightarrow{\aleph} & B^\vee. \end{array}$$

Next, we describe a condition on the map $\aleph$ that ensures that the pullback of the symplectic form by $\psi_\aleph$ tames a suitable class of cylindrical almost complex structures. In case of a single cut, if the $\aleph : [\frac{-\nu}{2}, \frac{\nu}{2}] \to [-\frac{1}{2}, \frac{1}{2}]$ is an increasing diffeomorphism, then $\psi_\aleph^\nu$ is a diffeomorphism for which $(\psi_\aleph^\nu)^*\omega_X$ tames any cylindrical almost almost complex structure that is locally strongly tamed (Definition 3.47 (c)). This is precisely the class of maps $\aleph$ used to define Hofer energy in [11].

To generalize the notion of Hofer energy to a multiple cut, we define an analogue of increasing maps for higher dimensional polytopes in Definition 7.3. The condition is that at any point, the derivative of the map $\aleph$ is a symmetric matrix with non-negative eigenvalues. In other words, we require a non-decreasing condition in a set of orthogonal directions.

To simplify notation, in this Chapter, we assume that the polyhedral decomposition $\mathcal{P}$ has a single zero-dimensional polytope, or in other words, the dual complex has a single top-dimensional polytope. As a result, the inner product on $\mathfrak{t}$ (in (3.10)) is assumed to be fixed throughout. Extensions to the general case do not present difficulties; in the proof of Proposition 7.1, which is the main result of the Chapter, we point out the necessary steps to extend the proof to the general case.

**Definition 7.3.** (Increasing maps) A map of complexes $\aleph : \nu B^\vee \to B^\vee$ is *increasing* resp. *strictly increasing* if for any polytope $P \in \mathcal{P}$ and a point $x \in \nu P^\vee$, the derivative $D(\aleph|\nu P^\vee)_x : \mathfrak{t}_P^\vee \to \mathfrak{t}_P^\vee$ is

(a) symmetric (that is, diagonalizable with a set of orthogonal eigenvectors with respect to the $\mathfrak{t}_P$-inner product (3.10)), and

(b) non-negative with norm at most 1 resp. norm at most 1 and invertible (that is, the eigenvalues $n_1, \dots, n_k$ lie in the interval $[0, 1]$ resp. $(0, 1]$.)

**Definition 7.4.** An almost complex structure $J$ is *weakly tamed* by a two-form $\omega$ if $\omega(v, Jv) \geq 0$ for all tangent vectors $v$.

Before stating the main result, we prove Lemma 7.5, which is a warm-up result dealing with the easier case of locally strongly tamed cylindrical almost complex structures (in contrast with the general result which deals with locally tamed cylindrical almost complex structures).

**Lemma 7.5.** *For an increasing map $\aleph : \nu B^\vee \to B^\vee$ of complexes, and any locally strongly tamed cylindrical almost complex structure $J^\nu$ on $X^\nu$, $(\psi^\nu_\aleph)^*\omega_X$ weakly tames $J^\nu$, where $\psi^\nu_\aleph : (X^\nu, J^\nu) \to (X, \omega_X)$ is the map of manifolds induced by $\aleph$.*

*Remark* 7.6. We describe the squashed area form $\psi^*_\aleph \omega_X$ induced by a squashing map $\aleph$, as it is used in the proofs of Lemmas 7.5 and 7.8. For any $P \in \mathcal{P}$, recall that

$$\pi_P : \Phi^{-1}(P^\blacksquare) \times \nu P^\vee \to \Phi^{-1}(P^\blacksquare)/T_P \tag{7.14}$$

is a $(T_P \times \nu P^\vee)$-fibration, and on $\Phi^{-1}(P^\blacksquare) \times \nu P^\vee \subset X^\nu$,

$$(\psi^\nu_\aleph)^*\omega_X = \omega_{X_P} + d\langle \aleph, \alpha_P \rangle,$$

where $\aleph_P := \aleph|(\nu P^\vee)$, and $\alpha_P \in \Omega^1(\Phi^{-1}(P^\blacksquare), \mathfrak{t}_P)$ is a $\mathfrak{t}_P$-valued connection one-form from the symplectic cylindrical structure (3.22). Indeed, $\aleph(\nu P^\vee) \subseteq P^\vee \subset \mathfrak{t}^\vee_P$, and therefore $\langle \aleph_P, \alpha_P \rangle$ is well-defined as a one-form. On the vertical sub-bundle $\ker(d\pi_P)$, the form $(\psi^\nu_{P,\aleph})^*\omega_X$ is equal to $\langle d\aleph_P \wedge \alpha_P \rangle$. On the horizontal sub-bundle $\ker(\alpha_P) \subset T\Phi^{-1}(P^\blacksquare)$, the form is $\omega_{X_P} + \aleph_P d\alpha_P$.

*Proof of Lemma* 7.5. We recall from Definition 3.47 that local strong tamedness implies that the fibers of the projection map $\pi_P$ in (7.14) are $J^\nu$-holomorphic, and the horizontal tangent sub-bundle $\ker(\alpha_P) \subset T\Phi^{-1}(P^\blacksquare)$ is $J^\nu$-invariant. On the vertical sub-bundle $\ker(d\pi_P)$, the form $(\psi^\nu_{P,\aleph})^*\omega_X$ is equal to $\langle d(\aleph|\nu P^\vee), \alpha_P \rangle$, which is taming because the increasing map $\aleph$ satisfies

$$\langle D(\aleph|\nu P^\vee) \wedge \alpha_P \rangle(v, J^\nu v) > 0, \quad \forall v \in \ker(d\pi_P). \tag{7.15}$$

(Note that (7.15) is equivalent to (7.16) below, since $J^\nu$ is strongly locally tamed.) On the horizontal sub-bundle $\ker(\alpha_P) \subset T\Phi^{-1}(P^\blacksquare)$, the form is $\omega_{X_P} + \aleph_P d\alpha_P$ which tames $J^\nu$, since $J^\nu$ is locally strongly tamed. ∎

*Remark* 7.7. The *increasing* condition of Definition 7.3 produces forms that are weakly taming for cylindrical almost complex structures that lie in a small $C^0$-neighborhood and which are locally tamed. If we restrict attention to cylindrical almost complex structures that are locally strongly tamed, that is, those whose underlying connection one-form is fixed, then the increasing condition may be replaced by the following condition on $\aleph$:

$$\langle D(\aleph|\nu P^\vee)_x v, v \rangle \geq 0 \quad \forall P \in \mathcal{P}, x \in \nu P^\vee, v \in T_x(\nu P^\vee), \tag{7.16}$$

which is equivalent to (7.15) used in the proof of Lemma 7.5. The condition (7.16) is a consequence of the *increasing* condition of Definition 7.3, and thus, is a weaker condition.

In the next result, which is the main result of this Section, we show that given a locally strongly tamed $\mathfrak{J}_0$, there is a $C^0$-neighborhood of locally tamed almost complex structures that are weakly tamed by $\psi_{\aleph}^*\omega_X$ for any $\psi_{\aleph}$ induced by an increasing map $\aleph$.

**Lemma 7.8.** *Suppose $\mathfrak{J}_0 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ is a locally strongly tamed cylindrical almost complex structure. There exists $\epsilon > 0$ and a $C^0$-neighbourhood*

$$U_{\mathfrak{J}_0} := \{\mathfrak{J} \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}) : \|\mathfrak{J} - \mathfrak{J}_0\|_{C^0} < \epsilon\}$$

*of $\mathfrak{J}_0$ such that the following is satisfied for any $\mathfrak{J}_1 \in U_{\mathfrak{J}_0}$ and any $\nu \in [1, \infty)$. Let $\aleph : \nu B^\vee \to B^\vee$ be a map of complexes, and let $\psi_{\aleph}^\nu : X^\nu \to (X, \omega_X)$ be the resulting map of manifolds.*

(a) *If $\aleph$ is increasing, then the form $(\psi_{\aleph}^\nu)^*\omega_X$ weakly tames $J_1^\nu$, where $J_1^\nu \in \mathcal{J}^{\mathrm{cyl}}(X^\nu)$ is obtained by gluing $\mathfrak{J}_1$ at cylindrical ends.*

(b) *If $\aleph$ is strictly increasing, then the form $(\psi_{\aleph}^\nu)^*\omega_X$ tames $J_1^\nu$.*

We point out that local tamedness is a $C^0$-open condition in $\mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$. Any $\mathfrak{J}_1 \in U_{\mathfrak{J}_0}$ is locally tamed. This can be seen, for example, by setting $\nu = 1$, in which case, $\aleph \equiv \mathrm{Id}$.

*Proof of Lemma 7.8.* We prove the first statement for increasing maps, since the second one is similar. We consider a polytope $P \in \mathcal{P}$, and prove the (Weakly taming) property of (Definition 7.4) in the $P$-cylindrical region

$$Z_{\mathbb{C}} := \Phi^{-1}(P^{\blacksquare}) \times \nu P^\vee \subset X^\nu.$$

We denote $Z := \Phi^{-1}(P^{\blacksquare})$. We recall that for a cylindrical almost complex structure $\mathfrak{J}$ with a family of neck-stretched almost complex structures $\{J_\nu\}_\nu$, the fibers of the projection

$$T_P \times \nu P^\vee \to Z_{\mathbb{C}} \xrightarrow{\pi_P} Z/T_P \subset X_P$$

are $J^\nu$-holomorphic. Furthermore, on $Z_{\mathbb{C}}$, $J^\nu$ is determined by its projection to $X_P$, denoted $D\pi_P(J)$ (see (3.18)), and the associated connection one-form $\alpha_{P,\mathfrak{J}} \in \Omega^1(Z, \mathfrak{t}_P)$ defined by the condition that the horizontal complement of $\ker(D\pi_P)$ given by

$$\ker(\alpha_{P,\mathfrak{J}}) \subset TZ \subset TZ_{\mathbb{C}}$$

is $J^\nu$-invariant (also see (3.19)). We recall that $\alpha_{P,\mathfrak{J}}$ is equal to $\alpha_P$ from the symplectic cylindrical structure (3.22) if $\mathfrak{J}$ is locally strongly tamed.

To prove the weakly taming property, we first consider perturbations to the horizontal projection of the almost complex structure $\mathfrak{J}_0$. The horizontal part of the two-form $\psi_{\aleph}^*\omega_X$ is $\omega_{X_P} + \aleph d\alpha_{\mathfrak{J}_0}$, which is a symplectic form since it occurs as a horizontal component of the symplectic form in (7.4); and it tames $D\pi_P(\mathfrak{J}_0)$ since $\mathfrak{J}_0$ is locally tamed (see Remark 3.48 (d)). Since $\aleph$ takes values in a compact set $P^\vee$, and tamedness on a symplectic manifold is a $C^0$-open condition, we conclude the following: There is a constant $\epsilon_H$ such that if

$$\|D\pi_P(\mathfrak{J}_1) - D\pi_P(\mathfrak{J}_0)\|_{C^0} < \epsilon_H \tag{7.17}$$

on $(Z/T_P)$ for some $\mathfrak{J}_1 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$, then $D\pi_P(\mathfrak{J}_1)$ is tamed by $\omega_{X_P} + \aleph \mathrm{d}\alpha_{\mathfrak{J}_0}$, see Lemma 3.49 for a proof.

Next, we study the effect of changing the connection associated to the almost complex structure. Suppose $\mathfrak{J}_1 \in \mathcal{J}^{\mathrm{cyl}}$ is a cylindrical almost complex structure whose horizontal projection $J_P := D\pi_P(\mathfrak{J}_1)$ is $C^0$-close to $D\pi_P(\mathfrak{J}_0)$ as in (7.17). Let $\mathfrak{J}_{10} \in \mathcal{J}^{\mathrm{cyl}}(Z_{\mathbb{C}})$ be a locally strongly tamed almost complex structure whose horizontal projection is the same as that of $\mathfrak{J}_1$, that is, $D\pi_P(\mathfrak{J}_{10}) = J_P$. Denote the $P$-connection forms by $\alpha_0 := \alpha_{\mathfrak{J}_{10}} = \alpha_{\mathfrak{J}_0}, \alpha_1 := \alpha_{\mathfrak{J}_1}$. The difference

$$A := \alpha_1 - \alpha_0$$

descends to a $\mathfrak{t}_P$-valued one-form on $X_P$. For a vector $(v, t) \in T_{\mathfrak{z},\tau} Z_{\mathbb{C}}$, where $\mathfrak{z} \in Z$, $\tau \in \nu P^\vee$, $v \in TZ$, $t \in \mathfrak{t}_P^\vee$, the difference between $\mathfrak{J}_1$ and $\mathfrak{J}_{10}$ is

$$(\mathfrak{J}_1 - \mathfrak{J}_{10})(v, t) = -(A(J_P v_P))_Z + J_F(A v_P)_Z, \quad v_P := D\pi_P(v),$$

where $J_F$ is the complex structure on the fibers of $\pi_P$, and for any $\xi \in \mathfrak{t}$, $\xi_Z \in \mathrm{Vect}(Z)$ is the vector field $\xi_Z(z) := \frac{d}{dt}(\exp(t\xi)z)|_{t=0}$ for any $z \in Z$. We write

$$(\psi_\aleph^* \omega_X)((v,t), \mathfrak{J}_1(v,t)) = (\psi_\aleph^* \omega_X)((v,t), \mathfrak{J}_{10}(v,t)) + (\psi_\aleph^* \omega_X)((v,t), (\mathfrak{J}_1 - \mathfrak{J}_{10})(v,t)). \tag{7.18}$$

By (7.17), there is a constant $c > 0$ such that the first term in the right-hand side of (7.18) is bounded on the $P$-region as

$$(\psi_\aleph^* \omega_X)((v,t), \mathfrak{J}_{10}(v,t)) \geq c|v_P|^2 + \langle D\aleph_\tau(\alpha_0(v)), \alpha_0(v)\rangle + \langle D\aleph_\tau(t), t\rangle. \tag{7.19}$$

In the second term in the right-hand side of (7.18), the difference $(\mathfrak{J}_1 - \mathfrak{J}_{10})(v,t)$ is in the fiber direction. The form $\psi_\aleph^* \omega_X$ in the fiber is $D\aleph \wedge \alpha_0$ and therefore,

$$(\psi_\aleph^* \omega_X)((v,t), (\mathfrak{J}_1 - \mathfrak{J}_{10})(v,t)) = \langle \alpha_0(v), D\aleph_\tau(A(v_P))\rangle + \langle D\aleph_\tau(t), A(J_P v_P)\rangle \tag{7.20}$$

By the increasing property, $D\aleph_\tau$ is diagonalizable with orthogonal eigenvectors and eigenvalues $n_1, \ldots, n_k \in [0,1]$. For any element $\xi \in \mathfrak{t}_P$ resp. $\mathfrak{t}_P^\vee$, we denote by $\xi_i \in \mathfrak{t}_P$ resp. $\mathfrak{t}_P^\vee$ the projection of $\xi$ to the $i$-th eigenspace. We write

$$D\aleph_\tau(\xi) = \textstyle\sum_{i=1}^k n_i \xi_i.$$

Using this eigen-decomposition, and the equations (7.18), (7.19) and (7.20), we get

$$\begin{aligned}(\psi_{\aleph}^{*}\omega_X)((v,t),\mathfrak{I}_1(v,t)) &\geq c|v_P|^2+\textstyle\sum_{i=1}^{k}n_i(|\alpha_0(v)_i|^2+|t_i|^2+\alpha_0(v)_i A(v_P)_i+t_i A(J_P v_P)_i)\\ &\geq \textstyle\sum_{i=1}^{k}n_i(\frac{c}{k}|v_P|^2+|\alpha_0(v)_i|^2+|t_i|^2+\alpha_0(v)_i A(v_P)_i+t_i A(J_P v_P)_i).\end{aligned}$$

The last two terms are bounded as

$$\alpha_0(v)_i A(v_P)_i \geq -\tfrac{1}{2}(|A|^2|v_P^2|+|\alpha_0(v)_i|^2),\quad t_i A(J_P v_P)_i \geq -\tfrac{1}{2}(|A|^2|v_P^2|+|t_i|^2)$$

where $|A| := \|A\|_{C^0}$. Therefore, $(\psi_{\aleph}^{*}\omega_X)((v,t),\mathfrak{I}_1(v,t))$ is non-negative if $|A|^2 \leq \frac{c}{2k}$, leading to the proof of the weak tamedness of $J_1^{\mathcal{V}}$. ■

## 7.2 Squashing maps

In the last section we saw *increasing maps* between complexes of polytopes induce two-forms that are weakly taming in the sense of Definition 7.4 for cylindrical almost complex structures in a neighborhood of a locally strongly tamed almost complex structure. In this section we construct a large class of increasing maps, called *squashing maps*. Squashing maps are continuous and piecewise smooth; and on each subset where the map is smooth, it is a composition of an orthogonal projection, a translation and a dilation. In the next section, Hofer energy for a multiple cut is defined as the supremum over the squashed areas of maps induced by squashing maps between polytopes.

*Remark* 7.9. In the rest of this Chapter, a polytope $P$ is assumed to be a subset of an affine space $V$. Two polytopes $P_1, P_2 \subset V$ are isomorphic if one is a translate of the other in $V$. A Euclidean metric on $V$ is inherited by polytopes in $V$.

**Definition 7.10.** (Squashing maps) Let $V$ be an affine space equipped with a Euclidean metric and let $Q \subset V$ be a compact top-dimensional polytope.

(a) (Undilated squashing map) An *undilated squashing map* $\aleph : V \to Q$ has

(i) an underlying polyhedral decomposition $\mathcal{Q}$ of $Q$, that is,

$$Q = \cup_{R\in\mathcal{Q}} R, \tag{7.21}$$

into polytopes $R$ (of all dimensions in the range $[0, \dim(V)]$ such that

- for any $R_1, R_2 \in \mathcal{Q}$, the interiors are disjoint, that is, $R_1^\circ \cap R_2^\circ = \emptyset$,
- and if $R \in \mathcal{Q}$ and $R_1 \subset R$ is a face, then, $R_1 \in \mathcal{Q}$;

(ii) and a decomposition

$$V = \cup_{R\in\mathcal{Q}} \tilde{R}$$

into top-dimensional polytopes $\tilde{R} \subset V$, such that $\aleph|\tilde{R}$ is an orthogonal projection onto $R$. That is, the fibers of $\aleph|\tilde{R}$ are polytopes perpendicular to $R$. The map $\aleph$ is continuous on $V$.

(b) (Squashing map) A *squashing map* $\aleph : V \to Q$ is a composition

$$\aleph = \delta_t \circ \aleph_0 \tag{7.22}$$

for some $t \geq 1$, where $\aleph_0 : V \to tQ$ is an undilated squashing map and $\delta_t : tQ \to Q$ scales by a factor of $\frac{1}{t}$.

(c) (Unpartitioned squashing map) We say that a squashing map $\aleph : V \to Q$ is *unpartitioned* if the underlying polyhedral decomposition $\mathcal{Q}$ is equal to the set of faces of $Q$.

*Remark* 7.11. A squashing map $\aleph : V \to Q$ is continuous, surjective and piecewise smooth.

*Remark* 7.12. (On unpartitioned squashing maps) An unpartitioned squashing map $\aleph : V \to Q$ is a translation composed with scaling on $\aleph^{-1}(Q^\circ)$, and therefore, $\aleph$ has a right inverse $\aleph_{inv} : Q \to V$. That is, $\aleph \circ \aleph_{inv} = \mathrm{Id}_Q$. An unpartitioned squashing map is uniquely determined by its right inverse $\aleph_{inv}$. See Figure 7.2.

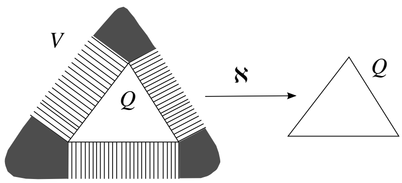



**Figure 7.2.** An unpartitioned undilated squashing map $\aleph : V \to Q$. Each of the solidly shaded regions is mapped to a vertex of $Q$, ruled regions are mapped to the sides of $Q$ by contracting each ruling to a point, and the blank region is mapped isometrically to the interior of $Q$.

Unpartitioned squashing maps are sufficient for most purposes. More general squashing maps are only used in the proof of Proposition 7.27, where we pass from maps in neck-stretched manifolds $X^\nu$ to maps in broken manifolds in the limit $\mathfrak{X}_P$. The rest of Section 7.2 is used only in the proof of Proposition 7.27. The following is a preliminary remark about general squashing maps (beyond the unpartitioned ones).

*Remark* 7.13. (A partial inverse of a squashing map) We continue Remark 7.12 where we defined a right inverse for unpartitioned squashing maps. For a general squashing map $\aleph : V \to Q$ with underlying polyhedral decomposition $\mathcal{Q}$ of $Q$, with top-dimensional polytopes $Q_1, \ldots, Q_k \in \mathcal{Q}^{(0)}$, a right inverse $\aleph_{inv}$ exists for the restriction $\aleph|(\cup_i \aleph^{-1}(Q_i^\circ))$. The right inverse $\aleph_{inv}|Q_i^\circ : Q_i^\circ \to \aleph^{-1}(Q_i^\circ)$ is a translation composed with scaling.

An undilated squashing map with an underlying polyhedral decomposition corresponds to a dual complex of $Q$, as we show in Lemma 7.14. Before stating the result, we point out that the polyhedral decomposition in (7.21) in the definition of squashing maps is a decomposition of a polytope $Q \subset V$, and not of the entire vector space $V$. This is in contrast with the polyhedral decomposition of Chapter 3 which was defined for the entire vector space. A decomposition of the vector space $V$ induces a decomposition of $Q \subset V$ as we now describe:

Let $V$ be a vector space and let $Q \subset V$ be a polytope of the same dimension as $V$. A polyhedral decomposition $\mathcal{P}$ of the vector space $V$ for which all top-dimensional polytopes $P \in \mathcal{P}^{(0)}$ intersect $Q^\circ$ induces a polyhedral decomposition $\mathcal{P}(Q)$ of $Q$ whose top-dimensonal polytopes are

$$\mathcal{P}(Q)^{(0)} := \{P \cap Q : P \in \mathcal{P}^{(0)}\},$$

and whose lower dimensional polytopes are faces of polytopes in $\mathcal{P}_Q^{(0)}$, see Figure 7.3.

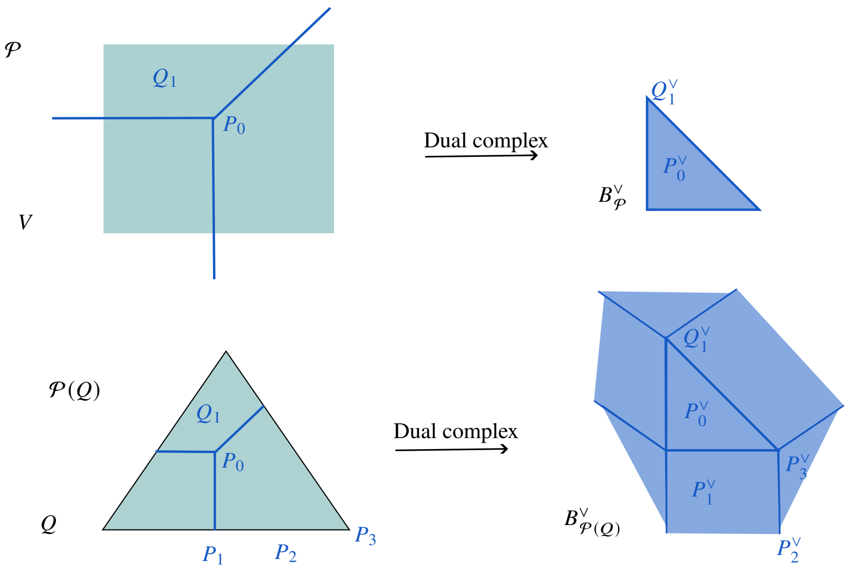



**Figure 7.3.** Left: A polyhedral decomposition $\mathcal{P}$ of $V$ induces a polyhedral decomposition $\mathcal{P}(Q)$ of $Q \subset V$. Right : The corresponding dual complexes.

In Lemma 7.14, we will construct a sequence of undilated squashing maps $\aleph_\nu : V \to Q$ whose underlying polyhedral decomposition of $Q$ is $\mathcal{P}(Q)$ described in the previous paragraph. The squashing maps will have the property that for any two top-dimensional polytopes $Q_i, Q_j$ of $Q$, the distance between $\aleph^{-1}(Q_i^\circ), \aleph^{-1}(Q_j^\circ)$ goes to infinity as $\nu \to \infty$.

**Lemma 7.14.** *(From a dual complex to squashing maps) Let $Q$, $V$, $\mathcal{P}$, $\mathcal{P}(Q)$ be as above. Suppose $B^\vee_{\mathcal{P}}$ is a dual complex of $\mathcal{P}$ that respects the metric on $V$ (as in Remark 3.25).*

(a) *There is a dual complex $B^\vee_{\mathcal{P}(Q)}$ of $\mathcal{P}(Q)$ that respects the metric on $V$ and for which $B^\vee$ is a subcomplex.*

(b) *For any $\nu \in \mathbb{R}_{>0}$, there is a squashing map $\aleph_\nu : V \to Q$ such that for any polytope $P \in \mathcal{P}(Q)$,*

$$\aleph_\nu^{-1}(P) \xrightarrow{\text{isomorphic}} P \times \nu P^\vee, \tag{7.23}$$

*where $P^\vee \subset V$ is the dual polytope of $P$ in $B^\vee_{\mathcal{P}(Q)}$.*

(c) *Let $P \in \mathcal{P}(Q)^{(0)}$ be a top-dimensional polytope, and for any $\nu$, let $i_\nu : P \to V$ be an embedding (which is a translation). For any $\nu$, there is a unique squashing map that satisfies (7.23) and the condition that $\aleph_\nu \circ i_\nu = \mathrm{Id}_P$. For any two top-dimensional polytopes $Q_i, Q_j \in \mathcal{P}(Q)^{(0)}$,*

$$d_V(\aleph^{-1}(Q_i^\circ), \aleph^{-1}(Q_j^\circ)) \to \infty \quad \text{as } \nu \to \infty. \tag{7.24}$$

*Proof.* To construct $B^\vee_{\mathcal{P}(Q)}$, we start with $B^\vee_{\mathcal{P}} \subset \mathfrak{t}^\vee$, and add polytopes corresponding to $P_0 \cap Q_0$ where $P_0 \in \mathcal{P}$ and $Q_0$ is a proper face of $Q$. First consider the case that $Q_0$ is a facet and $P_0$ is top-dimensional. Then, the one-dimensional polytope $(P_0 \cap Q_0)^\vee$ is a ray pointing in the outward normal direction to $Q_0$, whose starting point is $P_0^\vee \in B^\vee_{\mathcal{P}}$. Next, consider the general case when the face $Q_0 \subset Q$ is the intersection of facets $Q_1, \ldots, Q_k$, and the dual polytope of $P_0 \in \mathcal{P}$ in $B^\vee_{\mathcal{P}}$ has vertices $P_1, \ldots, P_\ell$. Then, we define

$$(P_0 \cap Q_0)^\vee := \text{Convex-hull}(\{(P_j \cap Q_i)^\vee\}_{i,j}),$$

which has the right dimension for the following reason: For any $j$, the convex hull of $C_j := \{P_j \cap Q_i\}_i$ is a cone spanned by the rays $(P_j \cap Q_i)^\vee$ with vertex $P_j^\vee$. Denote $C := C_j$, since for any $j'$, $C_{j'}$ is a translate of $C_j$. The convex hull of $\{P_j \cap Q_i\}_{i,j}$ is the product $C \times P_0^\vee$, and therefore, has codimension $k + \mathrm{codim}(P_0)$. Here, $P_0^\vee$ is the dual polytope of $P_0$ in $B^\vee_{\mathcal{P}}$.

To prove (b), we observe that for any $\nu > 0$ there is an isomorphism

$$\left(\sqcup_{P \in \mathcal{P}(Q)} (\nu P^\vee \times P)/\sim\right) \xrightarrow{i_\nu} V, \tag{7.25}$$

which is uniquely defined up to a translation in $V$. Here, $P^\vee$ is the dual polytope of $P$ in $B^\vee_{\mathcal{P}(Q)}$, and for any pair $Q \subset P$ with $\mathrm{codim}_P Q = 1$, $\sim$ identifies $(\nu P^\vee \times Q)$, which is a facet in both $(\nu P^\vee \times P)$ and $(\nu Q^\vee \times Q)$. In (7.25), each of the maps $i_\nu|(\nu P^\vee \times P)$ is a translation. The squashing map $\aleph_\nu$ is defined as projection to $P$ on the subset $i_\nu(\nu P^\vee \times P)$.

To prove (c), we observe that (7.25) is uniquely defined up to translation in $V$. The condition $\aleph_\nu \circ i_\nu = \mathrm{Id}_P$ cuts down this translation freedom and fixes a unique $\aleph_\nu$. The property (7.24) follows from the construction of $\aleph_\nu$, since the two sets being considered are separated by a dual polytope scaled up by $\nu$. This finishes the proof of the Lemma. ■

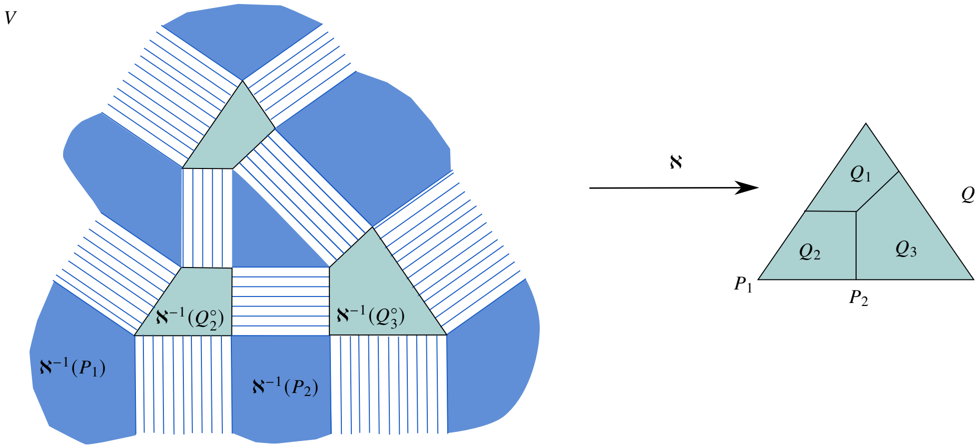



**Figure 7.4.** The undilated squashing map $\aleph : V \to Q$ corresponding to the dual complex $B^\vee_{\mathcal{P}(Q)}$ in Figure 7.3. In $V$, solidly shaded regions are mapped to points, ruled regions are mapped to lines by contracting each ruling to a point, and blank regions are mapped isometrically.

The following result constructs a polyhedral decomposition (with dual complex) of a polytope $Q$ that contains a prescribed top-dimensional piece. The polyhedral decomposition is used to construct squashing maps in the proof of Proposition 7.27.

**Lemma 7.15.** *Let $V$ be a vector space of dimension $n$ equipped with a Euclidean metric. Let $C \subset V$ be a $k$-dimensional cone, that is, $C$ is the convex hull of $k$ rays originating at $0 \in V$ and pointing in linearly independent directions. Let $C^\vee \subset V$ be an $(n-k)$-dimensional simplex that is orthogonal to $P$. Then, there is a polyhedral decomposition $\mathcal{P}$ of $V$ that contains $P \times P^\vee$ as a top-dimensional polytope, and which has a dual complex $B^\vee_{\mathcal{P}}$ respecting the metric on $V$.*

*Proof.* First, we construct a polyhedral decomposition $\mathcal{P}_{\mathrm{pre}}$ of $V$ that contains $C$ as a polytope, and whose dual complex is an $n$-simplex. Assuming that the cone $C$ is the convex hull of rays pointing in the directions $e_1, \ldots, e_k$, choose vectors $e_{k+1}, \ldots, e_{n+1}$ such that $V$ is the convex hull of the rays $\mathbb{R}_{\geq 0} e_i$, $i = 1, \ldots, n+1$, and (a translate of) $C^\vee$ is a cross-section of the cone spanned by the rays $\{\mathbb{R}_{\geq 0} e_i\}_{k+1 \leq i \leq n+1}$. Define a polyhedral decomposition

$$\mathcal{P}_{\mathrm{pre}} := \{P_I := \mathbb{R}_+ \langle x_i \rangle_{i \in I} \subset V : I \subset \{1, \ldots, n+1\}\}$$

of $V$, and observe that its dual complex (respecting the metric) is an $n$-simplex denoted by $B^\vee_{\mathcal{P}_{\text{pre}}}$. Define the polyhedral decomposition $\mathcal{P}$ so that its top-dimensional polytopes are

$$\tilde{C} := C^\vee + \mathbb{R}_{\geq 0}\langle e_i \rangle_{1\leq i\leq k} \simeq C \times C^\vee,$$

and $P\backslash\tilde{C}$ for all top-dimensional polytopes $P \in \mathcal{P}^{(0)}_{\text{pre}}$; see Figure 7.5. The polyhedral decomposition $\mathcal{P}$ has a dual complex $B^\vee_{\mathcal{P}}$, which is obtained by adding a vertex $\tilde{C}^\vee$ to the face $P^\vee_{1,\ldots,k}$ such that the convex hull of $\tilde{C}^\vee \cup P^\vee_{\{k+1,\ldots,n+1\}}$ is perpendicular to $P^\vee_{\{1,\ldots,k\}}$. The other polytopes in $B^\vee_{\mathcal{P}}$ are spanned by the given set of vertices, and can be determined combinatorially. ■

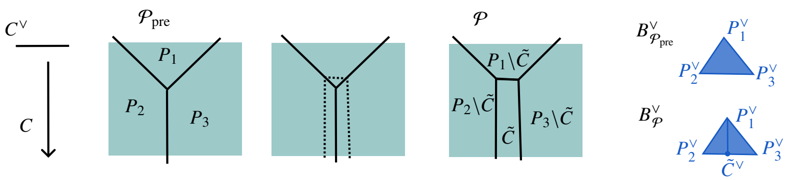



**Figure 7.5.** Constructing $\mathcal{P}$ from $\mathcal{P}_{\text{pre}}$ in Lemma 7.15. Here $n = 2$, $k = 1$.

## 7.3 Multi-directional Hofer energy

Hofer energy for a multiple cut is defined as the supremum over pullbacks of symplectic forms by squashing maps between manifolds that are induced by squashing maps between polytopes. The definition of Hofer energy uses squashing maps from $\omega$-complexes to $J$-complexes. In this section, we define these complexes for neck-stretched and broken manifolds, and prove some useful properties satisfied by Hofer energy.

In the last section, we defined squashing maps as maps from an affine space to a compact polytope. The definition extends naturally to maps between polytopes, and later, complexes.

**Definition 7.16.** (Squashing map between polytopes) Let $P, Q \subset V$ be polytopes, and let $Q$ be compact. A squashing map $\aleph : P \to Q$ is the restriction of a squashing map $\overline{\aleph} : V \to Q$ for which the subset $V_0 \subset V$ on which $\overline{\aleph}$ is a dilation is contained in $P$.

### 7.3.1 Hofer energy for neck-stretched manifolds

We recall from Section 7.1 that for a neck-stretched manifold $X^\nu$, the $\omega$-complex is the dual complex $B^\vee$ and the $J$-complex is $\nu B^\vee$.

**Definition 7.17.** (Hofer energy for neck-stretched manifolds) Let $\{X^\nu\}_\nu$ be a family of neck-stretched manifolds. For any $\nu \geq 1$, the *Hofer energy* of a map $u : C \to X^\nu$ is

$$E_{\text{Hof}}(u) = \sup_{\aleph:\nu B^\vee \to B^\vee} \int_C (\psi_\aleph \circ u)^* \omega_X,$$

where the supremum is over all maps of complexes $\aleph : \nu B^\vee \to B^\vee$ for which $\aleph|\nu P^\vee : \nu P^\vee \to P^\vee$ is a squashing map for each polytope $P \in \mathcal{P}$ (as in Definition 7.16); and

$$\psi_\aleph : X^\nu \to (X, \omega_X)$$

is the map induced by $\aleph$ as in (7.13). For any squashing map $\aleph$, the form $\psi_\aleph^* \omega_X$ is called a *squashed area form*. The squashing map $\aleph$ and the squashed area form are called *unpartitioned* if for each polytope $P$, the squashing map $\aleph|\nu P^\vee$ is unpartioned in the sense of Definition 7.10 (c). The supremum over unpartitioned squashed area forms is called *unpartitioned Hofer energy* and is denoted by

$$E^*_{\text{Hof}}(u) = \sup_{\aleph:\nu B^\vee \to B^\vee \text{ is unpartitioned}} \int_C (\psi_\aleph \circ u)^* \omega_X.$$

See Figures 7.6 and 7.7 for examples of squashing maps for neck-stretched manifolds. This ends the Definition.

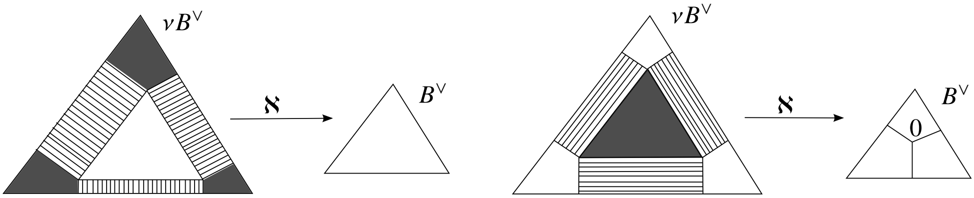



**Figure 7.6.** Examples of squashing maps $\aleph : \nu B^\vee \to B^\vee$. In both examples, solidly shaded regions are mapped to points, ruled regions are mapped to lines by contracting each ruling to a point, and blank regions are mapped isometrically. The map in the left is unpartitioned, the map in the right is not.

For any squashing map $\aleph$, the map of manifolds $\psi_\aleph : X^\nu \to X$ is continuous and piecewise-smooth with a finite number of pieces. Therefore, the integral computing squashed area is well-defined.

*Remark* 7.18. (When is squashed area non-degenerate) Given a squashing map $\aleph : \nu B^\vee \to B^\vee$, the squashed area form is non-degenerate on the subset $S \subset \nu B^\vee$ where $\aleph$ is dilation by a positive factor. For any polytope $P \in \mathcal{P}$, suppose

$$\aleph_P := \aleph|\nu P^\vee : \nu P^\vee \to P^\vee$$

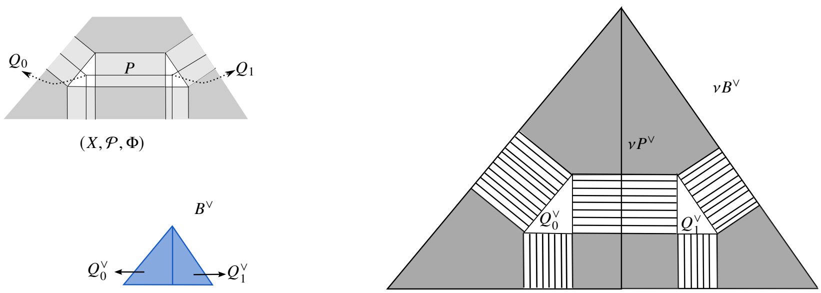



**Figure 7.7.** A squashing map $\aleph : \nu B^\vee \to B^\vee$ where there are two top-dimensional dual polytopes $Q_0^\vee, Q_1^\vee$ in $B^\vee$. The maps $\aleph|\nu Q_0^\vee, \aleph|\nu Q_1^\vee$ are equal on $\nu P^\vee$. The map $\aleph$ is unpartitioned.

has an underlying decomposition $\mathcal{Q}_P$ of $P^\vee$ (as in (7.21)). Then, for any top-dimensional polytope $Q \in \mathcal{Q}_P$, $\psi_\aleph^* \omega_X$ is non-degenerate on $\pi_{\nu B^\vee}^{-1}(\aleph_P^{-1}(Q^\circ))$.

*Remark* 7.19. For a compact curve $C$ with boundary, and a map $u : (C, \partial C) \to (X^\nu, L)$, the Hofer energy is equal to the area of $u$ :

$$E_{\mathrm{Hof}}(u) = \langle (\psi_\aleph \circ u)_* [C], [\omega_X] \rangle$$

for any squashing map $\aleph : \nu B^\vee \to B^\vee$. The quantity $E_{\mathrm{Hof}}$ is independent of $\aleph$, because any pair of squashing maps $\aleph_0, \aleph_1$ are isotopic via a family $\{\aleph_t\}_{t \in [0,1]}$ of continuous piecewise smooth maps, and therefore, the maps $(\psi_{\aleph_0} \circ u), (\psi_{\aleph_1} \circ u) : C \to (X, \omega_X)$ are isotopic.

We point out that squashing maps satisfy the increasing property (Definition 7.3), which we initially set out to achieve.

**Lemma 7.20.** *(Squashed area forms are weakly taming) Suppose $\mathfrak{J}_0 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ is a locally strongly tamed cylindrical almost complex structure, and suppose $U_{\mathfrak{J}_0} \subset \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ is a $C^0$-neighborhood of $\mathfrak{J}_0$ from Lemma 7.8. Then, for any $\nu$ and any squashing map $\aleph : \nu B^\vee \to B^\vee$, the squashed area form $\psi_\aleph^* \omega_X$ is weakly taming (Definition 7.4) for any $J^\nu \in \mathfrak{J}^{\mathrm{cyl}}(X^\nu)$ obtained by gluing $\mathfrak{J} \in U_{\mathfrak{J}_0}$ at cylindrical ends.*

*Proof.* A squashing map $\aleph : \nu B^\vee \to B^\vee$ satisfies the increasing property on $\aleph|\nu P^\vee$. Indeed, it is enough to check this property for squashing maps on vector spaces, which in turn, follows from the increasing property on undilated squashing maps, and finally, an undilated squashing map is piecewise smooth, and on each of these pieces, it is an orthogonal projection. ∎

The next result, which is a consequence of Lemma 7.20, says that the squashed area $(\psi_\aleph \circ u)^* \omega_X$ is pointwise non-negative if the map $u$ is pseudoholomorphic with respect to an almost complex structure that is close to a locally strongly tamed almost complex structure.

**Lemma 7.21.** *(Monotonicity of Hofer energy) Suppose $\mathfrak{J}_0 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ is a locally strongly tamed cylindrical almost complex structure, and suppose $U_{\mathfrak{J}_0} \subset \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ is a $C^0$-neighborhood of $\mathfrak{J}_0$ from Lemma 7.8. Let $u : C \to X^\nu$ be a map that is holomorphic with respect to the domain dependent almost complex structure $J^\nu : C \to U_{\mathfrak{J}_0}$. (Here, for any $z \in C$, $J^\nu(z)$ denotes both a broken almost complex structure $\mathfrak{J}_z$ on $\mathfrak{X}$ and the almost complex structure $J^\nu_z$ on $X^\nu$ obtained by gluing $\mathfrak{J}_z$ on the neck with neck length parameter $\nu$.) For any open subset $\Omega \subset C$,*

$$E_{\mathrm{Hof}}(u, \Omega) \leq E_{\mathrm{Hof}}(u, C).$$

For some results in Chapter 5, we consider maps that are holomorphic with respect to a locally strongly tamed almost complex structure, in which case, monotonicity of Hofer energy holds without restricting to a $C^0$-neighborhood. The relevant statement is as follows, and the proof is a consequence of Lemma 7.5.

**Lemma 7.22.** *(Monotonicity in the locally strongly tamed case) Suppose $\mathfrak{J}_0 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ is a locally strongly tamed cylindrical almost complex structure. Let $u : C \to X^\nu$ be a $J^\nu_0$-holomorphic map. Here, the almost complex structure $J^\nu_0$ on $X^\nu$ is obtained by gluing $\mathfrak{J}_0$ on the neck with neck length parameter $\nu$. For any open subset $\Omega \subset C$,*

$$E_{\mathrm{Hof}}(u, \Omega) \leq E_{\mathrm{Hof}}(u, C).$$

### 7.3.2 Hofer energy on a broken manifold

Hofer energy on broken manifolds is defined in a similar way as that on neck-stretched manifolds. The only new feature is that the $\omega$-complex and $J$-complex are different.

We describe the $\omega$-complexes for pieces of the broken manifold. For a polytope $P \in \mathcal{P}$, the $\omega$-complex of the cut space $X^\omega_P$ is the subset

$$\mathbb{B}^\vee_P := \pi_{B^\vee}(X^\omega_P) \subset B^\vee,$$

where $\pi_{B^\vee} : (X, \omega_X) \to B^\vee$ is the projection to the dual complex from (7.10) and we recall that the symplectic cut space $X^\omega_P = \Phi^{-1}(P)$ is a subset of $(X, \omega_X)$. (We recall that the symplectic cut space and symplectic broken manifolds have a superscript $\omega$, see the "important point about notation" in page 66.) The complex $\mathbb{B}^\vee_P$ may alternately be defined as

$$\mathbb{B}^\vee_P := \bigcup_{Q \in \mathcal{P}, \dim(Q)=0, Q \in P} i^{-1}_{\tilde{Q}}(P) \subset B^\vee, \tag{7.26}$$

where $i_{\tilde{Q}} : Q^\vee \to \mathrm{im}(\Phi) \subset \mathfrak{t}^\vee$ is the embedding from (3.13), noting that for a zero-dimensional polytope $Q$, $\tilde{Q} = Q^\vee$. See Figure 7.8. The space $\mathbb{B}^\vee_P$ inherits the structure of a complex from $B^\vee$. It is a union of polytopes

$$\mathbb{B}^\vee_P = \cup_{Q \in \mathcal{P} : Q \subseteq P} \mathbb{B}^\vee_{P,Q}, \quad \mathbb{B}^\vee_{P,Q} := \mathbb{B}^\vee_P \cap Q^\vee, \tag{7.27}$$

and for any pair $R \subset Q \subseteq P$, $\mathbb{B}^\vee_{P,Q}$ is identified to a face of $\mathbb{B}^\vee_{P,R}$. Note that $\dim(\mathbb{B}^\vee_P) = \dim(P)$, and $\dim(\mathbb{B}^\vee_{P,Q}) = \dim(P) - \dim(Q)$.

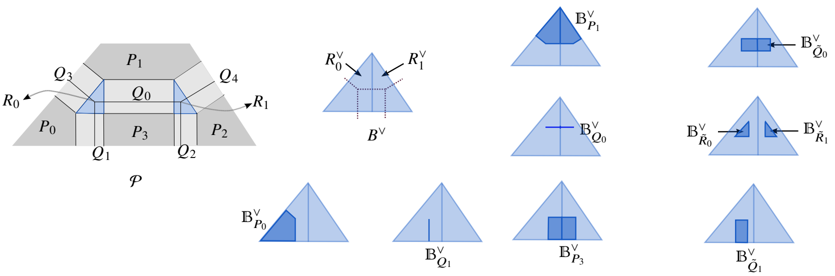



**Figure 7.8.** For a polyhedral decomposition $\mathcal{P}$ of a tropical manifold, $\mathbb{B}^\vee_P$ resp. $\mathbb{B}^\vee_{\tilde{P}}$ is the $\omega$-complex of the cut space $X_P$ resp. Broken manifold $\mathfrak{X}_P$ for $P \in \mathcal{P}$.

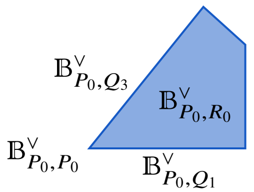



**Figure 7.9.** Polytopes in the complex $\mathbb{B}^\vee_{P_0}$ from Figure 7.8.

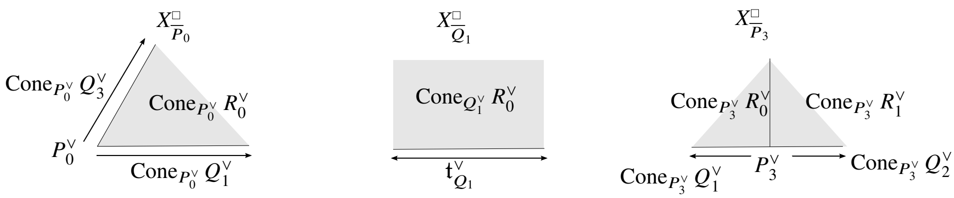



**Figure 7.10.** Some $J$-complexes for the polyhedral decomposition $\mathcal{P}$ in Figure 7.8.

We also associate an $\omega$-complex to thickened complexes $\tilde{P}$ in order to define Hofer energy for maps in the broken manifold $\mathfrak{X}_P$. We define $\mathbb{B}^\vee_{\tilde{P}}$ as a thickening of $\mathbb{B}^\vee_P$ :

$$\mathbb{B}^\vee_{\tilde{P}} := \mathbb{B}^\vee_P \times P^\vee,$$

where $P^\vee \subset \mathfrak{t}^\vee_P$ is small enough that there is an embedding $\mathbb{B}^\vee_{\tilde{P}} \longrightarrow B^\vee$ whose image is a tubular neighborhood of $\mathbb{B}^\vee_P \subset B^\vee$ (if needed we assume that $P^\vee$ is a scaling of the dual polytope of $P$). The space $\mathbb{B}^{\vee}_{\tilde{P}}$ inherits the structure of a complex from $B^{\vee}$, and thus, consists of polytopes

$$\mathbb{B}^{\vee}_{\tilde{P},Q} := \mathbb{B}^{\vee}_{P,Q} \times P^{\vee} \quad \forall Q \subseteq P.$$

The choice of the $\omega$-complexes $\mathbb{B}^{\vee}_{P}$, $\mathbb{B}^{\vee}_{\tilde{P}}$ is justified by the fact that the symplectic broken manifold or the symplectic cut space can be re-constructed from the complexes as

$$\begin{aligned}\left(\cup_{Q\subseteq P}\Phi^{-1}(Q^{\blacksquare}) \times \mathbb{B}^{\vee}_{\tilde{P},Q}\right)/\!\sim &= \Phi^{-1}(P^{\circ}) = (\mathfrak{X}^{\omega}_{P}, \omega_{\mathfrak{X}_P}),\\ \left(\cup_{Q\subseteq P}\Phi^{-1}(Q^{\blacksquare})/T_P \times \mathbb{B}^{\vee}_{P,Q}\right)/\!\sim &= \Phi^{-1}(P^{\circ})/T_P = (X^{\omega}_{P}, \omega_{X_P}),\end{aligned} \tag{7.28}$$

where $\sim$ is an identification on boundaries, which is a restriction of the equivalence relation in (3.28). The decomposition in (7.28) is a consequence of the decomposition of $(X, \omega_X)$ in (7.9).

Next, we describe the $J$-complexes. For any $P \in \mathcal{P}$, the $J$-complex for the broken manifold $X_P$ is

$$\mathrm{Cone}_{P^{\vee}} B^{\vee} := \left(\bigcup_{Q\subseteq P} \mathrm{Cone}_{P^{\vee}} Q^{\vee}\right)/\!\sim,$$

where, for any pair $Q_0 \subset Q_1$, the equivalence relation $\sim$ identifies $\mathrm{Cone}_{P^{\vee}} Q_1^{\vee}$ to a face of $\mathrm{Cone}_{P^{\vee}} Q_0^{\vee}$. The $J$-complex for the cut space $X_P$ is the corresponding normal cone

$$\mathrm{NCone}_{P^{\vee}} B^{\vee} := \left(\bigcup_{Q\subseteq P} \mathrm{NCone}_{P^{\vee}} Q^{\vee}\right)/\!\sim,$$

and thus $\mathrm{Cone}_{P^{\vee}} B^{\vee}$ is a product of orthogonal spaces

$$\mathrm{Cone}_{P^{\vee}} B^{\vee} = \mathrm{NCone}_{P^{\vee}} B^{\vee} \times \mathfrak{t}^{\vee}_{P}.$$

The almost complex broken manifold and cut spaces can be reconstructed from the $J$-complexes

$$\begin{aligned}\mathfrak{X}_P &= \left(\cup_{Q\subseteq P}\Phi^{-1}(Q^{\blacksquare}) \times \mathrm{Cone}_{P^{\vee}} Q^{\vee}\right)/\!\sim,\\ X_P &= \left(\cup_{Q\subseteq P}\Phi^{-1}(Q^{\blacksquare})/T_P \times \mathrm{NCone}_{P^{\vee}} Q^{\vee}\right)/\!\sim,\end{aligned} \tag{7.29}$$

where, for any facet $Q \subset R$, $\sim$ identifies the boundary component

$$\Phi^{-1}(Q^{\blacksquare}) \times \mathrm{Cone}_{P^{\vee}}(Q^{\vee}) \subset \Phi^{-1}(Q^{\blacksquare}) \times \mathrm{Cone}_{P^{\vee}}(Q^{\vee})$$

with the boundary component

$$\Phi^{-1}(Q^{\blacksquare}) \times \mathrm{Cone}_{P^{\vee}}(Q^{\vee}) \subset \Phi^{-1}(R^{\blacksquare}) \times \mathrm{Cone}_{P^{\vee}}(R^{\vee})$$

by the identity map. Here we use the viewpoint that broken manifolds are degenerate limits of neck-stretching, and therefore the decomposition in (7.29) is the limit of the $J$-decomposition in (7.11). As a consequence of the decomposition in (7.29), there are projection maps

$$\pi_{\mathrm{Cone}_{P^\vee} B^\vee} : \mathfrak{X}_P \to \mathrm{Cone}_{P^\vee} B^\vee, \quad \pi_{\mathrm{NCone}_{P^\vee} B^\vee} : X_P \to \mathrm{NCone}_{P^\vee} B^\vee \tag{7.30}$$

for all polytopes $P \in \mathcal{P}$.

**Definition 7.23.** Let $\mathfrak{X}_{\mathcal{P}}$ be a broken manifold and let $P \in \mathcal{P}$ be a polytope.

(a) (Squashing map for cut spaces) A *squashing map for the cut space $X_P$* is a map of complexes

$$\aleph : \mathrm{NCone}_{P^\vee} B^\vee \to \mathbb{B}^\vee_P$$

for which $\aleph(\mathrm{NCone}_{P^\vee} Q^\vee) \subset \mathbb{B}^\vee_{P,Q}$ for any polytope $Q \subseteq P$, and $\aleph : \mathrm{Cone}_{P^\vee} Q^\vee \to \mathbb{B}^\vee_{P,Q}$ is a squashing map of polytopes.

(b) (Squashing map for a broken manifold) A *squashing map* for the component $\mathfrak{X}_P$ of the broken manifold $\mathfrak{X}_{\mathcal{P}}$ is a map

$$\aleph = (\aleph^P, \aleph^{P^\vee}) : \mathrm{Cone}_{P^\vee} B^\vee \to \mathbb{B}^\vee_{\tilde{P}}$$

where $\aleph^P : \mathrm{NCone}_{P^\vee} B^\vee \to \mathbb{B}^\vee_P$ is a squashing map for the cut space $X_P$, and $\aleph^{P^\vee} : \mathfrak{t}^\vee_P \to P^\vee$ is a squashing map of polytopes. Note that $\aleph$ is itself a squashing map of complexes.

See Figure 7.11 for examples.

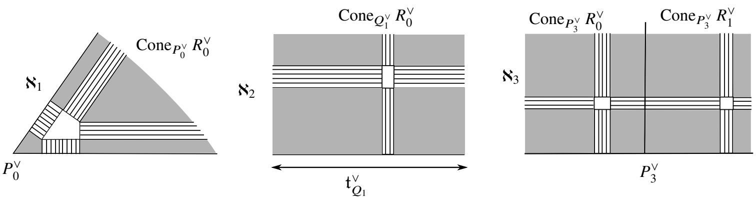



**Figure 7.11.** Squashing maps for a broken manifold. $\aleph_1 : \mathrm{Cone}_{P_0^\vee} R_0^\vee \to \mathbb{B}^\vee_{P_0}$, $\aleph_2 : \mathrm{Cone}_{Q_1^\vee} R_0^\vee \to \mathbb{B}^\vee_{Q_1}$, $\aleph_3 : \mathrm{Cone}_{P_3^\vee} B^\vee \to \mathbb{B}^\vee_{P_3}$ with $\omega$-complexes and $J$-complex from Figures 7.8, 7.10.

Via the decompositions (7.28), (7.29) of a cut space into a union of fibrations over polytopes, a squashing map $\aleph$ for the almost cut space $X_P$ induces a map of manifolds

$$\psi_\aleph : X_P \to X_P^\omega,$$

that is, piecewise, a smooth submersion onto the symplectic cut space $X_P^\omega$. Similarly for a component $\mathfrak{X}_P$ of an almost complex broken manifold $\mathfrak{X}$, a squashing map $\aleph$

induces a map of manifolds

$$\psi_{\aleph} : \mathfrak{X}_P \to \mathfrak{X}_P^{\omega}$$

onto the symplectic broken manifold $\mathfrak{X}_P$.

**Definition 7.24.** (Hofer energy for a broken manifold) Let $P \in \mathcal{P}$ be a polytope. The *P-Hofer energy* of a map $u : C \to \mathfrak{X}_P$ is

$$E_{P,\mathrm{Hof}}(u) = \sup_{\aleph} \int_C (\psi_{\aleph} \circ u)^* \omega_{\mathfrak{X}_P},$$

where the supremum is over all squashing maps $\aleph$ for $\mathfrak{X}_P$ (as in Definition 7.23). The *unpartitioned P-Hofer energy* of $u$ is defined as

$$E^*_{P,\mathrm{Hof}}(u) = \sup_{\aleph \text{ is unpartitioned}} \int_C (\psi_{\aleph} \circ u)^* \omega_{\mathfrak{X}_P},$$

where the squashing map $\aleph$ is said to be unpartitioned if for each $Q \subseteq P$, $\aleph | \mathrm{Cone}_{P^\vee} Q^\vee$ is unpartitioned (as in Definition 7.10 (c)). The $P$-Hofer energy resp. unpartitioned $P$-Hofer energy of a holomorphic map $u : C \to X_P$ to a cut space $X_P$, also denoted by $E_{P,\mathrm{Hof}}(u)$ resp. $E^*_{P,\mathrm{Hof}}$, is analogously defined.

The proof of the following monotonicity result is the same as the proof in the neck-stretched case (Lemma 7.21).

**Lemma 7.25.** *(Monotonicity of Hofer energy for broken manifolds) Suppose* $\mathfrak{J}_0 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ *is locally strongly tamed. Then there is a* $C^0$*-neighborhood* $U_{\mathfrak{J}_0} \subset \mathcal{J}^{\mathrm{cyl}}$ *of* $\mathfrak{J}_0$ *from Lemma 7.8 such that for any* $\mathfrak{J} \in U_{\mathfrak{J}_0}$, $P \in \mathcal{P}$ *and any map*

$$\psi_{\aleph} : (\mathfrak{X}_P, \mathfrak{J}) \to (\mathfrak{X}_P^{\omega}, \omega_{\mathfrak{X}_P})$$

*induced by a squashing map* $\aleph : \mathrm{Cone}_{P^\vee} B^\vee \to B_P^\vee$, $\mathfrak{J}|\mathfrak{X}_P$ *satisfies* $\psi_{\aleph}^* \omega_{\mathfrak{X}_P}(v, \mathfrak{J}v) \geq 0$ *for all* $v \in T\mathfrak{X}_P$.

Often, it is useful to have squashing area forms that are non-degenerate on a given compact subset of the cut space $X_P$.

**Lemma 7.26.** *For any* $P \in \mathcal{P}$ *and a compact subset* $K \subset X_P$*, there is an unpartitioned squashing map* $\aleph : \mathrm{NCone}_{P^\vee} B^\vee \to \mathbb{B}_P^\vee$ *such that the squashed area form* $\psi_{\aleph}^* \omega_{X_P}$ *is non-degenerate on* $K$.

*Proof.* The requirement in the Lemma is satisfied by a squashing map $\aleph$ that is a dilation on $\pi_{\mathrm{NCone}_{P^\vee} B^\vee}(K)$. Such a map exists by taking the dilation constant $t$ (from (7.22)) to be large enough. ■

**Proposition 7.27.** *(Limit of maps and Hofer energy) Let $\Omega \subset \mathbb{C}$ be a compact set, and let $u_\nu : \Omega \to X^\nu$ be such that there is a polytope $P \in \mathcal{P}$ and a sequence of translations $t_\nu \in \nu P^\vee$ such that*

$$d(t_\nu, \nu P_0^\vee) \to \infty, \quad \forall P_0 \supset P \tag{7.31}$$

*and the sequence of translated maps $\mathbb{e}^{-t_\nu} u : \Omega \to \mathfrak{X}_P$* [2] *converges uniformly on compact subsets to a limit $u : \Omega \to \mathfrak{X}_P$. Then,*

$$E^*_{P,\mathrm{Hof}}(u) \le \lim_\nu E_{\mathrm{Hof}}(u_\nu).$$

*Here $E^*_{P,\mathrm{Hof}}$ is unpartitioned $P$-Hofer energy from Definition 7.24.*

*Proof.* Consider an unpartitioned squashing map $\aleph : \mathrm{Cone}_{P^\vee} B^\vee \to \mathbb{B}^\vee_{\tilde{P}}$. To prove the result, we need to construct a sequence of squashing maps $\aleph_\nu : \nu B^\vee \to B^\vee$ for which

$$\lim_\nu u_\nu^*(\phi^*_{\aleph_\nu} \omega_X) = u^*(\phi^*_\aleph \omega_{\mathfrak{X}_P}). \tag{7.32}$$

In particular, we will construct $\aleph_\nu$ that fits into a diagram (whose commutativity we will describe later)

$$\begin{array}{ccc} \mathrm{Cone}_{P^\vee} B^\vee & \underset{\aleph_{inv}}{\overset{\aleph}{\rightleftarrows}} & \mathbb{B}^\vee_{\tilde{P}} \\ i_\nu \uparrow \downarrow i_\nu^{-1} & & \downarrow i \\ \nu B^\vee \supseteq \cup_{Q \subseteq P} \nu Q^\vee & \xrightarrow{\exists \aleph_\nu} & B^\vee, \end{array}$$

where

- $i_\nu := \mathbb{e}_P^{-t_\nu}$ from (3.60), the inverse $i_\nu^{-1}$ is defined on the image of $i_\nu$,
- $\aleph_{inv}$ is the right inverse of the unpartitioned squashing map $\aleph$ (see Remark 7.12),
- and $i : \mathbb{B}^\vee_{\tilde{P}} \hookrightarrow B^\vee$ is a tubular neighborhood of $\mathbb{B}^\vee_P$ in $B^\vee$.

By the hypothesis (7.31) on $t_\nu$, the images of $i_\nu$ exhaust $\mathrm{Cone}_{P^\vee} B^\vee$. We will define $\aleph_\nu$ so that there is a sequence of increasing open subsets $S_\nu \subset \mathrm{image}(i_\nu)$ that exhaust $\mathrm{Cone}_{B^\vee} P^\vee$ and on which the diagram commutes, that is,

$$\aleph = \aleph_\nu \circ i_\nu \quad \text{on } S_\nu. \tag{7.33}$$

This condition ensures that for any point $x \in \mathfrak{X}_P$ for which $\pi_{\mathrm{Cone}_{P^\vee} B^\vee}(x) \in S_\nu$, there is an equality of squashed forms

$$(\phi^*_{\aleph_\nu} \omega_X)|_{(\mathbb{e}^{-t_\nu})^{-1}x} = (\phi^*_\aleph \omega_{\mathfrak{X}_P})_x,$$

and consequently (7.32) holds.

[2] From (3.59), $e^{-t_\nu} : X^\nu_{\tilde{P}} \to \mathfrak{X}_P$ is an embedding of the $P$-cylindrical subset $X^\nu_{\tilde{P}} \subset X^\nu$.

We assume that $B^\vee$ has a single top-dimensional polytope, and therefore $B^\vee$ itself may be viewed as a polytope. The general case is a natural extension and is left to the reader.

The squashing map $\aleph_\nu$ is constructed using a polyhedral decomposition of the target space $\nu B^\vee$ that possesses a dual complex, as in Figures 7.3 and 7.4. By Lemma 7.15 and Lemma 7.14 (a), there exists a polyhedral decomposition $Q$ of $B^\vee$ for which $\mathbb{B}^\vee_{\tilde{P}}$ is a top-dimensional piece, and which has a dual complex $B^\vee_Q$ [3]. Note that Lemma 7.15 proves the result for a decomposition of a vector space, and Lemma 7.14 (a) extends the construction of a dual complex for the decomposition of a polytope. The squashing maps $\aleph_\nu$ are constructed by applying Lemma 7.14 (b) to the polyhedral decomposition induced on $B^\vee$ by $Q$. By Lemma 7.14 (b), (c) there is a unique family of squashing maps $\{\aleph_\nu\}_\nu$ for which

$$\aleph = \aleph_\nu \circ i_\nu^{-1} \quad \text{on } \aleph^{-1}(\mathbb{B}^{\vee,\circ}_{\tilde{P}}),$$

and such that for any top-dimensional polytope $P_0 \in Q$ other than $\mathbb{B}^\vee_{\tilde{P}}$,

$$d(\aleph_\nu^{-1}(P_0^\circ), \aleph_\nu^{-1}(\mathbb{B}^{\vee,\circ}_{\tilde{P}})) \to \infty, \tag{7.34}$$

see (7.24). By (7.34), the commutativity (7.33) holds on

$$S_\nu := \cup_{P_1 \in Q, P_1 \subseteq \mathbb{B}^\vee_{\tilde{P}}} \aleph_\nu^{-1}(P_1).$$

Informally, the reason for the commutativity on $S_\nu$ is that the squashing map sends $S_\nu$ to $\mathbb{B}^\vee_{\tilde{P}}$, and so $\aleph_\nu$ does not see the effect of other top-dimensional polytopes of $Q$ on $S_\nu$. Thus, the behavior of $\aleph_\nu$ is exactly like $\aleph$ with a domain translation given by $i_\nu$. See Figures 7.12 and 7.13 for examples. This finishes the proof of Proposition 7.27. ∎

**Proposition 7.28.** *(Hofer energy and quotients) Let $P \in \mathcal{P}$ be a polytope with* $\operatorname{codim}(P) > 0$. *Let $U_{\mathfrak{J}_0}$ be the $C^0$-neighborhood of cylindrical almost complex structures from Lemma 7.25 for which Hofer energy is monotonic. Let $u : C \to \mathfrak{X}_P$ be a holomorphic map with respect to a domain-dependent almost complex structure $J : C \to U_{\mathfrak{J}_0}$. Let $\pi_P : \mathfrak{X}_P \to X_P$ be the quotient under the action of $T_{P,\mathbb{C}}$. Then,*

(a) $E_{P,\mathrm{Hof}}(\pi_P \circ u) \le E_{P,\mathrm{Hof}}(u)$, *and*

(b) $E^*_{P,\mathrm{Hof}}(\pi_P \circ u) \le E^*_{P,\mathrm{Hof}}(u)$

*where $E_{P,\mathrm{Hof}}$ denotes $P$-Hofer energy, and the superscript $*$ refers to unpartitioned Hofer energy, see Definition 7.24.*

[3] The dual complex $B^\vee_Q$ is that of the polyhedral decomposition of $B^\vee$. The space $B^\vee$ itself happens to be a dual complex, but in this discussion it plays the role of the target space of a squashing map.

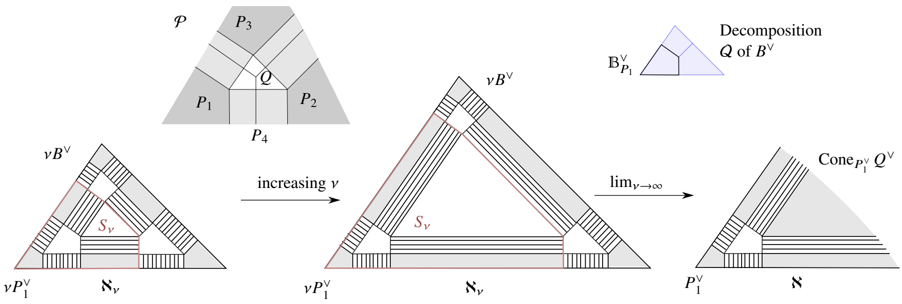



**Figure 7.12.** The squashing maps $\aleph_\nu : \nu B^\vee \to B^\vee$ (constructed in the proof of of Proposition 7.27) converge to $\aleph : \mathrm{Cone}_{P_1^\vee} Q^\vee \to \mathbb{B}^\vee_{P_1}$ as $\nu \to \infty$. Note that $\mathbb{B}^\vee_{P_1} = \mathbb{B}^\vee_{\tilde{P}_1}$ since $\mathrm{codim}(P_1) = 0$.

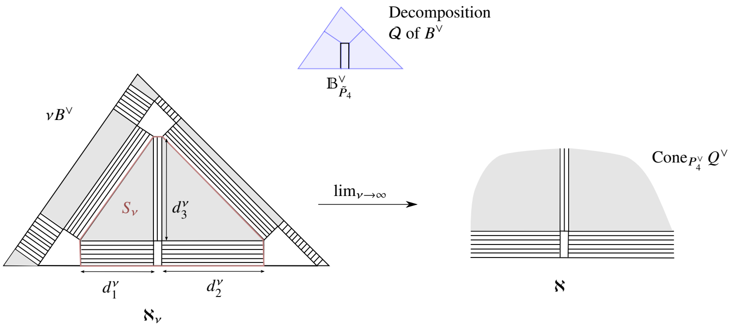



**Figure 7.13.** The squashing maps $\aleph_\nu : \nu B^\vee \to B^\vee$ (constructed in the proof of of Proposition 7.27) converge to $\aleph : \mathrm{Cone}_{P_4^\vee} Q^\vee \to \mathbb{B}^\vee_{\tilde{P}_4}$ as $\nu \to \infty$, and $d_1, d_2, d_3 \to \infty$. Here $\mathcal{P}$ and $B^\vee$ are from Figure 7.12.

*Proof.* We recall that a squashing map for $\mathfrak{X}_P$ is a product map

$$\aleph = (\aleph^P, \aleph^{P^\vee}) : \mathrm{Cone}_{P^\vee} B^\vee \to \mathbb{B}^\vee_{\tilde{P}}$$

where $\aleph^P : \mathrm{NCone}_{P^\vee} B^\vee \to \mathbb{B}^\vee_P$ is a squashing map for the cut space $X_P$, and $\aleph^{P^\vee} : \mathfrak{t}^\vee_P \to P^\vee$ is a squashing map of polytopes. The $P$-Hofer energy is the supremum over squashed areas induced by $\aleph_P$, and Hofer energy is the supremum over squashed areas induced by $\aleph$.

To prove the Proposition, consider a squashing map $\aleph^P : \mathrm{NCone}_{P^\vee} B^\vee \to \mathbb{B}^\vee_P$. Suppose $P^\vee$ is embedded in $\mathfrak{t}^\vee_P$ so that the image of $\pi_{\mathfrak{t}^\vee_P}(P)$ is the point $c_P \in \mathfrak{t}^\vee_P$. For a map

$$\aleph = (\aleph^P, c_P) : \mathrm{Cone}_{P^\vee} B^\vee \to \mathbb{B}^\vee_{\tilde{P}} \tag{7.35}$$

and any map $u : C \to \mathfrak{X}_P$, we have

$$(\pi_P \circ u)^*(\psi^*_{\aleph_P} \omega_{X_P}) = u^*(\psi^*_{\aleph} \omega_{\mathfrak{X}_P}). \tag{7.36}$$

Adding a translation to the squashing map $\aleph$ alters the squashed area by a bounded multiplicative factor: Let $v$ be a vertex of the polytope $P^\vee \subset \mathfrak{t}_P^\vee$. For the translated map

$$\aleph' : \mathrm{Cone}_{P^\vee} B^\vee \to \mathbb{B}_{\tilde{P}}^\vee, \quad \aleph' = \aleph + (0, v),$$

Lemma 3.49 shows that there is a constant $c > 0$ such that

$$\psi_{\aleph'}^* \omega_{\mathfrak{X}_P}(v, Jv) \geq c^{-1} \psi_{\aleph}^* \omega_{\mathfrak{X}_P}(v, Jv)$$

for any $J \in U_{\mathfrak{J}_0}$ and $v \in T\mathfrak{X}_P$.

Next, we construct a sequence of squashing maps that approximate $\aleph'$. (Note that $\aleph'$ itself is not a squashing map since it is not surjective.) Define

$$\tilde{\aleph}_\nu := (\aleph^P, \aleph_\nu)$$

where $\aleph_\nu : \mathfrak{t}_P^\vee \to P^\vee$ is an unpartitioned squashing map whose right inverse $\aleph_{\nu,inv} : P^\vee \to \mathfrak{t}_P^\vee$ is a translation such that the image $\aleph_{\nu,inv}(P^\vee)$ goes to $\infty$ in $\mathfrak{t}_P^\vee$ in the direction $-v$. For example, we may take $\aleph_{\nu,inv} := \aleph_{0,inv} - \nu v$ for any $\nu$. Consequently, there exist increasing open sets $U_\nu \subset \mathrm{Cone}_{P^\vee} B^\vee$ that exhaust $\mathrm{Cone}_{P^\vee} B^\vee$ such that

$$\aleph | U_\nu = (\aleph^P, c_P + v) = \aleph'.$$

The Lemma now follows: For any squashing map $\aleph^P$ of $X_P$, define a sequence of squashing maps $\tilde{\aleph}_\nu := (\aleph^P, \aleph_\nu)$ of $\mathfrak{X}_P$. We have

$$\begin{aligned} E_{\mathrm{Hof}}(u) &\geq \int_C (\psi_{\tilde{\aleph}_\nu} \circ u)^* \omega_{\mathfrak{X}_P} \geq \int_{C_\nu} (\psi_{\aleph'} \circ u)^* \omega_{\mathfrak{X}_P} \\ &\geq c \int_{C_\nu} (\psi_{\aleph} \circ u)^* \omega_{\mathfrak{X}_P} = \int_{C_\nu} (\psi_{\aleph_P} \circ \pi_P \circ u)^* \omega_{X_P}, \end{aligned}$$

where $C_\nu := (\pi_{\mathrm{Cone}_{P^\vee} B^\vee} \circ u)^{-1}(U_\nu)$. Since the sets $C_\nu$ exhaust the domain $C$, we conclude that for any squashing map $\aleph_P$ of $X_P$,

$$\int_C (\psi_{\aleph_P} \circ \pi_P \circ u)^* \omega_{X_P} \leq E_{\mathrm{Hof}}(u),$$

and consequently $E_{\mathrm{Hof}}(\pi_P \circ u) \leq E_{\mathrm{Hof}}(u)$. ∎

## 7.4 Removal of singularities

In this section, we prove the removal of singularities result, Proposition 7.1, for punctured holomorphic maps in a piece of a broken manifold. [4] We prove that any punctured

[4] An approach to proving this result by embedding the broken almost complex manifold into a compact symplectic manifold fails, because the almost complex manifold $X_P$ can not be

holomorphic map with finite Hofer energy lies in some $Q$-cylindrical region and the image of the projection to $X_Q$ lies in a compact set in the complement of relative divisors. The removal of singularities result applies on the projected map, and consequently on the original map.

Monotonicity for pseudoholomorphic maps is the main technical tool in the proof of Proposition 7.1. We state the monotonicity result (see for example [86, Proposition 3.12]).

**Proposition 7.29.** *(Monotonicity) Let $(X, \omega)$ be a compact symplectic manifold, $J_0$ an $\omega$-tamed almost complex structure, and let*

$$U_{J_0} := \{J \text{ is } \omega\text{-tamed} : \|J - J_0\|_{C^0} < \epsilon\}$$

*for some $\epsilon > 0$ be a $C^0$-neighborhood on the space of tamed almost complex structures. There exist constants $c, r_0 > 0$ such that for any $x \in X$, $0 < r \leq r_0$, a Riemann surface $C$ with boundary $\partial C$ and a pseudoholomorphic map $u : C \to X$ with respect to a domain-dependent almost complex structure $J : C \to U_{\mathcal{J}}$ whose image contains $x$ and $u(\partial C) \subset \partial B(x, r)$,*

$$\int_C u^*\omega \geq cr^2.$$

We first prove the removal of singularities result (Proposition 7.1) in the case of a single cut to serve as a warm-up for the more complicated proof in the case of multiple cuts.

*Proof of Proposition 7.1 in the case of a single cut.* We set up some notation first. We consider a single cut with polyhedral decomposition

$$\mathcal{P} = \{P_+ := (-\infty, 0], P_0 := \{0\}, P_- := [0, \infty)\},$$

and dual complex $B^\vee \cong [\frac{-\delta}{2}, \frac{\delta}{2}]$.

We first carry out the proof for maps whose target space is $X_{P_+}$, the case of $X_{P_-}$ being similar. We recall from (7.30) there is a projection to the $J$-complex

$$\pi_{B_J} : X_{P_+} \to \mathrm{Cone}_{P_+^\vee} \simeq (-\infty, 0].$$

We choose any unpartitioned squashing map (as in Definition 7.10 (c))

$$\mathrm{Cone}_{P_+^\vee} \simeq (-\infty, 0] \xrightarrow{\aleph} [\tfrac{-\delta}{2}, \tfrac{\delta}{2}] \simeq B^\vee.$$

embedded into the symplectic cut space $\overline{X}_P^\omega$ via increasing maps. For any vertex $Q \in \mathcal{P}$ of $P$, an increasing map may not exist in the $Q$-cylindrical corner of $X_P$ because the fixed $\mathfrak{t}$-inner product from (3.10) is not equal to the natural $\mathfrak{t}$-inner product at the $Q$-corner. The natural inner product is the one for which the edges $P_1 \subset P$ emanating from $Q$ form an orthogonal basis. Consequently, a Hofer energy bound in $X_P$ does not translate to a bound on symplectic area in $\overline{X}_P^\omega$.

Such a map is a translation on an interval $[\tau, \tau+\delta] \subset \mathbb{R}_-$ and a locally constant map on the complement.

**Definition 7.30.** (Crossing) A connected component $C \subset \mathrm{Cyl}$ of $(\pi_{B_J} \circ u)^{-1}([\tau, \tau+\delta])$ is called a *crossing* if $(\pi_{B_J} \circ u)(C)$ intersects both boundary components of $[\tau, \tau+\delta]$.

Step 1: We will establish a lower bound on the squashed area of any compact crossing $C$ by applying the monotonicity result Proposition 7.29. First, we recall that since any cylindrical almost complex structure $\mathfrak{J}$ on $\mathfrak{X}$ is the limit of neck-stretched almost complex structures $(X^\nu, J^\nu)$, $\mathfrak{J}$ corresponds to an $\omega_X$-tamed almost complex structure $J := J^1$ on $(X, \omega_X)$, and further, $\psi_{\aleph,*}\mathfrak{J} = J$ on subsets of $\mathfrak{X}$ where $\aleph$ is an isometry. We also note that $\psi_\aleph$ is an isometry on subsets of $\mathfrak{X}$ if $\aleph$ is an isometry on the correponding region in the $J$-complex. Since $\aleph$ is an isometry on the image of $\pi_{B_J} \circ u|C$, it is equivalent to work with

$$\psi_\aleph \circ u : C \to (X, \omega_X)$$

instead of $u : C \to \mathfrak{X}$. Choose $0 < \delta_1 < \min\{\frac{\delta}{2}, r_0\}$ where $r_0$ is the constant from the monotonicity result applied to $(X, \omega_X)$. Let $z_0 \in C$ be such that $\pi_{B_J}(u(z_0)) = \tau + \frac{\delta}{2}$. Applying the monotonicity result to the ball $B_{\delta_1}(\psi_\aleph \circ u(z_0)) \subset X$, we conclude there is a uniform constant such that for any compact crossing $C$,

$$\int_C u^*(\psi_\aleph^* \omega_X) \geq c. \tag{7.37}$$

Step 2: Next, we will show that the image $u(\mathrm{Cyl})$ is either contained in a compact subset of $X_{P_+}$ or after truncating the domain cylinder by a finite amount, the image of $u$ is contained in the $P_0$-cylindrical subset of $X_{P_+}$: After passing to a truncation of the domain cylinder, we may assume that the squashed area of the map is small enough to ensure that there are no compact crossings. If there is a non-compact crossing $C \subset \mathrm{Cyl}$, the image of $u$ is contained in a compact set. Indeed, there exists $\ell_0 \geq 0$ such that $C$ intersects the image $u(\{\ell\} \times S^1)$ for all $\ell \geq \ell_0$, and therefore the image $u(\mathrm{Cyl}(\ell_0))$ is contained within a radius $2\pi \|du\|_{L^\infty}$ of the compact subset $\{\tau \leq \pi_{B_J} \leq \tau + \delta\} \subset X_{P_+}$. Finally, if $u$ does not have any crossings, the image of $u$ is either contained in the compact subset $\{\pi_{B_J} \geq \tau\} \subset X_{P_+}$; or it is contained in $\{\pi_{B_J} \leq \tau + \delta\}$ which is in the $P_0$-cylindrical subset of $X_{P_+}$.

Step 3: In case the image of $u$ is contained in a compact subset of $X_{P_+}$, the result follows from the removal of singularities result for compact symplectic manifolds. Indeed, for any compact subset $K$ of $X_{P_+}$, by Lemma 7.26, there is a squashing area form $\omega_\aleph$ that is a symplectic form on $K$, and $\int_{\mathrm{Cyl}} u^*\omega_\aleph < E_{\mathrm{Hof}}(u)$.

Step 4: Next, we prove the result in the case when the image of $u$ is contained in the $P_0$-cylindrical end of $X_{P_+}$. The $P_0$-cylindrical end is a semi-infinite cylinder $Z_{P_0} \times$

$(-\infty, 0]$. The $(\omega_{X_{P_0}} - \frac{\epsilon}{2} d\alpha_{P_0})$-area of the projection $u_{P_0} := \pi_{P_0} \circ u$ is bounded by $E_{\text{Hof}}(u)$ because if we define the squashing map $\aleph_0$ so that

$$\aleph_0 \equiv \tfrac{-\epsilon}{2} \quad \text{on} \quad (-\infty, 0],$$

then

$$\int_{\text{Cyl}} u_{P_0}^*(\omega_{X_{P_0}} - \tfrac{\epsilon}{2} d\alpha_{P_0}) = \int_{\text{Cyl}} u^* \omega_{\aleph_0} \leq E_{\text{Hof}}(u).$$

By the removal of singularities result for compact symplectic manifolds applied to the projected map

$$u_{P_0} : \text{Cyl} \to (X_{P_0}, \omega_{X_{P_0}} - \tfrac{\epsilon}{2} d\alpha_{P_0}),$$

we conclude that $u_{P_0}$ extends holomorphically to

$$u_{P_0} : B_1 \to X_{P_0}.$$

Consider a holomorphic trivialization of the pullback bundle $u_{P_0}^*(Z_{P_0} \times \mathbb{R}) \to B_1$. The projection of $u$ to the fiber, denoted by

$$u_{\text{vert}} : B_1 \backslash \{0\} \to S^1 \times \mathbb{R}, \tag{7.38}$$

is holomorphic, and the $\mathbb{R}$-coordinate has an upper bound. Therefore, $u_{\text{vert}}$ extends over 0 to a holomorphic map in $\mathbb{P}^1$. Suppose $u_{\text{vert}}$ has a zero resp. pole of order $n \in \mathbb{Z}_{\geq 0}$ at $0 \in B_1$. Then the twisted map $\overline{u}(z) := z^{-n} u(z)$ resp. $z^n u(z)$ has the same projection to $X_{P_0}$ as $u$, the vertical component $\overline{u}_{\text{vert}}$ has a removable singularity, and therefore, $\overline{u}$ has a removable singularity. This proves (7.2) in Proposition 7.1.

Step 5: So far we have proved the result in the cases when the target space is $X_{P_\pm}$. Next, consider the case of a map $u : \text{Cyl} \to \mathfrak{X}_{P_0}$. As in the previous paragraph, the singularity at $\infty$ can be removed for the projected map $\pi_{P_0} \circ u : \text{Cyl} \to X_{P_0}$. To prove that the singularity can be removed for the vertical component $u_v$ (as in (7.38), it is enough to show that the $\mathbb{R}$-component has either an upper or a lower bound so that essential singularities are ruled out. This bound is a consequence of a lower bound on the squashed area $\psi^* \aleph$ for crossings. The details are exactly as in the case of $X_{P_\pm}$ and are therefore omitted. ∎

*Remark* 7.31. (The integrality of edge slopes) As part of the above proof, we have shown that if the punctured end has a slope $\mu \in \mathfrak{t}_{P_0}$, then $\mu$ is integral, and not just fractional. This fact relies on the torus actions being free in the neighborhood of cut loci (see Definition 1.2 and Figure 1.3). Indeed, in (7.38) we use the fact that $Z_{P_0} \times \mathbb{R} \to X_{P_0}$ is a $S^1 \times \mathbb{R}$-bundle, and so the $S^1$ factor in (7.38) is generated by an integral element in $\sqrt{-1}\mathfrak{t}_{P_0}$.

*Proof of Proposition* 7.1 *in the case of a multiple cut.* The proof is by induction on $\dim(P)$. By the induction hypothesis, we assume that the result holds for maps in the cut space $X_Q$ and the component $\mathfrak{X}_Q$ of the broken manifold for any $Q \in \mathcal{P}$ with $\dim(Q) < \dim(P)$.

We first consider the case when the target space is a cut space $X_P$, the case of a broken manifold $\mathfrak{X}_P$ is dealt with subsequently. For notational simplicity, we assume that there is only one zero-dimensional polytope $R \in \mathcal{P}$ contained in $P$, and so, $B^\vee = R^\vee$. We also denote $\mathfrak{t}_R = \mathfrak{t}$. The generalization is easy and is described in Step 3 in Page 215. We also assume that $\mathrm{codim}(P) = 0$ so that $\mathfrak{t}_P = \{0\}$. Otherwise, all instances of $\mathfrak{t}^\vee$ are replaced by $\mathfrak{t}^\vee/\mathfrak{t}_P^\vee$.

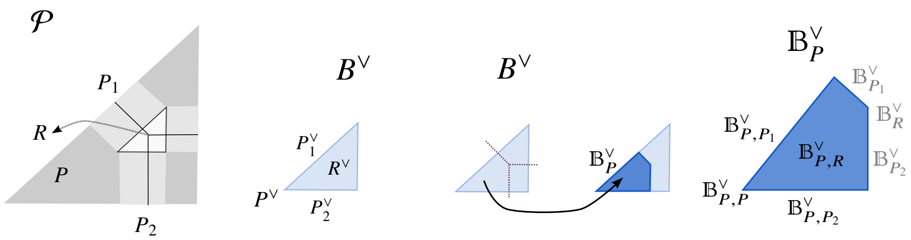



**Figure 7.14.** $\mathbb{B}_P^\vee$ is the $\omega$-complex of the cut space $X_P$. The faces of $\mathbb{B}_P^\vee$ with dark grey labels are subcomplexes, and those with light grey labels are the $\omega$-complexes of other cut spaces.

The proof is similar to that of a single cut. The new feature is that we now consider crossings defined with respect to various one-dimensional subspaces of $\mathfrak{t}^\vee$, which are defined as follows. Let $P_1, \dots P_k \in \mathcal{P}$ be facets of $P$. That is, for any $\lambda \in \{1, \dots, k\}$, $P_\lambda \subset P$ and $\dim(P_\lambda) = \dim(P) - 1$. For any $\lambda$, let $\mathfrak{p}_\lambda \in \mathcal{P}$ be the the one-dimensional face of $P$ that is transverse to $P_\lambda$, that is, $\mathfrak{p}_\lambda := \cap_{1 \le i \le k, i \ne \lambda} P_i$. Let

$$\pi_\lambda : \mathfrak{t}^\vee \to \mathbb{R} \tag{7.39}$$

be a linear projection that maps the codimension one polytope $\mathfrak{p}_\lambda^\vee \subset \mathfrak{t}^\vee$ to a constant. In particular, since we assumed $P^\vee \in \mathfrak{t}^\vee$ is the origin, we have

$$\pi_\lambda(\mathfrak{p}_\lambda^\vee) = 0.$$

Denote by

$$\pi_{B_J,\lambda} := \pi_\lambda \circ \pi_{B_J} : X_P \to \mathbb{R}$$

the composition of $\pi_{B_J} : X_P \to \mathrm{NCone}_{P^\vee} R^\vee \subset \mathfrak{t}^\vee$ with $\pi_\lambda$ from (7.39). The projection $\pi_\lambda$ has the useful property that the $P_\lambda$-cylindrical end in $X_P$ is

$$U_{P_\lambda}(X_P) = \{\pi_{B_J,\lambda} > 0\}, \tag{7.40}$$

and the complement $X_P \backslash U_{P_\lambda}(X_P)$ is equal to $\{\pi_{B_J,\lambda} = 0\}$.

We choose a point $\tau \gg 0$ in the interior of $\pi_\lambda(\mathrm{NCone}_{P^\vee} R^\vee) \subset \mathbb{R}$.

**Definition 7.32.** ($\mathfrak{p}_\lambda^\vee$-crossing) Given $\delta > 0$, a connected component $C \subset \mathrm{Cyl}$ of $(\pi_{B_J,\lambda} \circ u)^{-1}([\tau-\delta, \tau+\delta])$ is a $(\mathfrak{p}_\lambda^\vee, \delta)$-crossing if $\pi_{B_J,\lambda}(u(C))$ intersects both boundary components of $[\tau-\delta, \tau+\delta]$. If $\delta$ is clear from the context, we refer to $C$ as a $\mathfrak{p}_\lambda^\vee$-crossing.

We interrupt the proof of Proposition 7.1 (multiple cut case) to state and prove a technical Claim. The proof is continued in Page 215. ■

The following Claim is the technical heart of the proof of the removal of singularities result.

*Claim* 7.33. There are constants $\delta, c > 0$ and a squashing map $\aleph$ such that for any compact $(\delta, \mathfrak{p}_\lambda^\vee)$-crossing $C \subset \mathrm{Cyl}$, there is a lower bound

$$\int_C (\psi_\aleph \circ u)^* \omega_{X_P} \geq c.$$

*Proof of Claim* 7.33. Step 1: The first step is to fix the constant $\delta$ and the squashing map $\aleph$. Cut up the $\omega$-polytope $\mathbb{B}_P^\vee \subset \mathfrak{t}^\vee$ along a level set of $\pi_\lambda$ into two polytopes $\mathbb{B}_{P,\pm}^\vee$ so that the facet $\mathbb{B}_{P_\lambda}^\vee \subset \mathbb{B}_P^\vee$ is contained in $\mathbb{B}_{P,-}^\vee$, see Figure 7.15. Define

$$\delta := \tfrac{1}{2}\,\mathrm{length}(\pi_\lambda(\mathbb{B}_{P,+}^\vee)).$$

We will define $\aleph : \mathrm{NCone}_{P^\vee} R^\vee \to \mathbb{B}_P^\vee$ to be an unpartitioned undilated squashing map, and therefore $\aleph$ will possess a right inverse that is a translation

$$\aleph_{inv} : \mathbb{B}_P^\vee \to \overline{\aleph^{-1}(\mathbb{B}_P^{\vee,\circ})}.$$

In fact, $\aleph$ is fully determined by the map $\aleph_{inv}$, which we define by the condition

$$\aleph_{inv}(\mathbb{B}_{P,P}^\vee) := \{\pi_\lambda = \tau - \delta\} \cap \mathrm{NCone}_{P^\vee} P_\lambda^\vee. \tag{7.41}$$

We point out that both sides of the equation (7.41) are points. The squashing map $\aleph$ defined in this manner has the property that for any facet $\mathrm{NCone}_{P^\vee} Q^\vee$ of $\mathrm{NCone}_{P^\vee} R^\vee$ that contains $\mathrm{NCone}_{P^\vee} P_\lambda^\vee$ (that is, $Q \in \mathcal{P}$, $Q \subseteq P_\lambda$ and $\dim(Q) = 1$),

$$\aleph^{-1}(\mathbb{B}_{P,Q}^\vee) \subset \mathrm{NCone}_{P^\vee} Q^\vee. \tag{7.42}$$

Outline of the proof of the Claim: Compared to the single cut, the most significant difference in the multiple cut case is that the image of a crossing may not be contained in a subset of $\mathfrak{X}_P$ where $\psi_\aleph$ is an isometry. This corresponds to the fact that $\aleph$ is not an isometry on $\pi_\lambda^{-1}([\tau-\delta, \tau+\delta])$. The way around this issue is that if the image $u(C)$ is contained in a subset of $X_P$ where the map $\psi_\aleph$ is a projection whose fibers are $T_{Q_0,\mathbb{C}}$-orbits for some $Q_0 \in \mathcal{P}$ for which $Q_0 \subseteq P$, $Q_0 \nsubseteq P_\lambda$, Then we apply monotonicity result not on $u$ (as in the single cut proof) but on $u$ composed with a quotient by the

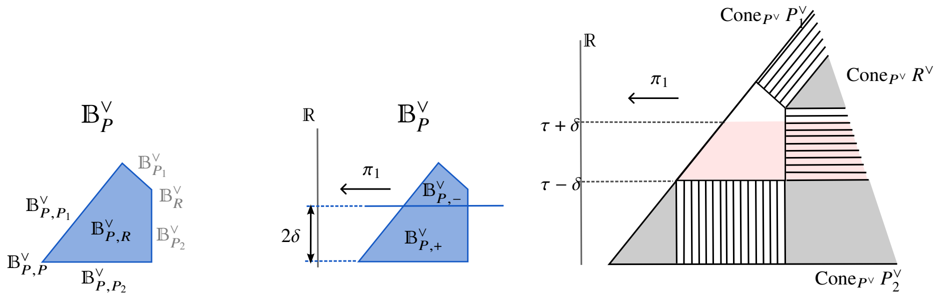



**Figure 7.15.** $\mathfrak{p}_\lambda^\vee$-crossing with $\lambda = 1$.

$T_{Q_0,\mathbb{C}}$-action. Observe that the fibers of $\psi_\aleph$ are $T_{Q_0,\mathbb{C}}$-orbits exactly if the map $\aleph$ is $\mathfrak{t}_{Q_0}$-*squashing*, that is, it is a projection map whose fibers are $\dim(\mathfrak{t}_{Q_0})$-dimensional and parallel to $\mathfrak{t}_{Q_0}^\vee$. We note that $\aleph$ is chosen so that in the subset $\pi_\lambda^{-1}([\tau-\delta,\tau+\delta])$, the $\mathfrak{t}_{P_\lambda}$ direction itself is not squashed (that is, $Q_0 \not\subseteq P_\lambda$); this feature is crucial in getting the lower bound on area. The details of the proof are as follows.

STEP 2: *Determining the polytope $Q_0 \in \mathcal{P}$ (from the proof outline) via a decomposition of the $J$-polytope.* The subset of the $J$-complex corresponding to the $\mathfrak{p}_\lambda^\vee$-crossing has a decomposition

$$\pi_\lambda^{-1}(\tau-\delta,\tau+\delta) \subset \bigcup_{Q\in\mathcal{P},\, Q\not\subseteq P_\lambda,\, Q\subseteq P} S_Q, \tag{7.43}$$

where $S_Q \subset \mathrm{NCone}_{P^\vee} R^\vee$ is the subset on which $\aleph$ is $T_Q$*-squashing*, that is, $\aleph$ projects to the direction perpendicular to $\mathfrak{t}_Q$. See Figure 7.16. The decomposition (7.43) exists because (7.42) rules out any other kind of squashing in $\pi_\lambda^{-1}(\tau-\delta,\tau+\delta)$.

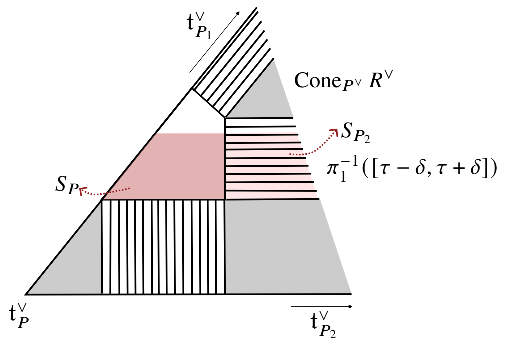



**Figure 7.16.** The $\mathfrak{p}_1^\vee$-crossing decomposes into $S_P$ and $S_{P_2}$.

The monotonicity result will be applied on small balls in the target space whose radius $\delta_1$ we determine next. Let $N := \dim(P)$. Choose a constant $0 < \delta_1 < \delta/(N+1)$

which is small enough that for any $x \in \pi_\lambda^{-1}([\tau - N\delta_1, \tau + N\delta_1]) \subset \mathrm{NCone}_{P^\vee} R^\vee$ and any polytope $Q \subseteq P$,

$$B_{\delta_1}(x) \cap \mathrm{NCone}_{P^\vee} Q^\vee \neq \emptyset \Longrightarrow B_{\delta_1}(x) \subset \bigcup_{Q_1 \in \mathcal{P}, Q_1 \supseteq Q} S_{Q_1}. \tag{7.44}$$

Consider a compact $\mathfrak{p}_\lambda^\vee$-crossing $C \subset \mathrm{Cyl}$. We call $(x \in X_P, Q \in \mathcal{P})$ an *admissible pair* if

- $x \in \pi_{B_J}(u(C))$ and $\pi_\lambda(x) \in [\tau - n_Q\delta, \tau + n_Q\delta]$, where $n_Q := \dim(Q)$; and
- $Q_x^\vee = \cap_{Q \in \mathcal{Q}} Q^\vee$, where

$$\mathcal{Q} := \{Q \in \mathcal{P} : Q \subseteq P, Q \nsubseteq P_\lambda, B_{\delta_1}(x) \cap \mathrm{NCone}_{Q^\vee} R^\vee \neq \emptyset\}.$$

Let

$$(x_0, Q_0) \tag{7.45}$$

be an admissible pair such that there is no admissible pair $(x, Q_x)$ such that $Q_0 \subset Q_x$. In other words, we choose an admissible pair $(x_0, Q_0)$ for which the torus $T_{Q_0,\mathbb{C}}$ is minimal. See Figure 7.17 for an example.

Step 3: *Applying the monotonicity result and obtaining the lower bound on squashed area.*
We will apply the monotonicity result (Proposition 7.29) to the map $u$ quotiented by $T_{Q_0,\mathbb{C}}$. Let

$$u_{Q_0} : C \dashrightarrow X_{Q_0}, \quad u_{Q_0} := u/T_{Q_0,\mathbb{C}}$$

be a map defined on the subset of $C$ that is mapped by $u$ to the $Q_0$-cylindrical end of $X_P$. Fix any $z_0 \in (\pi_{B_J,\lambda} \circ u)^{-1}(x_0)$, and let

$$C_1 := u_{Q_0}^{-1}(B_{\delta_1}(u_{Q_0}(z_0))).$$

We will now prove that

$$u_{Q_0}(\partial C_1) \subset \partial B_{\delta_1}(u_{Q_0}(z_0)), \tag{7.46}$$

which is a hypothesis of the monotonicity result. If (7.46) did not hold, there would be a point $z_1 \in \partial C_1$ such that $u(z_1)$ does not map to the $Q_0$-cylindrical end of $X_P$ (assuming $U_Q(X_P)$ is an open subset of $X_P$), and $\pi_{B,J}(u(z_1)) \in \mathrm{Cone}_{P^\vee} Q_1^\vee$ for some $Q_1 \in \mathcal{P}$ $Q_1 \supset Q$. Then, setting $x_1 := \pi_{B_J}(u(z_1))$, $(x_1, Q_{x_1})$ is an admissible pair for some $Q_{x_1} \in \mathcal{P}$ with $Q_{x_1} \subseteq Q_0$, where we use the observation that since $|\pi_\lambda(x_0) - \tau| < n_{Q_0}\delta_1$, we have $|\pi_\lambda(x_1) - \tau| < (n_{Q_0} + 1)\delta_1 \leq n_{Q_{x_1}}\delta_1$. We have thus contradicted the minimality of $T_{Q_0,\mathbb{C}}$ from (7.45), and shown that (7.46) indeed holds.

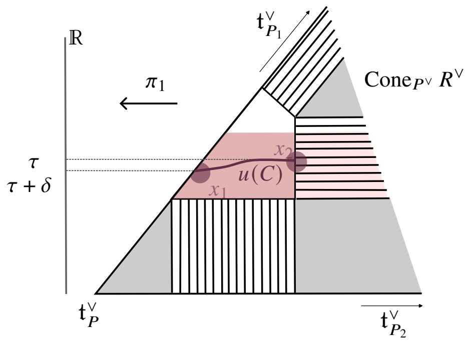



**Figure 7.17.** Assume the setting of Figure 7.16. For the map $u$, $(x_1, P_0)$, $(x_2, P_2)$ are admissible pairs, with $(x_1, P_0)$ being the pair chosen in (7.45).

To finish the proof of the result, we relate the squashed areas of $u$ and its projection $u_{Q_0}$, via the following commutative diagram

$$\begin{array}{ccc} U_{Q_0}(X_P) & \xrightarrow{\psi_\aleph} & X_P^\omega \\ {\scriptstyle /T_{Q_0,\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle /T_{Q_0,\mathbb{C}}} \\ X_{Q_0} & \dashrightarrow_{\psi_{\aleph,/\!/}} & X_{Q_0}^\omega . \end{array}$$

The maps denoted by dashed arrows are defined on subsets of the domain. However these subsets contain the images of $\psi_\aleph \circ u$ and $u/T_{Q_0,\mathbb{C}}$. The left downward arrow is a map between almost complex manifolds. The right downward arrow is a map between symplectic manifolds, and is defined by an orthogonal projection map $\mathbb{B}_{P,+}^\vee \to \mathbb{B}_{P,Q_0}$. The map $\psi_{\aleph,/}$ is induced by a squashing $\aleph_/ : \mathrm{NCone}_{Q_0^\vee} R^\vee \to \mathbb{B}_{Q_0}^\vee$ which is defined on a subset of the domain as follows: The restriction $\aleph|\aleph^{-1}(\mathbb{B}_{P,+}^\vee \cap \mathbb{B}_{Q_0}^\vee)$ orthogonally projects to $\mathbb{B}_{Q_0}^\vee$. Therefore, it descends to the $\mathfrak{t}_Q$-quotient of the domain, and yields the map $\aleph_/$. The map $\psi_{\aleph,/}$ is an isometry on the image of $(u/T_{Q_0,\mathbb{C}})|C_1$. Applying the monotonicity theorem on the map $\psi_{\aleph,/}(u/T_{Q_0,\mathbb{C}})|C_1$, we conclude

$$\int_{C_1} (\psi_{\aleph,/}(u/T_{Q_0,\mathbb{C}}))^* \omega_{X_{Q_0}} > c_{Q_0}\delta_1^2.$$

Finally, there is a constant $c(\mathfrak{X}) > 0$ such that

$$\int_{C_1} (\psi_\aleph \circ u)^* \omega_{X_P} \geq c \int_{C_1} (\psi_{\aleph,/}(u/T_{Q_0,\mathbb{C}}))^* \omega_{X_{Q_0}} > c c_{Q_0}\delta_1^2.$$

This leads to the lower bound on $\aleph$-squashed area of crossings, and finishes the proof of Claim 7.33. ∎

*Proof of Proposition 7.1 for multiple cuts, continued.* We recall that $P_1, \ldots, P_k \in \mathcal{P}$ are facets of $P$.

Step 1: *There is a constant $\ell_0$ such that for any facet $P_\lambda \in \mathcal{P}$ of $P$, one of the two possibilities occurs:*

- *Either the image $u(\mathrm{Cyl}(\ell_0))$ is contained in the $P_\lambda$-cylindrical end $U_{P_\lambda}(X_P)$, or*
- *the projection $\pi_{B_J,\lambda}(u(\mathrm{Cyl}(\ell_0)))$ has an upper bound.*

For a given $\lambda$, we first consider the case that $u$ has a non-compact crossing $C$. We may choose $\ell_\lambda$ so that $\{s\} \times S^1$ intersects $C$ for all $s > \ell_\lambda$. Therefore, $\pi_{B_J,P_\lambda} \circ u|\mathrm{Cyl}(\ell_\lambda)$ lies in a $2\pi\|du\|_{L^\infty}$-radius of $\pi^{-1}_{B_J,P_\lambda}([\tau - \delta_\lambda, \tau + \delta_\lambda)$, and therefore by (7.40) the image of $u|\mathrm{Cyl}(\ell_\lambda)$ lies in the $P_\lambda$-cylindrical end of $X_P$. Next, consider the case that $u$ does not have any non-compact crossings. Claim 7.33 implies that there is a constant $\ell_\lambda \geq 0$ such that $u|\mathrm{Cyl}(\ell_\lambda)$ does not have any compact crossings. Then, $\pi_{B_J,\lambda}(u(\mathrm{Cyl}(\ell_\lambda)))$ either has a lower bound of $\tau_\lambda - \delta_\lambda$ or an upper bound of $\tau_\lambda + \delta_\lambda$. In the former case, $u(\mathrm{Cyl}(\ell_\lambda))$ lies in the $P_\lambda$-cylindrical end. Finally, by taking $\ell_0 := \min_\lambda \ell_\lambda$, the condition stated in the beginning of the paragraph is satisfied.

Step 2: We use induction to finish the proof of the result for a map in a cut space $X_P$, and for which there is only one zero-dimensional polytope $R \in \mathcal{P}$ in $P$. First consider the case that for all facets $P_\lambda$, the projection $\pi_{B_J,\lambda}(u(\mathrm{Cyl}(\ell_0)))$ has an upper bound. Then, the image of $u$ is contained in a compact subset of $X_P$, and the proof of removal of singularity follows exactly as in Step 3 of the proof in the case of a single cut (see page 208). The other possibility is that for some $\lambda$, the image of $u|\mathrm{Cyl}(\ell_0)$ is contained in the $P_\lambda$-cylindrical end. Then, we may view $u$ as a map in $\mathfrak{X}_{P_\lambda}$ and apply the induction hypothesis to the map $\pi_{P_\lambda} \circ u : \mathrm{Cyl} \to X_{P_\lambda}$. For the proof of the induction hypothesis, it is enough to have a bound on squashed area corresponding to squashing maps on $\mathrm{NCone}_{P_\lambda^\vee} R^\vee$ which are restrictions of squashing maps $\mathrm{NCone}_{P^\vee} R^\vee$ with $\mathrm{NCone}_{P_\lambda^\vee} R^\vee$ being embedded in $\mathrm{NCone}_{P^\vee} R^\vee$ far enough from the faces corresponding $Q^\vee \subset P_\lambda^\vee$; such squashed areas are bounded by $E_{P,\mathrm{Hof}}(u)$.

Step 3: If the target of $u$ is a cut space $X_P$ and there are multiple zero-dimensional polytopes $R \in \mathcal{P}$ that are vertices of $P$, the only difference is in the definition of the projection map $\pi_\lambda$ from (7.39). First, observe that in the case of a single zero dimensional polytope $R$, the map $\pi_\lambda$ is actually defined on $R^\vee$ and $\mathrm{Cone}_{P^\vee} R^\vee$, both of which were viewed as subsets of $\mathfrak{t}^\vee$. In the general case, we define $\pi_{\lambda,R} : \mathfrak{t}_R^\vee \to \mathbb{R}$ for each $R$ so that $\pi_{\lambda,R}(\mathfrak{p}_{\lambda,R}) = 0$ (where $\mathfrak{p}^\vee_{\lambda,R} \subset R^\vee$ is the face corresponding to $\mathfrak{p}^\vee_\lambda$) and $\pi_{\lambda,R}|P^\vee_{\lambda,R}$ is the same for all $R \in P_\lambda$. The notion of $\mathfrak{p}^\vee_\lambda$-crossing can now be defined, and the rest of the proof carries over.

Step 4: Next, we consider the case that the target space of $u$ is a broken manifold $\mathfrak{X}_P$. We may assume that the projection $u/T_{P,\mathbb{C}}$ lies in a compact subset of $X_P$; otherwise by applying the preceding discussion to $u/T_{P,\mathbb{C}}$, we may conclude that its image lies on a cylindrical end of $X_P$ and hence the induction hypothesis is applicable. As in the previous paragraph, we conclude that $u/T_{P,\mathbb{C}}$ has a removable singularity, and we assume that $(u/T_{P,\mathbb{C}})(\infty) = x_P \in X_P$. Analogously to Step 4 of the proof in the case of a single cut (see page 208), we prove the result in $\mathfrak{X}_P$ by considering a holomorphic trivialization of the $T_{P,\mathbb{C}}$-bundle $(u/T_{P,\mathbb{C}})^*\mathfrak{X}_P \to B_1$ and projecting $u$ to the fiber direction in a neighborhood of the puncture at $0 \in B_1$. To finish the proof, it only remains to show that for any one-dimensional face $Q^\vee$ of $P^\vee$ (where $Q \in \mathcal{P}$), the projection of

$$\pi_{P,J}(u(\mathrm{Cyl})) \subset \mathrm{Cone}_{P^\vee} R^\vee \simeq \mathrm{NCone}_{P^\vee} R^\vee \times \mathfrak{t}_P^\vee$$

to $\mathfrak{t}_Q^\vee \simeq \mathbb{R}$ has a one-sided bound. This fact follows from a lower bound on squashed area of $\mathfrak{t}_Q$-crossings, which is proved in a similar manner to Claim 7.33. We use a squashing map $\aleph$ that is a product, that is

$$\aleph = (\aleph_1, \aleph_2), \quad \aleph_1 : \mathrm{NCone}_{P^\vee} R^\vee \to \mathbb{B}_P^\vee, \quad \aleph_2 : \mathfrak{t}_P \to P^\vee,$$

and where $\aleph_1$ is an isometry in the neighborhood of the image of $x_P \in X_P$. The rest of the details of the proof of the analogue of Claim 7.33 are left to the reader. This finishes the proof of Proposition 7.1. ■

The following is a version of the removal of singularities result for punctures on the boundary. This result does not involve any technical difficulties arising from neck-stretching because the Lagrangian submanifold does not intersect the neck regions.

**Proposition 7.34.** *(Removal of singularities on the boundary) Suppose*

$$u : (\mathbb{R}_+ \times [0,1], \mathbb{R}_+ \times \{0,1\} \to (X_{P_0}, L)$$

*is a map that is holomorphic with respect to a domain-dependent cylindrical almost complex structure taking values in $U_{\mathfrak{J}_0}$ (from Lemma 7.21), and*

$$E_{\mathrm{Hof}}(u) < \infty, \quad \|\mathrm{d}u\|_{L^\infty(\mathrm{Cyl})} < \infty.$$

*Then, $u$ extends to a holomorphic map*

$$u : (B_1 \cap \mathbb{H}, B_1 \cap \mathbb{R}) \to (\overline{X}_{P_0}, L).$$

*Proof.* By the uniform bound on the derivative, the image of $u$ is contained in a compact neighborhood $K \subset X_P$ of the Lagrangian $L$. By Lemma 7.26, there is a squashed area form $\omega_\aleph \in \Omega^2(X_P)$ that is symplectic on the closure of im(u) and that tames the almost complex structure. Therefore, the Proposition follows from the removal of singularity theorem for compact symplectic manifolds. ■

## 7.5 Hofer energy for Gromov compactness of broken maps

The results stated so far in this Chapter will be used in the proof that a sequence of breaking maps with bounded area converges to a broken map. In this section, we discuss analogs of the results which will be used to show that a sequence of broken maps with bounded area converges to a broken map. In components $\mathfrak{X}_P$ of the broken manifold, we use $P$-Hofer energy of maps.

The first result, which is an analog of Remark 7.19 for broken maps, is that for a component of a broken map in $\mathfrak{X}_P$, the $P$-Hofer energy is bounded by a constant that depends only on the topology of the map. In particular, we obtain a bound on Hofer energy for components of broken maps $u : C^\circ \to \mathfrak{X}_P$ whose projection to $X_P$ is in class $\beta \in \pi_2(\overline{X}_P, L)$, and whose tropical graph is $\mathcal{T}$. Note that both $\beta$ and $\mathcal{T}$ are part of the combinatorial type (Definition 6.14) of the map $u$, and that $\mathcal{T}$ determines the number of punctures in the domain of $u$, and the slope $\mathcal{T}(z_i) \in \mathfrak{t}_{\mathbb{Z}}$ of the map at each of the punctures $z_i$.

**Proposition 7.35.** *For any polytope $P \in \mathcal{P}$, a homotopy class $\beta \in \pi_2(\overline{X}_P, L)$ and a tropical graph $\mathcal{T}$ underlying a relative map, there is a constant $c$ such that for a holomorphic map $u : C^\circ \to \mathfrak{X}_P$ with tropical graph $\Gamma$, and whose projection $\pi_P \circ u$ is of class $\beta$,*

$$E_{P,\mathrm{Hof}}(u) \le c.$$

*Proof.* We first prove the result assuming that the $X$-inner product $g$ (see (3.10)) is rational, so that the compactification $\overline{\mathfrak{X}}_P$ of $\mathfrak{X}_P$ is an orbifold with symplectic form $\omega_{\mathfrak{X}_P}$. Furthermore,

$$E_{P,\mathrm{Hof}}(u) = \int_C u^* \psi_{\aleph}^* \omega_{\mathfrak{X}_P} =: \omega_{\mathfrak{X}_P}(u),$$

where $\aleph$ is any squashing form. The above quantity is independent of which $\aleph$ is used, because for any two choices $\aleph_1, \aleph_2$, the maps $\psi_{\aleph_1}$ and $\psi_{\aleph_2}$ are isotopic.

Under the splitting of $\mathfrak{t} = \mathfrak{t}_P \oplus \mathfrak{t}_P^\perp$, let $\mathcal{T}_g^{\mathrm{vert}}(z_i) \in \mathfrak{t}_P$ be the $\mathfrak{t}_P$-component of $\mathcal{T}_i$. Recall that the fiber of $\pi_P : \overline{\mathfrak{X}}_P \to \overline{X}_P$ is

$$P^\vee = \{x \in \mathfrak{t}_P^\vee : \langle \mu_i, x\rangle \le c_i, i = 1, \dots, N\},$$

where $c_i \in \mathbb{R}$ and $\mu_i \in \mathfrak{t}_{P,\mathbb{Z}}$, $i = 1, \dots, N$ are primitive outward pointing normals to the facets of $P^\vee$. Suppose the base symplectic form $\omega_{X_P}$ on $\overline{X}_P$ is given by reduction at the origin $0 \in \mathfrak{t}_P^\vee$. Then,

$$\begin{aligned}\langle u_*[C], \omega_{\mathfrak{X}_P}\rangle &= \langle(\pi_P \circ u)_*[C], \omega_{X_P}\rangle + 2\pi\textstyle\sum_{i=1}^{d(\bullet)}\sum_j c_j \cdot m(z_i, D_j)\\ &\le \langle(\pi_P \circ u)_*[C], \omega_{X_P}\rangle + \sum_i |\mathcal{T}_g^{\text{vert}}(z_i)|\\ &\le \langle(\pi_P \circ u)_*[C], \omega_{X_P}\rangle + c_g \sum_i |\mathcal{T}(z_i)| \end{aligned} \tag{7.47}$$

where $m(z_i, D_j)$ is the intersection multiplicity at $z_i$ with the toric divisor $D_j = \{\langle\mu_j, x\rangle = c_j\}$, $|\mathcal{T}^{\text{vert}}(z_i)|$ is the sum of the components of $\mathcal{T}^{\text{vert}}(z_i)$ for a fixed $\mathbb{Z}$-basis. Since the origin $0 \in \mathfrak{t}_P^\vee$ is in the interior of $P^\vee$, the constant $c_j$ is positive for all $j$. The Proposition follows in the case when the $X$-inner product is rational.

For a general $X$-inner product $g$, we consider a sequence $\{g_\nu\}_\nu$ of rational $X$-inner products uniformly converging to $g$. A squashing map with respect to the $X$-inner product $g$ is the limit of a squashing maps $\aleph_\nu$ with respect to the $X$-inner product $g_\nu$, and therefore, $E_{P,\text{Hof},g} \le \limsup_\nu E_{P,\text{Hof},g_\nu}$. For any $\nu$, the constant $c_{g_\nu}$ is uniformly bounded, since it is chosen so that it satisfies $|\mathcal{T}_{g_\nu}^{\text{vert}}(z_i)| \le c_{g_\nu}|\mathcal{T}(z_i)|$, and the Proposition follows. ■

The following is an analog of Proposition 7.27 for the convergence of a sequence of broken map components in $\mathfrak{X}_P$.

**Proposition 7.36.** *(Hofer energy and limits of maps, broken version) Let $\Omega \subset \mathbb{C}$ be an open domain, and let $u_\nu : \Omega \to \mathfrak{X}_P$ be such that there is a polytope $Q \subset P$ and a sequence of translations $t_\nu \in \text{Cone}_{P^\vee} Q^\vee$ such that*

$$d(t_\nu, \nu P_0^\vee) \to \infty, \quad \forall P \supseteq P_0 \supset Q, \tag{7.48}$$

*and the sequence of translated maps $e^{-t_\nu}u : \Omega \to \mathfrak{X}_Q$ converges uniformly on compact subsets to a limit $u : \Omega \to \mathfrak{X}_Q$. Then,*

$$E^*_{Q,\text{Hof}}(u) \le \lim_\nu E_{P,\text{Hof}}(u_\nu).$$

The proof is the same as that of Proposition 7.27.

# Chapter 8

# Gromov compactness

We prove two convergence results for maps in broken manifolds in this Chapter. The first result, Theorem 8.2, concerns the limit of holomorphic maps to the neck-stretched manifolds $X^\nu$ in the limit $\nu \to \infty$. The second result, Theorem 8.3, concerns the convergence behavior of broken maps. Both theorems require only an area bound on the sequence of holomorphic maps. In the breaking case, area of a map in $X^\nu$ refers to $\omega_X$-area (7.3). In the case of broken maps in Theorem 8.3, area refers to the sum of the $\omega_{X_P}$-areas (4.24) of the projections of the map components to cut spaces $X_P$.

We sketch the definition of Gromov convergence, with a more precise and detailed definition postponed to Section 8.1. As in symplectic field theory, convergence in neck-stretched manifolds is modulo target rescalings. Rescalings are analogous to *translations* defined in Section 3.5. Given $\nu \in \mathbb{R}_+$, the set of translations is parametrized by the dual complex $\nu B^\vee$. A translation $t \in \nu P^\vee \subset B^\vee$ is an embedding

$$\mathbb{e}^{-t} : X^\nu_{\tilde{P}} \to \mathfrak{X}_P \subset \mathfrak{X}$$

of the $P$-cylindrical subset $X^\nu_{\tilde{P}}$ of the neck-stretched manifold $X^\nu$. Gromov convergence of a sequence of maps $u_\nu : C_\nu \to X^\nu$ on neck-stretched manifolds to a broken map $u : C \to \mathfrak{X}$ encapsulates the following conditions:

(a) (Convergence of domains) The sequence of treed disks $C_\nu$ converges to a treed disk $C$;

(b) (Convergence of maps) for each component $C_v$ of $C$, there is a sequence of translations $t_\nu(v) \in \nu P(v)^\vee$ such that the translated maps $\mathbb{e}^{-t_\nu} u_\nu$ converges to a map $u_v$;

(c) (Thin cylinder convergence) for a tropical node $w$ in the domain $C$ of $u$ corresponding to an edge $e = (v_+, v_-)$, on the midpoints

$$m_\nu := (0,0) \in [-l_\nu/2, l_\nu/2] \times \mathbb{R}/2\pi\mathbb{Z} \subset C_\nu$$

of the sequence of annuli in $C_\nu$ converging to the node $w$, the translated evaluations $\mathbb{e}^{\frac{1}{2}(t_\nu(v_+)+t_\nu(v_-))} u_\nu(m_\nu)$ converge to the tropical evaluation (see (4.15)) of $u$ at the nodal point $w$.

The translation sequences yield tropical positions of the vertices in the tropical graph of the limit map, justifying the representation in Figure 1.9. For any $\nu$

$$\mathcal{T}(v) := \tfrac{t_\nu(v)}{\nu} \in P(v)^\vee, \quad v \in \mathrm{Vert}(\Gamma)$$

is a vertex position map for the combinatorial type $\Gamma$ of the limit map. For a Gromov-converging sequence of maps, area is automatically preserved in the limit, since the number of domain marked points is preserved in the limit, and this number is a constant multiple of area (see (5.2)). We remark that (Thin cylinder convergence) is not part of the definition of convergence of stable maps in smooth manifolds, since it can be deduced from convergence of maps. However, in case of neck-stretching, (Thin cylinder convergence) does not follow from (Convergence of maps). Moreover (Thin cylinder convergence) is used in the proof of surjectivity of gluing in Chapter 9.

In the statement of Gromov convergence for maps on neck-stretched manifolds, the perturbations on neck-stretched manifolds are obtained by gluing a perturbation datum on the broken manifold. A cylindrical divisor $\mathfrak{D}$ in a broken manifold $\mathfrak{X}$ can be glued to give a family of divisors $D^\nu$ in neck-stretched manifolds $X^\nu$. There is a natural correspondence of tamed cylindrical divisor-adapted almost complex structures

$$\rho_\nu : \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}, \mathfrak{D}) \to \mathcal{J}^{\mathrm{cyl}}(X^\nu, D^\nu). \tag{8.1}$$

**Definition 8.1.** (Breaking perturbation datum) Let $\underline{\mathfrak{p}}$ be a perturbation datum for the broken manifold $\mathfrak{X}$ adapted to a cylindrical divisor $\mathfrak{D}$. A *breaking perturbation datum* is a family of perturbation data $\{\underline{\mathfrak{p}}_\nu\}_{\nu\in[\nu_0,\infty]}$ for the neck-stretched manifolds $(X^\nu, D^\nu)$ satisfying the following property: There exists a constant $\nu(\Gamma)$ so that if $\nu > \nu(\Gamma)$ then

$$\mathfrak{p}_{\nu,\Gamma} := (\rho_\nu \circ \mathfrak{J}_\Gamma, F_\Gamma), \quad \text{where } \mathfrak{p}_\Gamma = (\mathfrak{J}_\Gamma, F_\Gamma).$$

The statement of the convergence theorem does not require perturbations be regular, but it does require perturbations to be adapted to a stabilizing pair $(D^\nu, J^\nu)$ on $X^\nu$. Such a pair is obtained by gluing a stabilizing pair $(\mathfrak{D}, \mathfrak{J})$ on the broken manifold (constructed in Section 5.4). We recall that by Definition 5.22 of a stabilizing pair $(D^\nu, J^\nu)$, any non-constant $J^\nu$-holomorphic sphere in $X^\nu$ intersects $D^\nu$ transversally at at least 3 points. The theorem is regarding a sequence of treed holomorphic disks in neck-stretched manifolds $X^\nu$ converging to a stable broken disk in a broken manifold $\mathfrak{X}$. We recall from Definition 4.14 that a *broken disk* is a broken map, one of whose components in the piece $X_{P_0} \subset \mathfrak{X}$ is a treed disk with the disk boundary and treed segments mapping to the Lagrangian $L \subset X_{P_0}$.

**Theorem 8.2.** *(Gromov convergence for breaking maps) Suppose* $(\mathfrak{J}_0, \mathfrak{D})$ *is a stabilizing pair for the broken manifold* $\mathfrak{X}$ *in the sense of Definition 5.22, and* $\underline{\mathfrak{p}}^\infty$ *is a perturbation datum on* $\mathfrak{X}$ *adapted to* $(\mathfrak{J}_0, \mathfrak{D})$*. Suppose* $\{\underline{\mathfrak{p}}^\nu\}_\nu$ *is a sequence of breaking perturbation data (as in Definition 8.1) on neck-stretched manifolds* $\{X^\nu\}_\nu$*. Let*

$$u_\nu : C_\nu \to X^\nu$$

*be a sequence of treed* $\underline{\mathfrak{p}}^\nu$*-holomorphic disks with uniformly bounded area* $\mathrm{Area}(u_\nu)$*. There is a subsequence of* $\{u_\nu\}_\nu$ *that Gromov converges to a* $\underline{\mathfrak{p}}^\infty$*-holomorphic stable* *broken disk $u : C \to \mathfrak{X}_{\mathcal{P}}$ modelled on a tropical graph $\Gamma$. The limit $u$ is unique up to domain reparametrizations and the action of the identity component of the tropical symmetry group $T_{\mathrm{trop},\mathcal{W}}(\Gamma)$ (see (4.35)).*

The second result of the Chapter concerns convergence of a sequence of broken maps with uniformly bounded area. After passing to a subsequence the maps $u_\nu$ have the same type, say $\Gamma$ with tropical structure $\mathcal{T}$. The limit $u$ typically has additional domain components. The tropical structure $\mathcal{T}'$ of the limit is related to $\mathcal{T}$ by a *tropical edge collapse morphism*, which is a morphism of graphs $\mathcal{T}' \xrightarrow{\kappa} \mathcal{T}$ that collapses a subset of edges of $\mathcal{T}'$, and the slopes of uncollapsed edges are the same in $\mathcal{T}$ and $\mathcal{T}'$. See Figure 8.1 for an example.

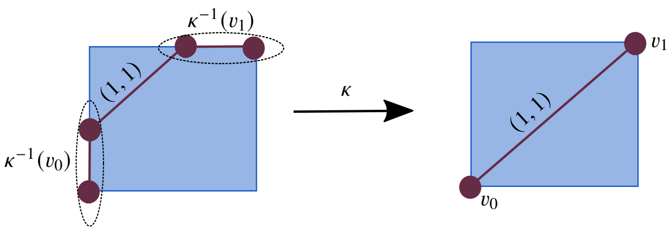



**Figure 8.1.** Tropical edge collapse morphism between tropical graphs.

**Theorem 8.3.** *(Gromov convergence for broken maps) Suppose $(\mathfrak{J}_0, \mathfrak{D})$ is a stabilizing pair for the broken manifold $\mathfrak{X}_{\mathcal{P}}$, and $\underline{\mathfrak{p}}$ is a perturbation datum on $\mathfrak{X}_{\mathcal{P}}$ adapted to $(\mathfrak{J}_0, \mathfrak{D})$. Suppose $u_\nu : C_\nu \to \mathfrak{X}_{\mathcal{P}}$ is a sequence of broken $\underline{\mathfrak{p}}$-holomorphic disks with uniformly bounded area.*

*After passing to a subsequence, the type of the broken disks $u_\nu$ is $\nu$-independent, and is equal to, say, $\Gamma$. The sequence $u_\nu$ Gromov converges to a broken $\underline{\mathfrak{p}}$-holomorphic disk $u : C \to \mathfrak{X}_{\mathcal{P}}$ of type $\Gamma'$ for which there is a tropical edge-collapse morphism $\Gamma' \to \Gamma$. The limit $u$ is unique up to domain reparametrizations and the action of the identity component of the tropical symmetry group $T_{\mathrm{trop},\mathcal{W}}(\Gamma)$ (see (4.35)).*

*If the edge-collapse $\Gamma' \to \Gamma$ is non-trivial (in the sense of Definition 8.31) then* $\dim_{\mathbb{C}}(T_{\mathrm{trop}}(\Gamma')) > \dim_{\mathbb{C}}(T_{\mathrm{trop}}(\Gamma))$.

*Remark* 8.4. In theorems 8.2 and 8.3, the limit broken map is uniquely determined if it is rigid. Indeed, for a rigid tropical graph $\Gamma$ the tropical symmetry group is finite.

## 8.1 Gromov convergence

In this Section, we define the notion of convergence in the compactness results, Theorem 8.2 and Theorem 8.3. Much of the Section is devoted to fixing identifications between domain curves that are close to each other in the compactified moduli space of stable curves. In addition to fixing such identifications in the complements of nodes, we also define a notion of when a sequence of annuli in nearby curves converges to a neighborhood of a node in a nodal curve; this notion is used in (Thin cylinder convergence), which is part of the statement of Gromov convergence. Finally, we define *translation sequences* which are sequences of rescalings of the target; the convergence of maps is modulo these rescalings.

To fix identifications between domain curves that are close to each other in the compactified moduli space of stable curves, we fix a choice of an exponential map in the neighborhood of nodes. Different choices lead to the same notion of convergence. We start by recalling a construction of deformations of a stable nodal curve. We describe the construction for rational stable nodal curves (without boundary). The extensions to treed disks are explained later in Remark 8.9. Let $\Gamma$ be the combinatorial type of a rational stable nodal curve with $d(\bullet)$ markings. The moduli space of curves $\mathcal{M}_\Gamma$ is a submanifold of the compactified moduli space $\mathcal{M}_{d(\bullet)}$ whose tubular neighborhood can be described as follows. For a stable curve $S$ in $\mathcal{M}_\Gamma$, let $\tilde{S}$ be the normalization of $S$ at the nodal points. For any edge $e \in \mathrm{Edge}_{\bullet,-}(\Gamma)$, we fix a map

$$\exp^S_{w_e^\pm} : (U(T_{w_e^\pm}\tilde{S}), 0) \to (U_{w_e^\pm}(\tilde{S}), w_e^\pm), \qquad U(T_{w_e^\pm}\tilde{S}) \subset T_{w_e^\pm}\tilde{S}, \quad U_{w_e^\pm}(\tilde{S}) \subset \tilde{S} \tag{8.2}$$

that biholomorphically maps a neighborhood $U(T_{w_e^\pm}\tilde{S})$ of the origin in the tangent space onto a neighborhood $U_{w_e^\pm}(\tilde{S})$ of the lift of the node $w_e$, satisfies $d\exp^S_{w_e^\pm}(0) = \mathrm{Id}$, and varies smoothly with $S$. The family of maps $\{\exp_{w_e^\pm} : e \in \mathrm{Edge}_{\bullet,-}(\Gamma)\}$ is fixed for the rest of this paper. Whenever we choose complex coordinates

$$z_\pm : (U_{w_e^\pm}(\tilde{S}), w_e^\pm) \to (\mathbb{C}, 0)$$

in neighborhoods of the node we assume that they are compatible with $\exp^S_{w_e^\pm}$. That is, $z_\pm$ is the composition of $\exp_{w^\pm}$ with a linear map on the tangent space. On an open neighborhood

$$U_{\mathcal{M}_\Gamma} \subset \mathcal{M}_{d(\bullet)}$$

of $\mathcal{M}_\Gamma$, there is a projection map

$$\pi_\Gamma : U_{\mathcal{M}_\Gamma} \to \mathcal{M}_\Gamma.$$

such that curves in a fiber $\pi_\Gamma^{-1}(S)$ are obtained by gluing the interior nodes of $S$ as follows. (See, for example, Siebert [79, Proposition 2.4] in the closed case; the open case is similar.)

**Definition 8.5.** Given a nodal sphere $S$ of type $\Gamma$, a *collection of gluing parameters* is a tuple

$$\delta = (\delta_e)_{e \in \mathrm{Edge}_-(\Gamma)} : \delta_e \in T_{w_e^+}\tilde{S} \otimes T_{w_e^-}\tilde{S}.$$

The *curve corresponding to a gluing parameter* $\delta$ is defined by

$$S^\delta := (S \backslash \cup_e U'_{w_e^\pm}(\tilde{S}))/\sim,$$
$$z_+ \sim z_- \Leftrightarrow (\exp^S_{w_e^+})^{-1}(z_+) \otimes (\exp^S_{w_e^-})^{-1}(z_-) = \delta_e. \quad (8.3)$$

Here $U'_{w_e^\pm}(\tilde{S}) \subset \tilde{S}$ is an open neighborhood of $w_e^\pm$ such that the boundary $\partial U'_{w_e^\pm}(\tilde{S})$ is identified to $\partial U_{w_e^\mp}(\tilde{S})$ (with reversed orientation) by the equivalence relation $\sim$. Thus, for any node $w_e$, the relation $\sim$ identifies the pair of annuli

$$U_{w_e^+}(\tilde{S}) \backslash U'_{w_e^+}(\tilde{S}) \xrightarrow{\sim} U_{w_e^-}(\tilde{S}) \backslash U'_{w_e^-}(\tilde{S}). \quad (8.4)$$

The resulting annulus in $S^\nu$ is called $\mathrm{Neck}_e(S^\nu)$, and

$$\mathrm{Neck}(S^\nu) := \cup_e \mathrm{Neck}_e(S^\nu) \subset S^\nu. \quad (8.5)$$

The gluing construction maps a neighborhood of zero in the space of gluing parameters to a neighbourhood of $S$ in the fiber $\pi_\Gamma^{-1}(S)$. Curves in the neighborhood $U_{\mathcal{M}_\Gamma}$ of $\mathcal{M}_\Gamma$ possess a *gluing parameter* function for every smoothed node

$$\delta = (\delta_e)_e : U_{\mathcal{M}_\Gamma} \to \prod_{e \in \mathrm{Edge}_{\bullet,-}(\Gamma)} T_{w_e^+}\tilde{S} \otimes T_{w_e^-}\tilde{S}, \quad S^\delta \mapsto \delta.$$

*Remark* 8.6. Suppose that at a node $w_e$ we choose a framing $\mathrm{fr} : T_{w_e^+}\tilde{S} \otimes T_{w_e^-}\tilde{S} \to \mathbb{C}$ as in (4.19) or complex coordinates

$$z_+ : (\tilde{S}, w_e^+) \to (\mathbb{C}, 0), \quad z_- : (\tilde{S}, w_e^-) \to (\mathbb{C}, 0).$$

Then the gluing parameter $\delta_e$ may be identified with a complex number.

*Remark* 8.7. (Identifications between nearby curves) Consider rational stable curves $S, S'$ with $[S] \in \mathcal{M}_\Gamma$ and $[S'] \in \mathcal{M}_{d(\bullet)}$ in the neighborhood of $S$. The complement of nodes $S^\circ$ can be identified to subsets of the curve $S'$ as follows. First, suppose that $S'$ is obtained by gluing $S$. That is, $S' \in \pi_\Gamma^{-1}(S)$. For a vertex $v \in \mathrm{Vert}(\Gamma)$, let $S'(v) \subset S'$ be the subset corresponding to the component $S_v \subset S$ including the necks $\mathrm{Neck}_e(S')$, $v \in e$. The subsets $S'(v)$ cover $S'$. By the gluing construction, there are natural inclusions

$$i_{S_v,S'} : S'(v) \to S_v \subset S. \quad (8.6)$$

Next, suppose $S'$ is not in $\pi_\Gamma^{-1}(S)$. Consider a neighborhood $U_{\Gamma,S} \subset \mathcal{M}_\Gamma$ of $S$ on which the universal curve $\mathcal{U}_\Gamma$ can be trivialized, so that the variation of the complex structure is given by a map

$$U_{\Gamma,S} \to \mathcal{J}(\mathcal{U}_\Gamma), \quad m \mapsto j(m) \quad (8.7)$$

for which $j(m)$ is $m$-independent in the neighborhood of special points. The trivialization of the universal curve $\mathcal{U}_\Gamma$ gives a diffeomorphism $\phi : \pi_\Gamma(S') \to S$. The maps $i_{S_v,S'}$ are defined as compositions

$$i_{S_v,S'} := \phi \circ i_{\pi_\Gamma(S')_v,S'}.$$

If a sequence $S_\nu$ of curves converges to $S$, then the images $i_{S_v,S_\nu}(S_\nu(v))$ exhaust the complement of nodes $S_v^\circ$ as $\nu \to \infty$.

*Remark* 8.8. (Uniqueness of identifications) The identifications between regions of nearby curves are unique in the following sense. Suppose a sequence $[S_\nu] \in \overline{\mathcal{M}}_{d(\bullet),d(\circ)}$ converges to a curve $[S]$. For any node $e = (v_+, v_-)$ in $S$, the identification in the neck region

$$i_{S_{v_\pm},S^\nu} : \mathrm{Neck}_e(S^\nu) \to S_{v_\pm} \tag{8.8}$$

is uniquely determined by the choice of the exponential map (8.2). On the complement of the neck region, let

$$\phi_\nu, \phi'_\nu : S^\nu \backslash \mathrm{Neck}(S^\nu) \to S$$

be two possible identifications given by trivializations of the universal curve. The maps $\phi_\nu, \phi'_\nu$ have the same image, and the maps $\phi'_\nu \circ \phi_\nu^{-1}$ converge to the identity uniformly in all derivatives.

*Remark* 8.9. (Identifications of nearby treed disks) The identifications between nearby nodal spheres extend to treed disks. Let $C$ be a stable treed disk of type $\Gamma$ with surface part $S$ and treed part $T$.

(a) (Glued treed disk) For a disk node $e \in \mathrm{Edge}^0_\circ(\Gamma)$ of zero length, $\ell(e) = 0$, a *gluing parameter* at the node $w_e \in C$ is an element

$$\delta_e \in (T_{w_e^+}\partial S \otimes T_{w_e^-}\partial S)_{\leq 0} \subset T_{w_e^+} S \otimes T_{w_e^-} S$$

where the orientation on $T_{w_e^+}\partial S \otimes T_{w_e^-}\partial S$ is derived from the orientation on the boundaries of components of $S$. Given gluing parameters

$$\delta := (\delta_e)_{e \in \mathrm{Edge}_{\bullet,-}(\Gamma) \cup \mathrm{Edge}^0_\circ(\Gamma)}$$

for all nodes in the surface component $S$, a glued surface $S^\delta$ is defined exactly as in (8.3). The glued treed curve is $C^\delta := S^\delta \cup T$, that is, the treed part in $C^\delta$ is the same as that of $C$.

(b) (Domain identifications) Let $C$ be a treed curve of type $\Gamma$, and let $\Gamma'$ be a treed disk type obtained by collapsing a subset of interior edges $e \in \mathrm{Edge}_{\bullet,-}(\Gamma)$ and zero length boundary edges $e \in \mathrm{Edge}^0_{\circ,-}(\Gamma)$. Let $C' = S' \cup T'$ be a treed disk of type $\Gamma'$ that is close to $C$. For any surface component $S_v \subset C$ corresponding to a vertex $v \in \mathrm{Vert}(\Gamma)$, there is a subset $S'(v) \subset S'$ and an identification

$$i_{S_v,S'} : S'(v) \to S_v$$

exactly as in (8.6).

Next, we identify punctured neighborhoods of interior nodes to annuli in nearby curves. These identifications are needed for stating the (Thin cylinder convergence) for tropical nodes in the definition of Gromov convergence. We make these identifications only for interior nodes since all tropical nodes are interior nodes.

Informally, we may restate the definition below as follows: A sequence of annuli converging to a node $w$ is obtained by gluing neighborhoods $U_{w^+}, U_{w^-}$ of the nodal lifts $w^+, w^-$. Given holomorphic coordinates in a neighborhood of the nodal $w^+, w^-$, a sequence of centered annuli converging to a node $w$ is given by gluing neighborhoods of $w^+, w^-$ that have the same radius.

**Definition 8.10.** (Annuli converging to a node) Let $C_\nu$ be a sequence of treed disks converging to a limit curve $C$ for which the arguments $\frac{\delta_e(C_\nu)}{|\delta_e(C_\nu)|}$ of the gluing parameters converge for all interior edges $e \in \mathrm{Edge}_{\bullet,-}(\Gamma)$. Let $w \in C$ be an interior node corresponding to the edge $e = (v_+, v_-) \in \mathrm{Edge}_{\bullet,-}(\Gamma)$.

(a) A sequence of *annuli* $A_\nu \subset C_\nu$ *converges to a node* $w$ in $C$ if there are open neighborhoods $U_{w^\pm} \subset S_{v_\pm}$ of the lifts $w^+, w^-$ of $w$ such that

$$A_\nu = i^{-1}_{S_{v_+},C_\nu}(U_{w^+}) \cap i^{-1}_{S_{v_-},C_\nu}(U_{w^-}).$$

We say that the sequence of annuli $A_\nu$ is obtained by *gluing the node* $w$.

(b) (Centered annuli converging to a node) Suppose the node $w \in C$ is equipped with complex coordinates

$$z_\pm : (U_{w^\pm}, w^\pm) \to (\mathbb{C}, 0)$$

on neighborhoods $U_{w^\pm} \subset C_{v_\pm}$ of its lifts $w^+, w^-$. A *sequence of centered annuli converging to the node* $w$ is a sequence of parametrized annuli

$$A_\nu := [-l'_\nu/2, l'_\nu/2] \times \mathbb{R}/2\pi\mathbb{Z} \hookrightarrow C_\nu$$

for which there exists $\epsilon > 0$ such that

$$A_\nu = i^{-1}_{S_{v_+},C_\nu}(\{|z_+| \le \epsilon\}) \cap i^{-1}_{S_{v_-},C_\nu}(\{|z_-| \le \epsilon\}),$$

and the map

$$A_\nu \xhookrightarrow{i_{S_{v_\pm},C_\nu}} C_{v_\pm} \xrightarrow{z_\pm} \mathbb{C} \quad \text{is equal to} \quad (s,t) \mapsto \exp(\mp(s+it) - (l_\nu + i\theta_\nu)/2).$$

Here $e^{-(l_\nu + i\theta_\nu)} \in \mathbb{C}^\times$ is the neck length parameter for the curve $C_\nu$ resulting from the choice of coordinates $z_\pm$.

The following definition describes rescalings of the target spaces required in the statement of convergence of components of the broken map. We recall from Definition 3.63 that a translation is a way of identifying a subset of a neck-stretched manifold $X^\nu$ to a subset of the broken manifold $\mathfrak{X}$. The set of translations is parametrized by $\nu B^\vee$ where $B^\vee$ is the dual complex of the neck-stretching. A translation sequence for a tropical graph consists of a translation sequence corresponding to each of the vertices of the tropical graph that are compatible with the edge slopes of the tropical graph.

**Definition 8.11.** (Translation sequence for a tropical graph) Suppose $\Gamma$ is a pre-tropical graph (as in Definition 4.6). A $\Gamma$-*translation sequence* consists of a collection of sequences

$$\{t_\nu(v) \in \nu B^\vee\}_\nu, \quad v \in \mathrm{Vert}(\Gamma)$$

such that the following conditions hold :

(a) (Polytope) For each vertex $v$

$$t_\nu(v) \in \nu P(v)^\vee \subset \nu B^\vee, \tag{8.9}$$

and for any polytope $P_0 \in \mathcal{P}$, $P_0 \supset P(v)$,

$$d_{B^\vee}(t_\nu(v), \nu P_0^\vee) \to \infty \quad \text{as} \quad \nu \to \infty.$$

(b) (Slope) For any node $e$ between vertices $v_+, v_-$, there is a sequence $l_\nu(e) \to \infty$ such that

$$t_\nu(v_+) - t_\nu(v_+) = \mathcal{T}(e) l_\nu. \tag{8.10}$$

*Remark* 8.12. The existence of a translation sequence $\{t_\nu\}_\nu$ on a pre-tropical graph $\Gamma$ implies that $\Gamma$ is a tropical graph, since, for any $\nu$,

$$\mathrm{Vert}(\Gamma) \ni v \mapsto \tfrac{t_\nu(v)}{\nu} \in \nu P(v)^\circ$$

is a tropical vertex position for $\Gamma$.

**Definition 8.13.** (Gromov convergence, multiple cuts) Let $C_\nu = S_\nu \cup T_\nu$ be a sequence of stable treed disks of type $\Gamma_0$. A sequence of holomorphic maps $u_\nu : (C_\nu, \underline{z}_\nu) \to X^\nu$ *(Gromov) converges* to a broken map $u : C = (S \cup T) \to \mathfrak{X}_\mathcal{P}$ of type $\Gamma$ and framing fr if the following are satisfied.

(a) (Convergence of domains) The sequence of treed disks $C_\nu$ converges to $C$ and for any tropical node $w_e$ of the limit map $u$ the arguments $\frac{\delta_e(C_\nu)}{|\delta_e(C_\nu)|}$ of the gluing parameters converge to a limit. Using the fixed holomorphic exponential map (8.2), let $S_\nu(v) \subset S_\nu$ be the subset corresponding to a vertex $v \in \mathrm{Vert}(\Gamma)$, and let

$$i_{v,\nu} := i_{S_v, C_\nu} : S_\nu(v) \to S_v, \quad S_\nu(v) \subset S_\nu,$$

be embeddings from (8.8) whose images $i_{v,\nu}(S_\nu(v))$ exhaust the complement of nodes $S_v^\circ$ as $\nu \to \infty$.

(b) (Convergence of maps) There is a $\Gamma$-translation sequence $\{t_\nu(v)\}_{v,\nu}$ such that for any irreducible surface component $S_v \subset S$, the sequence of maps

$$S_v^\circ \supset i_{v,\nu}(S_\nu(v)) \xrightarrow{\mathbb{e}^{-t_\nu(v)}(u_\nu \circ i_{v,\nu}^{-1})} \mathfrak{X}_{P(v)}$$

converges in $C^\infty_{\mathrm{loc}}(S_v^\circ)$ to $u_v : S_v^\circ \to \mathfrak{X}_{P(v)}$. The map $\mathbb{e}^{-t_\nu(v)} : X^\nu_{\tilde{P}(v)} \to \mathfrak{X}_{P(v)}$ is defined in (3.59). For each boundary edge $e$ in $\Gamma_0$, the map $u_\nu | T_{e,\nu}$ on the treed segment converges to a (possibly broken) treed segment in $u$.

(c) (Thin cylinder convergence) For a node $w$ in $C$ corresponding to a tropical edge $e = (v_+, v_-) \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, let

$$z_\pm : (U_{w^\pm}, w^\pm) \to (\mathbb{C}, 0)$$

be matching coordinates (see Definition 4.14 following (4.22)) on neighborhoods $U_{w^\pm} \subset S_{v_\pm}$ of the nodal point which respect the framing $\mathrm{fr}_e$. Let

$$A(l_\nu) := [-l_\nu/2, l_\nu/2] \times S^1 \subset S_\nu$$

be a sequence of centered annuli converging to the node $w$, see Definition 8.10. Then the sequence

$$x_\nu := \mathbb{e}^{-\frac{1}{2}(t_\nu(v_+)+t_\nu(v_-))} u_\nu(0,0) \in \mathfrak{X}_{P(e)}$$

converges to a limit $x_0$, and the components $u_{v_\pm}$ of the broken map are asymptotically close to

$$z_\pm \mapsto z_\pm^{\mathcal{T}(e)} x_0$$

in the sense of Remark 4.29.

*Remark* 8.14. (On the uniqueness of limits) The uniqueness of the limit, when it exists, is explained in Theorem 8.2. The Gromov limit does not depend on the various choices made in this section. In particular, the proof of uniqueness only depends on the (Convergence of domains) and (Convergence of maps). The latter item depends on the maps $i_{v,\nu}$ which are identifications between complements of neighborhoods of nodes in nodal curves, and complements of corresponding neck regions in nearby smooth curves. Different choices of these maps converge in the limit as pointed out in Remark 8.8.

We remind the reader that we should not expect to prove that the Gromov limit is independent of perturbation data which includes the choice of Donaldson divisors, since the definition of pseudoholomorphic maps is dependent on these choices. Later, in Chapter 10, we will prove that different choices of perturbation data produce homotopy-equivalent Fukaya algebras.

## 8.2 Horizontal convergence

In this section, we discuss a notion of convergence for a sequence of points in neck-stretched manifolds, called *horizontal convergence*, that is useful in the proof of convergence for breaking maps. It has the feature that any sequence of points $\{x_\nu \in X^\nu\}_\nu$ in the family of neck-stretched manifolds $X^\nu$ has a subsequence that horizontally converges.

**Definition 8.15.** (Horizontal convergence) A sequence of points $x_\nu \in X^\nu$ *horizontally converges* to a point $x \in X_P$ for a polytope $P \in \mathcal{P}$ if

- $x_\nu \in X^\nu_{\tilde{P}}$ for all $\nu$, and the sequence $\pi^\nu_P(x_\nu)$ converges to $x$, where $\pi^\nu_P : X^\nu_{\tilde{P}} \to X_P$ is the projection map $X^\nu_{\tilde{P}} \to X^\nu_P$ from (3.37) composed with the inclusion $X^\nu_P \to X_P$,
- and for any subsequence of $\{x_\nu\}_\nu$, the above condition is not satisfied for any polytope $P_0 \supset P$.

We present some results on horizontal convergence of sequences:

**Lemma 8.16.** *For any sequence $x_\nu \in X^\nu$, there exists a subsequence of $\{x_{\nu_k}\}_k$ that converges horizontally. There is a unique polytope $P \in \mathcal{P}$ for which the subsequence $\{x_{\nu_k}\}_k$ converges horizontally in $X_P$.*

*Proof.* Recall from (7.12) that there is a projection to the dual polytope $\pi_{\nu B^\vee} : X^\nu \to \nu B^\vee$ for all $\nu$. There is a polytope $P \in \mathcal{P}$ such that, after passing to a subsequence, $(x_\nu)_\nu$ satisfies the equation

$$\sup_\nu d(\pi_{\nu B^\vee}(x_\nu), \nu P^\vee) < \infty, \quad d(P_0^\vee, \pi_{\nu B^\vee}(x_\nu)) \to \infty \;\; \forall P_0 \supset P. \tag{8.11}$$

It is then clear that a subsequence of $(x_\nu)_\nu$ and the polytope $P$ satisfy the conclusions of the Lemma. ■

The following Lemma relates horizontal convergence to a property of translation sequences.

**Lemma 8.17.** *Suppose $P \in \mathcal{P}$ is a polytope with* $\operatorname{codim}(P) > 0$, *and suppose $x_\nu \in X^\nu$ is a sequence of points, and $t_\nu \in \nu P^\vee$ is a sequence of translations such that $e^{-t_\nu} x_\nu$ converges in $\mathfrak{X}_P$. Then, the following are equivalent:*

(a) *The sequence $x_\nu \in X^\nu$ converges horizontally in $X_P$.*

(b) *For any $P_0 \supset P$ (that is, $P$ is a proper face of $P_0$), $d(t_\nu, \nu P_0^\vee) \to \infty$.*

*Proof.* Firstly, we observe that the convergence of the translated sequence $e^{-t_\nu} x_\nu$ implies that $d(\pi_{\nu B^\vee}(x_\nu), \nu P^\vee)$ is uniformly bounded, and that $(\pi_P(x_\nu))_\nu$ converges to a limit in $X_P$. The condition in (b) then implies that a $(x_\nu)_\nu$ horizontally converges in $X_P$. The converse is similar and is left to the reader. ■

The next result may be seen as the horizontal convergence version of the Arzela-Ascoli theorem.

**Lemma 8.18.** *Suppose $C$ is a connected curve and $u_\nu : C \to X^\nu$ is a sequence of differentiable maps satisfying $\sup_\nu \|du_\nu\|_{L^\infty} < \infty$. There exists a subsequence of maps $\{u_{\nu_k}\}_k$ and a polytope $P \in \mathcal{P}$ such that*

(a) *for all $z \in C$ the sequence $u_{\nu_k}(z)$ converges horizontally in $X_P$,*

(b) *and there is a sequence of translations $t_\nu \in \nu P^\vee$ such that $\mathbb{e}^{-t_\nu} u_\nu$ converges uniformly in compact sets to a map $u : C \to \mathfrak{X}_P$.*

*For the subsequence $\{u_{\nu_k}\}_k$, the polytope $P$ is unique.*

*Proof.* First, assume that $C$ is compact. The proof of Lemma 8.16 carries over. Indeed, by the uniform bound on $\sup_\nu \|du_\nu\|_{L^\infty}$, the sequence $d(\pi_{\nu B^\vee}(u_\nu(z_0)), P^\vee)$ is uniformly bounded for a fixed point $z_0 \in C$ iff the sequence $d(\pi_{\nu B^\vee}(u_\nu(z)), P^\vee)$ is uniformly bounded for any $z \in C$. The result also holds for non-compact curves since they are exhausted by a sequence of compact curves.

For (b), we view $u_\nu$ as mapping to the $P$-cylinder, since for any translation $t_\nu \in \nu P^\vee \subset \mathfrak{t}_P^\vee$ there is an embedding

$$\mathbb{e}^{-t_\nu} : X^\nu_{\tilde{P}} \to \mathfrak{X}_P,$$

see (3.59). We choose a sequence of translations $t_\nu \in t_P^\vee$ so that $\pi_{\mathfrak{t}_P^\vee} \circ (\mathbb{e}^{-t_\nu} u_\nu)(z_0)$ is $\nu$-independent. The horizontal convergence of $u_\nu$ and the uniform bound on the derivatives of $u_\nu$ imply that for any compact set $K \subset C$, the images $\mathbb{e}^{-t_\nu} u_\nu(K)$ are contained in a compact set of $\mathfrak{X}_P$. The Arzela-Ascoli theorem then implies that the sequence $\mathbb{e}^{-t_\nu} \tilde{u}_\nu$ converges. ■

## 8.3 Breaking annuli

In the next proposition, called the *breaking annulus lemma*, we consider a sequence of maps on annuli with small areas and bounded Hofer energy that converge to a pair of maps on punctured disks, the maps on punctures being asymptotic to trivial cylinders. The main conclusion of the proposition is that there is a trivial cylinder $u_{\text{triv}}$ to which the sequence of annuli is asymptotically close. That is, the distance from $u_{\text{triv}}$ decays exponentially towards the middle of the cylinders. Furthermore, if the limit punctured disks lie in the components $\mathfrak{X}_{P_+}$ and $\mathfrak{X}_{P_-}$ of the broken manifold, where $P_+$, $P_-$ are polygons in the polyhedral decomposition $\mathcal{P}$, then, the trivial cylinder $u_{\text{triv}}$ lies in $\mathfrak{X}_{P_\cap}$, where $P_\cap := P_+ \cap P_-$, or in other words, the slope $\mu$ of $u_{\text{triv}}$ is in $\mathfrak{t}_{P_\cap}$.

In the proof of convergence of maps in neck-stretched manifolds to a broken map in Section 8.4, the breaking annulus lemma is used to show that a sequence of maps on annuli (as in the previous paragraph) converges to a neighborhood of a node in a broken map; in particular, to show that the *matching condition* is satisfied at nodes. The matching condition says that on the punctured neighborhoods of both lifts of the node, the map is asymptotic to the same trivial cylinder. In this setting, both punctured neighborhoods are asymptotic to $u_{\mathrm{triv}}$. The asymptotic closeness of the annuli to $u_{\mathrm{triv}}$ also implies that the convergence of the annuli satisfies the (Thin cylinder convergence) property. (Horizontal matching), which is one of the conclusions of the breaking annuli result, is a sub-condition of the node-matching condition.

The breaking annulus lemma also gives an approximation of the difference in the translation sequences in the convergence of maps on the two ends of the annuli (see (a)), which is used in constructing the tropical structure of the limit map in the proof of Gromov convergence in Section 8.4. This approximation follows from the fact that the maps on the annuli are well-approximated by trivial cylinders of the same length. The breaking annulus lemma is analogous to results in Parker [65, Lemmas 5.11, 5.13].

In the statement of the breaking annulus lemma, the hypothesis on small area is replaced by a bubble-free condition (Definition 8.19). For any $L > 0$, we denote by

$$A(L) := [-\tfrac{L}{2}, \tfrac{L}{2}] \times S^1 \cong \{z \in \mathbb{C} \mid |z| \in [e^{-L/2}, e^{L/2}]\}$$

the cylinder with length $L$ equipped with the product metric.

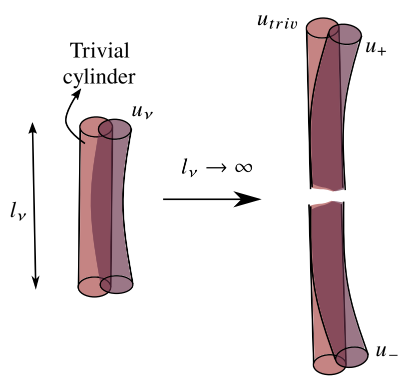



**Figure 8.2.** A sequence of annuli converging to a neighborhood of a node in a broken map.

**Definition 8.19.** (Bubble-free) A sequence of maps

$$u_\nu : A(l_\nu) \to X^\nu, \quad l_\nu \to \infty$$

satisfying $\sup_\nu \|du_\nu\|_{L^\infty} < \infty$ is *bubble-free* if, for any $P \in \mathcal{P}$, there is no non-constant map $v : \mathbb{R} \times S^1 \to X_P$ that is a limit of a subsequence of reparametrized maps

$$\pi_P \circ u_\nu(\cdot - r_\nu) \quad \text{with } r_\nu \in \mathbb{R}, \left| r_\nu \pm \tfrac{l_\nu}{2} \right| \to \infty.$$

**Proposition 8.20.** *(Breaking annulus lemma) Let $\mathfrak{J}_0$ be a cylindrical almost complex structure on $\mathfrak{X}$ for which Hofer energy is monotonic (see Lemma 7.21), and suppose that for any $\nu$ the almost complex structures $J_0^\nu$ is obtained by gluing $\mathfrak{J}_0$ on the necks. There are constants $0 < \rho < 1$, $c > 0$ such that the following hold. For a sequence $l_\nu \to \infty$, let*

$$u_\nu : A(l_\nu) \to X^\nu$$

*be a sequence of $J_0^\nu$-holomorphic maps satisfying*

(1) $\sup_\nu E_{\mathrm{Hof}}(u_\nu) < \infty$, $\sup_{z\in\mathbb{R}\times S^1,\nu} |du_\nu(z)| < \infty$, *where the domain annuli $A(l_\nu) := [-\frac{l_\nu}{2}, \frac{l_\nu}{2}] \times S^1$ are equipped with the product metric.*

(2) *The sequence $\{u_\nu\}_\nu$ is bubble-free.*

(3) *There exist polytopes $P_+, P_-$ such that the sequence of maps $u_\nu(\cdot \pm \frac{l_\nu}{2})$ converges horizontally in $X_{P_\pm}$.*

(4) *There exists a sequence of translations $t_\nu^\pm \in \nu P_\pm^\vee$ such that the sequence $\mathrm{e}^{-t_\nu^\pm} u_\nu(\cdot \pm \frac{l_\nu}{2})$ converges in $C^\infty_{\mathrm{loc}}$ to*

$$u_\pm : \mathbb{R}_\mp \times S^1 \to \mathfrak{X}_{P_\pm},$$

*and the map extends holomorphically over $\mp\infty$, possibly after passing to a finite cover in the orbifold case.*

*Then, there exists $\mu \in \mathfrak{t}_{P_\cap,\mathbb{Z}}$, $P_\cap := P_+ \cap P_-$ for which the following hold after passing to a subsequence of $\{u_\nu\}_\nu$.*

(a) *The sequence $t_\nu^+ - t_\nu^- - \mu l_\nu \in \mathfrak{t}_{P_\cap}^\vee$ is uniformly bounded. Here $\mu \in \mathfrak{t}_{P_\cap}$ is identified to an element in $\mathfrak{t}_{P_\cap}^\vee$ via (3.10).*

(b) *(Horizontal matching) The points $(\pi_{P_+} \circ u_+)(-\infty))$, $(\pi_{P_-} \circ u_-)(+\infty)$ lie in $X_{P_\cap} \subset \overline{X}_{P_\pm}$ and $(\pi_{P_+} \circ u_+)(-\infty)) = (\pi_{P_-} \circ u_-)(+\infty)$.* [1]

(c) *(Asymptotic decay) Let $\xi_\nu$ be a section defined by the relation*

$$u_\nu = \exp_{u_{\nu,\mathrm{triv}}} \xi_\nu, \quad u_{\nu,\mathrm{triv}}(s,t) := e^{\mu(s+it)} u_\nu(0,0).$$

*There exists $l \geq 0$ and a subsequence such that*

$$|\xi_\nu(s,t)| \leq c(e^{\rho(s-\frac{l_\nu}{2})} + e^{\rho(-s-\frac{l_\nu}{2})}), \quad \forall s \in \left[-\frac{l_\nu}{2} + l, \frac{l_\nu}{2} - l\right]. \tag{8.12}$$

**Lemma 8.21.** *Let $l_\nu \to \infty$ be a sequence. Suppose $u_\nu : [0, l_\nu] \times S^1 \to X^\nu$ is a sequence of maps that converges horizontally in $P_0 \in \mathcal{P}$, and for which there is a sequence of translations $t_\nu \in \nu P_0^\vee$ such that $\mathrm{e}^{-t_\nu} u_\nu$ converges uniformly on compact subsets to a map $u : \mathbb{R}_{\geq 0} \times S^1 \to \mathfrak{X}_{P_0}$, whose projection $\pi_{P_0} \circ u$ has a removable singularity at the*

[1] The point $(\pi_{P_\pm} \circ u_\pm)(\mp\infty) \in \overline{X}_{P_\pm}$ is defined, see (4.26).

*infinite end with* $u(\infty) \in X_{P_1} \subset \overline{X}_{P_0}$ *for some* $P_1 \subseteq P_0$*. Then, there is a subsequence* $r_\nu \to \infty$ *such that*

$$u_\nu(r_\nu, \cdot) : S^1 \to X^\nu$$

*converges horizontally in* $P_1$*.*

*Proof.* The removal of singularity for $u$ implies that there exists a point $x_0$ in the $P_1$-cylindrical end of $\mathfrak{X}_{P_0}$ and an element $\mu \in \mathfrak{t}_{P_1,\mathbb{Z}} \backslash (\cup_{P \supset P_1} \mathfrak{t}_P)$ such that

$$d(u(s,t), e^{\mu(-s-it)} x_0) \leq c e^{-|s|} \quad (s,t) \in \mathbb{R}_{\geq 0} \times S^1,$$

see Remarks 4.27 and 4.29. Then, there is a sequence $r_\nu \to \infty$ and translations $t_\nu \in \nu P_0^\vee$ such that

$$\mathrm{d}(u_\nu(r_\nu, \cdot), \mathfrak{e}^{t_\nu} e^{\mu(-r_\nu - it)} x_0) \to 0 \quad \text{as } \nu \to \infty,$$

where $\mathfrak{e}^{t_\nu}$ maps a subset of $\mathfrak{X}_{P_0}$ to $X^\nu_{\tilde{P}_0}$ (see (3.61)). It follows that $u_\nu(r_\nu, \cdot)$ horizontally converges to the constant map $\pi_{P_1}(x_0)$ in $X_{P_1}$. (Note that the values of $t_\nu$ are not relevant to the conclusion of horizontal convergence since $t_\nu \in \nu P_0^\vee \subset i\mathfrak{t}_{P_0}$, and for the purpose of horizontal convergence in $X_{P_1}$, we quotient the map by $T_{P_1,\mathbb{C}}$ which contains $T_{P_0,\mathbb{C}}$ as a subtorus.) ∎

**Lemma 8.22.** *Suppose* $l_\nu \to \infty$ *and* $u_\nu : [0, l_\nu] \times S^1 \to X^\nu_{\tilde{P}} \subset X^\nu$ *is a sequence of maps satisfying hypotheses (1), (2) of Proposition 8.20. Furthermore, assume that* $\pi_P \circ u_\nu$ *converges on compact subsets to a constant map* $u : \mathbb{R}_{\geq 0} \times S^1 \to X_P$ *with value* $x_0 \in X_P$*. Then, there is a compact subset* $K \subset X_P$ *that contains the image of* $(\pi_P \circ u_\nu)$ *for all* $\nu$*.*[2]

*Proof.* The Lemma is proved by applying the annulus Lemma 8.24 on a compact subset of $X_P$. The intervals

$$I_\nu := d_{\nu B^\vee}(\pi_{\nu B^\vee}(u_\nu([0, l_\nu] \times S^1), \nu P^\vee) \subseteq [0, \infty]$$

are connected, and the the conclusion in the Lemma is equivalent to

$$\sup_\nu \operatorname{length}(I_\nu) < \infty. \tag{8.13}$$

Let

$$\tau := d(\pi_{\mathrm{NCone}_{P^\vee} B^\vee} \circ u, P^\vee) = d(\pi_{\mathrm{NCone}_{P^\vee} B^\vee}(x_0), P^\vee), \tag{8.14}$$

so that $\tau \in I_\nu$ for all $\nu$. Let us assume that (8.13) does not hold. Consider any $\kappa > 0$. If (8.13) does not hold, after passing to a subsequence,

$$\tau + \kappa \in I_\nu \quad \forall \nu.$$

[2]For any $\nu$, there is a natural inclusion $X^\nu_P \to X_P$, see Lemma 3.50 and Definition 3.51.

For each $\nu$, let $s_\nu \in [0, l_\nu]$ be the least value for which there is a point $z_\nu = (s_\nu, \theta_\nu)$ in the domain of $u_\nu$ such that

$$d_{\nu B^\vee}(\pi_{\nu B^\vee}(u_\nu(z_\nu)), \nu P^\vee) = \tau + \kappa. \tag{8.15}$$

The minimality of $s_\nu$ implies that

$$d_{\nu B^\vee}((\pi_{\nu B^\vee} \circ u_\nu)([0, s_\nu] \times S^1), \nu P^\vee) \leq \tau + \kappa. \tag{8.16}$$

Because the limit $u$ is at a distance of $\tau$ from $P^\vee$, we have $s_\nu \to \infty$. We also have $l_\nu - s_\nu \to \infty$, otherwise, (8.16) together with the uniform bound on $\|du_\nu\|_{L^\infty}$ implies that the intervals $I_\nu$ are uniformly bounded. We rescale the domain centered at $s_\nu$. By the uniform bound on the derivatives $|du_\nu|$, after passing to a subsequence, the sequence $\pi_P \circ u_\nu(\cdot - s_\nu)$ converges uniformly on compact subsets to a limit $v : \mathbb{R} \times S^1 \to X_P$. By the bubble-free condition, $v$ is a constant map whose value is, say, $x_1 \in X_P$. We now arrive at a contradiction by applying the annulus Lemma 8.24 on compact symplectic manifolds to the sequence $\pi_P \circ u_\nu|[0, s_\nu]$. By (8.16), there is a compact set $K_0 \subset X_P$ that contains the images of $(\pi_P \circ u_\nu)([0, s_\nu] \times S^1)$. By Lemma 7.26, there exists a squashed area form $\omega_\aleph$ on $X_P$ that is a symplectic form on $K_0$. Proposition 8.25 on $(K_0, \omega_\aleph)$ implies that the sequence $(\pi_P \circ u_\nu)|([0, s_\nu] \times S^1)$ [3] converges to a pair of disks $(u, u')$ connected at a node. The second disk $u'$ is the same as $v$ which is a constant $x_1$. Furthermore, $x_1 \neq x_0$ because $d(\pi_{\mathrm{NCone}_{P^\vee} B^\vee}(x_0), P^\vee) = \tau$, and (8.15) implies that $d(\pi_{\mathrm{NCone}_{P^\vee} B^\vee}(x_1), P^\vee) = \tau + \kappa$, resulting in a contradiction. ∎

*Proof of Proposition* 8.20. Let $P \in \mathcal{P}$ be the polytope for which $x_0 := \pi_{P_-}(u_-(\infty)) \in X_P$. Then $P \subseteq P_-$. We will prove all the assertions of the Proposition with $P$ playing the role of $P_\cap$. In one of the final steps, we will show that $P$ is equal to $P_\cap$.

Step 1 : *There is a polytope $P \subset P_+ \cap P_-$ such that $(\pi_{P_\pm} \circ u_+)(\mp\infty) \in X_P$. Furthermore, there exists a constant $L \geq 0$ such that*

$$u_\nu(A(l_\nu - L)) \subset X^\nu_{\tilde{P}}, \tag{8.17}$$

*and a compact set $K \subset X_P$ that contains the image $\pi_P u_\nu(A(l_\nu - L))$ for all $\nu$.* [4]

Let $P \subset P_-$ be the polytope for which

$$(\pi_{P_-} \circ u_-)(\infty) \in X_P. \tag{8.18}$$

The convergence of $\pi_{P_-} \circ u_-(\cdot + \frac{l_\nu}{2})$ to $\pi_{P_-}(u_-)$, together with (8.18) implies that there exists $L_- \geq 0$ and a sequence $s_\nu$ such that

$$u_\nu([-\tfrac{l_\nu}{2} + L_-, s_\nu] \times S^1) \subset X_P. \tag{8.19}$$

[3] The image of $(\pi_P \circ u_\nu)$ lies in $X^\nu_P$, which is naturally embedded in $X_P$ by Definition 3.51 (a).

[4] For any $\nu$, there is a natural inclusion $X^\nu_P \to X_P$, see Lemma 3.50 and Definition 3.51.

We assume that for any $\nu$, $s_\nu$ is the largest value which satisfies (8.19). Also, Lemma 8.21 implies that there is a sequence $r_\nu$ such that

$$u_\nu(r_\nu,\cdot) \text{ horizontally converges in } P, \quad r_\nu + \tfrac{l_\nu}{2} \to \infty. \tag{8.20}$$

Automatically $r_\nu < s_\nu$. Suppose that the sequence $u_\nu(s_\nu,\cdot)$ converges horizontally in $P_1 \in \mathcal{P}$. The maximality of $s_\nu$ implies that $P_1 \supset P$.

*Claim* 8.23. The sequence $\frac{l_\nu}{2} - s_\nu$ is bounded.

*Proof.* Suppose $\frac{l_\nu}{2} - s_\nu \to \infty$. Then, the rescaled sequence $u_\nu(\cdot + s_\nu)$, after passing to a subsequence, converges horizontally in $P_1$ to a limit $v : \mathbb{R} \times S^1 \to X_{P_1}$. The bubble-free condition implies that $v$ is a constant map. Lemma 8.22 then implies that there is a compact set $K \subset X_{P_1}$ containing the images $(\pi_{P_1} \circ u_\nu)([-\frac{l_\nu}{2} + L_-, s_\nu] \times S^1)$. But since, $P \subsetneq P_1$, this contradicts (8.20) leading to the proof of the Claim. ■

Since $u_\nu(s_\nu,\cdot)$ converges horizontally in $P_1 \in \mathcal{P}$, the Claim implies that $P_1 = P_+$. Taking $L := \max\{L_-, \frac{l_\nu}{2} - s_\nu\}$, we obtain

$$u_\nu(A(l_\nu - L)) \subset X^\nu_{\tilde{P}}.$$

The sequence of maps $\pi_P \circ u_\nu$ converges to a limit $v_0 : \mathbb{R} \times S^1 \to X_P$, which by the bubble-free condition is a constant map. Lemma 8.22 applied to both $u_\nu|[0, l_\nu - L]$ and $u_\nu|[0, -(l_\nu - L)]$ shows that there is a compact subset $K \subset X_P$ that contains the images of $\pi_P \circ u_\nu(A(l_\nu - L))$.

Finally, since for any sequence $r'_\nu \in [-\frac{l_\nu}{2} + L, \frac{l_\nu}{2} - L]$, the maps $u_\nu(r'_\nu,\cdot)$ horizontally converge in $X_P$, Lemma 8.21 implies that $(\pi_{P_+} \circ u_+)(-\infty) \in X_P$.

Step 2 : *Determining the edge slope $\mu$.*
In this step we read off the slope $\mu$ of the edge from the topology of the cylinders $u_\nu$ and show that it is equal to the slopes of the limit maps at the nodal point. By Step 1, the images $\pi_P(u_\nu(A(l_\nu - L_0)))$ are contained in a compact subset $K$ of $X_P$. Choose a squashing area form $\omega_\aleph$ on $X_P$ that is a symplectic form on $K$; such a form exists by Lemma 7.26. The $\omega_\aleph$-area of $\pi_P \circ u_\nu$ is uniformly bounded, since

$$\int_{A(l_\nu - L)} (\pi_P \circ u_\nu)^* \omega_\aleph \le E_{\mathrm{Hof}}(\pi_P \circ u_\nu) \le E_{\mathrm{Hof}}(u_\nu),$$

where the last inequality is by Proposition 7.28. By the annulus lemma for compact symplectic manifolds (Proposition 8.25), the annuli $\pi_P \circ u_\nu$ converge to a pair of disks connected by an interior node $x_0 \in X_P$. Therefore, for some $L_1 > 0$, the images $\pi_P(u_\nu(A(l_\nu - L_1)))$ lie in a neighborhood $B_\epsilon(x_0) \subset X_P$. Consider a trivialization

$$B_\epsilon(x_0) \times T_{P,\mathbb{C}} \simeq \pi_P^{-1}(B_\epsilon(x_0)) \subset \mathfrak{X}_P.$$

of $\pi_P : \mathfrak{X}_P \to X_P$. Viewing the target space of $u_\nu$ as the product $B_\epsilon \times T_{P,\mathbb{C}}$, the homotopy class $(u_\nu)_*[A(l_\nu)] \in \pi_1(T_P)$ corresponds to an element $\mu_\nu \in \mathfrak{t}_{P,\mathbb{Z}}$. Let $\mu \in \mathfrak{t}_{P,\mathbb{Z}}$ be defined so that $u_-$ is asymptotically close to a trivial cylinder of slope $\mu$ at $\infty$ in the sense of Remark 4.29. Therefore $\mu_\nu = \mu$ for large $\nu$. Similarly $u_+$ is also asymptotically close to a trivial cylinder of slope $\mu$ at $-\infty$.

Step 3: *The sequence of twisted maps*

$$\overline{u}_\nu : A(l_\nu) \to \mathfrak{X}_P, \quad (s,t) \mapsto e^{-\mu(s+\frac{l_\nu}{2}+it)}(\mathfrak{e}^{-t_\nu^-}u_\nu)^{5}$$

*converges to a pair of disks connected at an interior point.*

Because the derivatives of the maps $(\overline{u}_\nu)_\nu$ are uniformly bounded, we expect the Gromov-Floer limit of the sequence to consist of two disks connected by a path of spheres. Each of the components is obtained by a sequence of rescalings of the cylinder of the form $(s,t) \mapsto (s + s_\nu, t)$ for some sequence $s_\nu \in \mathbb{R}$. The bubble-free condition rules out spheres in the limit, leading to the conclusion of Step 3. The proof is standard, except for the fact that we need to choose an appropriate notion of symplectic area.

The first component in this Gromov-Floer limit of $\overline{u}_\nu$ is a disk $\overline{u}_-$ which is a twisted version of $u_-$. This is seen as follows. The convergence of $\mathfrak{e}^{-t_\nu^-}u_\nu$ to $u_-$ implies that the sequence $\overline{u}_\nu(\cdot - \frac{l_\nu}{2})$ converges to

$$\overline{u}_- := e^{-\mu(s+it)}u_-. \tag{8.21}$$

The image of $\overline{u}_-$ is compact in $\mathfrak{X}_P$ since $u_-$ is asymptotically close to the $\mu$-cylinder $(s,t) \mapsto e^{\mu(s+it)}$. Since $\pi_P \circ u_-$ extends over $\infty$, the same is also true for $\overline{u}_-$. We denote $\overline{x}_0 := \overline{u}_-(\infty) \in \mathfrak{X}_P$.

Next, we finish the description of the Gromov-Floer limit. Choose any taming symplectic form $\omega_{\tilde{P}}$ defined in a neighborhood $U'_{\overline{x}_0} \subset \mathfrak{X}_P$ of $\overline{x}_0$. Let $U_{\overline{x}_0} \Subset U'_{\overline{x}_0}$ be a smaller open neighborhood whose closure is contained in $U'_{\overline{x}_0}$. Let $\kappa > 0$ be any constant. Since $\overline{u}_\nu(\cdot - \frac{l_\nu}{2})$ converges to $\overline{u}_-$, there exists a constant $r_0$, and a sequence $r_\nu$ such that

$$\begin{aligned} &\overline{u}_\nu([\tfrac{-l_\nu}{2} + r_0, r_\nu] \times S^1) \subset \overline{U}_{\overline{x}_0} \\ &\qquad \text{and } \omega_{\tilde{P}}(u_\nu, [\tfrac{-l_\nu}{2} + r_0, r_\nu] \times S^1) \le \omega_{\tilde{P}}(\overline{u}_-, [r_0, \infty) \times S^1) + \kappa. \end{aligned} \tag{8.22}$$

We assume that for each $\nu$, $r_\nu$ is the maximum value satisfying the above condition, and therefore $r_\nu + \frac{l_\nu}{2} \to \infty$. Applying the annulus Lemma for compact manifolds (Proposition 8.25) to the sequence of maps $u_\nu|([\frac{-l_\nu}{2} + r_0, r_\nu] \times S^1)$ with target space $(\overline{U}_{\overline{x}_0}, \omega_{\tilde{P}})$,

[5] By (3.59), there is an inclusion $\mathfrak{e}^{-t_\nu^-} : X_{\tilde{P}}^\nu \to \mathfrak{X}_P$.

we conclude that the sequence of annuli converges to a pair of disks $(\overline{u}_-, \overline{u}_+)$. Here, we note that the first map is the same as the map in (8.21). The second map

$$\overline{u}_+ : (-\infty, L_1) \times S^1 \to \mathfrak{X}_P$$

is the limit of rescaled maps

$$\overline{u}^+_\nu(s,t) := \overline{u}_\nu(\cdot + r_\nu) : [\tfrac{-l_\nu}{2} - r_\nu, \tfrac{l_\nu}{2} - r_\nu] \times S^1 \to \mathfrak{X}_P,$$

where

$$L_1 := \lim_\nu (\tfrac{l_\nu}{2} - r_\nu) \in (0, \infty]. \tag{8.23}$$

Proposition 8.25 also implies that the images of the components $\overline{u}_-$ and $\overline{u}_+$ connect at the nodal point, that is, $\overline{u}_-(\infty) = \overline{u}_+(-\infty)$. Finally, $L_1$ is finite, because otherwise, the sequence $\overline{u}_\nu(\cdot + r_\nu)$ converges to a sphere in $\mathfrak{X}_P$, contradicting the bubble-free assumption.

We have shown that the limit of the twisted maps is a pair of disks $(\overline{u}_-, \overline{u}_+)$. Since $L_1 = \lim_\nu(\frac{l_\nu}{2} - r_\nu)$ is finite, after truncating the domain cylinders by a $\nu$-independent amount, we may replace $r_\nu$ by $\frac{l_\nu}{2}$. The limit $\overline{u}_+$ will be altered by a domain reparametrization and $\overline{u}_+(-\infty) = \overline{u}_-(\infty)$ continues to hold.

Part (a) of the Proposition is now proved as follows. Both sequences of maps

$$\overline{u}^+_\nu(s,t) := e^{-\mu(s+it)}(\mathbb{e}^{-t^-_\nu - \mu l_\nu} u_\nu(s + \tfrac{l_\nu}{2}, t))$$

and $\mathbb{e}^{-t^+_\nu} u_\nu(\cdot + \frac{l_\nu}{2})$ converge on $\mathbb{R}_{\geq 0} \times S^1$, the former by our proof and the latter by the hypothesis. At the point $(s,t) = (0,0)$ the sequences $\overline{u}^+_\nu(s,t)$ and $\mathbb{e}^{-t^+_\nu} u_\nu(\cdot + \frac{l_\nu}{2})$ differ by a translation by $e^{t^+_\nu - t^-_\nu - \mu l_\nu}$. Since both sequences of points converge, we conclude that the limit

$$\delta := \lim_\nu (-\mu l_\nu - t^-_\nu + t^+_\nu)$$

exists (which proves (a)) and that

$$\overline{u}_+(s,t) = e^\delta e^{-\mu(s+it)} u_+(s,t).$$

Step 4 : *Proof of the decay estimate.*
We have shown that after truncating the domain cylinders by a $\nu$-independent amount, the sequence of the twisted maps $\overline{u}_\nu$ converge to a pair of disks $(\overline{u}_-, \overline{u}_+)$, and the images of the maps lie in a compact set $\overline{U}_{\overline{x}_0}$ with a taming symplectic form $\omega_{\tilde{P}}$. The sequence of domains can truncated again by a finite amount to ensure that $\omega_{\tilde{P}}(\overline{u}_\nu) < \hbar$, since the $\omega_{\tilde{P}}$-area on the sequence of cylinders converges to the $\omega_{\tilde{P}}$-area of the pair of disks $(\overline{u}_-, \overline{u}_+)$. We apply the annulus lemma for compact manifolds (Proposition 8.24) to the maps

$$\overline{u}_\nu : A(l_\nu - L) \to (\overline{U}_{\overline{x}_0}, \omega_{\tilde{P}}) \subset \mathfrak{X}_P.$$

The decay estimate for the twisted maps $\overline{u}_\nu$ implies the asymptotic decay estimate (8.12) for $u_\nu$ required by the Proposition.

Step 5 : *Proof of Horizontal Matching and $P = P_+ \cap P_-$.*
By Step 1, the nodal lifts are mapped to on $X_P$, that is,

$$(\pi_{P_\pm} u_\pm)(\mp\infty) \in X_P \subset \overline{X}_{P_\pm}.$$

Therefore at the nodal point, $u_\pm$ intersects all the divisors $\overline{X}_{Q_\pm}$ of $\overline{X}_{P_\pm}$ which contain $\overline{X}_P$, that is, $P \subseteq Q_\pm \subseteq P_\pm$. Consequently,

$$\mu \in \mathfrak{t}_P \backslash \cup_{Q \supset P} \mathfrak{t}_Q.$$

On the other hand $t_\nu^+ - t_\nu^- \in \mathfrak{t}_{P_\cap}$, where $P_\cap := P_+ \cap P_-$. Since $t_\nu^+ - t_\nu^- - \mu l_\nu$ is uniformly bounded, we conclude $\mu \in \mathfrak{t}_{P_\cap}$. Therefore $P_\cap = P$. Horizontal matching now follows from Step 2, where we showed that the sequence $(\pi_P \circ u_\nu)$ converges in $X_P$ to pair of annuli connected at a node. This finishes the proof of the breaking annulus lemma (Proposition 8.20). ∎

The proof of the breaking annulus lemma was based on the following results (Propositions 8.24 and 8.25) on compact symplectic manifolds.

**Proposition 8.24.** *(Annulus lemma on compact manifolds* [55, Lemma 4.7.3]*) Suppose $(X, \omega_X)$ is a compact symplectic manifold with a tamed almost complex structure $J$. There exists constants $0 < \rho < 1$, $\hbar > 0$, $c > 0$ such that the following holds for any $J$-holomorphic map $u : A(\ell) \to X$ with $E(u) \leq \hbar$. For $x = u(0,0)$, there is a map*

$$\xi : A(\ell - 1) \to T_x X \ \textit{ such that } \ u = \exp_x \xi$$

*on $A(\ell - 1)$ and*

$$|\xi(s,t)| \leq c(e^{\rho(s-\ell)} + e^{\rho(-s-\ell)}), \quad \forall s \in [-\ell + 1, \ell - 1]. \tag{8.24}$$

*The constants $\rho$, $\hbar$, $c$ depend continuously on $J$ with respect to the $C^2$-topology.*

**Proposition 8.25.** *(Convergence of long cylinders) Suppose $(X, \omega_X)$ is a compact symplectic manifold with a tamed almost complex structure $J$. Let $u_\nu : A(l_\nu) \to X$ be a sequence of holomorphic cylinders with a uniform bound on $\|du_\nu\|_{L^\infty}$ and the $\omega$-areas of $u_\nu$. Furthermore, the sequence $\{u_\nu\}_\nu$ satisfies the bubble-free condition: For any sequence $r_\nu$ with $|r_\nu \pm l_\nu/2| \to \infty$, if a subsequence of $\{u_\nu(\cdot - r_\nu)\}_\nu$ converges to a limit $v : \mathbb{R} \times S^1 \to X$, then $v$ is a constant map. After passing to a subsequence, $u_\nu(\cdot \pm \frac{l_\nu}{2})$ converges in $C^\infty_{\text{loc}}$ to*

$$u_\pm : \mathbb{R}_\mp \times S^1 \to X,$$

*the map $u_\pm$ extends holomorphically over $\mp\infty$, and $u_-(\infty) = u_+(-\infty)$.*

*Proof.* This Proposition is proved as part of the "bubbles connect" result in [55, Proposition 4.7.1]. ∎

We give an annulus lemma for sequences of holomorphic strips in $X^\nu$ whose boundaries lie on the Lagrangian submanifold $L$. The result is simpler than the breaking annulus lemma since the strips converge to a nodal disk in the complement of the relative divisors, and not a tropical node. The result is stated for maps whose domains are strips, defined as

$$A_\circ(\ell) := [-\tfrac{\ell}{2}, \tfrac{\ell}{2}] \times [0, 1]$$

for any $\ell > 0$. The result is a boundary version of the "bubbles connect" result in McDuff-Salamon [55, Proposition 4.7.1]. However, since the manifold $X_{P_0}$ has cylindrical ends, we need to use a Hofer energy bound.

**Proposition 8.26.** *(Annulus lemma with boundary) Suppose that the almost complex structures* $\mathfrak{J}_0 \in \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X})$ *and* $J_0^\nu \in \mathcal{J}^{\mathrm{cyl}}(X^\nu)$ *satisfy the conditions in the statement of Proposition 8.20. There are constants* $0 < \rho < 1$, $c > 0$ *such that the following hold. For a sequence* $l_\nu \to \infty$*, let*

$$u_\nu : A_\circ(l_\nu) \to X^\nu, \quad u_\nu(\partial A_\nu) \subset L$$

*be a sequence of* $J_0^\nu$*-holomorphic strips satisfying the following.*

(1) $\sup_\nu E_{\mathrm{Hof}}(u_\nu) < \infty$, $\sup_{z \in \mathbb{R}\times[0,1],\nu} |\mathrm{d}u_\nu(z)| < \infty$.

(2) *The sequence* $\{u_\nu\}_\nu$ *satisfies a disk bubble-free condition: For any sequence* $r_\nu$ *with* $|r_\nu \pm l_\nu/2| \to \infty$*, if a subsequence of* $\{u_\nu(\cdot - r_\nu)\}_\nu$ *converges to a limit* $v : \mathbb{R} \times [0,1] \to X$*, then* $v$ *is a constant map.*

(3) *The sequence of maps* $u_\nu(\cdot \pm \frac{l_\nu}{2})$ *converges in* $C^\infty_{\mathrm{loc}}$ *to a limit*

$$u_\pm : \mathbb{R}_\mp \times [0,1] \to X_{P_0},$$

*and the map extends holomorphically over* $\mp\infty$.

*Then,* $u_+(-\infty) = u_-(+\infty) \in L$.

*Proof.* There is a compact subset $K \subset X_{P_0}$ that contains the image all the strips $\{u_\nu\}_\nu$. Indeed, $L \subset X_{P_0}$ is compact and the images of the strips are contained in a $\sup_\nu \|du_\nu\|_{L^\infty}$-radius of $L$. By Lemma 7.26, there is a squashing area form $\omega_\aleph$ that is a symplectic form on $K$. The $\omega_\aleph$-area of the strips $u_\nu$ are uniformly bounded. Therefore, the target space may be seen as a compact symplectic manifold, and the proof is analogous to the "bubbles connect" result for long cylinders in [55, Proposition 4.7.1]. ∎

## 8.4 Proof of convergence for breaking maps

The arguments used in the proof of convergence of breaking maps are similar to those used in the proof of Gromov compactness for stable maps. The new features include a more refined analysis of sequences of long cylinders that converge to nodes of the limit broken maps. The decay estimate on such cylinders coming from the breaking annulus lemma (Proposition 8.20) allows us to prove (Thin cylinder convergence), and also prove the existence of a tropical graph, that is, the graph underlying the limit map is realizable (see Definition 4.6) in the dual complex. The stabilizing divisor plays a role in the proof, since, the choice of perturbation data implies that any non-constant limit component has a stable domain.

*Proof of Theorem* 8.2. Step 1 : *Domain components for the limit map.*
The sequence of domains $(C_\nu, z_\nu)$ converges to a stable treed nodal curve $(C, z)$ modelled on a tree $\Gamma$. The perturbation maps $\underline{\mathfrak{p}}_\nu$ converge to a perturbation datum $\underline{\mathfrak{p}}_\infty = (J_\infty, F_\infty)$ defined on $C$. In the next few steps, we will show that there are no additional domain components in the limit map.

In the rest of the proof, we focus on the convergence of maps on irreducible surface components. The proof of convergence of gradient trajectories on treed components does not involve any technical difficulties arising from multiple cutting, and are left to the reader.

Step 2 : *Boundedness of derivatives.*
In this step, we show a bound on the derivatives of the sequence of maps by ruling out bubble trees with unstable domains in the limit. In particular, we will show that for any $v \in \mathrm{Vert}(\Gamma)$, the derivative of

$$u_{v,\nu} := u_\nu \circ i_{v,\nu}^{-1} : C_v^\circ \to X^\nu$$

is uniformly bounded for all $\nu$. The norm on the derivative is with respect to the cylindrical metric on $X^\nu$. On the domain we use a metric on $C^\circ$ that is cylindrical (or strip-like) in the punctured neighborhood of nodal points and on the neck regions in $C_\nu$.

Assume by way of contradiction that the derivatives are not uniformly bounded. We will construct a sequence of rescaled maps which converges to a constant map with positive derivative, which cannot exist. After passing to a subsequence, there exists a sequence of points $z_\nu \in C_v^\circ$ and a point $z_\infty \in C_v$ for some $v \in \mathrm{Vert}(\Gamma)$ such that

$$z_\nu \to z_\infty, \quad |\mathrm{d}u_{v,\nu}(z_\nu)| \to \infty.$$

We first carry out the proof assuming that $z_\infty$ is not a nodal point and does not lie on the boundary $\partial C_v$, and thus $z_\infty \in C_v^\circ$. We apply Hofer's lemma 8.27 to the function $|\mathrm{d}u_{v,\nu}|$, $x = z_\nu$ and the constant $\delta = |\mathrm{d}u_\nu(z_\nu)|^{-1/2}$.

**Lemma 8.27.** *(Hofer's Lemma,* [55, Lemma 4.6.4]*) Suppose* $(X, d)$ *is a metric space,* $f : X \to \mathbb{R}_{\geq 0}$ *is a continuous function, and* $x \in X$*,* $\delta > 0$ *are such that the ball* $B_{2\delta}(x)$ *is complete. Then there exists a positive constant* $\epsilon \leq \delta$ *and a point* $\zeta \in B_{2\epsilon}(x)$ *such that*

$$\sup_{z \in B_\epsilon} f(z) \leq 2f(\zeta), \quad \epsilon f(\zeta) \leq \delta f(x).$$

We obtain another sequence $\zeta_\nu \in C_v^\circ$ converging to $z_\infty$, and a sequence of constants $\epsilon_\nu \to 0$ such that

$$c_\nu := |\mathrm{d}u_{v,\nu}(\zeta_\nu)| \to \infty, \quad \sup_{z \in B_{\epsilon_\nu}} |\mathrm{d}u_\nu(z)| \leq 2c_\nu, \quad c_\nu \epsilon_\nu \to \infty.$$

The rescaled maps are

$$\tilde{u}_\nu := u_{v,\nu}((\cdot - \zeta_\nu)/c_\nu) : B_{\epsilon_\nu c_\nu} \to X^\nu,$$

and $|d\tilde{u}_\nu| \leq 2$, $|d\tilde{u}_\nu(0)| = 1$.

Each of the rescaled maps converges to a limiting map with domain the affine line as follows. By Lemma 8.18 there is a polytope $P \in \mathcal{P}$ and a sequence of translations $t_\nu \in t_P^\vee$ so that after passing to a subsequence, the maps $\mathrm{e}^{-t_\nu}\tilde{u}_\nu$ converge in $C^\infty_{\mathrm{loc}}$ to a non-constant $J_{z_\infty}$-holomorphic limit $\tilde{u} : \mathbb{C} \to \mathfrak{X}_P$.

The limit map $\tilde{u} : \mathbb{C} \to \mathfrak{X}_P$ in the previous paragraph is asymptotic to a trivial cylinder at infinity: Hofer energy $E_{\mathrm{Hof}}(u_\nu, C_\nu)$ is equal to area (Remark 7.19) and therefore is uniformly bounded for all $\nu$. By monotonicity of Hofer energy (Lemma 7.21), the Hofer energy on a ball $E_{\mathrm{Hof}}(\tilde{u}_\nu, B_{\epsilon_\nu c_\nu})$ is uniformly bounded. By Proposition 7.27, $E_{\mathrm{Hof}}$ is preserved in the neck-stretching limit (in a certain sense), which implies that the quantity $E^*_{P,\mathrm{Hof}}(\tilde{u}, \mathbb{C})$ is finite. By Proposition 7.1, there exists $Q \subseteq P$ and $\mu \in \mathfrak{t}_Q$ such that at $\infty$, $\tilde{u}$ is asymptotically close to $z \mapsto z^\mu x_0$ for some $x_0 \in \mathfrak{X}_Q$.

Finally, we arrive at a contradiction by showing that the limit map is constant. The domain reparametrizations $\phi_{v,\nu}$ were derived from the stable map compactification. It follows from the definition of the compactification there is at most a single marked point $z_{i,\nu}$ that is contained in each of the regions $B_{\epsilon_\nu}(\zeta_\nu)$. Therefore, the projection $\pi_P \circ \tilde{u} : \mathbb{C} \to X_P$ either lies in the stabilizing divisor $D_P$, or it has at most one intersection with the divisor $D_P$. Since $(\mathfrak{J}_0, \mathfrak{D})$ is a stabilizing pair for $\mathfrak{X}$, and the perturbation $\underline{\mathfrak{p}}$ is adapted to $(\mathfrak{J}_0, \mathfrak{D})$. Neither possibility can happen if $\pi_P \circ \tilde{u}$ is non-constant. Therefore, $\pi_P \circ \tilde{u}$ is constant, and so the image of $\tilde{u}$ lies in a single toric fiber $V_{P^\vee}$. The image $\tilde{u}(\mathbb{C})$ does not intersect torus-invariant divisors of $V_{P^\vee}$, and therefore $\tilde{u}$ is a constant map.

We now consider the case that the sequence of points with increasing derivatives converges to an interior nodal point $w_e$. The sequence $z_\nu$ lies on the neck region

$$A_\nu := \left[\tfrac{-l_\nu}{2}, \tfrac{l_\nu}{2}\right] \times S^1 \subset C_\nu$$

obtained by gluing the node $w_e$, and $l_\nu \to \infty$ as $\nu \to \infty$. Since $z_\nu$ converges to the node $w_e$, there is a constant $\epsilon > 0$ such that the ball $B_\epsilon(z_\nu)$ is contained in $A_\nu$ for all $\nu$. The proof in the preceding few paragraphs using Hofer's Lemma and a rescaling can now be applied to the maps $u_\nu | B_\epsilon(z_\nu)$, because there are no marked points in $A_\nu$. We conclude that $|\mathrm{d}u_{v,\nu}|$ is uniformly bounded for all $\nu$.

Finally, if the sequence of points with increasing derivatives converges to a point on the boundary of the domain, then the same steps can be repeated to show the existence of a non-constant disk bubble with unstable domain, leading to a contradiction. The removal of singularities in this case is proved by Proposition 7.34.

STEP 3 : *Determining stable components of the limit map.*
In Step 2 we showed that the derivatives of the maps $u_{v,\nu}$ are uniformly bounded, where $v$ is any vertex of $\Gamma$. By Lemma 8.18, there is a polytope $P(v)$ and a sequence of translations $t_\nu(v) \in \nu P(v)^\vee \subset \mathfrak{t}^\vee_{P(v)}$ such that after passing to a subsequence, the maps $\mathbb{e}^{-t_\nu(v)} u_{v,\nu}$ converge in $C^\infty_{\mathrm{loc}}$ to a limit $u_v : C^\circ_v \to \mathfrak{X}_{P(v)}$. By arguments as in Step 2, the Hofer energy $E_{\mathrm{Hof}}(u_v, C^\circ_v)$ is bounded. By removal of singularities Proposition 7.1,

- $u_v$ has a removable singularity at the node $w$, or
- in a punctured neighborhood of the node $w$, $u_v$ is asymptotic to a vertical cylinder, and this latter case can only happen if $w$ is an interior point of the domain.

We obtain a $J_\infty$-holomorphic map $u_v : C^\circ_v \to \mathfrak{X}_{P(v)}$ that is adapted to the stabilizing divisor $\mathfrak{D}_{P(v)} \subset \mathfrak{D}$.

For later use, we remark that the convergence continues to hold if the sequence $t_\nu(v)$ is replaced by a sequence $t_\nu(v)'$ for which $\sup_\nu |t_\nu(v)' - t_\nu(v)| < \infty$. In that case, after passing to a subsequence, the limit $u_v$ would be replaced by $e^{-t} u_v$, where $t := \lim_\nu (t_\nu(v)' - t_\nu(v))$.

STEP 4 : *Determining the tropical structure and constructing translation sequences.*
We have so far determined the domain treed disk of the limit, and the limit map $u_v$ corresponding to each vertex. The construction above determines the polytope $P(v) \in \mathcal{P}$ for each component $v \in \mathrm{Vert}(\Gamma)$ of the limit. Furthermore, for every $v \in \mathrm{Vert}(\Gamma)$ the map $u_v$ is the limit of $\mathbb{e}^{-t_\nu(v)} u_\nu$ where $t_\nu(v) \in \nu P^\vee$ is a sequence of translations.

First, we observe that all boundary edges are internal edges, and are collapsed by the tropicalization morphism of (4.13). Vertices $v \in \mathrm{Vert}_\circ(\Gamma)$ corresponding to disk components map to a single vertex $v_\circ$ in the tropical graph and the translation sequence $t_\nu(v_\circ)$ is the point $P^\vee_0$.

Edge slopes of the tropical graph are determined using the breaking annulus lemma. For any node $w_e$ of $C$ corresponding to an interior node $e = (v_+, v_-) \in \mathrm{Edge}_{\bullet,-}(\Gamma)$, choose a complex coordinate

$$z^\pm_e : (U_\pm, w_e) \to (\mathbb{C}, 0) \tag{8.25}$$

in a neighborhood $U_\pm \subset C_{v_\pm}$ of the lift $w_e^\pm$ of the node, and let $A_{e,\nu} := A(l_\nu(e)) \subset C_\nu$ be a sequence of centered annuli converging to the neighborhoods $U_+, U_-$ of the nodal lifts $w_e^+$, $w_e^-$ (as in Definition 8.10). Since there are no markings $z_{i,\nu}$ on the annuli $A_{e,\nu}$, the maps $u_\nu|A_{e,\nu}$ are bubble-free in the sense of Definition 8.19. Indeed, if for some $P \in \mathcal{P}$ and some reparametrization of the domain, the sequence of projections $\pi_P \circ u_\nu|A_{e,\nu}$ converges to a map $v : \mathbb{R} \times S^1 \to X_P$, then $v$ extends to a map to $\overline{X}_P$ defined on an orbifold completion of $\mathbb{R} \times S^1$; such a limit map $v$ is constant since it either lies on the stabilizing divisor $D_P$ or does not intersect $D_P$ at all. Therefore the breaking annulus lemma (Proposition 8.20) is applicable on the sequence of annuli $u_\nu|A_{e,\nu}$, and we obtain a slope $\mathcal{T}(e) \in \mathfrak{t}_{P(e),\mathbb{Z}}$ corresponding to the edge $e$. The breaking annulus lemma implies that

$$\sup_\nu \{|t_\nu(v_+) - t_\nu(v_-) - \mathcal{T}(e) l_\nu(e)|\} < \infty, \tag{8.26}$$

which is the (Approximate Slope) condition in Definition 8.28.

The tropicalization of $\Gamma$ is determined as follows. An edge $e \in \mathrm{Edge}_{\bullet,-}(\Gamma)$ is an internal (non-tropical) edge exactly if $\mathcal{T}(e)$. In that case, the sequence is $|t_\nu(v_+) - t_\nu(v_-)|$ is uniformly bounded for all $\nu$. Therefore, we can replace $t_\nu(v_-)$ by $t_\nu(v_+)$ (or the other way around), and the convergence of Step 2 still holds. By performing such replacements for all internal edges, we may assume $t_\nu(v_+) = t_\nu(v_-)$ for all $e = (v_+, v_-) \in \mathrm{Edge}_{\mathrm{int}}(\Gamma)$. Therefore, $t_\nu$ descends to the tropical graph $\Gamma_{\mathrm{tr}}$ and is an approximate $\Gamma_{\mathrm{tr}}$-translation sequence.

Next, we obtain translation sequences and determine tropical vertex positions. Since $\{t_\nu(v) : v \in \mathrm{Vert}(\Gamma_{\mathrm{tr}})\}$ is an approximate $\Gamma_{\mathrm{tr}}$-translation sequence, by Lemma 8.30, there is a $\Gamma_{\mathrm{tr}}$-translation sequence $(t_\nu(v)')_{v,\nu}$ such that

$$\sup_{\nu,v} |t_\nu(v)' - t_\nu(v)| < \infty. \tag{8.27}$$

The convergence of maps in Step 2 continues to hold if $t_\nu$ is replaced by $t_\nu(v)'$. Any element of the translation sequence gives a tropical vertex position map. Indeed, since

$$t'_\nu = (t'_\nu(v))_{v \in \mathrm{Vert}(\Gamma_{\mathrm{tr}})}, \quad t'_\nu(v) \in \nu P(v)^\vee$$

satisfies the (Slope) condition (8.10),

$$\mathrm{Vert}(\Gamma_{\mathrm{tr}}) \ni v \mapsto \frac{t'_\nu(v)}{\nu}$$

is a tropical vertex position map.

Step 5: *Finishing the proof of convergence.*
To finish the proof of convergence it remains to show that the collection of limit maps satisfy matching conditions at nodes, and that (Thin cylinder convergence) is satisfied at tropical nodes. We first consider disk nodes. For a disk node $w_e \in C$ corresponding

to an edge $e = (v_+, v_-) \in \mathrm{Edge}_\circ(\Gamma)$ with length $\ell(e) = 0$, let $A_{e,\nu} \subset C_\nu$ be a sequence of strips converging to the node $w_e$. The maps $u_\nu | A_{e,\nu}$ are disk bubble-free since there are no markings $z_{i,\nu}$ on $A_{e,\nu}$; the reasoning is identical to the one used in page 242 to show that on the annuli converging to an interior node, maps are bubble-free. Therefore, the boundary version of the annulus lemma (Proposition 8.26) is applicable on $u_\nu | C_\nu$ and we conclude that node matching holds, that is, $u_{v_+}(w_e^+) = u_{v_-}(w_e^-) \in L$.

Next, we determine matching coordinates for tropical nodes and prove node matching using the breaking annulus Lemma. Matching coordinates at a tropical node $w_e$, $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ are obtained by applying a correction to the holomorphic coordinate chosen in (8.25). Multiplying a constant to the coordinates $z_e^+$, $z_e^-$ has the effect of adding a constant to the sequence of neck length parameters $l_\nu(e) + i\theta_\nu(e)$. We recall that as a consequence of (8.26) and (8.27) the $\Gamma$-translation sequence $t'_\nu$ satisfies

$$\sup_\nu \{|t'_\nu(v_+) - t'_\nu(v_-) - \mathcal{T}(e) l_\nu(e)|\} < \infty.$$

Since $t'_\nu$ satisfies the (Slope) condition (8.10), there is a sequence $l'_\nu(e) \to \infty$ for every $e \in \mathrm{Edge}_{\bullet,-}(\Gamma)$ such that

$$t'_\nu(v_+) - t'_\nu(v_-) = \mathcal{T}(e) l'_\nu(e), \quad \text{and} \quad \sup_\nu |l_\nu(e) - l'_\nu(e)| < \infty.$$

Therefore, we can adjust the coordinates $z_e^+$, $z_e^-$ by scalar multiplication so that a subsequence of neck length parameters $l_\nu(e) + i\theta_\nu(e)$ satisfies

$$\lim_\nu (t'_\nu(v_+) - t'_\nu(v_-) - \mathcal{T}(e) l_\nu(e)) = 0, \quad \lim_\nu \theta_\nu(e) = 0. \tag{8.28}$$

We will show that $(z_e^+, z_e^-)$ are matching coordinates at the node $w_e$ for the broken map $u$. For interior non-tropical nodes, we leave $(z_e^+, z_e^-)$ from (8.25) unchanged, and the following discussion is valid. To simplify calculations, we use logarithmic coordinates in the neighborhood of the node

$$C_\pm \supset U_\pm \backslash \{w_e^\pm\} \xrightarrow{\pm \ln z_e^\pm} \mathbb{R}_\mp \times S^1.$$

The annulus $A(l_\nu(e))$ is then identified to the limit curve by translations

$$A(l_\nu(e)) \to U_\pm, \quad (s,t) \mapsto s + it \mp \tfrac{1}{2}(l_\nu + i\theta_\nu).$$

By Step 2,

$$\mathrm{e}^{-t'_\nu(v_\pm)} u_\nu(s + it \pm \tfrac{1}{2}(l_\nu + i\theta_\nu)) \to u_{v_\pm} \quad \text{in } C^\infty_{\mathrm{loc}}(U_\pm \backslash \{w_e^\pm\}). \tag{8.29}$$

We apply the breaking annulus lemma on the maps $u_\nu$ on the annuli $A(l_\nu(e))$. The resulting decay estimate together with (8.29), (8.28) implies the convergence of the sequence

$$\mathrm{e}^{-\frac{1}{2}(t'_\nu(v_+) + t'_\nu(v_-))} u_\nu(0,0) \to x_0 \quad \text{in } \mathfrak{X}_{P(e)}.$$

It follows that $u_{v_\pm}$ are asymptotically close to the cylinder $(s,t) \mapsto e^{\mathcal{T}(e)(s+it)}x_0$ in the sense of 4.29. We conclude that $z_e^+, z_e^-$ are matching coordinates at $w_e$ and (Thin cylinder convergence) is satisfied. Note that we have shown node matching for all interior edges, both tropical and internal, in a unified way. In case of an internal node $w_e$, $\mathcal{T}(e) = 0$, and we have shown that $u_{v_\pm}(\mp\infty) = x_0 \in \mathfrak{X}_{P(v_\pm)}$.

Step 6 : *Uniqueness of the limit.*
The limit of the domain curves is unique up to reparametrization, because the limit is a stable curve. The identifications between subsets of $C^\nu$ to the limit curve are unique in the following sense (see Remark 8.8) : The neck regions in $C^\nu$ are parametrized in a unique way, and the difference between any two choices of identifications of the complement of the neck in $C^\nu$ to $C$ converge uniformly to identity as $\nu \to \infty$. Let $\Gamma$ be the combinatorial type of the limit curve $C$. For every vertex $v \in \mathrm{Vert}(\Gamma)$, the polytope $P(v)$ in the limit map is uniquely determined as follows: Suppose there is a translation sequence $\{t'_\nu(v) \in P'(v)^\vee : v \in \mathrm{Vert}(\Gamma)\}_\nu$ for which Gromov convergence holds. Then the property

$$d_{B^\vee}(t_\nu(v), P_0^\vee) \to \infty \quad \forall P_0 \in \mathcal{P}, P_0 \supset P'(v)$$

of a translation sequence implies, by Lemma 8.17, that the maps $u_{\nu,v}$ horizontally converge in $P'(v)$. However, $P'(v) = P(v)$ because the polytope of horizontal convergence is unique by Lemma 8.18.

Translation sequences are well-determined up to uniformly bounded perturbations as follows : Suppose $t_\nu, t'_\nu$ are two distinct translation sequences, such that the sequence $\mathfrak{e}^{-t_\nu}u_\nu$ resp. $\mathfrak{e}^{-t'_\nu}u_\nu$ converges to a broken map $u$ resp. $u'$ . Then for all vertices $v \in \mathrm{Vert}(\Gamma)$, there is a uniform bound

$$\sup_\nu |t_\nu(v) - t'_\nu(v)| < \infty,$$

because both the sequences $\mathfrak{e}^{-t_\nu(v)}u_\nu, \mathfrak{e}^{-t'_\nu(v)}u_\nu$ converge pointwise in $C_v^\circ$. After passing to a subsequence, we may assume that there exists a limit

$$t(v) := \lim_\nu t_\nu(v) - t'_\nu(v).$$

Then, for each vertex $v$, $u_v = e^{t(v)}u'_v$. Since $u_v$ and $u'_v$ satisfy matching conditions at nodes we conclude that $t$ is an element of $T_{\mathrm{trop},\mathcal{W}}(\Gamma)$, which is the identity component of $T_{\mathrm{trop}}(\Gamma)$. We have thus shown that the limit is unique up to the action of $T_{\mathrm{trop},\mathcal{W}}(\Gamma)$.

■

**Definition 8.28.** (Approximate translation sequence) Suppose $\Gamma$ is a pre-tropical graph (as in Definition 4.6). An *approximate* $\Gamma$*-translation sequence* consists of sequences $\{t_\nu(v) \in \nu P(v)^\vee\}_\nu$ for each $v \in \mathrm{Vert}(\Gamma)$ such that

- (Approximate Slope) For any edge $e = (v_+, v_-) \in \mathrm{Edge}_-(\Gamma)$, there exists a sequence $l_\nu(e) \to \infty$ such that

$$\sup_\nu (t_\nu(v_+) - t_\nu(v_+) - \mathcal{T}(e) l_\nu) < \infty.$$

The differences appearing in the (Approximate Slope) condition will be referred to later using the following notation:

**Definition 8.29.** On a tropical graph $\Gamma$ define the *discrepancy* function on any edge $e = (v_+, v_-) \in \mathrm{Edge}_{\bullet,-}(\Gamma)$ as

$$\mathrm{Diff}_e : \oplus_{\mathrm{Vert}(\Gamma)} \mathfrak{t}^\vee_{P(v)} \to \mathfrak{t}^\vee_{P(e)} / \langle \mathcal{T}(e) \rangle, \quad (t_v)_{v \in \mathrm{Vert}(\Gamma)} \mapsto (t_{v_+} - t_{v_-}) \mod \mathcal{T}(e).$$

**Lemma 8.30.** *(From approximate to exact translation sequences) Suppose* $\Gamma$ *is a pre-tropical graph and* $t_\nu$ *is an approximate* $\Gamma$*-translation sequence. Then, after passing to a subsequence, there is a* $\Gamma$*-translation sequence* $\bar{t}_\nu$ *such that* $|\bar{t}_\nu(v) - t_\nu(v)|$ *is uniformly bounded for all* $\nu$, $v \in \mathrm{Vert}(\Gamma)$. *Consequently,* $\Gamma$ *is a tropical graph.*

*Proof.* The (Approximate Slope) condition says that the sequences of discrepancies $(\mathrm{Diff}_e(t_\nu))_\nu$ are uniformly bounded. Via uniformly bounded adjustments to $t_\nu$, we aim to make this quantity vanish for all edges.

We give an algorithm that transforms $t_\nu$ into a bounded sequence $t^k_\nu \in \oplus_{\mathrm{Vert}(\Gamma)} \mathfrak{t}^\vee_{P(v)}$, and will prove later that $t_\nu - t^k_\nu$ is an exact translation sequence. The algorithm is as follows:

Step 1: *Relativisation*. In this step, we replace $t_\nu$ by

$$t^0_\nu(v) := t_\nu(v) - \nu \lim_\nu (t_\nu(v)/\nu) \in \mathfrak{t}^\vee_{P(v)}.$$

The limit in the right-hand side exists after passing to a subsequence because the original translation sequences $t_\nu$ lie in $\nu B^\vee$ and $B^\vee$ is compact. For any $v \in \mathrm{Vert}(\Gamma)$, the discrepancies across edges are preserved:

$$\mathrm{Diff}_e(t_\nu) = \mathrm{Diff}_e(t^0_\nu). \tag{8.30}$$

Step 2: *Subtracting fastest growing sequences*. By a sequence of further transformations, we will change $t^0_\nu$ to a bounded sequence $t^k_\nu \in \mathfrak{t}^\vee$. At each step, the sequence $t^i_\nu$ is replaced by $t^{i+1}_\nu$ defined as follows. Choose a vertex $v_0 \in \mathrm{Vert}(\Gamma)$ for which the rate of increase of the sequence $|t^i_\nu(v_0)|$ is the maximum. That is, for all $v \in \mathrm{Vert}(\Gamma)$, $\lim_\nu |t^i_\nu(v)|/|t^i_\nu(v_0)|$ is finite. Such a vertex can indeed be chosen, because after passing to a subsequence, the limit $\lim_\nu |t^i_\nu(v_i)|/|t^i_\nu(v_j)|$ exists in $[0, \infty]$ for any pair of vertices. Now, define

$$r^i_\nu := |t^i_\nu(v_0)|,$$

and

$$t_\nu^{i+1}(v) := t_\nu^i(v) - r_\nu^i \lim_\nu \frac{t_\nu^i(v)}{r_\nu^i} \in \mathfrak{t}_{P(v)}^\vee. \tag{8.31}$$

We stop the iteration when the sequence $t_\nu^i(v)$ corresponding to every vertex is bounded, and suppose the final sequence is q $t_\nu^k$.

The process terminates in a finite number of steps. Indeed, notice that $t_\nu^{i+1}(v_0) = 0$ for all $\nu$. The number of vertices $v \in \mathrm{Vert}(\Gamma)$ for which $t_\nu^{i+1}(v)$ vanishes is at least one more than the number of vertices $v$ for which $t_\nu^i(v)$ vanishes.

The iterations of the algorithm preserve the discrepancies across edges: For all tropical edges $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$

$$\mathrm{Diff}_e(t_\nu^{i+1}) = \mathrm{Diff}_e(t_\nu^i). \tag{8.32}$$

Indeed, (8.31) implies

$$\mathrm{Diff}_e(t_\nu^{i+1}) = \mathrm{Diff}_e(t_\nu^i) - r_\nu^i \lim_\nu \frac{\mathrm{Diff}_e(t_\nu^i)}{r_\nu^i},$$

and the second term in the right-hand-side vanishes because $\mathrm{Diff}_e(t_\nu^i)$ is uniformly bounded and $r_\nu^i \to \infty$ as $\nu \to \infty$.

We claim that $t_\nu - t_\nu^k$ is an exact translation sequence. For all vertices $v$ the (Polytope) condition (8.9) $(t_\nu - t_\nu^k)(v) \in P(v)^\vee$ is satisfied because $t_\nu(v) \in P(v)^\vee$ and $t_\nu^k(v) \in \mathfrak{t}_{P(v)}^\vee \simeq TP(v)^\vee$. The (Slope) condition is satisfied because $\mathrm{Diff}_e(t_\nu) - \mathrm{Diff}_e(t_\nu^k) = 0$ by (8.30) and (8.32). The last statement that $\Gamma$ is a tropical graph follows from Remark 8.12. ■

## 8.5 Convergence for broken maps

In this section, we prove Theorem 8.3 on Gromov compactness for broken maps. The limit map may have additional components because of bubbling and consequently the tropical graph of the limit map may have additional vertices. The tropical graph of the limit map is related to the tropical graph of the maps in the sequence by a *tropical edge collapse* relation defined below. We show that such bubbling happens only in families whose dimension is at least two, and so does not occur in the zero-dimensional moduli spaces we use to define the Fukaya algebra.

**Definition 8.31.** (Collapsing edges tropically) A *tropical edge collapse* is a morphism of tropical graphs $\Gamma' \xrightarrow{\kappa} \Gamma$ that collapses a subset of edges $\mathrm{Edge}(\Gamma') \setminus \mathrm{Edge}(\Gamma)$ in $\Gamma'$ inducing a surjective map on the vertex sets

$$\kappa : \mathrm{Vert}(\Gamma') \to \mathrm{Vert}(\Gamma),$$

and satisfies the following conditions:

(a) for any vertex $v \in \mathrm{Vert}(\Gamma')$, $P(v) \subseteq P(\kappa(v))$; and
(b) the edge slope is unchanged for uncollapsed edges, i.e. if $\mathcal{T}, \mathcal{T}'$ are the edge slope functions for $\Gamma$, $\Gamma'$, then $\mathcal{T}(\kappa(e)) = \mathcal{T}'(e)$ for any uncollapsed edge $e \in \mathrm{Edge}(\Gamma')$.

Since the edge slope function $\mathcal{T}'$ extends $\mathcal{T}$, we often use the same notation for both. A tropical edge collapse $\Gamma' \xrightarrow{\kappa} \Gamma$ is *trivial* if no edge is collapsed, and $P_{\Gamma'}(v) = P_\Gamma(\kappa(v))$ for all $v$.

The definition of the tropical edge collapse morphism $\kappa : \Gamma' \to \Gamma$ is meaningful even if $\Gamma'$ is a pre-tropical graph (Definition 4.6). This extension of terminology is often used in proofs at points when the realizability of $\Gamma'$ has not yet been ascertained.

**Definition 8.32.** (Tropical edge collapse for a disk type) Let $\Gamma_1, \Gamma_2$ be combinatorial types of treed disks that are equipped with a tropical structure given by tropicalization maps

$$\mathrm{tr}_1 : \Gamma_1 \to \Gamma_{1,\mathrm{tr}}, \quad \mathrm{tr}_2 : \Gamma_2 \to \Gamma_{2,\mathrm{tr}}.$$

An edge collapse morphism $\Gamma_1 \to \Gamma_2$ is a *tropical edge collapse* if it is a lift of a tropical edge collapse map $\Gamma_{1,\mathrm{tr}} \to \Gamma_{2,\mathrm{tr}}$ between the tropical graphs.

*Example* 8.33. In Figure 4.5, collapsing the middle edge in $\Gamma_2$ gives a tropical edge collapse morphism $\kappa : \Gamma_2 \to \Gamma_1$. See Figure 8.1 for another example of a tropical edge collapse morphism.

*Example* 8.34. Figure 6.2 lists the types of broken maps whose gluing has a certain fixed homology class. The tropical graphs in Figure 6.2 are labelled $\Gamma_1, \dots, \Gamma_7$. For every even $i$ there are edge collapse morphisms $\Gamma_i \to \Gamma_{i-1}$ and $\Gamma_i \to \Gamma_{i+1}$.

### 8.5.1 Relative translations

In the definition of convergence of broken maps, the role of translation sequences is played by *relative translation sequences*, introduced in Definition 8.35 below. Components of a relative translation correspond to maps of broken manifolds that rescale the coordinates on cylindrical ends. For a pair $Q \subseteq P$ of polytopes, we recall that (3.51) gives an embedding

$$i_Q^{\tilde{P}} : U_Q(\mathfrak{X}_P) \to \mathfrak{X}_Q \tag{8.33}$$

where $U_Q(\mathfrak{X}_P) \subset \mathfrak{X}_P$ is the $Q$-cylindrical end of $\mathfrak{X}_P$ from (3.44). If $Q = P$, then $U_Q(\mathfrak{X}_P) = U_P(\mathfrak{X}_P) = \mathfrak{X}_P$ and the map in (8.33) is the identity map. A relative translation $t \in \mathrm{Cone}_{P^\vee} Q^\vee$ gives an embedding

$$e^{-t} : U_Q(\mathfrak{X}_P) \to \mathfrak{X}_Q \tag{8.34}$$

defined as $i_Q^{\tilde{P}}$ in (8.33) composed with a translation by $-t$ in the $\mathfrak{t}_Q^\vee$-coordinate in $\mathfrak{X}_Q \simeq Z_Q \times \mathfrak{t}_Q^\vee$. Here, we assume $\mathrm{Cone}_{P^\vee} Q^\vee \subset \mathfrak{t}_Q^\vee$ by fixing a point in $P^\vee$ to be the origin.

For a sequence of converging broken maps, each with tropical graph $\Gamma$, certain components may escape into cylindrical ends in the limit, or in the compactified setting, sink into (intersections of) relative divisors. We examine the convergence of such map components after applying a translation sequence, which in the compactified setting, is a sequence of rescalings of the target space. In order for the translated maps to converge, the translation sequences must go to infinity (in the sense of Definition 8.36 below). These translations must be thought of as happening at a much smaller scale, compared to the tropical graph $\Gamma$. In other words, for the map component corresponding to a vertex $v$ in $\Gamma$, the tropical position $v_0$ of the limit of a sequence of translated maps should be seen as being much closer to $v$, compared to other vertices $v'$ of $\Gamma$. In fact, one may think of vertex positions in the tropical graph $\Gamma'$ of the limit map as $\mathcal{T}_\Gamma + \epsilon \mathcal{T}_{\mathrm{rel}}$, where $\mathcal{T}_\Gamma$ is a vertex position map on $\Gamma$, $\mathcal{T}_{\mathrm{rel}}$ is an element in the translation sequence for the convergence of the maps, and $\epsilon$ is an infinitesimal. [6] However, we avoid the use of infinitesimals in our statements, since for any $\mathcal{T}_\Gamma$ and $\mathcal{T}_{\mathrm{rel}}$, there exists $t_0 > 0$ such that for any $t \in (0, t_0)$, $\mathcal{T}_\Gamma + t\mathcal{T}_{\mathrm{rel}}$ is a vertex position map for $\Gamma'$.

**Definition 8.35.** (Relative translations) Suppose $\kappa : \Gamma' \to \Gamma$ is a tropical edge-collapse morphism, where $\Gamma$ is a tropical graph and $\Gamma'$ is a pre-tropical graph. For any vertex $v$ of $\Gamma'$, let

$$\begin{aligned}\mathrm{Cone}(\kappa, v) &:= \mathrm{Cone}_{P(\kappa(v))^\vee}(P(v)^\vee) \\ &:= \{\alpha(t - t_0) \in \mathfrak{t}^\vee : t \in P(v)^\vee, t_0 \in P(\kappa v)^\vee, \alpha \in \mathbb{R}_{\geq 0}\} \quad (8.35)\end{aligned}$$

be the cone in the polytope $P(v)^\vee$ based at points in $P(\kappa(v))^\vee$ (as in Definition (3.3)), and $\mathrm{Cone}(\kappa, v) \subset \mathfrak{t}_{P(v)}^\vee$ by fixing a point in $P(\kappa(v))^\vee$ to be the origin. A *relative translation* or a $(\Gamma', \Gamma)$*-translation* is an element

$$\mathcal{T}_{\Gamma',\Gamma} = (\mathcal{T}_{\Gamma',\Gamma}(v) \in \mathrm{Cone}(\kappa, v))_{v \in \mathrm{Vert}(\Gamma')}$$

satisfying

$$\mathcal{T}_{\Gamma',\Gamma}(v_+) - \mathcal{T}_{\Gamma',\Gamma}(v_-) \in \begin{cases} \mathbb{R}_{\geq 0}\mathcal{T}(e), & e \in \mathrm{Edge}(\Gamma') \setminus \mathrm{Edge}(\Gamma), \\ \mathbb{R}\mathcal{T}(e), & e \in \mathrm{Edge}(\Gamma) \end{cases} \quad (8.36)$$

for any edge $e = (v_+, v_-)$ in $\Gamma'$. The space of $(\Gamma', \Gamma)$-translations is denoted by $w(\Gamma', \Gamma)$.

[6] The last few sentences tell the reader how to think about translation sequences. These statements can not be stated formally because, although a tropical graph is realizable in the dual complex as in Definition 4.6, the vertex position map is not part of the data of a tropical map.

Relative translations appearing in the convergence of broken maps *go to infinity* in the following sense:

**Definition 8.36.** (Relative translation sequence going to infinity) Suppose $\kappa : \Gamma' \to \Gamma$ is a tropical edge collapse, where $\Gamma$ is a tropical graph and $\Gamma'$ is a pre-tropical graph. A $(\Gamma', \Gamma)$-translation sequence

$$t_\nu(v) \in \mathrm{Cone}(\kappa, v) = \mathrm{Cone}_{P(\kappa v)^\vee}(P(v)^\vee), \quad v \in \mathrm{Vert}(\Gamma')$$

(where $\mathrm{Cone}(\kappa, v)$ is as defined in (8.35)) *goes to infinity* if

(a) (Polytope) For any vertex $v \in \mathrm{Vert}(\Gamma')$ and a polytope $Q \in \mathcal{P}$ such that $P(v) \subset Q \subseteq P(\kappa(v))$,
$$d(t_\nu(v), \mathrm{Cone}_{P(\kappa v)^\vee}(Q^\vee)) \to \infty.$$

(b) (Slope for collapsed edge) For any edge $e \in \mathrm{Edge}(\Gamma_{\kappa^{-1}(v)})$ connecting vertices $v_+$, $v_-$, the sequence $l_\nu \in \mathbb{R}$ defined as
$$t_\nu(v_+) - t_\nu(v_-) = \mathcal{T}(e) l_\nu$$
goes to infinity, that is, $l_\nu \to \infty$. (Note that the existence of $l_\nu$ follows from the Slope condition (8.36) on relative translations.)

The following result says that the existence of a $(\Gamma', \Gamma)$-translation sequence implies that $\Gamma'$ is a tropical graph, in that it has a vertex position map. The proof is straightforward and is left to the reader.

**Lemma 8.37.** *Suppose $\kappa : \Gamma' \to \Gamma$ is a tropical edge-collapse morphism, where $\Gamma$ is a tropical graph and $\Gamma'$ is a pre-tropical graph. Suppose $\{\mathcal{T}_\nu\}_\nu$ is a sequence of $(\Gamma', \Gamma)$-translations going to infinity. Then, $\Gamma'$ is a tropical graph. For any vertex position $\mathcal{T}_\Gamma$ of $\Gamma$ and a large enough $\nu$, there is a constant $t_0 > 0$ such that*

$$\mathrm{Vert}(\Gamma') \ni v \mapsto \mathcal{T}_\Gamma(\kappa(v)) + t\mathcal{T}_\nu(v)$$

*is a vertex position for $\Gamma'$ for all $t \in (0, t_0)$.*

*Remark* 8.38. (The space of relative translations is a cone) Given a tropical edge collapse morphism $\kappa : \Gamma' \to \Gamma$ between tropical graphs $\Gamma'$, $\Gamma$, the space of relative translations $w(\Gamma', \Gamma)$ is a cone

$$w(\Gamma', \Gamma) = \mathrm{Cone}_{\overline{\mathcal{W}}(\Gamma)} \overline{\mathcal{W}}(\Gamma') \tag{8.37}$$

based at the polytope $\overline{\mathcal{W}}(\Gamma)$ (as in Definition 3.3). Here, we recall from Remark 4.8 that the closures of the spaces of vertex positions $\overline{\mathcal{W}}(\Gamma')$ and $\overline{\mathcal{W}}(\Gamma)$ are polytopes, and

$$\overline{\mathcal{W}}(\Gamma) \hookrightarrow \overline{\mathcal{W}}(\Gamma'), \quad \mathcal{T} \mapsto (v \mapsto \mathcal{T}(\kappa(v))$$

is a face. We obtain (8.37) by observing that for any vertex position maps $\mathcal{T}_{\Gamma'} \in \mathcal{W}(\Gamma')$ and $\mathcal{T}_\Gamma \in \mathcal{W}(\Gamma)$, and $\alpha \geq 0$, the scaled difference

$$\text{Vert}(\Gamma') \ni v \mapsto \alpha(\mathcal{T}_{\Gamma'}(v) - \mathcal{T}_\Gamma(\kappa v)) \in \mathfrak{t}^\vee_{P(v)}$$

is a $(\Gamma', \Gamma)$-translation. If $\Gamma$ is rigid, $\mathcal{W}(\Gamma)$ is a single point, and the space $w(\Gamma', \Gamma)$ of relative translations is a cone based at a single point.

*Example* 8.39. For the tropical edge collapse $\kappa : \Gamma' \to \Gamma$ in both Figure 4.5 and Figure 8.1, the tropical graph $\Gamma$ is rigid, and the space of vertex positions of $\Gamma'$ is (linearly) isomorphic to the interval $(0, 1)$, where 0 corresponds to the vertex position where the edges collapsed by $\kappa$ have length 0, and therefore, by (4.11), is not a vertex position for $\Gamma'$; and 1 corresponds to some vertex $v$ of $\Gamma'$ lying in a lower dimensional polytope $Q^\vee \subset P^\vee(v)$, and so, is not a legitimate vertex position of $\Gamma'$. In both cases the space of relative translations is

$$w(\Gamma', \Gamma) = \text{Cone}_0[0, 1] = \mathbb{R}_{\geq 0}.$$

### 8.5.2 Defining Gromov convergence of broken maps

The definition of Gromov convergence for broken maps is largely similar to the definition of convergence of breaking maps (Definition 8.13) with the following two differences:

- The definition of convergence for breaking maps is given for a sequence of irreducible maps in neck-stretched manifolds. For convergence of broken maps, we need to consider maps with multiple components. Thus, in the definition below, we give the conditions under which a sequence of broken maps with tropical graph $\Gamma$ converges to a broken map with tropical graph $\Gamma'$, where the tropical graphs are related by an edge collapse $\kappa : \Gamma' \to \Gamma$. For each vertex $v \in \text{Vert}(\Gamma)$ of the tropical graph, the sequence of smooth maps $u_{v,\nu}$, with markings corresponding to nodal lifts $w_e \in C_v$, converges to a limit map $u_v$. The limit map $u_v$ may have multiple components, corresponding to vertices that are collapsed to $v$ by the morphism $\kappa$. That is, $u_v = u|(\cup_{v' \in \kappa^{-1}(v)} C_{v'})$.
- Translations occurring in the convergence of breaking maps are replaced by relative translations for the case of convergence of broken maps. The corresponding maps of manifolds $\mathfrak{e}^{-t}$ in the breaking case (see Definition 3.63) are replaced by $e^{-t}$ (see (8.34)) in the case of broken maps.

**Definition 8.40.** (Gromov convergence for broken maps) Suppose $\Gamma', \Gamma$ are combinatorial types of stable treed disks that are equipped with a tropical structure. Suppose $\Gamma' \xrightarrow{\kappa} \Gamma$ is a tropical edge collapse morphism which induces a vertex map $\kappa$ : $\text{Vert}(\Gamma') \to \text{Vert}(\Gamma)$. A sequence of broken maps $u_\nu : C_\nu \to \mathfrak{X}$ of type $\Gamma$ *(Gromov)* *converges* to a limit broken map $u : C \to \mathfrak{X}$ of type $\Gamma'$ if the following conditions are satisfied.

(a) (Convergence of domains) The sequence of treed disks $C_\nu$ converge to the treed disk $C$ and for any tropical node $w_e$, $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma')$ that is collapsed by $\kappa$, the arguments $\frac{\delta_e(C_\nu)}{|\delta_e(C_\nu)|}$ of the gluing parameters converge to a limit. Let $S_\nu(v) \subset (S_\nu)_{\kappa(v)} \subset C_\nu$ be the subset corresponding to a vertex $v \in \mathrm{Vert}(\Gamma')$, and let

$$i_{v,\nu} := i_{S_v, S_{\nu,\kappa(v)}} : S_\nu(v) \to S_v, \quad S_\nu(v) \subset S_\nu,$$

be embeddings from (8.8) whose images $i_{v,\nu}(S_\nu(v))$ exhaust $S_v^\circ$ as $\nu \to \infty$. Here

$$S_v^\circ := S_v \backslash \{w_e : v \in e, \ \ e \in \mathrm{Edge}_-(\Gamma') \text{ is collapsed by } \kappa\}.$$

(b) (Convergence of maps) There is a $(\Gamma', \Gamma)$-translation sequence $(t_\nu(v))_{v \in \mathrm{Vert}(\Gamma')}$ going to infinity in the sense of Definition 8.36, such that for any vertex $v \in \mathrm{Vert}(\Gamma')$, the sequence of maps

$$S_v^\circ \supset i_{v,\nu}(S_\nu(v)) \xrightarrow{e^{-t_\nu(v)}(u_\nu \circ i_{v,\nu}^{-1})} \mathfrak{X}_{P(v)}$$

converges in $C^\infty_{\mathrm{loc}}(S_v^\circ)$ to $u_v : S_v^\circ \to \mathfrak{X}_{P(v)}$. The map $e^{-t_\nu(v)} : U_{P(v)}(\mathfrak{X}_{P(\kappa v)}) \to \mathfrak{X}_{P(v)}$ is defined on the $P(v)$-cylindrical end $U_{P(v)}(\mathfrak{X}_{P(\kappa v)}) \subset \mathfrak{X}_{P(\kappa v)}$, see (8.34). For each boundary edge $e \in \mathrm{Edge}_\circ(\Gamma)$, the maps $u_\nu | T_{e,\nu}$ on the treed segment converge to a (possibly broken) treed segment in $u$.

(c) For a node $w$ in $C$ corresponding to a tropical edge $e$ collapsed by $\kappa$, that is, $e = (v_+, v_-) \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma') \backslash \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, let

$$z_\pm : (U_{w^\pm}, w^\pm) \to (\mathbb{C}, 0)$$

be matching coordinates (see Definition 4.14 following (4.22)) on neighborhoods $U_{w^\pm} \subset S_{v_\pm}$ of the nodal point which respect the framing $\mathrm{fr}_e$. Let

$$A(l_\nu) := [-l_\nu/2, l_\nu/2] \times S^1 \subset S_\nu$$

be a sequence of centered annuli converging to the node $w$, see Definition 8.10. Then the sequence

$$x_\nu := e^{-\frac{1}{2}(t_\nu(v_+) + t_\nu(v_-))} u_\nu(0,0) \in \mathfrak{X}_{P(e)}$$

converges to a limit $x_0$, and the components $u_{v_\pm}$ of the broken map are asymptotically close to

$$z_\pm \mapsto z_\pm^{\mathcal{T}(e)} x_0$$

in the sense of Remark 4.29.

### 8.5.3 From an area bound to a bound on types

The following result is the first ingredient in the proof of Theorem 8.3, which shows that there is a finite number of tropical graphs corresponding to broken maps that satisfy an area bound.

**Proposition 8.41.** *(Finite number of tropical graphs underlying broken maps) For any $E > 0$, $d(\circ) \geq 1$ there are a finite number of tropical graphs $\Gamma$ that occur as tropical graphs of broken maps $u : C \to \mathfrak{X}$ (see Definition 6.14) of area at most $E$ and $d(\circ)$ boundary leaves.*

*Proof.* Consider a broken map $u : C \to \mathfrak{X}$ of type $\Gamma$ and area at most $E_0$, whose edge slopes are given by a map $\mathcal{T} : \mathrm{Edge}(\Gamma) \to \mathfrak{t}_{\mathbb{Z}}$. We first consider the case when the $X$-inner product (see (3.10)) is rational, so that the compactification $\overline{\mathfrak{X}}_P$ of $\mathfrak{X}_P$ is an orbifold for any $P \in \mathcal{P}$. We discuss the generalization at the end of the proof.

Step 1: *Uniform bound on the number of vertices.*
The number of interior markings in $\Gamma$ is bounded by $kE_0$, where $k$ is the degree of the stabilizing divisor from (5.1). Since $u$ is adapted to the stabilizing divisor (as in Definition 6.10 (d)), all the surface components $S_v, v \in \mathrm{Vert}(\Gamma)$ in its domain $C$ are stable. Given a uniform bound on the number of interior and boundary markings on the type of domain curve, we obtain a uniform bound on the number irreducible surface components.

Step 2: *Uniform bound on the sum of vertical components of edge slopes.*
Let $u_v : S_v \to \overline{\mathfrak{X}}_P$ be a component of the broken map corresponding to a vertex $v \in \mathrm{Vert}(\Gamma)$, and $P := P(v)$. By the balancing property (4.30), the sum of the edge slopes projected to $\mathfrak{t}_P^\vee$ is

$$\sum_{e \ni v} \pi_{\mathfrak{t}_P^\vee}(\mathcal{T}(e)) = c_1((\pi_P \circ u_v)^* Z_{P(v)} \to \overline{X}_P)).$$

The right-hand side, which is the pairing of $(\pi_P \circ u)_*[C]$ and the Chern class $c_1(Z_P \to \overline{X}_P)$ has an $E_0$-dependent bound. Indeed, for any $\epsilon \in (0, 1)$, there is a constant $k_0$ such that for any domain-dependent almost complex structure $J \in B_\epsilon(J_{\overline{X}_P})_{C^0}$ and a $J$-holomorphic sphere $u : \mathbb{P}^1 \to \overline{X}_P$,

$$\int_{\mathbb{P}^1} u^* c_1(Z_P \to \overline{X}_P) \leq k_0 \int_{\mathbb{P}^1} u^* \omega_{X_P} \leq k_0 E_0.$$

This estimate is similar to the one in Lemma 5.5 and the proof is the same – by choosing a two-form on $\overline{X}_P$ representing the Chern class and bounding it pointwise by $\omega_{X_P}$.

Step 3: *Uniform bound on the horizontal components of edge slopes.*
For a vertex $v$ and an incident edge $e$, the horizontal component of the slope $\mathcal{T}(e)$ is

the sum of intersection multiplicities at the node $w_e$ with horizontal relative divisors of $\overline{X}_P$:

$$\pi^{\perp}_{\mathfrak{t}^{\vee}_P}(\mathcal{T}(e)) = \textstyle\sum_{Q \subset P} m_{w_e}(u_{v,P}, \overline{X}_Q)\nu_Q, \quad u_{v,P} := \pi_P \circ u_v : S_v \to \overline{X}_P,$$

where the sum ranges over facets $Q \in \mathcal{P}$ of $P$, and $\nu_Q$ is the normal vector of the facet $Q \subset P$. For any relative divisor $\overline{X}_Q \subset \overline{X}_P$, the sum $\sum_{e \ni v} m_{w_e}(u_v, \overline{X}_Q)$ is bounded by $c\omega_{X_P}(u_{v,P})$ for a uniform constant $c(\overline{X}_P, \overline{X}_Q)$. The proof is similar to the vertical case, by expressing the intersection number with any divisor as an integral of a two-form.

Step 4: *Finishing the proof.*
We will show that the tropical edge slopes $\mathcal{T}(e)$ of $\Gamma$ are uniformly bounded in $\mathfrak{t}^{\vee}$ for the set of all broken maps with area at most $E_0$. So far we have shown that (Step 3) the horizontal components of the edge slopes are uniformly bounded, and the vector sum of the vertical slope components of the edges incident on any vertex are uniformly bounded. From here, we conclude that for a vertex $v$ and an edge $e_0$ incident on $v$:

$$\exists c(E) : |\mathcal{T}(e_0)| \leq \sum_{e \ni v, e \neq e_0} |\mathcal{T}(e)| + c(E). \tag{8.38}$$

Recall that $\Gamma$ is a tree, any edge $e \in \mathrm{Edge}_{\bullet}(\Gamma)$ is oriented so that it points away from the root vertex. The slope of any incoming edge can be bounded by the slope of outgoing edges by (8.38). Applying (8.38) iteratively, we conclude that for any edge $e$ in $\Gamma$

$$|\mathcal{T}(e)| \leq c(E)|\,\mathrm{Vert}(\Gamma)|,$$

where the constant $c(E)$ is the same as the one in (8.38). The Proposition now follows from the bound on the number of vertices in Step 1 in cases when the $X$-inner product is rational.

For a general $X$-inner product $g$, we consider a sequence $\{g_\nu\}_\nu$ of rational $X$-inner products uniformly converging to $g$. A squashing map with respect to the $X$-inner product $g$ is the limit of a squashing maps $\aleph_\nu$ with respect to the $X$-inner product $g_\nu$, and therefore,

$$E_{P,\mathrm{Hof},g} \leq \limsup_{\nu} E_{P,\mathrm{Hof},g_\nu}.$$

The Proposition then follows from the uniform bound on $E_{P,\mathrm{Hof},g_\nu}$ for all rational $g_\nu$ proved in the preceding paragraphs. ■

### 8.5.4 A translation sequence from an approximate translation sequence

Another ingredient in the proof of Gromov compactness for broken maps is a result that produces a relative translation sequence, starting from an *approximate relative translation sequence* via a sequence of adjustments that is uniformly bounded. We first define the approximate sequences that we require.

**Definition 8.42.** (Approximate $(\Gamma', \Gamma)$-translation sequence) Suppose the tropical graph $\Gamma$ is obtained by collapsing edges in $\Gamma'$ and the induced map on the vertex set is $\kappa : \mathrm{Vert}(\Gamma') \to \mathrm{Vert}(\Gamma)$. Then, an *approximate* $(\Gamma', \Gamma)$*-translation sequence* consists of a sequence

$$t_\nu(v) \in \mathrm{Cone}(\kappa, v),$$

for every $v \in \mathrm{Vert}(\Gamma')$ satisfying the (Polytope) and (Slope for collapsed edges) conditions in Definition 8.36 and the following weakened version of the (Slope for uncollapsed edges) condition.

- (Approximate slope for uncollapsed edges) For an edge $e$ of $\Gamma'$ that is not collapsed in $\Gamma$,

$$t_\nu(v_+) - t_\nu(v_-) \mod \mathcal{T}(e)$$

is a bounded sequence in $\mathfrak{t}^\vee/\mathbb{R}\mathcal{T}(e)$. (Recall that $\mathrm{Cone}(\kappa, v_\pm)$ is embedded in $\mathfrak{t}^\vee$ with vertex $\mathcal{T}(v_\pm)$ mapped to the origin in $\mathfrak{t}^\vee$, and identify $\mathfrak{t}$ to $\mathfrak{t}^\vee$ via a pairing (3.10).)

An approximate $(\Gamma', \Gamma)$-translation sequence can be adjusted by a uniformly bounded amount to produce an actual $(\Gamma', \Gamma)$-translation sequence.

**Lemma 8.43.** *(From an approximate to an exact* $(\Gamma', \Gamma)$*-translation sequence) Let* $\kappa : \Gamma' \to \Gamma$ *be a tropical edge collapse, and let* $\{t_\nu\}_\nu$ *be an approximate* $(\Gamma', \Gamma)$*-translation sequence. There exists a* $(\Gamma', \Gamma)$*-translation sequence* $\{\bar{t}_\nu\}_\nu$ *such that*

$$\sup_\nu |\bar{t}_\nu(v) - t_\nu(v)| < \infty$$

*for all* $v \in \mathrm{Vert}(\Gamma')$.

*Proof.* The proof is by replicating the iteration in Step 2 of the proof of Lemma 8.30. At the start, we set $t_\nu^0 := t_\nu$. At the $(i+1)$-th step, we construct $t_\nu^{i+1}$ as follows. As in the proof of Lemma 8.30, there exists a vertex $v_0 \in \mathrm{Vert}(\Gamma')$ such that the sequence $|t_\nu^i(v)|$ has the fastest growth rate. That is, for all $v \in \mathrm{Vert}(\Gamma')$, $\lim_\nu |t_\nu^i(v)|/|t_\nu^i(v_0)|$ is finite. Define

$$t_\nu^{i+1}(v) := t_\nu^i(v) - |t_\nu^i(v_0)| \lim_\nu \frac{t_\nu^i(v)}{|t_\nu^i(v_0)|}.$$

For the sequences $\{t_\nu^{i+1}(v)\}_\nu$, $v \in \mathrm{Vert}(\Gamma')$, the quantity

$$\pi^\perp_{\mathcal{T}(e)}(t_\nu^{i+1}(v_+) - t_\nu^{i+1}(v_-)), \quad e = (v_+, v_-) \in \mathrm{Edge}(\Gamma')$$

vanishes for collapsed edges, and is uniformly bounded for uncollapsed edges. Furthermore, for any vertex $v$, $t_\nu^{i+1}(v) \in \mathfrak{t}^\vee_{P(v)}$. After, say, $k$ steps, the sequence $|t_\nu^k(v)|$ is uniformly bounded for all vertices $v$. Then, $\bar{t} := t - t^k$ is an exact $(\Gamma', \Gamma)$-translation sequence. ∎

### 8.5.5 Proof of Gromov compactness for broken maps

The notion of horizontal convergence extends in a natural way to broken manifolds, though it is easier to state in this case.

**Definition 8.44.** (Horizontal convergence in a broken manifold) Let $P \in \mathcal{P}$ be a polytope. A sequence of points $x_\nu \in \mathfrak{X}_P$ horizontally converges in $X_Q$ for a polytope $Q \subseteq P$ if the sequence $\pi_P(x_\nu) \in X_P$ converges to a point $x \in X_Q \subseteq \overline{X}_P$.

*Remark* 8.45. Analogs of Lemma 8.17 and 8.18 hold for horizontal convergence in broken manifolds. An analog of the breaking annulus lemma (Proposition 8.20) also holds for broken maps. For all these results, we make the following substitutions in the original result to obtain the broken version for maps in $\mathfrak{X}_P$: a translation sequence is replaced by a relative translation sequence going to infinity, any instance of the translation map $\mathbb{e}^{-t}$ is replaced by $e^{-t}$ where $t \in \mathrm{Cone}_{P^\vee} B^\vee$ (defined in (8.34)), and Hofer energy is replaced by $P$-Hofer energy. As an example, we state the analog of Lemma 8.17, leaving the rest to the reader.

**Lemma 8.46.** *(Broken analog of Lemma 8.17) Let $P \in \mathcal{P}$ be a polytope. Suppose $x_\nu \in \mathfrak{X}_P$ is a sequence of points, and $t_\nu \in \mathrm{Cone}_{P^\vee} Q^\vee$ is a sequence of translations for a proper face $Q \subset P$, such that $e^{-t_\nu} x_\nu$ converges in $\mathfrak{X}_Q$. Then, the following are equivalent:*

(a) *The sequence $x_\nu \in X_P$ converges horizontally in $X_Q$ (in the sense of Definition 8.44).*

(b) *For any $P \supseteq P_0 \supset Q$ (that is, $Q$ is a proper face of $P_0$), $d(t_\nu, \nu P_0^\vee) \to \infty$.*

*Proof of Theorem* 8.3. After passing to a subsequence, the tropical graph $\Gamma$ underlying the maps $u_\nu$ is $\nu$-independent. Indeed by Proposition 8.41, there is a finite number of tropical graphs that underlie broken maps with area $< E$. By Proposition 7.35, the Hofer energy for a map $u : (C, \partial C) \to (\mathfrak{X}, L)$ can be read off from its area and its intersection data with relative divisors, which is provided by the slopes $\mathcal{T}(e)$ of the edges of the tropical graph $\Gamma$. Consequently, the Hofer energy $E_{\mathrm{Hof}}(u_\nu)$ of the maps $u_\nu$ is uniformly bounded.

We first find the limit map at each of the vertices of the tropical graph $\Gamma$. For each vertex $v \in \mathrm{Vert}(\Gamma)$, we assume that the domain curve $C_{\nu,v}$ for $u_{\nu,v}$ has marked points corresponding to lifts of nodes $e \in \mathrm{Edge}_-(\Gamma)$, $v \in e$, in addition to the marked points corresponding to leaves $e \in \mathrm{Edge}_\to(\Gamma)$. The proof of breaking maps carries over verbatim to the sequence of maps $u_{\nu,v}$. Indeed, all the results used in the proof of convergence of breaking maps have analogs for the case of broken maps; see Propositions 7.35, 7.36, Remark 8.45 and Lemma 8.46. The conclusion is that there is a limit map $u_v$ modelled on a tropical graph $\Gamma_v$, and the convergence is via a translation sequence

$t_\nu(v')$ going to infinity, $v' \in \Gamma_v$. The convergence includes the following statements:

$$P(v') \subseteq P(v); t_\nu(v') \in \mathrm{Cone}_{P(v)^\vee} P(v')^\vee \subset \mathfrak{t}^\vee_{P(v)} \simeq \sqrt{-1}\mathfrak{t}_{P(v)},$$

where the inclusion is obtained by fixing a point in $P(v)^\vee$ as the origin of the cone; the sequence of maps $e^{-t_\nu(v')}u_{\nu,v}$ lie in the $P(v')$-cylindrical end of $\mathfrak{X}_{P(v)}$, and converge uniformly on compact subsets to a limit $u_{v'}$ in $\mathfrak{X}_{P(v')}$. For any tropical edge $e = (v_+, v_-)$ of $\Gamma$, the limit of the nodal lifts $w^e_{\nu,\pm} \in S_{\nu,v_\pm}$ lies on some a domain component of $u_v$, corresponding to a vertex $v_{\pm,e}$ of the graph $\Gamma_{v_\pm}$. By connecting the graphs $\{\Gamma_v\}_{v\in \mathrm{Vert}(\Gamma)}$ by edges $(v_{+,e}, v_{-,e})$ for all edges $e$ of $\Gamma$, we obtain a pre-tropical graph $\Gamma'$ for which there is an edge collapse morphism $\kappa : \Gamma' \to \Gamma$. We will subsequently show that $\Gamma'$ is a tropical graph and $\kappa$ is a tropical edge collapse morphism.

Next, we simultaneously prove that the limit map $u$ satisfies matching conditions on the edges of $\Gamma$, and that $\Gamma'$ is a tropical graph. As a first step, we show that the translation sequence $\{t_\nu(v) : v \in \mathrm{Vert}(\Gamma')\}$ in the previous paragraph is an approximate $(\Gamma', \Gamma)$-translation sequence in the sense of Definition 8.42. The convergence to the maps $u_v$ implies that for any $\nu$, $t_\nu$ satisfies the slope condition (8.36) for edges collapsed by $\kappa$, that is, for edges $e \in \mathrm{Edge}(\Gamma') \setminus \mathrm{Edge}(\Gamma)$. Therefore, it remains to prove that the (Approximate slope for uncollapsed edges) condition is satisfied. Consider an edge $e$ of $\Gamma$, that is incident on $v_+$, $v_- \in \mathrm{Vert}(\Gamma')$, and the nodal point corresponding to $e$ is the pair $w^\pm_e$ on $C_{v_\pm}$. The edge matching condition for broken maps (4.18) implies

$$(\pi^\perp_{\mathcal{T}(e)} \circ u_\nu)(w^e_{\nu,+}) = (\pi^\perp_{\mathcal{T}(e)} \circ u_\nu)(w^e_{\nu,-}) \quad \text{in } \mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}. \tag{8.39}$$

On a sequence of converging relative maps, the projected tropical evaluations of the relative marked points converge, that is,

$$(\pi^\perp_{\mathcal{T}(e)} \circ (e^{-t_\nu(v)}u_\nu))(z_{i,\nu}) \to (\pi^\perp_{\mathcal{T}(e)} \circ u)(z) \quad \text{in } \mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}.$$

This convergence is a consequence of the convergence of the marked points $z_{i,\nu}$ on the domain curve and the convergence of maps $e^{-t_\nu(v)}u_\nu$. Therefore, for the edge $e = (v_+, v_-)$,

$$d((\pi^\perp_{\mathcal{T}(e)} \circ (e^{-t_\nu(v_+)}u_\nu))(w^e_{\nu,+}), (\pi^\perp_{\mathcal{T}(e)} \circ (e^{-t_\nu(v_-)}u_\nu))(w^e_{\nu,-})) \to \\ d((\pi^\perp_{\mathcal{T}(e)} \circ u)(w^e_+), (\pi^\perp_{\mathcal{T}(e)} \circ u)(w^e_-)),$$

where $d$ is the cylindrical metric distance in $\mathfrak{X}_{P(e)}/T_{\mathcal{T}(e),\mathbb{C}}$. Using (8.39), the sequence

$$\pi^\perp_{\mathcal{T}(e)}(t_\nu(v_+) - t_\nu(v_-)) \in \mathfrak{t}_\mathbb{C}/\mathfrak{t}_{\mathcal{T}(e),\mathbb{C}} \tag{8.40}$$

is bounded, implying that $\{t_\nu(v) : v \in \mathrm{Vert}(\Gamma')\}$ is an approximate $(\Gamma', \Gamma)$-translation sequence. An approximate $(\Gamma', \Gamma)$-translation sequence can be adjusted by a uniformly bounded amount to produce an actual $(\Gamma', \Gamma)$-translation sequence by Lemma 8.43.

After making such an adjustment the quantity in (8.40) vanishes. Then by (8.39) the limit map $u$ satisfies the edge matching condition (4.18) for edges in $\Gamma$, and $u$ is therefore a broken map. Furthermore, we now have that $\{t_\nu\}_\nu$ is a relative translation sequence going to infinity (as in Definition 8.36), and then, Lemma 8.37 implies that $\Gamma'$ is a tropical graph, and therefore, the limit map $u = (u_v)_{v \in \mathrm{Vert}(\Gamma')}$ is a broken map.

The proof of uniqueness of the limit is similar to the case of breaking maps: The domain curve is uniquely determined, and any two $(\Gamma', \Gamma)$-translation sequences $t_\nu$, $t'_\nu$ differ by a uniformly bounded amount : $\sup_\nu |t_\nu(v) - t'_\nu(v)| < \infty$. After passing to a subsequence, the limit

$$t_\infty(v) := \lim_\nu (t_\nu(v) - t'_\nu(v))$$

exists. The tuple $(t_\infty(v))_v$ is an unsigned version of a $(\Gamma', \Gamma)$-translation since $t_\infty(v) \in \mathrm{Cone}(\kappa, v)$ and for any edge $e = (v_+, v_-) \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma')$,

$$t_\infty(v_+) - t_\infty(v_-) \in \mathbb{R}\mathcal{T}(e).$$

Therefore, we obtain a tropical symmetry transformation $e^{-t_\infty} \in T_{\mathrm{trop}}(\Gamma')$ that relates the limit maps $\lim_\nu e^{-t_\nu} u_\nu$, $\lim_\nu e^{-t'_\nu} u_\nu$. Consequently, the limit is unique up to the action of the identity component $T_{\mathrm{trop},\mathcal{W}}(\Gamma')$ of the tropical symmetry group.

It remains to prove the last statement of the Theorem. Suppose the tropical edge collapse $\kappa : \Gamma' \to \Gamma$ is non-trivial. Then, either a tropical edge $e_0$ of $\Gamma'$ is collapsed by $\kappa$, or there is a vertex $v_0$ of $\Gamma'$ for which $P(v_0) \subsetneq P(\kappa(v_0))$. In both cases, for large enough $\nu$, the translation $(t_\nu(v))_{v \in \mathrm{Vert}(\Gamma')}$ generates a subgroup $T_{t_\nu}$ of $T_{\mathrm{trop}}(\Gamma')$ that is not contained in $T_{\mathrm{trop}}(\Gamma)$. Indeed, in the first case of a collapsed tropical edge $e_0 = (v_+, v_-)$

$$t_\nu(v_+) - t_\nu(v_-) \in \mathbb{R}_+\mathcal{T}(e_0)$$

and in the second case where $P(v_0) \subsetneq P(\kappa(v_0))$ we have

$$t_\nu(v_0) \notin P(\kappa(v_0))^\vee.$$

Therefore $\dim_{\mathbb{C}}(T_{\mathrm{trop}}(\Gamma')) > \dim_{\mathbb{C}}(T_{\mathrm{trop}}(\Gamma))$. ∎

## 8.6 Boundaries of rigid strata

In the moduli space of maps, a stratum is *rigid* if it does not deform to another type; that is, it does not occur in the boundary of a compactification of a different stratum. To count isolated maps under generic perturbations one must restrict attention to rigid strata. In this section, we formally define rigid strata of broken and unbroken maps, and describe the codimension one boundary components of rigid strata.

For unbroken maps, rigid types are those whose domain type is a top-dimensional stratum in the moduli space of treed disks. We recall that top-dimensional strata in the moduli space of treed disks correspond to disk types which do not have any interior nodes, and at boundary nodes, the length of the treed segment is finite, see Page 103. Rigidity also includes the condition that the map has simple intersections with the stabilizing divisor, because maps with non-simple intersections can be deformed to maps with multiple simple intersections.

**Definition 8.47.** (Rigid types of unbroken maps) The combinatorial type $\Gamma$ of a (unbroken) treed holomorphic map is *rigid* if the only edges $e \in \mathrm{Edge}_-(\Gamma)$ corresponding to nodes are boundary edges $e \in \mathrm{Edge}_\circ(\Gamma)$ of finite non-zero length $\ell(e) \in (0,\infty)$, and intersection multiplicity $m_{w_e}(u, D_P)$ of $u$ of type $\Gamma$ at $T_e \cap S$ with the stabilizing divisor $\mathfrak{D} = (D_P, P \in \mathcal{P})$ is 1.

Broken maps necessarily have tropical nodes, therefore in the definition of rigidity of broken maps, the *no interior nodes* condition of the unbroken case is replaced by the condition that the tropical graph is rigid.

**Definition 8.48.** (Rigid types of broken maps) The combinatorial type $\Gamma$ of a broken treed holomorphic map is *rigid* if the tropical graph $\mathcal{T}(\Gamma)$ is rigid, and the only internal (non-tropical) edges $e \in \mathrm{Edge}_{\mathrm{int}}(\Gamma)$ are boundary edges $e \in \mathrm{Edge}_\circ(\Gamma)$ of finite non-zero length $\ell(e) \in (0,\infty)$, and intersection multiplicity $m_{w_e}(u, D_P)$ of $u$ of type $\Gamma$ at $T_e \cap S$ with the stabilizing divisor $\mathfrak{D} = (D_P, P \in \mathcal{P})$ is 1.

*Remark* 8.49. The rigidity for types of broken maps includes the additional condition of the rigidity of the tropical graph because any non-rigid tropical graph $\Gamma$ is related to a rigid tropical graph $\Gamma_0$ by a tropical edge collapse morphism, and therefore tropical symmetry orbits of broken maps modelled on $\Gamma$ occur in the compactification of the moduli space of maps modelled on $\Gamma_0$.

Next, we address the question of which strata of broken maps occur in the compactification of a one-dimensional moduli space $\mathcal{M}_\Gamma^{\mathrm{brok}}(\underline{x})$ where $\Gamma$ is a rigid type. The following result proves that the compactification does not contain broken maps of *crowded type* (Definition 6.26). Roughly speaking, being of crowded type means that there are no ghost components with more than one interior marking. We recall that interior markings correspond to intersections with the stabilizing divisors, and in the context of broken maps, a ghost component is a domain component on which the map is *horizontally constant* (Definition 4.17). Once this is established, we prove in Proposition 8.51 that the only strata that occur in the codimension one boundary are those that contain either a broken edge, or an edge with length zero.

**Proposition 8.50.** *(No crowded strata in the compactification) Let $\underline{\mathfrak{p}}$ be a regular perturbation datum. Let $\Gamma$ be a type of an uncrowded rigid broken map and let $\underline{x} \in (\mathcal{I}(L))^{d(\circ)+1}$ be a tuple of limit points of boundary leaves such that the expected* *dimension of the moduli space* $\mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$ *is at most* 1. *Then, for any curve* $u$ *in the compactification* $\overline{\mathcal{M}}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$*,*

- *there are no horizontally constant components that contain interior markings, and*
- *$u$ has simple intersections with the stabilizing divisor.*

*Proof.* First, we point out that a horizontally constant component containing an interior marking can not be a disk component. Indeed, the Lagrangian $L$ is contained in a top-dimensional cut space $X_{P_0}$ and the torus $T_{P_0,\mathbb{C}}$ is trivial. Therefore, a horizontally constant disk component $u_v : S_v \to X_{P_0}$ is in fact constant and maps to a point in $L$. Since $L$ is disjoint from the stabilizing divisor there are no interior markings on $S_v$.

Next, we rule out horizontally constant spherical components that contain markings. Suppose the map $u \in \mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$ is horizontally constant on a component $S_v$, for a vertex $v$ of $\Gamma$, and suppose $S_v$ has an interior marking $z_e$. Since $\Gamma$ is an uncrowded stratum, $S_v$ has no other interior marking besides $z_e$. The component has at most two nodes – otherwise moving the marking $z_e$ on the component $S_v$ gives a two-dimensional family of regular maps adapted to the stabilizing divisor. By rigidity of the type $\Gamma$, all edges $e$ corresponding to interior nodes $w_e$ are tropical edges. Since $u|_{S_v}$ is horizontally constant, the balancing property (4.29) implies that there are exactly two edges $e_1, e_2 \in \mathrm{Edge}(\Gamma)$ incident on $v$, and they have the same slope $\mathcal{T}(e_1) = \mathcal{T}(e_2)$. This means the type $\Gamma$ has a non-trivial tropical symmetry group $T_{\mathrm{trop}}(\Gamma)$, contradicting the rigidity of $\Gamma$.

Next, we consider a map in a boundary of the moduli space. Suppose $\{u_\nu\}_\nu$ is a sequence in $\mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$ that converges to a limit $u : C \to \mathfrak{X}$ of type $\Gamma'$. First we prove the proposition assuming that the limit $u$ is uncrowded. Then $u$ is a regular map adapted to the stabilizing divisor. Since the index of $u_\nu$ is at most 1, standard arguments (see the proof of Proposition 8.51, for example) imply that the type $\Gamma$ of $u_\nu$ is obtained from the limit type $\Gamma'$ by the morphism (Making an edge length finite/non-zero). In particular, the tropical graph associated to $\Gamma'$ is the same as that of $\Gamma$. By the argument in the previous paragraph, there are no markings in horizontally constant components of the limit map $u$.

Next, consider the case that the limit $u$ has a crowded component. Starting from $u$, we obtain a map $u'$ of uncrowded type as follows: Forgetting all but one interior marking on each of the crowded components yields a map $u'$ adapted to the stabilizing divisor. If in this process, a domain component $S_v \subset S$ (belonging to a crowded component of $u$) becomes unstable, it is collapsed. Indeed, the vertex $v$ has valence 1 or 2, and in either case, there is an obvious way to drop $v$ from the graph. Denote by $\Gamma_s$ the resulting type of the map. In $\Gamma_s$, the remaining interior marking, corresponding to the leaf $e$, is assigned a multiplicity of $\mu_{\mathfrak{D}}(e)$ plus the sum of multiplicities $\mu_{\mathfrak{D}}(e')$ of all the forgotten leaves $e' \neq e$. The limit $u'$ is $\mathfrak{p}_{\Gamma_s}$-holomorphic because of the (Locality axiom) in Definition 6.4. Indeed, forgetting markings changes the type of the limit curve, but it does not affect the perturbation datum $\mathfrak{p}_\Gamma$ on the other curve components on which the map is horizontally non-constant. Therefore, $u'$ is regular. If no component is collapsed in $\Gamma' \to \Gamma_s$, then, the expected dimension of the type $\Gamma_s$ is the same as that of $\Gamma'$. In this case, $u$ is an uncrowded map with a marking in a horizontally constant component. This possibility has been ruled out in the last paragraph. If a component is collapsed, the expected dimension of $\Gamma_s$ is at least two lower than that of $\Gamma$, and therefore, the map $u$ does not exist.

Finally, we show that for a map $u$ in the compactification $\overline{\mathcal{M}}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$, the intersections with the stabilizing divisor are simple. We assume that $u$ is the limit of a sequence of maps $\{u_\nu\}_\nu$ in $\mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$. Since $\Gamma$ is rigid, the maps $u_\nu$ have simple intersections with $\mathfrak{D}$ and each intersection is an interior marking. The map $u$ can have a non-simple intersection with $\mathfrak{D}$ only if two sequences interior markings $z_{e,\nu}, z_{e',\nu}$ on the maps $\{u_\nu\}_\nu$ coincide in the limit, which means that $u$ has a ghost component containing the two markings. Since we have shown that $u$ is uncrowded, we have arrived at a contradiction. ∎

**Proposition 8.51.** *(Boundary strata) Let $\underline{\mathfrak{p}}$ be a regular perturbation datum. Suppose $\Gamma$ is a rigid type for a broken map and $\underline{x} \in (\mathcal{I}(L))^{d(\circ)}$ is a tuple of limit points of boundary leaves such that the expected dimension $i(\Gamma, \underline{x})$ is $\leq 1$. The moduli space $\mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$ admits a compactification $\overline{\mathcal{M}}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$. The boundary strata*

$$\overline{\mathcal{M}}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x}) \backslash \mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$$

*correspond to types $\Gamma'$ with a single broken boundary trajectory $e \in \mathrm{Edge}_\circ(\Gamma')$, $\ell(e) = \infty$ or a single boundary edge with length zero $\ell(e) = 0$.*

*Proof.* For a sequence of maps $u_\nu : C \to \mathfrak{X}$ in the moduli space $\mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$, a subsequence converges to a limit $u : C \to \mathfrak{X}$ by Theorem 8.3. By Proposition 8.50, $u$ is uncrowded and therefore, $u$ is regular and adapted to the stabilizing divisor. Interior nodes $w_e$ corresponding to non-tropical edges $e \notin \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ are ruled out in $u$ for dimension reasons using the index relation (6.30). We next claim that the tropical type of $u$ is $\Gamma$. If not, it is of type $\Gamma'$, and there is a non-trivial tropical edge collapse morphism $\Gamma' \to \Gamma$. The last statement in Theorem 8.3 implies that $\dim(T_{\mathrm{trop}}(\Gamma')) \geq 2$. By Proposition 6.22, the expected dimension of the moduli space of broken maps is unaffected by collapsing tropical edges. Therefore, the index of $u$ is at most 1. However, the positive dimensionality of the tropical symmetry group $T_{\mathrm{trop}}(\Gamma')$, and the fact that the action of the tropical symmetry group does not have infinitesimal stabilizers implies that $i^{\mathrm{brok}}(u) \geq 2$, leading to a contradiction. We conclude that the tropical type of $u$ is the same as that of the sequence $u_\nu$. The only other phenomenon which occurs in the limit of $\{u_\nu\}_\nu$ is the formation of a boundary node $w \in C$ corresponding to an edge $e$ of length $\ell(e)$ zero, or the length of a boundary edge $\ell(e)$ going to zero or infinity. ∎

# Chapter 9

# Gluing

In this Chapter, we show that a rigid broken map can be glued at nodes to produce a family of unbroken maps, one in each neck-stretched manifold. A fixed perturbation datum $\underline{\mathfrak{p}}$ on a broken manifold $\mathfrak{X}$ can be glued in a natural way to produce a perturbation datum $\underline{\mathfrak{p}}^\nu$ for $X^\nu$ which is equal to $\underline{\mathfrak{p}}$ away from the neck regions. With respect to these perturbation data, we construct a bijection between rigid broken map and rigid maps in neck-stretched manifolds $X^\nu$ with sufficiently large neck lengths. Denote by $\mathcal{M}^{\text{brok},<E}(\mathfrak{X}, L)_0$ the union of strata $\mathcal{M}_\Gamma^{\text{brok},<E}(\mathfrak{X}, L, \underline{x})$in the moduli space of broken maps that have index $i^{\text{brok}}(\Gamma, \underline{x}) = 0$.

**Theorem 9.1.** *(Gluing) Suppose that* $u^0 : C \to \mathfrak{X}$ *is a regular broken disk of index zero. There exists* $\nu_0 > 0$ *such that if* $\nu \geq \nu_0$ *then there exists a family of unbroken disks* $u_\nu : C^\nu \to X^\nu$ *of index zero, with the property that* $\lim_{\nu \to \infty} [u_\nu] = [u^0]$. *For any area bound* $E > 0$ *there exists* $\nu_0$ *such that for* $\nu \geq \nu_0$ *the correspondence* $[u] \mapsto [u_\nu]$ *defines an orientation-preserving bijection between the rigid moduli spaces* $\mathcal{M}^{<E}(X^\nu, L)_0$ *and* $\mathcal{M}^{\text{brok},<E}(\mathfrak{X}, L)_0$ *for* $\nu \geq \nu_0$.

*Remark* 9.2. The gluing operation is defined on broken maps, and not on tropical symmetry orbits of broken maps. In fact if the tropical symmetry group is non-trivial (it is necessarily finite for rigid broken maps), gluing different elements in the tropical symmetry orbit produces distinct sequences of unbroken maps. The convergence result also distinguishes between rigid maps in the same tropical symmetry orbit. Indeed, in Theorem 8.2, if the limit map is rigid then the limit is uniquely determined up to domain reparametrization.

Similar to other gluing theorems in pseudoholomorphic curves, the proof of Theorem 9.1 is an application of a quantitative version of the implicit function theorem for Banach manifolds. The steps are: construction of an approximation solution called the pre-glued map; construction of an approximate inverse to the linearized operator; quadratic estimates; application of the contraction mapping principle, and surjectivity of the gluing construction. Through the proof of the gluing theorem, the notation $c$ denotes a $\nu$-independent constant whose value may be different in every occurrence. The fact that gluing preserves the orientation of rigid moduli spaces is discussed in Remark 9.6.

## 9.1 The approximate solution

Given a broken map, we start by constructing a family of approximate solutions, called *pre-glued maps*, with one map in each neck-stretched manifold $X^\nu$. These maps are obtained by gluing the broken map in the neck region using a cut-off function, and therefore, they are not holomorphic in the neck regions.

The pre-glued family for a broken map is constructed using tropical vertex positions of the broken map. We recall that a rigid broken map is modelled on a rigid tropical graph $\Gamma$, whose vertex positions $\{\mathcal{T}(v) : v \in \mathrm{Vert}(\Gamma)\}$ are uniquely determined. These positions determine the neck lengths for the approximate solution as follows. For any edge $e = (v_+, v_-)$ of $\Gamma$, there exists $l_e > 0$ such that

$$\mathcal{T}(v_+) - \mathcal{T}(v_-) = l_e \mathcal{T}(e), \tag{9.1}$$

where $\mathcal{T}(e) \in \mathfrak{t}_{\mathbb{Z}}$ is the slope of the edge.

The domain of any map in the glued family is a treed disk obtained by replacing tropical nodes in $C$ with necks and leaving the tree part in $C$ unchanged: For any $\nu \in \mathbb{R}_+$ the curve $C_\nu$ has a neck of length $\nu l_e$ in place of a tropical node $w_e$ corresponding to $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ in $C$. Denote by

$$\Gamma_{\mathrm{glue}} \tag{9.2}$$

the type of the glued curve. Recall that the lift of a tropical node $w_e$ in $C$ has matching coordinates (see (Matching at tropical nodes) in Definition 4.14) in the neighbourhoods $U_e^+, U_e^-$ of the lifts $w_e^+, w_e^-$, which are denoted by

$$S_{v_\pm} \supset (U_e^\pm, w_e^\pm) \xrightarrow{z_e^\pm} (\mathbb{C}, 0). \tag{9.3}$$

The coordinates can be chosen to be compositions of the complex exponential map (8.2) and linear functions from tangent spaces to $\mathbb{C}$. The glued curve $C_\nu$ is obtained from $C$ by deleting a small disk in $U_e^\pm$ for every edge $e$ in $\Gamma$, and gluing the remainder of the neighbourhoods $U_e^\pm$ using the identification $z_e^+ \sim e^{-\nu l_e} z_e^-$, and leaving the tree part $T$ unchanged. So the surface and tree parts of $C_\nu$ are

$$C_\nu = S_\nu \cup T.$$

For future use in the proof we point out that the punctured neighbourhoods $U_e^\pm \backslash \{w_e^\pm\}$ have matching logarithmic coordinates

$$(s_e, t_e) : U_e^- \backslash \{w_e^-\} \to [0, \infty) \times S^1, \quad (s_e, t_e) : U_e^+ \backslash \{w_e^+\} \to (-\infty, 0] \times S^1$$

given by $(s_e, t_e) := \pm \ln(z_e^\pm)$.

Translated maps can be glued at the nodal points to yield a sequence of approximate solutions for the holomorphic curve equation in neck-stretched manifolds. We recall that for a rigid tropical graph, the translation sequences are unique and are given by

$$t_\nu : \mathrm{Vert}(\Gamma) \to \nu B^\vee, \quad v \mapsto \nu \mathcal{T}(v). \tag{9.4}$$

Translation sequences give identifications between broken manifolds and neck-stretched manifolds. For any $\nu$, $v \in \mathrm{Vert}(\Gamma)$ the map

$$\mathbb{e}^{t_\nu(v)} : \mathfrak{X}_{P(v)} \to X^\nu_{\tilde{P}} \subset X^\nu$$

is defined in (3.61). The complement of the neck region in $S^\nu$ is a disjoint union

$$S^\nu \backslash \mathrm{Neck}(S^\nu) = \cup_{v \in \mathrm{Vert}(\Gamma)} S^\nu(v).$$

On $S^\nu(v)$, the approximate solution is defined as the translation of the map $u^0_v$, that is,

$$u^{\mathrm{pre}}_\nu := \mathbb{e}^{t_\nu(v)} u^0_v \quad \text{on } S^\nu(v). \tag{9.5}$$

On the treed part $T \subset C^\nu$, we have

$$u^{\mathrm{pre}}_\nu := u^0|_T \quad \text{on } T \subset C^\nu.$$

Next, we define $u^{\mathrm{pre}}_\nu$ on the neck region $\mathrm{Neck}_e(S^\nu)$ corresponding to a tropical edge $e = (v_+, v_-) \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$. The matching condition on $u^0$ at the tropical node $w_e$ implies that there is a point $x_e \in \mathfrak{X}_{P(e)}$ such that the map $u_{v_\pm}$ is asymptotically close to the trivial cylinder

$$u^{\mathrm{vert}}_e : \mathbb{R}_\pm \times S^1 \to \mathfrak{X}_{P(e)}, \quad (s,t) \mapsto e^{\mathcal{T}(e)(s+it)} x_e \tag{9.6}$$

near the nodal lift $w^\pm_e \in C^\pm_v$. That is, if we define $\zeta^\pm_e \in T_{u^{\mathrm{vert}}_e} \mathfrak{X}_{P(e)}$ by the condition

$$u^0_{v_\pm} = \exp_{u^{\mathrm{vert}}_e} \zeta^\pm_e, \tag{9.7}$$

then

$$\| D^k \zeta^\pm_e(s,t) \| \le c e^{-|s|} \tag{9.8}$$

for any $k \ge 0$. Indeed, in an almost complex compactification of the target space $\mathfrak{X}_{P(v_\pm)}$ of $u^0_{v_\pm}$ as in (6.19), the map extends smoothly over $\mp\infty$, from where we obtain the decay (9.8) in cylindrical domain coordinates. In (9.7) we note that the target space of the map $u^0_{v_\pm}$ resp. $\exp_{u^{\mathrm{vert}}_e} \zeta^\pm_e$ is $\mathfrak{X}_{P(v_\pm)}$ resp. $\mathfrak{X}_{P(e)}$, but the relation (9.7) is well-defined because near the node the image of $u_{v_\pm}$ lies in the $P(e)$-cylindrical end of $\mathfrak{X}_{P(v_\pm)}$ which is canonically identified to $\mathfrak{X}_{P(e)}$. Define a cylinder in $\mathrm{Neck}_e(C_\nu)$ corresponding to $u^{\mathrm{vert}}_e$ by

$$u^{\mathrm{vert}}_{e,\nu} := \mathbb{e}^{\frac{1}{2}(t_\nu(v_+) + t_\nu(v_-))} u^{\mathrm{vert}}_e : [\tfrac{-\nu l_e}{2}, \tfrac{\nu l_e}{2}] \times S^1 \to X^\nu.$$

The approximate solution on $\mathrm{Neck}_e(C_\nu)$ is the trivial cylinder $u^{\mathrm{vert}}_{e,\nu}$ corrected by a section obtained by patching $\zeta_e^+, \zeta_e^-$ defined as

$$u^{\mathrm{pre}}_\nu(s,t) := \exp_{u^{\mathrm{vert}}_{e,\nu}}(\zeta^\nu_e(s,t)) : [\tfrac{-\nu l_e}{2}, \tfrac{\nu l_e}{2}] \times S^1 \to X^\nu,$$
$$\zeta^\nu_e(s,t) := \beta(-s)\zeta_e^-(s + \tfrac{\nu l_e}{2}, t) + \beta(s)\zeta_e^+(s - \tfrac{\nu l_e}{2}, t). \quad (9.9)$$

where

$$\beta \in C^\infty(\mathbb{R}, [0,1]), \qquad \begin{cases} \beta(s) = 0 & s \le 0 \\ \beta(s) = 1 & s \ge 1 \end{cases} \quad (9.10)$$

is the cutoff function from (6.15). In other words, in the cylinder in $C_\nu$ corresponding to an edge $e$, one translates the domain on both ends by an amount $\frac{\nu l_e}{2}$, and then patches the map together using the cutoff function and geodesic exponentiation. The definition (9.5) of $u^{\mathrm{pre}}_\nu$ in the complement of the neck, and its definition (9.9) on the neck together give a smooth map $u^{\mathrm{pre}}_\nu : C_\nu \to X^\nu$. Indeed, using the relation $t_\nu(v_+) - t_\nu(v_-) = \nu l_e$, we conclude that the expression in (9.9) is equal to $\mathbb{e}^{t_\nu(v_\pm)} u^0_{v_\pm}$ in the neighborhood of the boundary component $\frac{\pm \nu l_e}{2} \times S^1$.

*Remark* 9.3. The number $\nu \in \mathbb{R}_+$ is called the *map gluing parameter* to differentiate it from the parameter used for gluing nodes in a curve. In standard gluing results, the map gluing parameter $\nu$ is taken to be the neck length in the domain $C_\nu$ of the glued map $u^\nu$. In our setting, the map gluing parameter is the neck length $\nu$ in the target space $X^\nu$. The domain neck lengths, approximately equal to $\nu l_e$, are allowed to vary in the Picard iteration argument. Note that resolving nodes $w_e$ corresponding to tropical edges $e$ does not increase the index of the map. As a result gluing an index zero broken map $u$ produces an isolated map $u^\nu$ in each $X^\nu$.

## 9.2 Fredholm theory for glued maps

We define a map between suitable Banach spaces whose zeroes describe pseudoholomorphic curves close to the approximate solution. Pseudoholomorphic maps are zeroes of the section

$$\mathcal{F}_\nu := (\overline{\partial}, \mathrm{ev}) : \mathcal{M}_{\Gamma_{\mathrm{glue}}} \times \mathrm{Map}(C_\nu, X^\nu) \to$$
$$\Omega^{0,1}(S_\nu, u^*TX) \oplus \Omega^1(T, u^*TL) \oplus X(\Gamma_{\mathrm{glue}}) \quad (9.11)$$

defined in (6.35). The components of the map consist of $\overline{\partial}$ operators on surfaces, shifted gradient operators on treed segments, and various evaluation maps. In this section we define metrics/norms in each of the spaces. We recall the operator $\mathcal{F}_\nu$ later in the section.

We describe a metric on the moduli space of treed disks. Recall that the domain of $u$ is $C$, which is a curve of type $\Gamma$ with surface part $(S, j_S)$ and tree part $T$, and recall from (9.2) that the type $\Gamma_{\text{glue}}$ of the glued curve is given by collapsing all the interior edges in $\Gamma$. In a neighborhood $U_{\mathcal{M}_\Gamma} \subset \overline{\mathcal{M}}_{\Gamma_{\text{glue}}}$ of $\mathcal{M}_\Gamma$, there is a projection map

$$\pi_\Gamma : U_{\mathcal{M}_\Gamma} \to \mathcal{M}_\Gamma$$

such that any curve $C' \in U_{\mathcal{M}_\Gamma}$ is obtained by gluing at tropical nodes of the curve $\pi_\Gamma(C')$ using the coordinates (9.3). We remark that the domain curves $C_\nu$ of the approximate solutions which were constructed by gluing the tropical nodes in $C$ lie in the neighborhood $U_{\mathcal{M}_\Gamma}$, and $\pi_\Gamma(C_\nu) = C$. The subset $U_{\mathcal{M}_\Gamma} \cap \mathcal{M}_{\Gamma_{\text{glue}}}$ close to the boundary stratum $\mathcal{M}_\Gamma$ is equipped with a metric

$$g_{\Gamma_{\text{glue}}} : T(\mathcal{M}_{\Gamma_{\text{glue}}} \cap U_{\mathcal{M}_\Gamma})^{\otimes 2} \to \mathbb{R} \tag{9.12}$$

that is cylindrical in the fibers of $\pi_\Gamma$. That is, each fiber of $\pi_\Gamma$ is isometric to a product of cylinders $\Pi_{e \in \text{Edge}_{\text{trop}}(\Gamma)}(\mathbb{R} \times S^1)$ parametrized by gluing parameters $(s_e, t_e)$.

In the neighborhood of each of the glued curves, we give a description of the complex structures on the surface components. As a preliminary step, we describe the complex structures on curves close to the domain $C$ of $u$. Let the complex curve $(S, j_S)$ be the surface part of $C$. A trivialization

$$\mathcal{S}_\Gamma | U_{\Gamma,C} \simeq S \times U_{\Gamma,C} \tag{9.13}$$

of the universal curve $\mathcal{S}_\Gamma \to \mathcal{M}_\Gamma$ in a neighborhood $U_{\Gamma,C} \subset \mathcal{M}_\Gamma$ of $C$ yields a family of complex structures (as in (4.8))

$$U_{\Gamma,C} \to \mathcal{J}(S), \quad m \mapsto j_\Gamma(m).$$

We write $j_\Gamma$ as a sum

$$j_\Gamma(m) = j_S + \Delta j_\Gamma(m)$$

and assume that the trivialization in (9.13) is chosen so that $\Delta j_\Gamma(m) = 0$ in neighborhoods of special points. Consider a glued treed disk $C_\nu \in \mathcal{M}_{\Gamma_{\text{glue}}}$ whose surface part $(S_\nu, j_{S_\nu}) \subset C_\nu$ is obtained by gluing the tropical nodes in $S \subset C$. In a neighborhood $U_{C_\nu} \subset \mathcal{M}_{\Gamma_{\text{glue}}}$, choose a trivialization of the universal curve $\mathcal{S}_{\Gamma_{\text{glue}}} \to \mathcal{M}_{\Gamma_{\text{glue}}}$ so that the induced family of complex structures

$$U_{C_\nu} \to \mathcal{J}(S_\nu), \quad m \mapsto j_\nu(m), \tag{9.14}$$

satisfies the following : The function $j_\nu$ is a sum

$$j_\nu = j_{S_\nu} + \Delta j_\bullet(m) + \Delta j^\nu_{\text{neck}}(m)$$

where

- $j_{S_\nu}$ is the complex structure on the glued curve $S_\nu$ and is constant over $U_{C_\nu}$,
- the function $m \mapsto \Delta j_\bullet(m)$ is supported in the complement of the neck regions of $S_\nu$ and is equal to $\Delta j_\Gamma(\pi_\Gamma(m))$,
- and $\Delta j^\nu_{\text{neck}}(m)$ is supported in the neck regions of $S_\nu$, and the support is contained in a uniformly bounded neighborhood the boundary of the neck, that is, there is a $\nu$-independent constant $L$ such that

$$\operatorname{supp}(\Delta j^\nu_{\text{neck}}) \subset \bigcup_{e \in \text{Edge}_{\text{trop}}(\Gamma)} \{s : \tfrac{\nu l_e}{2} - L \le |s| \le \tfrac{\nu l_e}{2}\} \times S^1 \subset A(l_\nu) \subset S_\nu. \quad (9.15)$$

Furthermore, there is a $\nu$-independent constant $c$ such that on the neck region corresponding to any edge $e \in \text{Edge}_{\text{trop}}(\Gamma)$ and for any $\nu$

$$\|\Delta j^\nu_{\text{neck}}(m)\|_{C^1} \approx c|\delta_e(m) - \nu l_e|, \quad (9.16)$$

where $\delta_e(m) \in \mathbb{R}_+ \times S^1$ is the gluing parameter used at the node $e$ to smoothen the node $w_e$ in the curve $\pi_\Gamma(m)$.

Indeed, (9.16) can be satisfied by making suitable choices, as it amounts to choosing a diffeomorphism from a cylinder of length $\delta_e(m)$ to a cylinder of length $\nu l_e$ that is a translation map outside a length $L$ sub-cylinder, and whose derivative is bounded as in (9.16). Such a choice of the trivialization of the universal curve ensures that there is a $\nu$-independent constant $c$ such that for any two curves represented by $m_1, m_2$

$$|\Delta j^\nu(m_2) - \Delta j^\nu(m_1)| \le c d_{g_{\Gamma_{\text{glue}}}}(m_1, m_2). \quad (9.17)$$

These uniform estimates are used in the proof of the quadratic estimate in Section 9.5.

The second space in the domain of (9.11) is a space of $W^{1,p}_{\text{loc}}$ maps

$$\operatorname{Map}(C_\nu, X^\nu)_{1,p} \subset \operatorname{Map}(S_\nu, X^\nu)_{1,p} \times \operatorname{Map}(T, X^\nu)_{1,p}$$

defined by requiring that the maps from $S_\nu$ and $T$ agree on the intersection $S_\nu \cap T$. We recall that $T$ is the disjoint union of treed segments $\cup_e T_e$ corresponding to boundary edges $e \in \text{Edge}_\circ(\Gamma)$ with positive lengths; and $T_e = [0,1]$ if the segment is finite, and $\mathbb{R}_{\ge 0}, \mathbb{R}_{\le 0}$ or $\mathbb{R}$ if $\ell(e) = \infty$. The tangent space of $\operatorname{Map}(C_\nu, X^\nu)_{1,p}$ at a map $u : C_\nu \to X^\nu$ is the space of sections

$$\Omega^0(C_\nu, u^*TX) = \Omega^0(S_\nu, (u|_S)^*TX^\nu) \oplus \Omega^0(T, (u|_T)^*TL). \quad (9.18)$$

As in Abouzaid [1, 5.38] the first summand in (9.18) is equipped with a weighted Sobolev norm based on the decomposition of the section into a part constant on the neck and the difference on the neck corresponding to each edge $e \in \text{Edge}_{\text{trop}}(\Gamma)$ described as follows. Denote by

$$(s_e, t_e) \in [-\nu l_e/2, \nu l_e/2] \times S^1$$

the coordinates on the neck region created by the gluing at the node corresponding to the edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$. Let

$$\lambda \in (0,1)$$

be a *Sobolev weight*. Define a *Sobolev weight function*

$$\kappa_\nu : C_\nu \to [0,\infty), \quad \kappa_\nu := \textstyle\sum_{e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)} \beta(\nu l_e/2 - |s_e|)(\nu l_e/2 - |s_e|). \tag{9.19}$$

Here, the function $\beta(\nu l_e/2 - |s_e|)$ is extended by zero outside the neck region corresponding to $e$. As $\nu \to \infty$, $\kappa_\nu$ converges to the Sobolev weight function $\kappa$ defined on the punctured curve $C - \{w_e : e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)\}$ in (6.16). Given a section

$$\xi = (\xi_S, \xi_T) \in \Omega^0(C_\nu, u^*TX)$$

define

$$\begin{aligned} \|\xi\|_{1,p,\lambda} &:= \|\xi_S\|_{1,p,\lambda} + \|\xi_T\|^p_{1,p} \\ \|\xi_S\|_{1,p,\lambda} &:= \Big( \textstyle\sum_e \|\xi_{S,e}(0,0)\|^p + \displaystyle\int_{C_\nu} (\|\nabla \xi_S\|^p \\ &+ \|\xi_S - \textstyle\sum_e \beta(\nu l_e/2 - |s_e|) \mathbb{T}^u_s \xi_{S,e}(0,0)\|^p) \exp(\kappa_\nu \lambda p) \mathrm{d}\,\mathrm{Vol}_{C_\nu} \Big)^{1/p} \end{aligned} \tag{9.20}$$

where $\xi_{S,e}$ is the restriction of $\xi_S$ to the neck region $[\frac{-\nu l_e}{2}, \frac{\nu l_e}{2}] \times S^1$ corresponding to the edge $e$, and $\mathbb{T}^u_s$ is parallel transport from $u(0,t)$ to $u(s,t)$ along $u(s',t)$, $s' \in [0,s]$. Let $\Omega^0(C_\nu, u^*TX)_{1,p,\lambda}$ be the Sobolev completion of $W^{1,p}_{\mathrm{loc}}$ sections with finite norm (9.20); these are sections whose difference from a covariant-constant section on the neck has an exponential decay behavior governed by the Sobolev constant $\lambda$.

The target space of (9.11) is a space of $(0,1)$-forms, which we equip with a weighted $L^p$ norm. For a $(0,1)$-form $\eta \in \Omega^{0,1}(S_\nu, u^*TX^\nu)$ define

$$\|\eta\|_{0,p,\lambda} = \left( \int_{S_\nu} \|\eta\|^p \exp(\kappa_\nu \lambda p) \mathrm{d}\,\mathrm{Vol}_{S_\nu} \right)^{1/p}.$$

Next, we describe a $\overline{\partial}$-operator pulled back by an exponential map on which the implicit function theorem will be applied. Pointwise geodesic exponentiation defines a map (using Sobolev multiplication estimates)

$$\exp_{u_\nu^{\mathrm{pre}}} : \Omega^0(C_\nu, (u_\nu^{\mathrm{pre}})^*TX^\nu)_{1,p,\lambda} \to \mathrm{Map}_{1,p}(C_\nu, X^\nu) \tag{9.21}$$

where $\mathrm{Map}_{1,p}(C_\nu, X^\nu)$ denotes maps of class $W^{\mathrm{loc}}_{1,p}$ from $C_\nu$ to $X^\nu$. We define

$$\begin{aligned} &\overline{\partial} : U_{C_\nu} \times \Omega^0(C_\nu, (u_\nu^{\mathrm{pre}})^*TX^\nu)_{1,p,\lambda} \to \Omega^{0,1}(S_\nu, (u_\nu^{\mathrm{pre}})^*TX^\nu)_{0,p} \oplus \Omega^1(T, (u_\nu^{\mathrm{pre}})^*TL), \\ &(m,\xi) \mapsto \mathbb{T}^{-1}_{u_\nu^{pre},\xi} \left( \overline{\partial}_{j(m),J} (\exp_{u_\nu^{\mathrm{pre}}} \xi)|_{S_\nu}, \left( \frac{1}{\lambda_e} \frac{d}{ds} + \mathrm{grad}_F \right) (\exp_{u_\nu^{\mathrm{pre}}} \xi)|_T \right), \end{aligned} \tag{9.22}$$

where

$$\mathbb{T}_{u_\nu^{\text{pre}},\xi} : \Omega^{0,1}(S_\nu, (u_\nu^{\text{pre}})^* TX^\nu)_{0,p,\lambda} \oplus \Omega^1(T, (u_\nu^{\text{pre}})^* TL)_{0,p} \\ \to \Omega^{0,1}(S_\nu, (\exp_{u_\nu^{\text{pre}}}(\xi))^* TX^\nu)_{0,p,\lambda} \oplus \Omega^1(T, (\exp_{u_\nu^{\text{pre}}}(\xi))^* TL)_{0,p}$$

is the parallel transport defined using an almost-complex connection, and we recall that $U_{C_\nu} \subset \mathcal{M}_{\Gamma_{\text{glue}}}$is a neighborhood of $C_\nu$; and

$$\lambda_e := \begin{cases} \ell_e(m), & e \in \text{Edge}_{\circ,-}^{(0,\infty)}(\Gamma), \\ 1, & e \in \text{Edge}_\circ^\infty(\Gamma). \end{cases}$$

In (9.22), it is necessary to scale the derivative $\frac{d}{ds}$ by $\lambda_e$ because we took $T_e = [0,1]$ for any finite treed segment.

In order to construct local models for moduli of adapted tree disks, we require that the treed disks $C_\nu$ have a collection of interior leaves $e_1, \ldots, e_{d(\bullet)}$ and

$$(\exp_{u_\nu^{\text{pre}}}(\xi))(e_i) \in D, \quad i = 1, \ldots, n.$$

Additionally, we require matching conditions at boundary nodes and lifts of $S_\nu \cap T_\nu$. Using notation from the proof of transversality (Theorem 6.28), these constraints may be incorporated into $\mathcal{F}_\nu$ from (9.11) to produce a map

$$\mathcal{F}_\nu : U_{C_\nu} \oplus \Omega^0(S_\nu, (u_\nu^{\text{pre}})^* TX^\nu)_{1,p,\lambda} \oplus \Omega^0(T, (u_\nu^{\text{pre}})^* TL)_{1,p} \\ \to \Omega^{0,1}(C_\nu, (u_\nu^{\text{pre}})^* TX^\nu)_{0,p,\lambda} \oplus \Omega^1(T, (u_\nu^{\text{pre}})^* TL)_{0,p} \oplus TX(\Gamma_{\text{glue}})/\Delta(\Gamma_{\text{glue}}).$$

whose zeroes correspond to *adapted* pseudoholomorphic maps near the pre-glued map $u_\nu^{\text{pre}}$.

## 9.3 Error estimate

We estimate the failure of the approximate solution to be an exact solution in the Banach norms of the previous section. To derive the estimate, we split the treed disk $C_\nu$ into neck regions corresponding to tropical nodes in $C$, namely

$$\text{Neck}_e(S_\nu) := \{(s_e, t_e) \in [-\tfrac{\nu l_e}{2}, \tfrac{\nu l_e}{2}] \times S^1\} \subset S_\nu, \quad \forall e \in \text{Edge}_{\text{trop}}(\Gamma);$$

and its complement

$$C_\nu^\bullet := C_\nu \setminus \cup_e \text{Neck}_e(S_\nu), \quad \text{and} \quad C_\nu^\bullet = S_\nu^\bullet \cup T.$$

The one-form $\mathcal{F}_\nu(0)$ has contributions from the difference between the perturbation data $P(C)$ and $P(C_\nu)$ on the complement of the neck regions, and from the cutoff function and the difference between $J_u$ and $J_{u_\nu^{\mathrm{pre}}}$ on the neck regions :

$$\|\mathcal{F}_\nu(0)\|_{L^{p,\lambda}(S_\nu)} = \|\overline{\partial}_{J_{u_\nu^{\mathrm{pre}}}} u^\nu_{\mathrm{pre}}\|_{L^{p,\lambda}(S^\bullet_\nu)} + \sum_{e\in \mathrm{Edge}_\circ(\Gamma)} \|\tfrac{1}{\lambda_e} + (\mathrm{grad}_F)_{u_\nu^{\mathrm{pre}}}\|_{L^p(T_e)}$$
$$+ \textstyle\sum_e \|\overline{\partial}\exp_{e^{(s+it)\mathcal{T}(e)}x_e}(\beta(-s_e)\zeta_e^-(s_e+\nu l_e/2, t_e)$$
$$+\beta(s_e)\zeta_e^+(s_e-\nu l_e/2,t_e))\|_{L^{p,\lambda}(\mathrm{Neck}_e(S_\nu))}. \quad (9.23)$$

The first two terms may not vanish because the perturbation is domain dependent: in the complement of the neck regions, the map $u_\nu^{\mathrm{pre}}$ is $P(C)$-holomorphic but not $P(C_\nu)$-holomorphic. For any metric $d_{\overline{\mathcal{M}}}$ on the compactified moduli space $\overline{\mathcal{M}}_{\Gamma_{\mathrm{glue}}}$ of treed disks, the distance between the domain curves is bounded as

$$d_{\overline{\mathcal{M}}}(C_\nu, C) \le c \max_{e\in\mathrm{Edge}_{\mathrm{trop}}(\Gamma)} \exp(-\nu l_e).$$

Therefore, the distance between the domain-dependent perturbations has a similar bound. On the complement of the necks $u_\nu^{\mathrm{pre}}$ is $J(C)$-holomorphic, so

$$\|\overline{\partial}_{J_{u_\nu^{\mathrm{pre}}}} u^\nu_{\mathrm{pre}}\|_{L^{p,\lambda}(S^\bullet_\nu)} \le c\|J(C)-J(C_\nu)\|_{L^\infty} \le c \max_{e\in\mathrm{Edge}_{\mathrm{trop}}(\Gamma)} \exp(-\nu l_e). \quad (9.24)$$

The last term in the right-hand side of (9.23) is equal to

$$\textstyle\sum_e \|(D\exp_{e^{(s+it)\mathcal{T}(e)}x_e}(\mathrm{d}\beta(-s_e)\zeta_e^-(s_e+\nu l_e/2,t_e)+\mathrm{d}\beta(s_e)\zeta_e^+(s_e-\nu l_e/2,t_e))+$$
$$(\beta(-s_e)\mathrm{d}\zeta_e^-(s_e+\nu l_e/2,t_e)+\beta(s_e)\mathrm{d}\zeta_e^+(s_e-\nu l_e/2,t_e)))^{0,1}\|_{L^{p,\lambda}(\mathrm{Neck}_e(S_\nu))}.$$

On the neck regions, the almost complex structure is domain-independent since it is equal to the background almost complex structure (see (6.4)). Holomorphicity of $u$ implies that the terms are non-zero only in the support of $d\beta_e$ which is contained in the interval

$$[-1,1]\times S^1 \subset [-\tfrac{\nu l_e}{2}, \tfrac{\nu l_e}{2}]\times S^1 = \mathrm{Neck}_e(S_\nu).$$

Both $\zeta_e^\pm$ and its derivative decay at the rate of $e^{-s_e}$ on the cylindrical end $\mathbb{R}_+\times S^1$ in $S^\circ$, see (9.8). As a result both the difference between $J_u$ and $J_{u_\nu^{\mathrm{pre}}}$, and the terms containing $d\beta$ are bounded by $ce^{-l_e\nu/2}$ where $c$ is a constant independent of $\nu$. The Sobolev weight function (9.19) has a multiplicative factor of $e^{\lambda l_e \nu/2}$, and therefore,

$$\|\mathcal{F}_\nu(0)\| \le c\textstyle\sum_{e\in\mathrm{Edge}_{\mathrm{trop}}(\Gamma)} e^{-(1-\lambda)l_e\nu/2}, \quad (9.25)$$

with $c$ a constant independent of $\nu$. (See Abouzaid [1, 5.10]).

## 9.4 Uniform right inverse

In this Section, we construct a uniformly bounded right inverse for the linearized operator of the approximate solution. Using the right inverses of the pieces of the broken map, we construct an approximate inverse. For large neck lengths, the error of the approximate inverse is shown to be small enough that an actual inverse can be constructed.

First, for any element in the target space of the linearized operator of the approximate solution, we give a nearby element in the target space of the linearized operator of the broken map. We denote the linearized operator of the approximate solution by

$$D_{u_\nu^{\text{pre}}} := D\mathcal{F}_\nu([C_\nu], 0).$$

Given an element

$$\eta = (\eta_S, \eta_T) \in \Omega^{0,1}(S_\nu, (u_\nu^{\text{pre}}|S_\nu)^* TX^\nu)_{L^{p,\lambda}} \oplus \Omega^1(T, (u_\nu^{\text{pre}}|T)^* TL)_{L^p}$$

in the target space of $D_{u_\nu^{\text{pre}}}$ one obtains an element in the target space of the linearized operator $D_{u^0}$ of the broken map

$$\tilde{\eta} = (\eta_v)_{v\in \text{Vert}(\Gamma)} \oplus \eta_T, \quad \eta_v \in \bigoplus_{v\in\text{Vert}(\Gamma)} \Omega^{0,1}(S_v^\circ, (u_v^0)^* T\overline{\mathfrak{X}}^\circ_{P(v)})$$

as follows. The element $\tilde{\eta}$ is equal to $\eta$ in the tree components $T \subset C$ and in the complement of the neck region on the surface components $S_v \subset C$. On the neck region, for an edge $e = (v_+, v_-) \in \text{Edge}_{\text{trop}}(\Gamma)$, $\tilde{\eta}$ is defined by restricting $\eta|\,\text{Neck}_e(S_\nu)$ to half the neck and extending by zero :

$$\eta_{v,+} : (-\infty, 0] \times S^1 \to \mathfrak{X}_{P(e)}, \quad (s,t) \mapsto \begin{cases} \mathbb{T}_e^{+,\nu}\eta\left(s + \frac{\nu l_e}{2}, t\right), & s \geq \frac{-\nu l_e}{2} \\ 0, \quad s < \frac{-\nu l_e}{2}, \end{cases}$$

and

$$\eta_{v,-} : [0, \infty) \times S^1 \to \mathfrak{X}_{P(e)}, \quad (s,t) \mapsto \begin{cases} \mathbb{T}_e^{-,\nu}\eta\left(s - \frac{\nu l_e}{2}, t\right), & s \leq \frac{\nu l_e}{2} \\ 0, \quad s > \frac{\nu l_e}{2}, \end{cases} ,$$

where

$$\mathbb{T}_e^{\pm,\nu} : \Gamma((u_\nu^{\text{pre}})^* T\mathfrak{X}_{P(e)}) \to \Gamma((u_v^{\pm})^* T\mathfrak{X}_{P(e)}) \tag{9.26}$$

is parallel transport along the path

$$\exp_{e^{(s+it)\mathcal{T}(e)}x_e}(\rho\zeta_e^\nu(s \pm \tfrac{\nu l_e}{2}, t) + (1-\rho)\zeta_e^{\pm}(s,t)), \quad \rho \in [0,1],$$

$\zeta_\nu^e$ is defined in (9.9) and $\zeta_e^{\pm}$ is defined in (9.7).

We apply the inverse of the linearization of the broken map to the element constructed in the previous paragraph. Since the broken map $u^0$ is regular and isolated, its linearized operator is bijective. We recall that the linearized operator is a map of Banach spaces (see (6.39))

$$D_{u^0} : T_m\mathcal{M}_\Gamma \times \mathrm{Map}(C, \mathfrak{X})_{1,p,\lambda} \to \Omega^{0,1}(S, (u^0|S)^*T\mathfrak{X}) \\ \oplus \Omega^1(T, (u^0|T)^*TL) \oplus \mathrm{ev}_\Gamma^* T\mathfrak{X}/T\Delta.$$

Bijectivity of $D_{u^0}$ implies there is an inverse $(m, \xi)$ for the element $(\tilde{\eta}, 0 \in \mathrm{ev}_\Gamma^* T\mathfrak{X}/T\Delta)$. We write $\xi = ((\xi_v)_{v\in\mathrm{Vert}(\Gamma)}, \xi_T)$. The vanishing of the last term in $D_{u^0}(m, \xi)$ means that $\xi$ satisfies matching conditions at tropical and boundary nodes, and the interior markings $z_i$ satisfy the divisor constraint :

$$\xi(z_i) \in T_{u_\pm(z_i)}\mathfrak{D}.$$

The matching at tropical nodes implies that for any tropical edge $e = (v_+, v_-)$, the limit of $\xi_{v_+}, \xi_{v_-}$ at the cylindrical end $e$ is equal :

$$\xi_{v_+,e} = \xi_{v_-,e} =: \xi_e \in T_{x_e}\mathfrak{X}_{P(e)}.$$

We now define the approximate inverse by patching the inverse of the linearization of the broken map. We denote the approximate inverse of $D_{u_\nu^{\mathrm{pre}}}$ by $Q^\nu$. On the complement of the neck regions in $C_\nu$ we define

$$Q^\nu(\eta) := \xi \quad \text{on } C_\nu \setminus \cup_e \mathrm{Neck}_e(S_\nu);$$

and on the neck region corresponding to a tropical edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ we patch the solutions $\xi_{v_\pm}$ together using a cutoff function:

$$\begin{aligned} Q^\nu\eta &:= \beta\left(-s + \tfrac{1}{4}\nu l_e\right)((\mathbb{T}_e^{-,\nu})^{-1}\xi_{v_-}(s + \nu l_e/2) - \mathbb{T}_e^\nu\xi_e) \\ &\quad + \beta\left(s + \tfrac{1}{4}\nu l_e\right)((\mathbb{T}_e^{+,\nu})^{-1}\xi_{v_+}(s - \nu l_e/2)) - \mathbb{T}_e^\nu\xi_e) + \mathbb{T}_e^\nu\xi_e \\ &\quad \in \Omega^0(C_\nu, (u_\nu^{\mathrm{pre}})^*TX)_{1,p,\lambda}, \end{aligned} \tag{9.27}$$

where $\mathbb{T}_e^{\pm,\nu}$ is defined in (9.26) and $\mathbb{T}_e^\nu$ is the parallel transport from $\Gamma((u_{\nu,e}^{\mathrm{vert}})^*T\mathfrak{X}_{P(e)})$ to $\Gamma((u_\nu^{\mathrm{pre}})^*T\mathfrak{X}_{P(e)})$.

The approximate inverse $Q^\nu$ is uniformly bounded for all $\nu$: It follows easily from the construction of $Q^\nu$ that the operations $\eta \mapsto \tilde{\eta}, \tilde{\eta} \mapsto \xi$ are uniformly bounded operators for all $\nu$, and that on the complement of the neck regions there is a uniform bound on the norm of the operator $\xi \mapsto (Q_\nu\eta)|C_\nu^\bullet$. On the neck $\mathrm{Neck}_e(S_\nu)$ corresponding to a tropical edge $e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$ there is a uniform bound

$$\|((Q^\nu\eta) - \mathbb{T}_e\xi_e)e^{\kappa_\nu\lambda}\|_{L^p(\mathrm{Neck}_e(S_\nu))} \le c\|\xi\|_{W^{1,p,\lambda}}.$$

However, we need a $\kappa_\nu$-weighted Sobolev norm bound on $((Q^\nu\eta) - \mathbb{T}_e(Q^\nu\eta)(0,0)_e)$, which follows from the observation

$$|Q_\nu(0,0)_e - \xi_e| \le |\xi_{v_+}(\tfrac{-\nu l_e}{2},0)_e - \xi_e| + |\xi_{v_-}(\tfrac{\nu l_e}{2},0)_e - \xi_e| \le ce^{-\lambda\nu l_e/2}\|\eta\|_{L^{p,\lambda}},$$

which implies that

$$\|(\mathbb{T}_e Q^\nu(0,0)_e - \mathbb{T}_e\xi_e)e^{\kappa_\nu\lambda}\| \le c\|\eta\|_{L^{p,\lambda}}.$$

We have thus shown that there is a $\nu$-independent constant $c$ such that

$$\|Q^\nu\| \le c.$$

Next, we give an error estimate for the approximate inverse. We need to bound the quantity $D_{u^\nu_{\text{pre}}}Q^\nu\eta - \eta$. On the complement of the neck regions of $C_\nu$ (including treed segments), this quantity is bounded by the difference in the domain-dependent perturbation on $(C,j)$ and $(C_\nu, j^\nu)$. Similar to (9.24) in the estimate of $\mathcal{F}^\nu(0)$, we have the bound

$$\|D_{u^\nu_{\text{pre}}}Q^\nu\eta - \eta\|_{L^{p,\lambda}(C^\bullet_\nu)} \le c \max_{e\in \text{Edge}_{\text{trop}}(\Gamma)} \exp(-\nu l_e). \tag{9.28}$$

We bound $\|D_{u^\nu_{\text{pre}}}Q^\nu\eta - \eta\|_{L^{p,\lambda}}$ on $\text{Neck}_e(S_\nu)$ corresponding to an edge $e = (v_+, v_-) \in \text{Edge}_{\text{trop}}(\Gamma)$. We drop the parallel transport notation in the analysis, noting that both the transport map and its derivative contribute smooth multiplicative factors that decay as $ce^{\nu l_e/2-|s|}$ on the neck. We recall that $\text{Neck}_e(S_\nu) \simeq \{(s,t) \in [\frac{-\nu l_e}{2}, \frac{\nu l_e}{2}] \times S^1\}$, and analyze the half $\{s \ge 0\}$, the analysis on the other half being similar. On $\{s \ge 0\}$, we have

$$\eta = \eta_{v_+} = D_{u^0}\xi_{v_+}, \quad Q^\nu\eta = \xi_{v_+} + \beta(-s + \tfrac{\nu l_e}{4})(\xi_{v_-} - \mathbb{T}_e\xi_e),$$

and therefore,

$$D_{u^\nu_{\text{pre}}}Q^\nu\eta - \eta = D_{u^\nu_{\text{pre}}}(\beta(-s + \tfrac{\nu l_e}{4})(\xi_{v_-} - \mathbb{T}_e\xi_e)) + (D_{u^0_{v_+}} - D_{u^\nu_{\text{pre}}})\xi_{v_+}. \tag{9.29}$$

We have

$$\|(D_{u^0_{v_+}} - D_{u^\nu_{\text{pre}}})\xi_{v_+}\|_{L^{p,\lambda}(S_\nu)} \le ce^{-\nu l_e/2}\|\xi_{v_+}\|_{W^{1,p,\lambda}(S^\circ_{v_+})} \le ce^{-\nu l_e/2}\|\eta\|_{L^{p,\lambda}(C_\nu)} \tag{9.30}$$

since

$$d_{X^\nu}(u^0_{v_+}(s - \frac{\nu l_e}{2}, t), u^\nu_{\text{pre}}(s,t)) \le ce^{-\nu l_e/2}.$$

Since $D_{u^0_{v_-}}\xi_{v_-} = 0$ on $\{s \ge 0\}$, by a similar bound to (9.30) on $(D_{u^0_{v_-}} - D_{u^\nu_{\text{pre}}})\xi_{v_-}$, we have

$$\|D_{u^\nu_{\text{pre}}}\xi_{v_-}\|_{L^{p,\lambda}(\{s\ge 0\})} \le ce^{-\nu l_e/2}\|\xi_{v_-}\|_{W^{1,p,\lambda}(S^\circ_{v_-})} \le ce^{-\nu l_e/2}\|\eta\|_{L^{p,\lambda}(C_\nu)}. \tag{9.31}$$

From (9.29), (9.30), (9.31) we conclude

$$\|D_{u^\nu_{\mathrm{pre}}} Q^\nu \eta - \eta\|_{L^{p,\lambda}} \le \|\beta(D_{u^\nu_{\mathrm{pre}}}(\mathbb{T}_e \xi_e))\| + \|d\beta(\xi_{v_-} - \mathbb{T}_e \xi_e)\| + ce^{-\nu l_e/2}\|\eta\|. \quad (9.32)$$

To bound the first term in the right-hand side of (9.32) we observe that $\mathbb{T}_e \xi_e$ is the parallel transport of a covariant constant section on the trivial cylinder $u_e^{\mathrm{ver}}$, and therefore,

$$\|D_{u^\nu_{\mathrm{pre}}}(\mathbb{T}_e \xi_e)\| = \|(D_{u^\nu_{\mathrm{pre}}} - D_{u_e^{\mathrm{ver}}})(\mathbb{T}_e \xi_e)\| \le ce^{-\nu l_e/2}|\xi_e| \le ce^{-\nu l_e/2}\|\eta\|_{L^{p,\lambda}}.$$

It remains to bound the second term which is supported in the unit interval $[\frac{\nu l_e}{4}, \frac{\nu l_e}{4} + 1] \times S^1$ in the neck $[\frac{-\nu l_e}{2}, \frac{\nu l_e}{2}] \times S^1$. In this interval, the Sobolev weight in the curve $S_{v_-}$ and the glued curve $S_\nu$ differ by $\frac{\nu l_e}{2}$ :

$$\kappa_\nu(s,t) = \kappa_{S^\circ_{v_-}}(s + \nu l_e/2) - \nu l_e/2, \quad \mp s \ge \nu l_e/2.$$

Therefore there is a $\nu$-independent constant $c$ such that

$$\|\mathrm{d}\beta(s - \nu l_e/4)\xi_{v_-}\|_{L^{p,\lambda}(S_\nu)} \le ce^{-\lambda \nu l_e/2}\|\xi_{v_-}\|_{W^{1,p,\lambda}(S^\circ_v)} \le ce^{-\lambda\nu l_e/2}\|\eta\|_{L^{p,\lambda}(S_\nu)}. \quad (9.33)$$

From (9.28), (9.33), one obtains an estimate as in Fukaya-Oh-Ohta-Ono [36, 7.1.32], Abouzaid [1, Lemma 5.13]: For some constant $c > 0$, for any $\nu$

$$\|D_{u_\nu^{\mathrm{pre}}} Q^\nu - \mathrm{Id}\,\| < c \min_{e \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)} (\exp(-\lambda \nu l_e/2), \exp(-(1-\lambda)\nu l_e/2)). \quad (9.34)$$

It follows that for $\nu$ sufficiently large an actual inverse may be obtained from the Taylor series formula

$$D^{-1}_{u_\nu^{\mathrm{pre}}} = Q^\nu (D_{u_\nu^{\mathrm{pre}}} Q^\nu)^{-1} = Q^\nu \sum_{k \ge 0} (I - Q^\nu D_{u_\nu^{\mathrm{pre}}})^k.$$

The approximate inverse $Q^\nu$ is uniformly bounded for all $\nu$. For large enough $\nu$, (9.34) implies that $\|D_{u_\nu^{\mathrm{pre}}} Q^\nu - \mathrm{Id}\| \le \frac{1}{2}$, and so,

$$\|D^{-1}_{u_\nu^{\mathrm{pre}}}\| \le 2\|Q^\nu\| \le c. \quad (9.35)$$

## 9.5 Uniform quadratic estimate

We obtain a uniform quadratic estimate for the non-linear terms in the map cutting out the moduli space locally. We will prove that there exists a constant $c$ such that for all $\nu$

$$\|D_{(m_1,\xi_1)}\mathcal{F}_\nu(m_2,\xi_2) - D_{u_\nu^{\mathrm{pre}}}(m_2,\xi_2)\| \le c\|(m_1,\xi_1)\|_{1,p,\lambda}\|(m_2,\xi_2)\|_{1,p,\lambda}. \quad (9.36)$$

We prove the quadratic estimate for the $\overline{\partial}$ term on surface components. The other terms in the operator $\mathcal{F}_\nu$ are left to the reader as the proof is similar.

As a preliminary step we prove a quadratic estimate on a simpler operator. Define

$$\mathcal{G} : W^{1,p,\lambda}(\Omega^0(S_\nu, (u_\nu^{\text{pre}})^* TX^\nu)) \to L^{p,\lambda}(\Omega^1(S_\nu, (u_\nu^{\text{pre}})^* TX^\nu)),$$
$$\xi \mapsto \mathbb{T}_\xi^{-1} d(\exp_{u_\nu^{\text{pre}}} \xi).$$

*Claim* 9.4. There is a $\nu$-independent constant $c$ such that

$$\|D_{\xi_1}\mathcal{G}(\xi_2) - D_0\mathcal{G}(\xi_2)\|_{0,p,\lambda} \le c\|\xi_1\|_{1,p,\lambda}\|\xi_2\|_{1,p,\lambda}$$

if $\|\xi_1\|_{1,p,\lambda}$ is small enough.

*Proof of Claim.* For $x \in X^\nu$, $\xi_1, \xi_2 \in T_x X^\nu$, define

$$\overline{\mathbb{T}}^x_{-\xi_1} : T_x X \to T_x X, \quad \xi_2 \mapsto \mathbb{T}_{\xi_1}^{-1} \tfrac{\mathrm{d}}{\mathrm{d}\tau} \exp_x(\xi_1 + \tau\xi_2)|_{\tau=0}.$$

It extends to a map on sections. For a map $u : S_\nu \to X^\nu$, and $\xi_1 \in \Gamma(S_\nu, u^* TX^\nu)$ define

$$\overline{\mathbb{T}}^u_{-\xi_1} : \Gamma(S_\nu, u^*TX^\nu) \to \Gamma(S_\nu, u^*TX^\nu), \quad \xi_2 \mapsto (z \mapsto \overline{\mathbb{T}}^{u(z)}_{-\xi_1(z)}(\xi_2(z))).$$

We have

$$\begin{aligned} D_{\xi_1}\mathcal{G}(\xi_2) - D_{u_\nu^{\text{pre}}}\mathcal{G}(\xi_2) &= \tfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{T}^{-1}_{\xi_1+\tau\xi_2}\mathrm{d}_z(\exp_{u_\nu^{\text{pre}}}(\xi_1 + \tau\xi_2)) - \tfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{T}^{-1}_{\tau\xi_2}\mathrm{d}_z(\exp_{u_\nu^{\text{pre}}}(\tau\xi_2)) \\ &= \nabla_z(\overline{\mathbb{T}}^{u_\nu^{\text{pre}}}_{-\xi_1}\xi_2 - \xi_2), \end{aligned} \tag{9.37}$$

where in the second line $\nabla_z$ is differentiation along the domain curve $S_\nu$. In the second equality we switch the order of differentiation between $\frac{\mathrm{d}}{\mathrm{d}\tau}$ and $\mathrm{d}_z$. For any $z \in S_\nu$ we have a uniform pointwise estimate

$$\begin{aligned} |D_{\xi_1}\mathcal{G}(\xi_2) - D_{u_\nu^{\text{pre}}}\mathcal{G}(\xi_2)| &\le |\nabla_z(\overline{\mathbb{T}}^{u(z)}_{-\xi_1(z)} - \mathrm{Id})| \cdot |\xi_2(z)| + |\overline{\mathbb{T}}^{u_\nu^{\text{pre}}(z)}_{-\xi_1(z)} - \mathrm{Id}| \cdot |(\nabla_z\xi_2)(z)| \\ &\le c((|du_\nu^{\text{pre}}(z)| \cdot |\xi_1(z)| + |\nabla_z\xi_1(z)|) \cdot |\xi_2(z)| + \\ &\quad |\xi_1(z)| \cdot |\nabla_z\xi_2(z)|) \end{aligned}$$

where the constant $c$ is $\nu$-independent. Indeed, such a uniform constant exists because the complement of the neck regions is compact and identical for all $\nu$; and the neck regions have a cylindrical metric, and only the lengths of the cylinders vary with $\nu$. Then,

$$\begin{aligned} \|D_{\xi_1}\mathcal{G}(\xi_2) - D_{u_\nu^{\text{pre}}}\mathcal{G}(\xi_2)\| &\le c(\|\mathrm{d}u_\nu^{\text{pre}}\|_{L^\infty} \cdot \|\xi_1\|_{L^\infty} \cdot \|\xi_2\|_{L^{p,\lambda}} + \|\nabla_z\xi_1(z)\|_{L^{p,\lambda}} \cdot \|\xi_2\|_{L^\infty} \\ &\quad + \|\xi_1\|_{L^\infty} \cdot \|\nabla_z\xi_2(z)\|_{L^{p,\lambda}}) \le c\|\xi_1\|_{1,p,\lambda}\|\xi_2\|_{1,p,\lambda}, \end{aligned}$$

where for the last inequality, we use the fact that $\|du_\nu^{\mathrm{pre}}\|_{L^\infty}$ is uniformly bounded for all $\nu$ by construction, and the following bound from Sobolev embedding : For any section $\xi \in W^{1,p,\lambda}(S_\nu)$ there is a $\nu$-independent constant $c$ such that $\|\xi\|_{L^\infty} \le c\|\xi\|_{W^{1,p,\lambda}}$. This proves the Claim. ■

We obtain the quadratic estimate (9.36) for $\mathcal{F}_\nu$ by adapting the proof of the above claim. We note that, compared to $\mathcal{G}$, the operator $\mathcal{F}_\nu$ additionally consists of a projection to (0, 1)-forms :

$$\mathcal{F}_\nu : (m, \xi) \mapsto \mathbb{T}_\xi^{-1} \pi^{0,1}_{j(m),J(\xi)} \mathrm{d}(\exp_{u_\nu^{\mathrm{pre}}} \xi),$$

where we abbreviate $J_{\exp_{u_\nu^{\mathrm{pre}}} \xi}$ as $J(\xi)$. The analog of (9.37) is

$$\begin{aligned}
&D_{(m_1,\xi_1)}\mathcal{F}_\nu(\xi_2) - D_{u_\nu^{\mathrm{pre}}}\mathcal{F}_\nu(\xi_2)\\
&\quad = \tfrac{\mathrm{d}}{\mathrm{d}\tau} \mathbb{T}^{-1}_{\xi_1+\tau\xi_2} \pi^{0,1}_{j_\nu(m_1+\tau m_2),J(\xi_1+\tau\xi_2)} \mathrm{d}_z(\exp_{u_\nu^{\mathrm{pre}}}(\xi_1 + \tau\xi_2))|_{\tau=0}\\
&\qquad - \tfrac{\mathrm{d}}{\mathrm{d}\tau} \mathbb{T}^{-1}_{\tau\xi_2} \pi^{0,1}_{j_\nu(\tau m_2),J(\tau\xi_2)} \mathrm{d}_z(\exp_{u_\nu^{\mathrm{pre}}}(\tau\xi_2))|_{\tau=0}\\
&\quad = \pi^{0,1}_{j_\nu(m_1),J(\xi_1)} \nabla_z(\overline{\mathbb{T}}^u_{-\xi_1}\xi_2) + (\nabla_z(\pi^{0,1}_{j_\nu(m_1),J(\xi_1)}))(\overline{\mathbb{T}}^u_{-\xi_1}\xi_2)\\
&\qquad - \pi^{0,1}_{j(C_\nu),J(0)}(\nabla_z\xi_2) - (\nabla_z\pi^{0,1}_{j(C_\nu),J(0)})\xi_2, \qquad (9.38)
\end{aligned}$$

where $\tau m_2$ resp. $m_1 + \tau m_2$, $\tau \in [0, 1]$ is a path in a neighborhood $U_{C_\nu}$ of $C_\nu$ in the moduli space $\mathcal{M}_{\mathrm{glue}}$. The estimate (9.36) can be obtained from the above expression in a similar way to the proof of Claim 9.4. We point out that the metric $g_{C_\nu}$ on $U_{C_\nu}$ is cylindrical on the non-compact ends of $\mathcal{M}_{\mathrm{glue}}$, and with respect to this metric $\|j_\nu(m_1) - j_\nu(m_2)\|_{C^1} \le c d_{g_{\Gamma_{\mathrm{glue}}}}(m_1, m_2)$, see (9.17).

## 9.6 Picard iteration

We apply the implicit function theorem to obtain an exact solution for a glued holomorphic map. We recall the following version of the Picard Lemma [55, Proposition A.3.4].

**Lemma 9.5.** *(Picard Lemma) Let $X$ and $Y$ be Banach spaces, $U \subset X$ be an open set containing* 0*, and $f : U \to Y$ be a smooth map. Suppose $df(0)$ is invertible with inverse $Q : Y \to X$. Suppose $c$ and $\epsilon > 0$ are constants such that $\|Q\| \le c$, $B_\epsilon \subset U$, and*

$$\|df(x) - df(0)\| \le \tfrac{1}{2c} \quad \forall x \in B_\epsilon(0).$$

*Suppose $f(0) \le \frac{\epsilon}{4c}$. Then, there is a unique point $x_0 \in B_\epsilon$ satisfying $f(x_0) = 0$.*

Picard's Lemma and the estimates (9.25), (9.35), (9.36) imply the existence of a solution $(m(\nu), \xi(\nu))$ to the equation

$$\mathcal{F}^{\nu}(m(\nu), \xi(\nu)) = 0$$

for each $\nu$. The map

$$u_{\nu} := \exp_{u_{\nu}^{\mathrm{pre}}}(\xi(\nu))$$

is a $(j(m(\nu)), J^{\nu})$-holomorphic map to $X^{\nu}$. Additionally, there is a $\nu$-independent constant $\epsilon > 0$ such that $(m(\nu), \xi(\nu))$ is the unique zero of $\mathcal{F}^{\nu}$ in an $\epsilon$-neighbourhood of $((C_{\nu}, j^{\nu}), u_{\nu}^{\mathrm{pre}})$ with respect to the $g_{\Gamma_{\mathrm{glue}}}$-norm on $m(\nu)$ and the weighted Sobolev norm $W^{1,p,\lambda}$ on $\xi(\nu)$.

## 9.7 Surjectivity of gluing

We show that the gluing construction gives a bijection between rigid maps in neck-stretched manifolds and rigid broken maps. Note that any family $\{u'_{\nu} : C'_{\nu} \to X^{\nu}\}_{\nu}$ converges to a broken map $u : C \to \mathfrak{X}$ by Theorem 8.2. To prove the bijection we must show that any such family of maps is in the image of the gluing construction. Since the implicit function theorem used to construct the gluing gives a unique solution in a neighbourhood, it suffices to show that the maps $u'_{\nu}$ are close, in the Sobolev norm used for the gluing construction, to the approximate solution $u_{\nu}^{\mathrm{pre}}$ defined by (9.9).

We first show that the domain curves of the converging sequence of maps are close enough to the domains of the approximate solution with respect to the cylindrical metric $g_{\Gamma_{\mathrm{glue}}}$ from (9.12). In the definition of Gromov convergence, the convergence of domains implies that $C'_{\nu} \to C$ in the compactified moduli space $\overline{\mathcal{M}}_{\Gamma_{\mathrm{glue}}}$, which implies

$$\pi_{\Gamma}(C'_{\nu}) \to C \quad \text{in } \mathcal{M}_{\Gamma}. \tag{9.39}$$

We additionally need to prove that the distance $d_{g_{\Gamma_{\mathrm{glue}}}}(C'_{\nu}, C_{\nu}) \to 0$ where the metric $g_{\Gamma_{\mathrm{glue}}}$ is cylindrical in the non-compact ends of $\overline{\mathcal{M}}_{\Gamma_{\mathrm{glue}}}$. By assumption the limit map $u$ does not have any tropical symmetry. Therefore, the translation sequence $t_{\nu}$ is uniquely determined by the tropical graph of $u$ and coincides with the translations used for gluing. By (Thin cylinder convergence), for any tropical edge $e = (v_+, v_-) \in \mathrm{Edge}_{\mathrm{trop}}(\Gamma)$, the gluing parameters $l'_{\nu}(e) + i\theta'_{\nu}(e)$ of the curves $C'_{\nu}$ satisfy

$$\lim_{\nu \to \infty} \theta'_{\nu}(e) = 0, \qquad \lim_{\nu \to \infty} (t_{\nu}(v_+) - t_{\nu}(v_-) - \mathcal{T}(e) l'_{\nu}(e)) = 0$$

The gluing parameter of $C_{\nu}$ at the edge $e$ is $\nu l_e$, which satisfies the relation

$$t_{\nu}(v_+) - t_{\nu}(v_-) = \mathcal{T}(e) \nu l_e.$$

Therefore, $l'_\nu(e) - \nu l_e \to 0$, and $d_{g_{\Gamma_{\text{glue}}}}(C_\nu, C'_\nu) \to 0$ as $\nu \to \infty$. In addition there is a biholomorphism

$$\phi_\nu : (C_\nu, j_\nu([C'_\nu])) \to C'_\nu, \tag{9.40}$$

where $j_\nu$ is defined in (9.14).

Next, we show that the maps in the converging sequence are close enough to the approximate solutions. Via the identification (9.40), we view $u'_\nu$ as a map on $C_\nu$. We need to bound the section $\xi'_\nu \in \Omega^0(C_\nu, (u^{pre}_\nu)^* TX^\nu)$ defined by the equation $u'_\nu = \exp_{u^{pre}_\nu} \xi'_\nu$ in the weighted Sobolev norm (9.20). Consider the neck region in $C_\nu$ corresponding to an edge $e$ with coordinates

$$(s_e, t_e) \in [-\tfrac{\nu l_e}{2}, \tfrac{\nu l_e}{2}] \times \mathbb{R}/2\pi\mathbb{Z}.$$

Denote the midpoint of the neck as

$$0_e := \{(s_e, t_e) = (0, 0)\} \in C_\nu.$$

In the neck region, the maps $u^{pre}_\nu$ and $u'_\nu$ are equal to a vertical cylinder perturbed by a quantity that decays exponentially in the middle of the cylinder. The vertical cylinder is determined by $u^{pre}_\nu(0_e)$ resp. $u'_\nu(0_e)$. The sequence $u^{pre}_\nu(0_e)$ converges to $x_e$ because of the asymptotic decay of the sections $\zeta^\pm$. The sequence $u'_\nu(0_e)$ converges to $x_e$ by (Thin cylinder convergence). Indeed, since the complex structure $\Delta j^\nu(C'_\nu)$ is standard on a truncation $[-\frac{\nu l_e}{2} - L, \frac{\nu l_e}{2} - L] \times S^1$ of the neck (see (9.15)), and the mid point of the cylinder is preserved by the biholomorphism $\phi$ in (9.40), (Thin cylinder convergence) is applicable with the coordinates $(s_e, t_e)$. On the neck region, the section $\xi'_\nu$ and its derivatives decay exponentially :

$$|D^k \xi'_\nu(s_e, t_e)| \le c e^{-(\nu l_e/2 - |s_e|)}, \quad k \in \{0, 1\}.$$

This inequality follows from the decay of the terms $\zeta^\pm_\nu$ in the definition of $u^{pre}_\nu$, and the breaking annulus lemma applied to $u'_\nu$. Consequently $\|\xi'_\nu\|_{W^{1,p,\lambda}}$ can be made small enough by taking a large $\nu$ and shrinking the neck by a fixed amount: that is, we decrease the cylinder length to $\nu l_e - C$ where $C$ is a constant independent of $\nu$. Next we consider the complement of the neck regions. Here, the sequences $u'_\nu$ and $u^{pre}_\nu$ uniformly converge to $u$. So, by taking $\nu$ large enough the maps $u^{pre}_\nu$ and $u'_\nu$ are $W^{1,p}$-close enough in the complement of the neck regions.

*Remark* 9.6. (Gluing preserves orientations) From the definition of orientation of moduli spaces in Remark 6.29 it is easily seen that gluing preserves orientation in the special case that the Cauchy-Riemann operator on each surface component of the broken map $u^0$ is complex linear. It remains to consider the effect of the interpolation from the linearized operator $D_{u^0}$ resp. $D_{u_\nu}$ to a complex linear operator $\overline{\partial}_{u^0}$ resp. $\overline{\partial}_{u_\nu}$. This interpolation can be accomplished by an interpolation on the Cauchy-Riemann operator one ach component $u|_{S_v}$ to a complex-linear operator; the interpolations can be then be glued to obtain a interpolation of $D_{u_v}$ to a complex linear operator. It follows that the gluing sign is the same as in the complex-linear case, which is to say, positive.

## 9.8 Tubular neighbourhoods and true boundary

In Section 8.6 we proved a convergence result that the boundaries of one-dimensional moduli spaces of rigid broken maps consist of maps that either have a length zero boundary edge, or a broken boundary edge. Theorem 9.7 is the converse gluing result, and implies that given a broken map $u$ containing a boundary edge that is broken or has zero length, the map $u$ indeed occurs as the codimension one boundary of the expected moduli spaces. The convergence and gluing results, together, allow us to identify the true boundary of one-dimensional moduli spaces. A stratum containing a zero length edge is actually a *fake boundary stratum* as it is the boundary of two different rigid strata with opposite induced boundary orientations. Thus, the *true boundary strata* are those that have a broken boundary edge, see Remark 9.8.

**Theorem 9.7.** *Let $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$ be a coherent regular perturbation datum for all types $\Gamma$. Suppose $\Gamma$ is a type of broken treed disks and $\underline{x} \in \mathcal{I}(L)^{n+1}$ is a set of limits for boundary leaves such that $i(\Gamma, \underline{x}) = 1$.*

(a) *(Tubular neighbourhoods) If a type $\Gamma_{\text{glue}}$ of broken maps is obtained from a type $\Gamma$ by collapsing an edge $e \in \text{Edge}_{\circ,-}(\Gamma')$ with $\ell(e) = 0$, or by making an edge length finite/non-zero, then the stratum $\mathcal{M}^{\text{brok}}_\Gamma(L, \underline{x})$ is a codimension one boundary of in $\overline{\mathcal{M}}^{\text{brok}}_{\Gamma_{\text{glue}}}(L, \underline{x})$ and has a tubular neighborhood in it; and*

(b) *(Orientations) the orientations are compatible with the morphisms (Cutting edges) and (Collapsing an edge/Making an edge finite/non-zero) in the following sense:*

   (i) *Suppose $\Gamma$, $\Gamma_0$, $\Gamma_c$ are types of broken maps related by the morphisms*

$$\Gamma \xrightarrow{\text{Make } \ell(e) \text{ zero}} \Gamma_0 \xrightarrow{\text{Collapse } e} \Gamma_c,$$

   *and $i^{\text{brok}}(\Gamma, \underline{x}) = i^{\text{brok}}(\Gamma_c, \underline{x}) = 1$, $i^{\text{brok}}(\Gamma_0, \underline{x}) = 0$. Then, the boundary orientation induced by $\mathcal{M}^{\text{brok}}_\Gamma(L, \underline{x})$ on $\mathcal{M}^{\text{brok}}_{\Gamma_0}(L, \underline{x})$ is the opposite of the boundary orientation induced by $\mathcal{M}^{\text{brok}}_{\Gamma_c}(L, \underline{x})$ on $\mathcal{M}^{\text{brok}}_{\Gamma_0}(L, \underline{x})$.*

   (ii) *Suppose $\Gamma_f$, $\Gamma$, $\Gamma_1$, $\Gamma_2$ are types of broken maps and $\underline{x}_1 \in \mathcal{I}(L)^{d_\circ(\Gamma_1)+1}$, $\underline{x}_2 \in \mathcal{I}(L)^{d_\circ(\Gamma_2)+1}$ are labels such that there are morphisms*

$$(\Gamma_f, \underline{x}) \xleftarrow{\text{Make } \ell(e) \text{ finite}} (\Gamma, \underline{x}) \xrightarrow{\text{Cut } e} (\Gamma_1, \underline{x}_1) \times (\Gamma_2, \underline{x}_2),$$

*and* $i^{\text{brok}}(\Gamma_f, \underline{x}) = 1$, $i^{\text{brok}}(\Gamma, \underline{x}) = i^{\text{brok}}(\Gamma_1, \underline{x}_1) = i^{\text{brok}}(\Gamma_2, \underline{x}_2) = 0$. *Then, there is an isomorphism*

$$\mathcal{M}_{\Gamma}^{\text{brok}}(L, \underline{x}) \simeq \mathcal{M}_{\Gamma_1}^{\text{brok}}(L, \underline{x}_1) \times \mathcal{M}_{\Gamma_1}^{\text{brok}}(L, \underline{x}_2),$$

*and the boundary orientation on* $\mathcal{M}_{\Gamma}^{\text{brok}}(L, \underline{x})$ *induced by* $\mathcal{M}_{\Gamma_f}^{\text{brok}}(L, \underline{x})$ *is related to the product orientation by a sign* $(-1)^{\circ}$ *that depends only on the domain type* $\Gamma$ *and the labels* $\underline{x}$.

*Outline of proof.* The tubular neighborhood is constructed by gluing and there are different cases depending on the morphism:

CASE 1 : *Collapsing a boundary edge e of zero length.*
The proof is by gluing a boundary node, which is non-tropical since the Lagrangian is disjoint from relative divisors. The gluing proof is on the same lines as the gluing of tropical nodes in Theorem 9.1, so we point out the differences, referring the reader to [36, Chapter 7.1] for the full proof.

- Firstly, we choose strip-like coordinates on a neighborhood of the boundary node $w_e$ and we use weighted Sobolev spaces on these strips, see Definition 7.1.3 in [36]. A difference from our Fredholm set-up for broken maps in Section 6.3 is that we do not need cylindrical coordinates in the target space since there are no relative divisors.
- The next difference is that the map gluing parameter $\nu$ refers to the neck length parameter in the glued curve (see Remark 9.3), and therefore in the domain of the operator $\mathcal{F}_\nu$ in (9.11) the component $\mathcal{M}_{\Gamma_{\text{glue}}}$ is replaced by a subspace where the neck length corresponding to the node $w_e$ is fixed to be $\nu$. The proof of 'surjectivity of gluing' also simplifies, since it is enough to show convergence of the domain curves away from the neck as in (9.39) and the arguments following (9.39) in that paragraph are not necessary.

  Cases 2 and 3 do not involve gluing of surface components.

CASE 2 : *Making a boundary edge length* $\ell(e)$ *non-zero.*
The proof structure of Theorem 9.1 can be used in this case also. However, the map gluing parameter $\nu$ corresponds to the edge length $\ell(e)$ in the glued curve and goes to zero in the limit. The domain curve for the glued map $u_\nu : C_\nu \to \mathfrak{X}$ is allowed to vary among the set of curves of type $\Gamma_{\text{glue}}$ with $\ell(e) = \nu$. The approximate solution $u_\nu^{\text{pre}} : C_\nu \to \mathfrak{X}$ is defined to be equal to $u$, and the map $u_\nu^{\text{pre}}$ is defined to be constant on the treed segment $T_e$, so the error $\|D\mathcal{F}_\nu(u_\nu^{\text{pre}})\|$ is $\leq c\nu$. The rest of the steps are analogous to the proof of Theorem 9.1.

Case 3 : *Making a boundary edge length $\ell(e)$ finite.*
Similar to Case 2, the map gluing parameter $\nu$ is equal to the length $\ell(e)$ of the edge $e$, but unlike Case 2, it goes to infinity in the limit. The domain curve for the glued map $u_\nu : C_\nu \to \mathfrak{X}$ is allowed to vary among the set $\Gamma_{\text{glue}}$ curves with $\ell(e) = \nu$. The approximate solution $u_\nu^{\text{pre}} : C_\nu \to \mathfrak{X}$ is defined to be equal to $u$ on all components except the treed segment $T_e$, where it is defined by pre-gluing the Morse trajectory (see (2.66) in Schwarz [73]) at the edge $e$ in the map $u$. The norm of the error is bounded by $ce^{-\nu}$ and is derived in a similar way to Section 9.3. The rest of the steps are analogous to the proof of Theorem 9.1.

For the orientations result, part (bi) follows from the definition of orientations in Remark 6.29 and the sign computation for part (bii) is carried out in Seidel's book [76, (12.25)]. ■

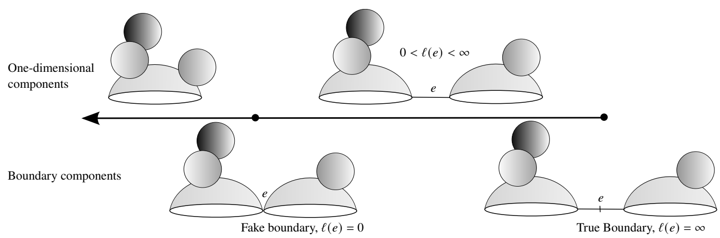



**Figure 9.1.** True and fake boundary strata of a one-dimensional component of the moduli space of treed holomorphic disks. The sphere components lie in different pieces of the tropical manifold.

*Remark* 9.8. (True and fake boundary strata) There are two types of strata that occur as the codimension one boundary of a one-dimensional moduli space – one with a boundary edge of length zero, and the second with a boundary edge containing a breaking. The first is a *fake* boundary, and the second one is a *true* boundary as we explain: If a type $\Gamma_0$ of broken maps has a boundary edge of length zero, we may make the edge length $\ell(e)$ non-zero or we may collapse the edge (by disk gluing) to produce the following types

$$\Gamma \xrightarrow{\text{Make } \ell(e) \text{ zero}} \Gamma_0 \xrightarrow{\text{Collapse } e} \Gamma_c.$$

Let $\underline{x}$ be a tuple of input and output such that $i^{\text{brok}}(\Gamma, \underline{x}) = 0$. By Theorem 9.7, the moduli space $\mathcal{M}_{\Gamma_0}^{\text{brok}}(\underline{x})$ is the boundary of the one-dimensional moduli spaces $\mathcal{M}_{\Gamma_c}^{\text{brok}}(\underline{x})$ and $\mathcal{M}_{\Gamma}^{\text{brok}}(\underline{x})$ with opposite orientations. So, $\mathcal{M}_{\Gamma_0}^{\text{brok}}(\underline{x})$ does not represent a component in the topological boundary of the compactified moduli space

$$\bigcup_{\Gamma:\Gamma \text{ is rigid, } i^{\text{brok}}(\Gamma,\underline{x})=1} \overline{\mathcal{M}}_{\Gamma}^{\text{brok}}(L, \underline{x}).$$

This is the fake boundary in Figure 9.1. The only (true) boundary components of one-dimensional strata thus consist of maps with a single broken Morse trajectory, see Figure 9.1.

## Chapter 10

# Broken Fukaya algebras

In this Chapter we describe $A_\infty$ algebra structures defined by counting treed holomorphic disks on broken and unbroken manifolds, and show that they are equivalent up to $A_\infty$ homotopy.

## 10.1 $A_\infty$ algebras

The set of treed holomorphic disks has the structure of an $A_\infty$ algebra. $A_\infty$ algebras were introduced by Stasheff [80] in order to capture algebraic structures on the space of cochains on loop spaces. We follow the sign convention in Seidel [75]. A $\mathbb{Z}_2$*-graded* $A_\infty$ *algebra* consists of a $\mathbb{Z}_2$-graded vector space $A$ together with for each $d \geq 0$ a multilinear degree zero *composition map*

$$m^d : A^{\otimes d} \to A[2-d]$$

satisfying the $A_\infty$ *associativity equations* [75, (2.1)]

$$0 = \sum_{j,k\geq 0, j+k\leq d} (-1)^{j+\sum_{i=1}^{j}|a_i|} m^{d-k+1}(a_1, \ldots, a_j, \\ m^k(a_{j+1}, \ldots, a_{j+k}), a_{j+k+1}, \ldots, a_d) \quad (10.1)$$

for any $d \geq 0$ and any tuple of homogeneous elements $a_1, \ldots, a_d$ with degrees $|a_1|, \ldots, |a_d| \in \mathbb{Z}_2$. The notation $[2-d]$ indicates a degree shift of $2-d$, so that the degree of $m^d(a_1, \ldots, a_d)$ is $\sum_i |a_i| + 2 - d \in \mathbb{Z}_2$. The signs appearing in (10.1) are the *shifted Koszul signs*, that is, the Koszul signs for the shifted grading in which the structure maps have degree one as in Kontsevich-Soibelman [46]. The first of these associativity relations is $m^1(m^0(1)) = 0$, and the second one is

$$m^2(m^0(1), a) - (-1)^{|a|} m^2(a, m^0(1)) + m^1(m^1(a)) = 0, \quad \forall a \in A \qquad (10.2)$$

which may be intepreted as saying that $m^1$ is a differential if $m^0(1)$ vanishes. A *strict unit* for $A$ is an element $1_A \in A$ such that

$$m^2(1_A, a) = a = (-1)^{|a|} m^2(a, 1_A), \quad m^d(\ldots, 1_A, \ldots) = 0, \forall d \neq 2. \qquad (10.3)$$

A *strictly unital* $A_\infty$ algebra is an $A_\infty$ algebra that admits a strict unit.

The element $m^0(1) \in A$ (where $1 \in \mathbb{R}$ is the unit) is called the *curvature* of the algebra. The $A_\infty$ algebra $A$ is *flat* if the curvature vanishes. The *cohomology* of a flat

$A_\infty$ algebra $A$ is defined by

$$H(m^1) = \frac{\ker(m^1)}{\operatorname{im}(m^1)}.$$

The algebra structure on $H(m^1)$ is given by

$$[a_1 a_2] = (-1)^{|a_1|}[m^2(a_1, a_2)]. \tag{10.4}$$

The $A_\infty$ algebras in our work are *curved*, that is $m^0(1)$ typically does not vanish.

The coefficient ring of our $A_\infty$ algebras are Novikov rings defined as follows. Let $q$ be a formal variable and $\Lambda$ the *universal Novikov field* of formal sums

$$\Lambda = \left\{ \sum_{i \in \mathbb{Z}_{\geq 0}} c_i q^{\alpha_i} \;\middle|\; c_i \in \mathbb{C},\ \alpha_i \in \mathbb{R}, \quad \alpha_i \to \infty \right\}.$$

The subring

$$\Lambda_{\geq 0} := \left\{ \sum_{i \in \mathbb{Z}_{\geq 0}} c_i q^{\alpha_i} \in \Lambda \;\middle|\; \alpha_i \geq 0 \right\}$$

with only the non-negative exponents is called the *Novikov ring*. Denote by $\Lambda_{>0}$ the subring with only positive exponents, and by

$$\Lambda^\times = (\mathbb{C} - \{0\}) + \Lambda_{>0} \subset \Lambda_{\geq 0}$$

the subgroup of formal power series with invertible leading coefficient and non-negative exponents. The Novikov ring has natural adic topology, in which a sequence of elements converges iff it converges in the quotient by any subring $q^E \Lambda_{\geq 0}$ for any $E \in \mathbb{R}$.

Cohomology can be defined for a strictly unital $A_\infty$ algebra if the curvature is a multiple of the unit : $m^0(1) \in \Lambda_{\geq 0} 1_A$, because in this case, $(m^1)^2 = 0$ by (10.2). More generally, the cohomology exists for any solution to the projective Maurer-Cartan equation [36]. The projective Maurer-Cartan equation for $b \in A$ is

$$m^0(1) + m^1(b) + m^2(b, b) + \ldots \in \Lambda_{\geq 0} 1_A. \tag{10.5}$$

A solution $b \in A$ of odd degree to the equation (10.5) is called a *weakly bounding cochain* and the set of all the odd solutions is denoted $MC(A)$. Given a weakly bounding cochain $b \in MC(A)$, the *deformed composition map* is defined by

$$m_b^n(a_1, \ldots, a_n) = \sum_{i_1, \ldots, i_{n+1}} m^{n+i_1+\ldots+i_{n+1}}(\underbrace{b, \ldots, b}_{i_1}, a_1, \underbrace{b, \ldots, b}_{i_2}, a_2, b, \ldots, b, a_n, \underbrace{b, \ldots, b}_{i_{n+1}}) \tag{10.6}$$

over all possible combinations of insertions of the element $b \in A^+$ between (and before and after) the elements $a_1, \ldots, a_n$. The maps $m_b^n$ define an $A_\infty$ structure on $A$ if $b$ has odd degree. By the weakly bounding cochain condition $m_b^0(1) \in \Lambda_{\geq 0} 1_A$, and the $A_\infty$ relations imply

$$(m_b^1)^2(a) = m_b^2(m_b^0(1), a) - m_b^2(a, m_b^0(1)) = 0.$$

Consequently, the cohomology

$$H(m_b^1) = \ker(m_b^1)/\mathrm{im}(m_b^1)$$

is well-defined. The function

$$W : MC(A) \to \Lambda_{\geq 0}, \quad b \mapsto W(b), \tag{10.7}$$

where $m_b^0(1) = W(b) 1_A$ is called the *potential* of the $A_\infty$ algebra $A$.

An $A_\infty$ algebra $A$ with coefficients in $\Lambda_{\geq 0}$ is *convergent* if $m^0(1) \in \Lambda_{>0}$. This property ensures that the expression $m_b^0$ in the Maurer-Cartan equation (10.5) is well-defined for any $b \in A$ with positive $q$-valuation.

One also has homotopy notions of algebra morphisms. Let $A_0, A_1$ be $A_\infty$ algebras. An $A_\infty$ *morphism* $\mathcal{F}$ from $A_0$ to $A_1$ consists of a sequence of linear maps

$$\mathcal{F}^d : A_0^{\otimes d} \to A_1[1-d], \quad d \geq 0$$

such that the following holds:

$$\sum_{i+j \leq d} (-1)^{i + \sum_{j=1}^{i} |a_j|} \mathcal{F}^{d-j+1}(a_1, \ldots, a_i, m_{A_0}^j(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_d) =$$
$$\sum_{i_1 + \ldots + i_m = d} m_{A_1}^m(\mathcal{F}^{i_1}(a_1, \ldots, a_{i_1}), \ldots, \mathcal{F}^{i_m}(a_{i_1 + \ldots + i_{m-1}+1}, \ldots, a_d)) \tag{10.8}$$

where the first sum is over integers $i, j$ with $i + j \leq d$, the second is over partitions $d = i_1 + \ldots + i_m$. The first relation in the family of relations in (10.8) is

$$\mathcal{F}^1(m_{A_0}^0) = m_{A_1}^0 + m_{A_1}^1(\mathcal{F}^0) + m_{A_1}^2(\mathcal{F}^0, \mathcal{F}^0) + \ldots$$

An $A_\infty$ morphism $\mathcal{F}$ is *unital* if and only if

$$\mathcal{F}^1(1_{A_0}) = 1_{A_1}, \quad \mathcal{F}^k(a_1, \ldots, a_i, 1_{A_0}, a_{i+2}, \ldots, a_k) = 0 \tag{10.9}$$

for every $k \geq 2$ and every $0 \leq i \leq k-1$, where $1_{A_0}$ resp. $1_{A_1}$ is the strict unit in $A_0$ resp. $A_1$. The *composition* of $A_\infty$ morphisms $\mathcal{F}_0, \mathcal{F}_1$ is defined by

$$(\mathcal{F}_0 \circ \mathcal{F}_1)^d(a_1, \ldots, a_d) = \sum_{i_1 + \ldots + i_m = d} \mathcal{F}_0^m(\mathcal{F}_1^{i_1}(a_1, \ldots, a_{i_1}),$$
$$\ldots, \mathcal{F}_1^{i_m}(a_{d-i_m+1}, \ldots, a_d)), \quad d \geq 0. \tag{10.10}$$

There is also a natural notion $A_\infty$ homotopy equivalence of morphisms. Homotopies are defined as pre-natural transformations of $A_\infty$ morphisms, which we define following Seidel [76] and Charest-Woodward [17]: Given $A_\infty$ algebras $A_0$, $A_1$, the set of morphisms is an $A_\infty$ category denoted by $Q$ whose objects are $A_\infty$ morphisms $\mathcal{F} : A_0 \to A_1$, and whose morphisms are *pre-natural transformations*. For $A_\infty$ morphisms $\mathcal{F}_0, \mathcal{F}_1 : A_0 \to A_1$, a pre-natural transformation $\mathcal{T} = (\mathcal{T}^d)_{d\geq 0} \in \mathrm{Hom}_Q(\mathcal{F}_0, \mathcal{F}_1)$ is a family of maps

$$\mathcal{T}^d : A_0^{\otimes d} \to A_1.$$

Pre-natural transformations are equipped with composition maps $m_Q^e$ for all $e \geq 1$ defined as follows: For any $d \geq 0$,

$$\begin{aligned}(m_Q^1 \mathcal{T})^d(a_1, \ldots, a_d) = &\sum_{k,l} \sum_{i_1,\ldots,i_l} (-1)^{\dagger} m_{A_2}^l(\mathcal{F}_0^{i_1}(a_1, \ldots, a_{i_1}), \mathcal{F}_0^{i_2}(a_{i_1+1}, \ldots, a_{i_1+i_2}), \ldots, \\ &\mathcal{T}^{i_k}(a_{i_1+\ldots+i_{k-1}+1}, \ldots, a_{i_1+\ldots+i_k}), \mathcal{F}_1^{i_{k+1}}(a_{i_1+\ldots+i_k+1}, \ldots,), \ldots, \mathcal{F}_1^{i_l}(a_{d-i_l}, \ldots, a_d)) \\ &- \sum_{i,e} (-1)^{i+\sum_{j=1}^{i}|a_j|+|\mathcal{T}|-1} \mathcal{T}^{d-e+1}(a_1, \ldots, a_i, m_{A_1}^e(a_{i+1}, \ldots, a_{i+e}), a_{i+e+1}, \ldots, a_d),\end{aligned} \tag{10.11}$$

where the sign $\dagger$ is from [17, p82]. For $A_\infty$ morphisms $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 : A_0 \to A_1$, and pre-natural transformations $\mathcal{T}_0 \in \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_1)$, $\mathcal{T}_1 \in \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$, the binary composition $m_Q^2(\mathcal{T}_0, \mathcal{T}_1) \in \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_2)$ is defined as

$$\begin{aligned}(m_Q^2(\mathcal{T}_1, \mathcal{T}_2))^d(a_1, \ldots, a_d) = &\sum_{m,k,l} \sum_{i_1,\ldots,i_m} (-1)^{\ddagger} m_{A_2}^l(\mathcal{F}_0^{i_1}(a_1, \ldots, a_{i_1}), \ldots, \mathcal{F}_0^{i_{k-1}}(\ldots), \\ &\mathcal{T}_1^{i_k}(a_{i_1+\ldots+i_{k-1}+1}, \ldots, a_{i_1+\ldots+i_k}), \mathcal{F}_1^{i_{k+1}}(\ldots), \ldots, \mathcal{F}_1^{i_{l-1}}(\ldots), \\ &\mathcal{T}_2^{i_l}(a_{i_1+\ldots+i_{l-1}+1}, \ldots, a_{i_1+\ldots+i_l}), \mathcal{F}_2^{i_{l+1}}(\ldots), \ldots, \mathcal{F}_2^{i_l}(a_{d-i_l}, \ldots, a_d)),\end{aligned} \tag{10.12}$$

where the sign $\ddagger$ is from [17, p82]. The higher order composition maps $m_Q^e$ are defined analogously.

Finally, we define a homotopy between $A_\infty$ morphisms. A pre-natural transformation $\mathcal{T}$ between $A_\infty$ morphisms $\mathcal{F}_0, \mathcal{F}_1 : A_0 \to A_1$ is a *homotopy* if

$$\mathcal{F}_1 - \mathcal{F}_0 = m_Q^1(\mathcal{T}). \tag{10.13}$$

Furthermore, if $\mathcal{F}_1 = (\mathcal{F}_{1,d})_{d\geq 0}$ is only a collection of maps from $A_0^{\otimes d}$ to $A_1[1-d]$ and $\mathcal{F}_0$ satisfies the $A_\infty$ morphism axiom then so does $\mathcal{F}_1$. Homotopy is an equivalence relation in $Q$, with transitivity seen as follows: If $\mathcal{T}_0 \in \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_1)$ and $\mathcal{T}_1 \in \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ are homotopies, then,

$$\mathcal{T}_1 \circ \mathcal{T}_0 := \mathcal{T}_0 + \mathcal{T}_1 + m_Q^2(\mathcal{T}_0, \mathcal{T}_1) \in \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_2) \tag{10.14}$$

is a homotopy, see [76, p16]. The $A_\infty$ algebras $A_0$, $A_1$ are *homotopy equivalent* if there are $A_\infty$ morphisms $\mathcal{F}_0 : A_0 \to A_1$, $\mathcal{F}_1 : A_0 \to A_1$ such that $\mathcal{F}_0 \circ \mathcal{F}_1 : A_1 \to A_1$ and $\mathcal{F}_1 \circ F_0 : A_0 \to A_0$ are both homotopic to the identity $A_\infty$ morphism.

## 10.2 Composition maps

In this section, we describe $A_\infty$ algebras, called *Fukaya algebras* whose composition maps are given by counts of treed holomorphic maps, both in an ordinary symplectic manifold and in a broken manifold. The boundary of the disks map to a Lagrangian, which in the broken case, is contained in the complement of relative divisors.

Lagrangians will be equipped with additional data called *brane structures*. We assume Lagrangians are compact, connected and oriented. A brane structure consists of :

- a relative spin structure (see [36, Chapter 44] for the definition),
- and a local system, which is an element

$$\mathrm{Hol}_L \in \mathcal{R}(L) = \mathrm{Hom}(\pi_1(L), \Lambda^\times).$$

Typically, a grading in the sense of Seidel [74] is included in the definition of brane; however, since gradings are needed only while working with multiple Lagrangians, we do not use them in this paper.

The Fukaya algebra of a Lagrangian brane involves counts of disks that are holomorphic with respect to coherent perturbation data $\underline{\mathfrak{p}} := (\mathfrak{p}_\Gamma)_\Gamma$ for all types $\Gamma$ of stable treed disks. The data $\underline{\mathfrak{p}}$ consists of perturbations of a background almost complex structure $J_0$ on the manifold $X$ and a Morse function $F : L \to \mathbb{R}$ on the Lagrangian. Theorem 6.28 constructs coherent perturbations for a broken manifold $\mathfrak{X}$; for the corresponding result on a smooth manifold, see [17, Theorem 4.20]. We first define composition maps giving a (unbroken) Fukaya algebra without strict units called the *geometric Fukaya algebra*; we later explain how to upgrade the construction to include strict units.

**Definition 10.1.** (Geometric Fukaya algebra) For a Lagrangian brane $L$ and coherent perturbation datum $\underline{\mathfrak{p}} := (\mathfrak{p}_\Gamma)_\Gamma$ on the manifold $X$, the *geometric Fukaya algebra* is an $A_\infty$ algebra consisting of the space of *Floer cochains* over the Novikov ring $\Lambda_{\geq 0}$

$$CF^{\mathrm{geom}}(L, \underline{\mathfrak{p}}) := \bigoplus_{d \in \mathbb{Z}_2} CF^d(L, \underline{\mathfrak{p}}), \quad CF^d(L, \underline{\mathfrak{p}}) := \bigoplus_{x \in \mathcal{I}_d(L)} \Lambda_{\geq 0}\langle x \rangle$$

where $\mathcal{I}_d(L)$ is the set of index $d$ critical points of the Morse function $F : L \to \mathbb{R}$, see Definition 4.5; and equipped with *composition maps*

$$m^{d(\circ)} : (CF^{\mathrm{geom}}(L))^{\otimes d(\circ)} \to CF^{\mathrm{geom}}(L), \quad d(\circ) \geq 0$$

defined on generators $x_i \in \mathrm{crit}(F)$ by

$$m^{d(\circ)}(x_1,\ldots,x_{d(\circ)}) = \sum_{\Gamma, x_0, u \in \mathcal{M}_\Gamma(X,L,\mathfrak{p}_\Gamma,\underline{x})_0} w(u) x_0 \tag{10.15}$$

where the sum is over all rigid types $\Gamma$ of maps from Definition 8.47 with $d(\circ)$ incoming boundary edges, and

$$w(u) := (-1)^{\heartsuit}(d_\bullet(\Gamma)!)^{-1}\,\mathrm{Hol}_L([\partial u])\epsilon(u) q^{A(u)}. \tag{10.16}$$

The factors in (10.16) are as below:

(a) $\heartsuit = \sum_{i=1}^{d(\circ)} i|x_i|$;

(b) $\mathrm{Hol}_L([\partial u]) \in \Lambda^\times$ is the evaluation of the local system $\mathrm{Hol}_L \in \mathcal{R}(L)$ on the homotopy class of loops $[\partial u] \in \pi_1(L)$ defined by going around the boundary of each disk component in the treed disk once;

(c) $d_\bullet(\Gamma)$ the number of interior markings on the map $u$; and

(d) $\epsilon(u) \in \{\pm 1\}$ is the orientation sign, as in Remark 6.29.

This ends the Definition.

In the count of disks (10.15), we only consider rigid types, because types of maps that are not rigid occur in the boundary of a compactification of a larger dimensional stratum.

The $A_\infty$ relation follows from the description of the boundary of the moduli space. We recall from Section 9.8, that the *true boundary* of one-dimensional moduli spaces of treed holomorphic disks consists of maps with a broken edge. Indeed, as a stratified space, the closure of the one-dimensional moduli spaces have zero-dimensional strata which include both strata where edge lengths have become zero and strata where edge lengths have become infinity; however, only the latter types form topological boundary strata, since the length-zero strata lie in the closures of two one-dimension strata either by deforming to positive length or performing the gluing construction for holomorphic disks.

**Theorem 10.2.** *($A_\infty$ algebra for a Lagrangian in a symplectic manifold) For a coherent perturbation system $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$ on the manifold $(X,\omega_X)$, the maps $(m^{d(\circ)})_{d(\circ)\geq 0}$ on $CF^{\mathrm{geom}}(L)$ satisfy the axioms of a convergent (possibly curved) $A_\infty$ algebra $CF^{\mathrm{geom}}(L)$.*

*Proof.* We prove the $A_\infty$ associativity relations by counting the ends of one-dimensional moduli spaces of treed holomorphic disks. We perform the count for a fixed number of interior and boundary markings, and fixed limits on the input and output treed segments, and then sum over all choices. Consider integers $d(\circ), d(\bullet) \geq 0$ and a tuple $\underline{x} \in (\mathcal{I}(L))^{d(\circ)+1}$ of inputs and outputs. We denote the one-dimensional component of the moduli space of treed disks of rigid type with $d(\circ)$ boundary inputs and $d(\bullet)$ interior markings by

$$\mathcal{M}_{d(\bullet),d(\circ)}(L,\underline{x})_1 := \cup_{\Gamma: i(\Gamma,\underline{x})=1} \mathcal{M}_\Gamma(L,\underline{x}),$$

where $\Gamma$ ranges over rigid types containing $d(\bullet)$ interior markings and $d(\circ)$ boundary inputs. By the unbroken version of Proposition 8.51 and Remark 9.8 (obtained by replacing broken maps with unbroken maps in the statement; this was proved in [17, Remark 4.22]), the true boundary of $\mathcal{M}_{d(\bullet),d(\circ)}(L,\underline{x})_1$ consists of disks with a single broken edge, that is, an $e \in \mathrm{Edge}_{\circ,-}$ with $\ell(e) = \infty$. The boundary points of one-dimensional moduli spaces occur in pairs, and therefore,

$$\sum_{u \in \mathcal{M}_\Gamma(L,\underline{x})} \epsilon_\partial(u)(d(\bullet)!)^{-1} = 0, \tag{10.17}$$

where, in the summation, $\Gamma$ ranges over all types in the true boundary of the moduli space $\mathcal{M}_{d(\bullet),d(\circ)}(L,\underline{x})_1$, and the sign $\epsilon_\partial(u) \in \{\pm 1\}$ is given by the induced orientation on the boundary of $\mathcal{M}_{d(\bullet),d(\circ)}(L,\underline{x})_1$. Consider one such type $\Gamma$, and suppose $\Gamma_+, \Gamma_-$ are the treed disk types obtained by cutting the broken edge, each containing $d_\pm(\bullet) := d_\bullet(\Gamma_\pm)$ interior markings. For any critical point $x \in \mathcal{I}(L)$, denote by $(\underline{x},x)_+$, $(\underline{x},x)_-$ the input-output labelling on $\Gamma_+, \Gamma_-$ where both ends of the broken edge are labelled $x$. There is a bijection

$$\mathcal{M}_\Gamma(L,\underline{x}) \simeq \bigcup_{x\in\mathcal{I}(L)} \mathcal{M}_{\Gamma_+}(L,(\underline{x},x)_+))_0 \times \mathcal{M}_{\Gamma_-}(L,(\underline{x},x)_-))_0. \tag{10.18}$$

However, any map in the right-hand side corresponds to $\binom{d(\bullet)}{d_+(\bullet)}$ maps in the true boundary of $\mathcal{M}_{d(\bullet),d(\circ)}(L,\underline{x})_1$ as follows: There are $\binom{d(\bullet)}{d_+(\bullet)}$ bijections

$$f : \{1,\ldots,d_+(\bullet)\} \sqcup \{1,\ldots,d_-(\bullet)\} \to \{1,\ldots,d(\bullet)\}$$

whose restrictions to $\{1,\ldots,d_+(\bullet)\}$, $\{1,\ldots,d_-(\bullet)\}$ are order-preserving. Given a pair in $(u_+,u_-)$ in the right-hand side of (10.18), each bijection $f$ corresponds to a map $u$ in the true boundary of $\mathcal{M}_{d(\bullet),d(\circ)}(L,\underline{x})_1$, where the $i$-th interior marking in $u_\pm$ is labelled $f(i)$ in $u$. Therefore, the expression in (10.17) is equal to

$$0 = \sum_{(\Gamma_+,\Gamma_-),x} \left\{ \left( \sum_{u_+\in\mathcal{M}_{\Gamma_+}(L,(\underline{x},x)_+)} \epsilon(u_+)(d_\bullet(\Gamma_+)!)^{-1} \right) \cdot \left( \sum_{u_-\in\mathcal{M}_{\Gamma_-}(L,(\underline{x},x)_-)} \epsilon(u_-)(d_\bullet(\Gamma_-)!)^{-1} \right) \frac{\epsilon_\partial(u)}{\epsilon(u_+)\epsilon(u_-)} \right\}, \tag{10.19}$$

where $x$ ranges over critical points in $\mathcal{I}(L)$ and $(\Gamma_+,\Gamma_-)$ range over all pairs of types obtained by cutting an edge in a treed disk type occurring in the boundary of $\mathcal{M}_{d(\bullet),d(\circ)}(L,\underline{x})_1$.

The sign contribution $\frac{\epsilon_\partial(u)}{\epsilon(u_+)\epsilon(u_-)}$ is equal to the difference in the orientation of the moduli space of treed disks with a broken edge when viewed as a boundary of a larger stratum, and when viewed as a product, and is equal to the shifted Koszul sign in the $A_\infty$ associativity relation (10.1) by [76, (12.25)]. Adding the equations (10.19) corresponding to all $d(\bullet) \geq 0$, we obtain the associativity relation on the maps $(m^d)_{d\geq 0}$ on $CF^{\text{geom}}(L)$ corresponding to $d = d(\circ)$. The convergence property follows from the fact that any treed disk with no incoming edges must, by stability, have a disk component with interior markings. Indeed, the disk components that are furthest from the outgoing edge have a single adjoining boundary edge, and so must intersect the stabilizing divisor. By (5.2), the area of the holomorphic disk is proportional to the number of its intersections with the stabilizing divisor, and as a result, the $q$-valuation of $m^0(1)$ is positive. ∎

The broken Fukaya algebra is defined analogously by counts of broken disks.

**Definition 10.3.** (Geometric broken Fukaya algebra) Let $\mathfrak{X}$ be a broken manifold, let $L \subset \mathfrak{X}$ be a Lagrangian brane that is contained in a single piece of $\mathfrak{X}$ and does not intersect relative divisors, and let $\underline{\mathfrak{p}} := (\mathfrak{p}_\Gamma)_\Gamma$ be coherent perturbation data on $\mathfrak{X}$ (constructed by Theorem 6.28). The *geometric broken Fukaya algebra* is an $A_\infty$ algebra consisting of the space of *Floer cochains* over the Novikov ring $\Lambda_{\geq 0}$

$$CF^{\text{geom}}_{\text{brok}}(L, \underline{\mathfrak{p}}) := \bigoplus_{d\in\mathbb{Z}_2} CF^d_{\text{brok}}(L, \underline{\mathfrak{p}}), \quad CF^d_{\text{brok}}(L, \underline{\mathfrak{p}}) := \bigoplus_{x\in\mathcal{I}_d(L,\underline{\mathfrak{p}})} \Lambda_{\geq 0}\langle x\rangle,$$

where $\mathcal{I}_d(L)$ is as in Definition 10.1; and equipped with *composition maps*

$$m^{d(\circ)}_{\text{brok}} : (CF^{\text{geom}}_{\text{brok}}(L, \underline{\mathfrak{p}}))^{\otimes d(\circ)} \to CF^{\text{geom}}_{\text{brok}}(L, \underline{\mathfrak{p}}), \quad d(\circ) \geq 0$$

defined on generators $x_i \in \text{crit}(F)$ by

$$m^{d(\circ)}_{\text{brok}}(x_1, \dots, x_{d(\circ)}) = \sum_{x_0,\Gamma,u\in\mathcal{M}^{\text{brok}}_\Gamma(\mathfrak{X},L,D,\underline{x})_0} w(u)x_0. \tag{10.20}$$

Here, $x_0$ ranges over critical points of the Morse function $F$, the combinatorial type $\Gamma$ of the broken map $u$ ranges over all rigid types (see Definition 8.48) with $d(\circ)$ boundary inputs, and $w(u)$ is as in (10.16). The orientation sign $\epsilon(u) \in \{\pm 1\}$, which is a factor in $w(u)$, is determined as in Remark 6.29.

**Theorem 10.4.** *($A_\infty$ algebra for a Lagrangian in a broken manifold) For any coherent perturbation system $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$ on the broken manifold $\mathfrak{X}$, the maps $(m^{\text{brok}}_{d(\circ)})_{d(\circ)\geq 0}$ on $CF^{\text{geom}}_{\text{brok}}(L)$ satisfy the axioms of a convergent (possibly curved) $A_\infty$ algebra $CF^{\text{geom}}_{\text{brok}}(L)$.*

We sketch two proofs. The first proof is an almost verbatim repeat of the proof of the unbroken $A_\infty$ relation in Theorem 10.2 from the description of the true boundary of the moduli space in Proposition 8.51. The combinatorial factors arising from the distribution of interior markings are accounted exactly as in the unbroken case. Another proof uses the fact that the structure maps for the broken Fukaya algebra are the limits of those in the unbroken case, by Theorem 10.31, and is completed after Corollary 10.32.

## 10.3 Homotopy units

In the Fukaya algebra constructed in the previous section, a homotopy unit construction can be applied to produce a strictly unital $A_\infty$ algebra. Recall that the Morse function $F : L \to \mathbb{R}$ used in the construction of $CF^{\text{geom}}(L)$ is assumed to have a unique maximum point denoted $x^\blacktriangledown \in \text{crit}(F)$. In an idealized situation where domain-dependent perturbations are not required, $\langle x^\blacktriangledown \rangle$ is a strict unit for $CF^{\text{geom}}(L)$. This is because a boundary marked point mapping to the unstable locus of $x^\blacktriangledown$ is an empty constraint, and such a marking can be forgotten without affecting the disk. In our setting, marked points can not be forgotten because domain-dependent perturbations depend on them. The homotopy unit construction is a way of enhancing the Fukaya algebra so that the perturbation system admits forgetful maps, so that the algebra admits a strict unit and the potential of (10.7) is well-defined.

We outline the idea of the homotopy units construction: For the Fukaya algebra to have a unit, we would like the perturbations on the Morse function to be domain-independent on the treed segments asymptotic to the maximum point $x^\blacktriangledown \in \text{crit}(F)$. However, we can not impose such a condition on the perturbation, because the perturbation depends only on the "domain" which does not include the information about which critical point a Morse trajectory asymptotes to. One can not include the critical point label into the domain data, because Morse trajectories may break in the limit, and the label at the breaking point is not known beforehand. Therefore, we add a new kind of boundary marking to the domain, which is "forgettable", and will serve as a unit. The treed segment at a forgettable leaf is required to asymptote to the maximum of the Morse function $F$, but is labelled $x^\triangledown$ (to distinguish it from $x^\blacktriangledown$). We also add 'weighted boundary markings", labelled $x^{\blacktriangledown}$, to the domain which give us a way of homotoping between treed segments with domain-dependent perturbations asymptoting to $x^\blacktriangledown$ and those with domain-independent perturbations asymptoting to $x^\triangledown$. The data of the domain now includes the information about whether a leaf is forgettable, weighted or unforgettable. This new kind of domain is called a *weighted treed disk* and is defined below after the statement of the main Theorem.

**Theorem 10.5.** *(Homotopy unit construction) Suppose $\underline{\mathfrak{p}}$ is a coherent perturbation datum for treed holomorphic disks, and suppose $CF^{\text{geom}}(L) := CF^{\text{geom}}(L, \underline{\mathfrak{p}})$ is the $A_\infty$ algebra whose composition maps count $\underline{\mathfrak{p}}$-holomorphic disks. Then there exists a* *convergent strictly unital $A_\infty$ structure on the vector space*

$$CF(L) := CF^{\text{geom}}(L) \oplus \Lambda_{\geq 0} x^{{\color{gray}\blacktriangledown}}[1] \oplus \Lambda_{\geq 0} x^{\triangledown}, \tag{10.21}$$

*with gradings*

$$|x^{\triangledown}| = 0, \quad |x^{{\color{gray}\blacktriangledown}}| = -1,$$

*whose composition maps count weighted $\underline{\mathfrak{p}}^{\text{wt}}$-holomorphic disks (see Definition 10.8), where $\underline{\mathfrak{p}}^{\text{wt}}$ is an extension of the perturbation datum $\underline{\mathfrak{p}}$ to weighted disks; and in the resulting $A_\infty$ algebra,*

(a) *$x^{\triangledown}$ is a strict unit,*

(b) *$CF^{\text{geom}}(L) \subset CF(L)$ is a $A_\infty$ sub-algebra, and*

(c)
$$m^1(x^{{\color{gray}\blacktriangledown}}) = x^{\blacktriangledown} - x^{\triangledown} \mod \Lambda_{>0}. \tag{10.22}$$

The theorem is proved later in the section after defining weighted treed holomorphic disks. We remark that the expression (10.22) for $m^1(x^{{\color{gray}\blacktriangledown}})$ is similar to that in Fukaya-Oh-Ohta-Ono [36, (3.3.5.2)].

The condition that $x^{\triangledown}$ is a strict unit determines all $A_\infty$ structure maps involving occurrences of $x^{\triangledown}$. In the following geometric construction of a homotopy unit, the axioms are designed keeping this fact in mind.

**Definition 10.6.** (a) (Weightings) A *weighting* of a treed disk $C = S \cup T$ of type $\Gamma$, with $S \neq \emptyset$, consists of a partition of the boundary semi-infinite edges

$$\text{Edge}^{\blacktriangledown}(\Gamma) \sqcup \text{Edge}^{{\color{gray}\blacktriangledown}}(\Gamma) \sqcup \text{Edge}^{\triangledown}(\Gamma) = \text{Edge}_{\circ,\to}(\Gamma)$$

into *unforgettable* resp. *weighted* resp. *forgettable*, and a *weight* on semi-infinite edges $\rho : \text{Edge}_{\circ,\to}(\Gamma) \to [0,\infty]$ satisfying

$$\rho(e) \in \begin{cases} \{0\} & e \in \text{Edge}^{\blacktriangledown}(\Gamma) \\ [0,\infty] & e \in \text{Edge}^{{\color{gray}\blacktriangledown}}(\Gamma) \\ \{\infty\} & e \in \text{Edge}^{\triangledown}(\Gamma). \end{cases}$$

The weighting $\rho$ satisfies the following axiom:

(Outgoing edges axiom) A disk output $e_0 \in \text{Edge}_{\circ}(\Gamma)$ can be weighted only if the disk has exactly one weighted input $e_1 \in \text{Edge}^{{\color{gray}\blacktriangledown}}(\Gamma)$, all the other inputs $e_i \in \text{Edge}(\Gamma), i \neq 1$ are forgettable, and there are no interior leaves, $\text{Edge}_{\bullet}(\Gamma) = \emptyset$. In this case, the output $e_1$ has the same weight $\rho(e_1) = \rho(e_0)$ as the weighted input $e_0$. A disk output $e_0$ can be forgettable only if all the inputs are forgettable, and there are no interior leaves. In all the other cases, the output of a disk is unforgettable.

In the exceptional case that the treed disk $C$ is an infinite tree segment and does not have surface components, the only possible labels are

$$\blacktriangledown \to \triangledown, \quad \blacktriangledown \to \blacktriangledown, \quad \text{or} \quad \blacktriangledown \to \blacktriangledown. \tag{10.23}$$

In the first two cases, the input has weight $\rho(e)$ equal to $\infty$ resp. 0.

(b) (Stability) A weighted treed disk $C = S \cup T$ with $S \neq \emptyset$ is *stable* if $C$ is stable as a treed disk. In case $S = \emptyset$ and $C$ is an infinite segment, then $C$ is stable iff the labels are $\blacktriangledown \to \triangledown$ or $\blacktriangledown \to \blacktriangledown$.

(c) (Isomorphism) Two weighted treed disks $C$ and $C'$ are isomorphic if there is an isomorphism of treed disks $\phi : C \to C'$, the edge labels are identical, and the following is true.

  (i) If the output edge is not weighted, then the weights on the inputs of $C_1$ and $\phi(C_1)$ are equal;

  (ii) if the output edge $e_0$ is weighted, then the weights on the inputs are equal up to scalar multiplication, i.e.

$$\exists \lambda \in (0, \infty) : \ \forall e \in \mathrm{Edge}_{\circ,\to}(C) \backslash \{e_0\} \quad \rho(e) = \lambda \rho'(\phi(e)). \tag{10.24}$$

Consequently, if the output edge is weighted, since there is exactly one incoming edge by the (Outgoing edges axiom), graphs with different weights may be isomorphic.

(d) (Combinatorial type) The *type* of a weighted treed disk is given by the type of the treed disk, and the labels $\{\triangledown, \blacktriangledown, \blacktriangledown\}$ at the inputs and outputs, and whether the weight at any vertex is zero, infinite or neither. Thus the combinatorial type of a weighted treed disk includes a partition of weighted edges

$$\mathrm{Edge}^{\blacktriangledown}(\Gamma) = \mathrm{Edge}_0^{\blacktriangledown}(\Gamma) \cup \mathrm{Edge}_{(0,\infty)}^{\blacktriangledown}(\Gamma) \cup \mathrm{Edge}_\infty^{\blacktriangledown}(\Gamma)$$

into edges of weight 0, non-zero finite, and infinity.

The moduli space $\mathcal{M}_\Gamma$ of weighted treed disks can be identified with

$$\begin{cases} \mathcal{M}_{\Gamma'} \times [0, \infty]^{|\mathrm{Edge}^{\blacktriangledown}(\Gamma)|}, & \text{if the output label is not } \blacktriangledown \\ \mathcal{M}_{\Gamma'}, & \text{if the output edge is } \blacktriangledown, \end{cases}$$

where $\Gamma'$ is the type of treed disk obtained by forgetting the weighting. If the type $\Gamma$ is $\blacktriangledown \to \triangledown$ resp. $\blacktriangledown \to \blacktriangledown$, then $\mathcal{M}_\Gamma$ is a point.

The (Cutting edges) morphism has some additional features for weighted treed disks.

**Definition 10.7.** (Cutting edges in weighted treed disks) Given a type $\Gamma$ of a weighted treed disk, we say that the weighted disk types $\Gamma_+, \Gamma_-$ (here $\Gamma_+$ contains the root of $\Gamma$)

are produced by cutting $\Gamma$ at an infinite edge $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ if $\Gamma_+, \Gamma_-$ (with weights forgotten) are the treed disk types produced by cutting the edge $e$ in $\Gamma$ as in Definition 6.3, and labels and weights assigned as follows: Assuming that $e_\pm \in \mathrm{Edge}_{\circ,\to}(\Gamma_\pm)$ are the pair of leaves created by the cutting, the label ($\blacktriangledown$, $\blacktriangledown$ or $\triangledown$) and the weight at $e_+$ are the same as that of $e_-$, and the label and weight at $e_-$ is determined by the (Outgoing edges axiom) applied to $\Gamma_-$.

There is a new type of morphism for weighted disk types, wherein we cut a weighted incoming edge of weight 0 or $\infty$.

(Cutting a weighted input edge) Suppose $e \in \mathrm{Edge}^{\blacktriangledown}(\Gamma)$ is an input, and $\rho(e) = 0$ resp. $\infty$. Cutting $e$ produces two types : $\Gamma_-$ is an infinite segment $\blacktriangledown \to \blacktriangledown$ resp. $\blacktriangledown \to \triangledown$, and $\Gamma_+$ is $\Gamma$ with $e$ as an unforgettable resp. forgettable edge, see Figure 10.1. Additionally, if $\Gamma$ has a weighted output $e_0$, we make the output edge $e_0$ in $\Gamma_+$ unforgettable resp. forgettable.

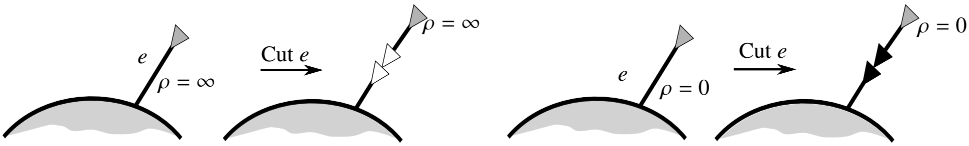



**Figure 10.1.** Cutting a weighted input edge. The arrows on each input and output edge indicate whether the weighting is infinite (white) finite and non-zero (grey) or zero (black).

Perturbation data defined on the moduli space of weighted treed disks are required to be coherent with respect to (Collapsing edges), (Making an edge length finite or non-zero), (Making an edge weight finite or non-zero), (Cutting edges), (Cutting a weighted input edge), and the (Locality axiom). The perturbation vanishes on infinite segments $\blacktriangledown \to \blacktriangledown$ and $\blacktriangledown \to \triangledown$. Additionally, the following coherence conditions are satisfied:

(a) (Making an edge weight finite/non-zero) If the weighted treed disk type $\Gamma$ is obtained from $\Gamma'$ by making the weight of an edge $e \in \mathrm{Edge}^{\blacktriangledown}(\Gamma')$ finite or non-zero, then $\mathfrak{p}_{\Gamma'}$ is the pullback of $\mathfrak{p}_\Gamma$ by the inclusion of the universal moduli space $\overline{\mathcal{U}}_{\Gamma'} \to \overline{\mathcal{U}}_\Gamma$.

(b) (Forgetting edges) Suppose $e$ is an input edge in a weighted treed type $\Gamma$ that is either forgettable or weighted with infinite weight, and $\Gamma'$ is the type obtained by forgetting $e$. Then $\mathfrak{p}_\Gamma$ is the pullback of $\mathfrak{p}_{\Gamma'}$.

**Definition 10.8.** (Weighted treed holomorphic disks) Let $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$ be a coherent perturbation datum for weighted treed disks. A *weighted* $\mathfrak{p}_\Gamma$*-holomorphic disk* is a map $u : C \to X$ that is a $\mathfrak{p}_\Gamma$-holomorphic disk, and additionally satisfies the following:

(Label axiom) A treed input or output segment labelled $\blacktriangledown$ resp. $\triangledown$ asymptotes to the maximum point $x^{\blacktriangledown} \in \mathrm{crit}(F)$.

An weighted holomorphic disk $u : C \to X$ is *stable* if for any component of the domain $C_v$, either the map $u_v$ is non-constant, or the domain $C_v$ is stable in the sense of weighted treed disks. The new feature of stability for weighted holomorphic disks is that a stable map $u$ may be constant on an infinite tree segment labelled $\blacktriangledown \to \blacktriangledown$ or $\blacktriangledown \to \triangledown$. The *combinatorial type* of a weighted treed holomorphic disk consists of the type of the domain weighted treed disk, together with the combinatorial type data of the treed holomorphic map, namely homology classes of the maps on surface components, and intersection multiplicities with the stabilizing divisor at the markings. The type $\Gamma$ of a weighted treed holomorphic disk is *rigid* if all the edges $e \in \mathrm{Edge}_-(\Gamma)$ are boundary edges with finite non-zero length, and in case of weighted inputs or output, the weight $\rho(e)$ is finite and non-zero.

There is an additional notion of equivalence for holomorphic weighted treed disks. Two weighted treed holomorphic disks $u$, $u'$ are equivalent, if $u$ has a weighted leaf $e$ with weighting $\rho(e_i) = \infty$ resp. 0, and $u'$ is obtained from $u$ by attaching to $e$ a constant trajectory $u'' : \mathbb{R} \to L$ with weighted incoming $e^-$ and forgettable resp. unforgettable outgoing edge $e^+$. See Figure 10.1. This ends the Definition.

The expanded set of labels on the ends of treed segments is denoted by

$$\hat{\mathcal{I}}(L) := \mathcal{I}(L) \cup \{x^{\blacktriangledown}, x^{\triangledown}\}, \tag{10.25}$$

where we recall from (6.2) that $\mathcal{I}(L)$ is the set of critical points of the Morse function on the Lagrangian $L$.

**Proposition 10.9.** *(Transversality for weighted treed holomorphic disks) Given a regular coherent perturbation datum $\underline{\mathfrak{p}}$ for treed holomorphic disks, $\underline{\mathfrak{p}}$ extends to a regular perturbation datum $\underline{\mathfrak{p}}^{\mathrm{wt}}$ on weighted treed holomorphic disks such that the following holds: For an uncrowded type $\Gamma$ of weighted holomorphic maps, and a prescribed tuple of inputs $\underline{x} := (x_1, \ldots, x_{d(\circ)}) \in \hat{\mathcal{I}}(L)^{d(\circ)}$ and an output $x_0 \in \hat{\mathcal{I}}(L)$ respecting the (Label axiom) and for which $i(\Gamma, \underline{x}) \leq 1$, the moduli space*

$$\mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$$

*of weighted $\underline{\mathfrak{p}}$-holomorphic treed disks with limits $\underline{x}$ is a smooth manifold of expected dimension. The moduli space $\mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$ is compact if $i(\Gamma, \underline{x}) = 0$, and if $i(\Gamma, \underline{x}) = 1$, $\mathcal{M}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})$ has a compactification whose boundary consists of configurations with a boundary node with length $0$ or $\infty$, or a weighted edge with weight $0$ or $\infty$.*

*Proof of Proposition 10.9.* We extend the perturbation datum $\underline{\mathfrak{p}} = \{\mathfrak{p}_\Gamma\}_\Gamma$ on domain curves that only contain unforgettable leaves to those that have other kinds of leaves as well. First, consider domain types whose output is unforgettable. Suppose that the domain type $\Gamma$ has a single weighted/forgettable input leaf $e$, since the other cases follow inductively. Let $\Gamma_0$ resp. $\Gamma_\infty$ be the domain type obtained by making the weight of $e$ in $\Gamma$ zero resp. infinity. By the (Cutting a weighted edge) morphism, on the subset $\{\rho(e) = 0\}$ resp. $\{\rho(e) = \infty\}$ of $\mathcal{M}_\Gamma$, the perturbation $\tilde{\mathfrak{p}}_\Gamma$ is given by $\tilde{\mathfrak{p}}_{\Gamma_0}$ resp. $\tilde{\mathfrak{p}}_{\Gamma_\infty}$. The perturbation $\tilde{\mathfrak{p}}_{\Gamma_0}$ is equal to $\mathfrak{p}_{\Gamma_0}$. The perturbation $\tilde{\mathfrak{p}}_{\Gamma_\infty}$ is defined via the (Forgetting edges) axiom, which means that the perturbation is independent of forgettable treed segment on the domain curve. By standard arguments, the perturbation $\tilde{\mathfrak{p}}_\Gamma$ can be extended over $\{\rho(e) \in (0, \infty)\}$ while satisfying regularity.

For domain curves whose output is weighted or forgettable, we define the perturbation to be domain-independent. Indeed, on such domains maps are constant and lie in the maximum point in $\text{crit}(F) \subset L$; and strata with only unforgettable leaves and root do not occur in the compactification of strata where the output is forgettable/weighted.

Next we prove the statement on compactification. We analyze the cases where a sequence of weighted maps of type $\Gamma$ that are non-constant on either the surface or tree part converges to a limit that has a breaking on a weighted/forgettable leaf $e$, and the map is non-constant on both segments incident at the breaking. The other cases are covered either by the compactification result for treed holomorphic maps without weights; or, if the maps in the sequence are constant, by straightforward formal arguments.

If $e$ is weighted, for dimension reasons, the infinite segment $(-\infty, \infty)$ in the broken segment maps to the maximum point in $L$, and therefore, the map is constant on this segment. If $e$ is forgettable, by the (Forgetting edges) axiom, the only case to be considered is when $i(\Gamma, \underline{x}) = 1$ and $\Gamma$ is of the form shown in Figure 10.2. The map is

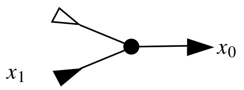



**Figure 10.2.** A type of map. The black arrowheads mean that the input/output is unforgettable. The white attowhead means that the input/output is forgettable.

constant on the surface, and the Morse index of $x_0, x_1 \in \text{crit}(F)$ differ by 1. We may assume the domain-dependent Morse perturbation on the unforgettable edges is small enough that the ends of the moduli space are given by the breaking of the unforgettable input or output edge as in Figure 10.3 below. This rules out the possibility of breaking

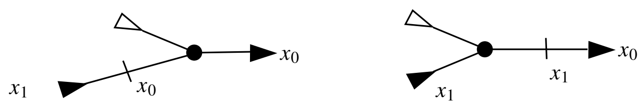



**Figure 10.3.** Breaking an edge

on the forgettable edge. ∎

For a coherent perturbation system $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$ for weighted treed types define composition maps on the generators of the Fukaya algebra

$$CF(L, \underline{\mathfrak{p}}) := CF^{\mathrm{geom}}(L, \underline{\mathfrak{p}}) \oplus \Lambda_{\geq 0}\langle x^{\blacktriangledown}\rangle \oplus \Lambda_{\geq 0}\langle x^{\triangledown}\rangle$$

as

$$m^{d(\circ)}(x_1, \dots, x_{d(\circ)}) = \sum_{x_0, u \in \overline{\mathcal{M}}_\Gamma(L, \underline{\mathfrak{p}}, \underline{x})_0} w(u) x_0 \qquad (10.26)$$

where

$$w(u) = (-1)^{\heartsuit} (d(\bullet)!)^{-1} \operatorname{Hol}_L([\partial u]) q^{A(u)} \epsilon(u), \qquad \heartsuit = \sum_{i=1}^{d(\circ)} i|x_i|.$$

The sum ranges over all rigid types $\Gamma$ of weighted treed holomorphic disks as in Definition 10.8. That is, $\Gamma$ is a rigid type for broken maps (Definition 8.48) after forgetting the weighting, and additionally, the weight for any weighted input is finite non-zero in the type $\Gamma$. Rigidity includes the latter condition because a weighted edge with weight 0 or $\infty$ may be deformed to a weighted edge with finite non-zero weight.

We now prove the main Theorem of this section.

*Proof of Theorem* 10.5. By Proposition 10.9, the coherent perturbation data $\underline{\mathfrak{p}}$ for treed holomorphic disks extend to coherent data $\underline{\mathfrak{p}}^{\mathrm{wt}}$ for weighted treed holomorphic disks. We first prove properties (b)-(c), and then proceed to show the $A_\infty$ associativity relations. The geometric part $CF^{\mathrm{geom}}(L)$ is a sub-algebra because if the inputs to a treed disk are unforgettable, the output is also unforgettable, proving (b).

The element $x^{\triangledown}$ is a strict unit for the following reasons. For $d(\circ) > 2$, we have

$$m^{d(\circ)}(\dots, x^{\triangledown}, \dots) = 0$$

because the input $x^{\triangledown}$ is an empty constraint, and can be forgotten because the perturbation satisfies the (Forgetting edges) axiom. The term $m^1(x^{\triangledown})$ is also zero : any disk that is counted has interior markings, and therefore, placing the marked point $x^{\triangledown}$ adds one to the dimension, and therefore the moduli space is not zero-dimensional. Finally, by the same argument, $m^2(x^{\triangledown}, y)$ and $m^2(y, x^{\triangledown})$ do not count any disk with interior markings. The only contributions are from constant disks. We conclude that both terms are equal to $\pm y$, for any generator $y$, proving (a).

For (c), we recall from (10.23) that the infinite tree segments with labels $\blacktriangledown \to \blacktriangledown$, $\blacktriangledown \to \triangledown$ are rigid configurations with orientations +1 and −1 (see [17, Remark 4.24]). These contribute $x^{\blacktriangledown} - x^{\triangledown}$ to $m^1(x^{\blacktriangledown})$. The other terms in $m^1(x^{\blacktriangledown})$ necessarily arise from configurations with at least one non-constant disk component, by the definition of stability, and so have positive $q$-valuation.

To show that the $A_\infty$ associativity relationships are satisfied, we consider one-dimensional moduli space of weighted maps of rigid type. By Proposition 10.9, the true boundary strata contain one of the following configurations : a configuration with a weight 0 or $\infty$ at a weighted input which is equivalent to the broken configuration in Figure 10.1, a boundary node with a broken segment, or a broken Morse trajectory. These configurations exactly correspond to the terms in the $A_\infty$ associativity relations, and so $CF(L)$ is an $A_\infty$ algebra. In particular, the former boundary configuration contributes $m^i(\dots, x^{\blacktriangledown}, \dots)$, $m^i(\dots, x^{\triangledown}, \dots)$ to terms of the form $m^i(\dots, m^1(x^{\blacktriangledown}, \dots))$ in the associativity relation. Any other kind of contribution to a term of the form $m^i(\dots, m^j(\dots), \dots)$ in the associativity relation, comes from a non-constant disk breaking off, that is, the map is constant on either the surface or the tree part. ■

We recall from [17] a useful criterion for a Lagrangian brane $L$ to be weakly unobstructed.

**Lemma 10.10.** *(A criterion for unobstructedness* [17, Lemma 4.43]*) Suppose for a Fukaya algebra $CF(L)$ of a Lagrangian submanifold $L \subset X$, $m^0(1) = Wx^{\blacktriangledown}$ for some $W \in \Lambda_{\geq 0}$ and every non-constant disk has positive Maslov index. Then, $b := Wx^{\blacktriangledown} \in MC(L)$ and $m_b^0(1) = Wx^{\triangledown}$.*

## 10.4 Quilted disks

Morphisms between Fukaya algebras are defined by counts of quilted holomorphic disks. The domains of such disks are *quilted disks*, which are ordinary disks with a *quilting circle* as defined in Ma'u-Woodward [53]. In this Section, we describe moduli spaces of treed nodal quilted disks.

**Definition 10.11.** (Quilted disks)

(a) A *quilted disk* $(S, Q, \underline{z})$ is a complex disk $S \simeq \mathbb{D}^2 \subset \mathbb{C}$ with a collection $\underline{z}$ of interior and boundary markings, and a *quilting circle*, which is a circle $Q \subset S$ tangent to the 0-th boundary marking $z_0$. Two quilted disks $(S_0, Q_0, \underline{z}^0)$, $(S_1, Q_1, \underline{z}^1)$ are *isomorphic* if there is a biholomorphism which maps each marking in $S_0$ to the corresponding marking in $S_1$, and which maps the quilting circle $Q_0 \subset S_0$ to the quilting circle $Q_1 \subset S_1$.

(b) (Treed quilted disk) A *treed quilted disk* is a treed nodal disk $C$, a subset of whose components are quilted components. The subset of quilted components satisfies the following:
   - A path in $C$ from any (non-root) boundary or interior marking $z_e$, $e \in \text{Edge}_{\rightarrow} \backslash \{0\}$ to the root marking $z_0$ intersects exactly one quilted disk.
   - (Equal lengths condition) For the disk component $S_{v_0}$ containing the outgoing leaf, and quilted components $S_{v_1}, S_{v_2}$, the sum of lengths $\ell(e)$ of treed segments $T_e$ connecting $S_{v_0}$ to $S_{v_k}$, $k \in \{1, 2\}$ is independent of $k$.

(c) (Combinatorial types) The combinatorial type $\Gamma$ of a quilted treed disk is the combinatorial type of the treed disk (without the quilting datum) together with a subset of the boundary vertex set

$$\text{Vert}^q(\Gamma) \subset \text{Vert}_\circ(\Gamma),$$

corresponding to quilted disks, and are called *quilted vertices*.

(d) (Stability) A treed quilted disk $C = S \cup T$ is stable if the automorphism group of any surface component $S_v, v \in \text{Vert}(\Gamma)$ is trivial, and there are no treed segments both whose ends are infinite. This means a quilted disk is stable if it has at least two special points.

*Remark* 10.12. (Interior markings on quilted disks) An interior point in a quilted disk component may lie inside, outside or on the quilting circle; the relative position of interior points with respect to the quilting circle does not affect the combinatorial type of the treed quilted disk. [1]

*Remark* 10.13. (Quilted disks via affine structures) A quilted disk may alternately be defined as a disk $S \simeq \mathbb{D}^2 \subset \mathbb{C}$ with markings $\underline{z}$, and a biholomorphism

$$\phi : (S \backslash \{z_0\}, \partial S) \xrightarrow{\phi} (\mathbb{H}, \partial\mathbb{H}),$$

which we call an *affine structure*. Two affine structures $\phi_0$, $\phi_1$ are equivalent if there exists $\xi \in \mathbb{R}$ such that $\phi_1(z) = \phi_0(z) + \xi$. Two quilted disks $(S_0, \underline{z}^0, \phi_0)$, $(S_1, \underline{z}^1, \phi_1)$ are *isomorphic* if there exists $\xi \in \mathbb{R}$ such that, defining $\tau(z) := z + \xi$, the biholomorphism $\phi_1^{-1} \circ \tau \circ \phi_0 : S_0 \to S_1$ maps each marking in $S_0$ to the corresponding marking in $S_1$. A choice of a quilting circle is equivalent to an affine structure, by taking the quilting circle to be $Q = \{\text{Im}(z) = 1\}$.

The moduli space of stable quilted disks with interior and boundary markings is homeomorphic to a compact cell complex. More precisely, the moduli space is a compact topological space admitting a Thom-Mather stratification into smooth submanifolds with toric singularities. As the interior and boundary markings go to infinity, the markings bubble off onto either quilted disks or unquilted disks or spheres. The case of combined boundary and interior markings is a straight-forward generalization of Ma'u-Woodward [53]; see also Bottman-Oblomkov [10] where a generalization to markings on arbitrary number of quilting circles is constructed. In the case with no

[1] In [17], the compactification of the moduli space of quilted disks contains configurations with "quilted spheres". Quilted spheres arise in the compactification only if we impose a requirement that markings can not lie on dark components, that is, on unquilted components lying below the quilting circle.

interior markings, the moduli spaces are cell complexes called the "multiplihedra" introduced by Stasheff [80]. We denote by

$$\mathcal{M}^{q}_{d(\bullet),d(\circ)}$$

the moduli space of stable marked quilted treed disks with $d(\circ)$ boundary leaves and $d(\bullet)$ interior leaves.

*Remark* 10.14. (Shadings on a treed quilted disk) In Figures, we represent quilted disks using shadings. The subset of a quilted disk lying inside resp. outside the quilting circle is shaded dark resp. light. An unquilted surface component $S_v$ of a treed quilted disk is shaded light resp. dark if the path from $v$ to the root vertex contains resp. does not contain a quilted vertex. See Figures 10.4, 10.5 for examples.

*Example* 10.15. Figure 10.4 depicts the moduli space $\mathcal{M}^{q}_{0,2}$ of treed quilted disks with two incoming leaves, one outgoing leaf, and no interior marking; and Figure 10.5) depicts the moduli space $\mathcal{M}^{q}_{1,0}$ of treed quilted disks with one outgoing leaf and one interior marking. In both cases, there are 3 open strata, interspersed with four strata of codimension one. Figure 10.4 shows the type of each of these strata. The hashes on the line segments $T_e$ indicate breakings. We justify the compactification of an open stratum in the moduli space of quilted disks with one interior marking. Let $\Gamma$ be a type of quilted disk with a single interior marking, a boundary output and no boundary inputs, and no boundary nodes (fourth figure from the left in Figure 10.5). The stratum $\mathcal{M}^{q}_{\Gamma} \subset \mathcal{M}^{q}_{1,0}$ is an open interval $(0,\infty)$ parametrizing disks with markings

$$z^{\circ}_{0,\nu} = \infty, \quad z^{\bullet}_{0,\nu} = i\nu, \quad \nu \in (0,\infty), \tag{10.27}$$

and quilting circle $\{\mathrm{Im}(z) = 1\}$, where the disk is identified to $\mathbb{H} \cup \{\infty\}$. In the limit $\nu \to \infty$ resp. $\nu \to 0$, we obtain the strata to the left resp. right of $\Gamma$ in Figure 10.5. In particular, in the limit $\nu \to \infty$, the limit of the markings in (10.27) gives the quilted disk in the limit, and the unquilted dark disk is given by the reparametrization $z \mapsto \frac{z}{\nu}$, so that the markings in the sequence are $z^{\circ}_{0,\nu} = \infty$, $z^{\bullet}_{0,\nu} = i$, and the quilting circle is $\{\mathrm{Im}(z) = \frac{1}{\nu}\}$. This gives rise to a limit disk which is fully in the dark region. This finishes the example.

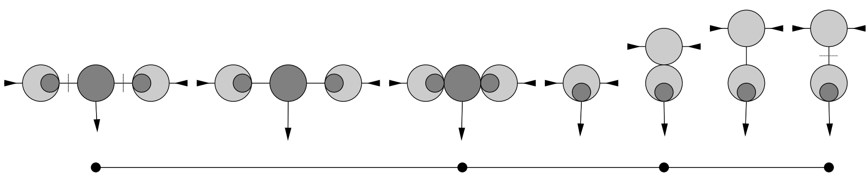


**Figure 10.4.** Moduli space $\mathcal{M}^{q}_{0,2}$ of stable quilted treed disks.

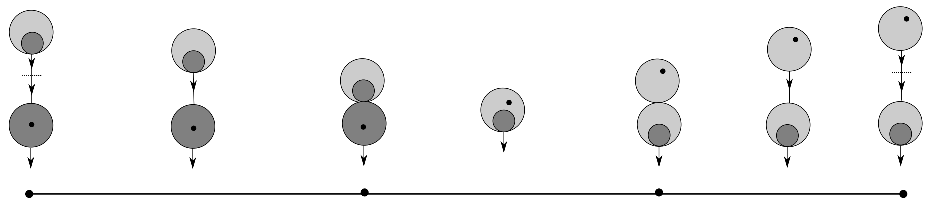


**Figure 10.5.** Moduli space $\mathcal{M}^q_{1,0}$ of stable quilted treed disks.

As in the unquilted case, the top-dimensional cells in $\mathcal{M}^q_{d(\bullet),d(\circ)}$ consist of strata $\mathcal{M}_\Gamma$ which do not have any interior edges, that is, $\mathrm{Edge}_{\bullet,-}(\Gamma) = \emptyset$, and all boundary edges $e \in \mathrm{Edge}_{\circ,-}(\Gamma)$ have finite non-zero length, and the dimension of these strata is equal to

$$d(\circ) + 2d(\bullet) - 1.$$

The true boundary of $\mathcal{M}^q_{d(\bullet),d(\circ)}$ consists of strata $\mathcal{M}_\Gamma$ where either

- $\Gamma$has a single broken edge $e \in \mathrm{Edge}_\circ(\Gamma)$, or
- $\Gamma$ has a collection of broken edges $e_1, \ldots, e_k$, such that the types $\Gamma_0, \ldots, \Gamma_k$ obtained by disconnecting $\Gamma$ at the breakings at $e_1, \ldots, e_k$ consist of an unquilted disk type $\Gamma_0$, all whose vertices have negative distance from the quilting circle in $\Gamma$, and a collection $\Gamma_1, \ldots, \Gamma_k$ of quilted disk types.

For example, the combinatorial types in Figure 10.6 occur in the codimension one stratum of $\mathcal{M}^q_{0,4}, \mathcal{M}^q_{0,5}$.

The following is a useful quantity defined on treed quilted disks.

**Definition 10.16.** (Distance to the quilting circle) Let $C = S \cup T$ be a treed quilted disk of type $\Gamma$. Given a point $x \in C$ and a surface component $S_v \subset C$, define the *distance* between $x$ and $S_v$ as

$$\overline{d}(x, S_v) := \ell(\gamma \cap T) = \sum_{e \in \mathrm{Edge}_{\circ,-}(\Gamma)} \ell(\gamma \cap T_e) \in [0, \infty], \tag{10.28}$$

where $\gamma$ is any non self-intersecting path from $x$ to any point in $S_v$. For a point $z \in C$, the *distance to the quilting circle* is the signed distance to the closest quilted surface component, that is,

$$d(z) := \pm\overline{d}(z, S_v) \in [-\infty, \infty], \tag{10.29}$$

where $S_v, v \in \mathrm{Vert}^q(\Gamma)$ is the closest quilted surface component to $z$ with respect to $\overline{d}$ in (10.28), and the sign is + resp. − if $z$ is above resp. Below the quilted disk components (that is, further from resp. closer to the root than the quilted disk components). Note that $d$ is constant on any surface component of $C$. Therefore, the distance $d(v)$ is defined for any vertex $v$ in the graph underlying $C$. The distance function $d$ is zero on a quilted disk component and the spheres attached to it. To define perturbations, $d$ is composed with an increasing diffeomorphism

$$\delta : [-\infty, \infty] \to [0, 1] \tag{10.30}$$

which is assumed to be fixed.

*Remark* 10.17. (Morphisms for quilted disk types) Morphisms of graphs (Cutting an edge, collapsing edges, making edge lengths finite or non-zero) from Definition 6.3 induce morphisms of moduli spaces of stable quilted treed disks as in the unquilted case. The new feature is that (Cutting edges) is done such that one of the pieces is quilted and the other unquilted. This implies that output edges of quilted disks are cut simultaneously, and therefore the output has a disconnected type.

For example, in Figure 10.6, in the left picture, one can cut the $e$ at the breaking to obtain an unquilted disk with positive distance from the quilting circle, and a quilted disk. In the picture to the right, the edges $e_1$ and $e_2$ get cut simultaneously to yield an unquilted disk with negative distance from the quilting circle and a disconnected type consisting of two quilted disks.

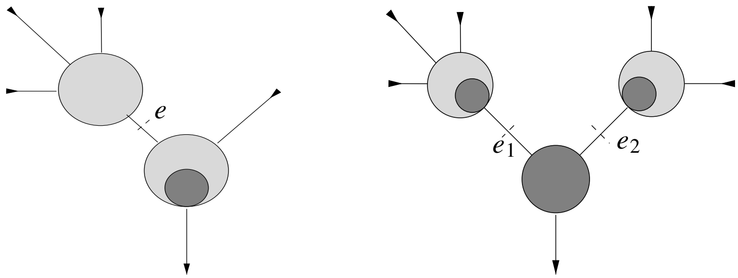



**Figure 10.6.** A quilted treed disk with edges of infinite length.

In a similar vein, the morphisms (Collapsing edges) and (Making an edge length finite or non-zero) may involve several edges instead of a single one. For example, in the quilted disk with three boundary edges of length zero shown in Figure 10.7, there

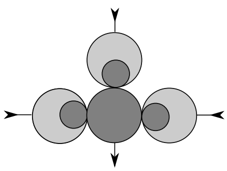


**Figure 10.7.** A quilted disk with length zero edges.

is no (Collapsing edges) or (Making an edge length finite or non-zero) morphism for one of the edges alone; these operations can only be performed for all three edges simultaneously in order to respect the (Equal lengths condition).

For any combinatorial type $\Gamma$ of quilted disk there is a *universal quilted treed disk* $\overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$ which is a cell complex whose fiber over $[C]$ is isomorphic to $C$. The universal disk splits into surface and tree parts $\overline{\mathcal{U}}_\Gamma = \overline{\mathcal{S}}_\Gamma \cup \overline{\mathcal{T}}_{\circ,\Gamma} \cup \overline{\mathcal{T}}_{\bullet,\Gamma}$, where the last two sets are the boundary and interior parts of the tree respectively. This ends the Remark.

Weights can be added to the inputs and output of quilted treed disks as in the case of treed disks. We suppose there is a partition of the boundary markings

$$\mathrm{Edge}^{\blacktriangledown}(T) \sqcup \mathrm{Edge}^{\triangledown}(T) \sqcup \mathrm{Edge}^{\blacktriangledown}(T) = \mathrm{Edge}_{\circ,\to}(T)$$

into *weighted* resp. *forgettable* resp. *unforgettable* edges as in the unquilted case. The outgoing edge axiom is the same as in the unquilted case. In the quilted case, the trees in Figure 10.8 are stable. Isomorphism of weighted quilted disks is the same as the unquilted case, and therefore, the moduli space with a single weighted leaf and no markings is a point.

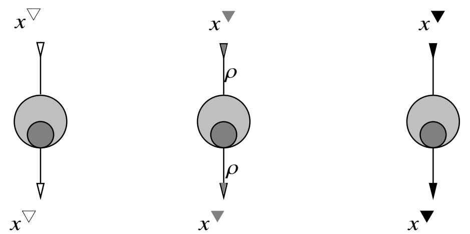



**Figure 10.8.** Unmarked stable treed quilted disks.

Orientations of the moduli space of quilted treed disks are defined as follows. Each main stratum of $\mathcal{M}^q_{d(\bullet),d(\circ)}$ can be oriented using the isomorphism of the stratum made of quilted treed disks having a single disk with $\mathbb{R}$ times $\mathcal{M}_{d(\bullet),d(\circ)}$, the extra factor corresponding to the quilting parameter. The boundary of the moduli space is naturally isomorphic to a union of moduli spaces:

$$\partial\overline{\mathcal{M}}^q_{d(\bullet),d(\circ)} \cong \bigcup_{\substack{i,j \\ I_1\cup I_2=[d(\bullet)]}} \left(\overline{\mathcal{M}}^q_{|I_1|,d(\circ)-i+1} \times \overline{\mathcal{M}}_{|I_2|,i}\right)$$
$$\cup \bigcup_{\substack{r\geq 1, I_1\cup \ldots I_r=[d(\bullet)], \\ m_1+\cdots+m_r=d(\circ)}} \left(\overline{\mathcal{M}}_{|I_0|,r} \times \prod_{j=1}^r \overline{\mathcal{M}}^q_{|I_j|,m_j}\right). \quad (10.31)$$

Here $[d(\bullet)] := \{1,\ldots,d(\bullet)\}$. In the first union, $j$ is the index of the attaching leaf in the quilted tree and so ranges from 1 to $n-i+1$. The first union ranges over all partitions of the set $[d(\bullet)]$ into $I_1$, $I_2$. The second union ranges over partitions of $[d(\bullet)]$ into $I_0, \ldots, I_r$. By construction, for the facet of the first type, the sign of the inclusions of boundary strata are the same as that for the corresponding inclusion of boundary facets of the moduli space $\overline{\mathcal{M}}_{d(\bullet),d(\circ)}$ of unquilted disks, that is, $(-1)^{i(n-i-j)+j}$. For facets of the second type, the gluing map is

$$(0,\infty) \times \mathcal{M}_{r,m_0} \times \prod_{j=1}^{r} \mathcal{M}^q_{|I_j|,m_j} \to \mathcal{M}^q_{d(\bullet),d(\circ)}$$

given for boundary markings by

$$(\delta, x_1, \ldots, x_r, (x_{1,j} = 0, x_{2,j}, \ldots, x_{m_j,j})_{j=1}^r) \mapsto$$
$$(x_1, x_1 + \delta^{-1}x_{2,1}, \ldots, x_1 + \delta^{-1}x_{m_1,1}, \ldots, x_r, x_r + \delta^{-1}x_{2,r}, \ldots, x_r + \delta^{-1}x_{m_r,r}). \tag{10.32}$$

This map views the markings as lying in the affine half plane $\mathbb{H} \subset \mathbb{C}$; an interior $x \in I_j$ is mapped to $x_j + \delta^{-1}x$; and the map is well-defined for $\delta$ that is large enough to ensure that the ordering of the boundary markings is preserved. This map changes orientations by $\sum_{j=1}^r (r-j)(m_j - 1)$; in case of non-trivial weightings, $m_j$ should be replaced by the number of incoming markings plus non-trivial but finite weightings on the $j$-th component.

## 10.5 Quilted pseudoholomorphic disks

In this Section and the next, we prove that the Fukaya algebra of a Lagrangian submanifold is independent of the choice of perturbation data up to homotopy equivalence. Given $A_\infty$ algebras defined using perturbation data $\underline{\mathfrak{p}}_0$, $\underline{\mathfrak{p}}_1$ whose underlying stabilizing divisor has the same degree, in this Section, we construct an $A_\infty$ morphism between them. The $A_\infty$ morphism is defined by counts of quilted treed holomorphic disks. The perturbation system corresponding to such disks is defined by extending the perturbation systems $\underline{\mathfrak{p}}_0$, $\underline{\mathfrak{p}}_1$; that is, the perturbation system for quilted holomorphic disks restricts to $\underline{\mathfrak{p}}_0$ and $\underline{\mathfrak{p}}_1$ on light and dark unquilted components respectively.

The following is the main result. In the statement of the result, a unital $A_\infty$ morphism is defined in (10.9), a convergent $A_\infty$ morphism is defined in Remark 10.19, and perturbation morphisms are from Definition 10.20.

**Proposition 10.18.** *Suppose $\underline{\mathfrak{p}}^0$, $\underline{\mathfrak{p}}^1$ are regular perturbation data that are defined using stabilizing pairs $(J^0, D^0)$ and $(J^1, D^1)$, which are connected by a path of stabilizing pairs $\{(J^t, D^t)\}_{t\in[0,1]}$. There exists a coherent perturbation datum $\underline{\mathfrak{p}}^{01}$ which induces a convergent unital $A_\infty$ morphism*

$$\phi : CF(L, \underline{\mathfrak{p}}^0) \to CF(L, \underline{\mathfrak{p}}^1),$$

*which is a convergent unital $A_\infty$ homotopy equivalence.*

Proposition 10.18 is used in Section 10.7 to show that the Fukaya algebra of a Lagrangian in a neck-stretched manifold is independent of the neck length parameter up to homotopy equivalence. The $A_\infty$ morphism $\phi$ required by Proposition 10.18 is constructed later in this Section after defining quilted holomorphic disks. The fact that $\phi$ is a homotopy equivalence is proved in Section 10.6 as part of Corollary 10.30.

*Remark* 10.19. (Convergent $A_\infty$ morphisms) An $A_\infty$ morphism $\mathcal{F} : A_0 \to A_1$ between $A_\infty$ algebras with Novikov coefficients is said to be *convergent* if $\mathcal{F}^0$ has positive $q$-valuation. A convergent $A_\infty$ morphism $\mathcal{F} : A_0 \to A_1$, induces a well-defined map $MC(\mathcal{F}) : MC(A_0) \to MC(A_1)$ on the space of Maurer-Cartan solutions and a map on cohomology $H(\mathcal{F}) : H(A_0, b) \to H(A_1, \mathcal{F}(b))$ for any $b \in MC(A_0)$, see [17, Lemma 5.2, 5.3]. In Proposition 10.18, the $A_\infty$ morphism $\phi$ being convergent means that $\phi^0(1) \in \Lambda_{>0}\langle \hat{\mathcal{I}}(L)\rangle$.

**Definition 10.20.** Given perturbation data

$$\underline{\mathfrak{p}}^0 = (J^0_\Gamma, F^0_\Gamma)_\Gamma, \quad \underline{\mathfrak{p}}^1 = (J^1_\Gamma, F^1_\Gamma)_\Gamma$$

on unquilted treed disks with respect to stabilizing divisors $D^0$ and $D^1$ that have the same degree and a path $\{D^t\}_{t\in[0,1]}$ of Donaldson divisors, a *perturbation morphism* $\underline{\mathfrak{p}}^{01}$ from $\underline{\mathfrak{p}}^0$ to $\underline{\mathfrak{p}}^1$ for the type $\Gamma$ of a quilted treed disk consists of

(a) a domain-dependent Morse function

$$F^{01}_\Gamma : \overline{\mathcal{T}}_{\circ,\Gamma} \to \mathbb{R},$$

which for $k = 0, 1$, is equal to the domain-independent Morse function $F^k$ on the neighbourhood $\overline{\mathcal{T}}_{\circ,\Gamma} - \overline{\mathcal{T}}^{\mathrm{cp}}_{\circ,\Gamma}$ of the endpoints for which the distance to the quilting circle $\delta \circ d$ is $k$, where $F^k : L \to \mathbb{R}$ is the background Morse function for $F^k_\Gamma$ for $k = 0, 1$;

(b) and a domain-dependent almost complex structure

$$J^{01}_\Gamma : \overline{\mathcal{S}}_\Gamma \to \bigsqcup_{t\in[0,1]} \{J \in \mathcal{J}(X, D^t) : J \text{ is } \omega\text{-tamed}\},$$

with the property that

- on the curve $\mathcal{S}_{\Gamma,m} \subset \overline{\mathcal{S}}_\Gamma$ associated to any point $m \in \mathcal{M}_\Gamma$, $J^{01}_\Gamma$ is locally constant on $\mathcal{S}_{\Gamma,m} - \mathcal{S}^{\mathrm{cp}}_{\Gamma,m}$, where the compact set $\mathcal{S}^{\mathrm{cp}}$ is as defined in Section 6.1; and on a component $\mathcal{S}_{v,m} \subset \mathcal{S}_{\Gamma,m}$ corresponding to $v \in \mathrm{Vert}(\Gamma)$, $J^{01}_\Gamma$ is adapted to the divisor $D^{\delta\circ d(v)}$ where $d$ is the distance from the quilting circle function (10.29), that is, $J^{01}_\Gamma(\mathcal{S}_{v,m}) \subset \mathcal{J}(X, D^{\delta\circ d(v)})$.

- Furthermore, for $k = 0, 1$, denoting by $\Gamma_k \subset \Gamma$ the sub-tree where the distance from the quilting circle $\delta \circ d$ is $k$, $J^{01}_\Gamma$ is equal to the domain-dependent complex structure $J^k_{\Gamma_k}$ on the (unquilted) treed disk component of type $\Gamma_k$.

A collection of perturbation morphisms $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$ defined on moduli spaces of quilted disks is *coherent* if it is compatible with the (Cutting edges), (Making an edge length/weight finite or non-zero) and (Forgetting edges) morphisms on weighted quilted disk types as in Remark 10.17, and satisfies the (Locality axiom) from Definition 6.5.

**Definition 10.21.** (Holomorphic quilted treed disk) Let $\underline{\mathfrak{p}} = (\mathfrak{p}_\Gamma)_\Gamma$ be a coherent perturbation morphism. A holomorphic quilted treed disk is a map $u : C \to X$ where $C$ is a quilted disk of type $\Gamma$, and $u$ is $\mathfrak{p}_\Gamma$-holomorphic in the sense of ordinary perturbed holomorphic disks (see Definition 6.12), and *adapted* to a family of divisors $\{D^t\}_{t\in[0,1]}$ in the sense that

- each interior marking $z_i$ maps to $D^{\delta\circ d(z_i)}$, where $d$ is the distance from the quilting circle (10.29) and $\delta$ is from (10.30),
- and for $t \in [0, 1]$, each component of $u^{-1}(D^t) \cap (\delta \circ d)^{-1}(t)$ contains a marking.

If the domain quilted treed disk $C$ is unstable, we obtain a stable quilted treed disk $C'$ by collapsing unstable surface and tree components, and we denote the collapsing map by $f : C \to C'$. Pulling back by $f$ we obtain a datum on $C$, still denoted $(J^{01}_\Gamma, F^{01}_\Gamma)$. A quilted holomorphic treed disk $u : C \to X$ is *stable* if every (surface or tree) component of $C$ on which $u$ is non-constant is stable in the sense of weighted quilted disks. The *type* of a quilted holomorphic disk consists of the combinatorial type of the domain quilted treed disk, and the tangency and homology data of the map components as in the type of a treed holomorphic disk, see Definition 6.14.

*Remark* 10.22. (Symplectic area and the number of markings) For a quilted treed disk $C = S \cup T$, on any surface component $S_v \subset S$, the function $\delta \circ d|S_v$ is constant and therefore a quilted holomorphic disk $u : C \to X$ is adapted to the divisor $D^{\delta\circ d(S_v)}$ on the component $S_v$. Furthermore, the divisor $D^{\delta\circ d(S_v)}$ is holomorphic with respect to the background almost complex structure $J^{\delta\circ d(S_v)}$, and therefore all intersections between the map $u|S_v$ and the divisor $D^{\delta\circ d(S_v)}$ are positive. The intersection number of $u$ with $D^\delta$ is $k\omega[u]$, where the divisor $D_0$ (and $D_1$) is Poincaré dual to $k\omega$.

*Remark* 10.23. (Quilted holomorphic disks for a path of perturbations)

(a) Often a perturbation morphism is constructed using a generic path of perturbations $\mathfrak{p}^t_\Gamma = (J^t_\Gamma, F_\Gamma)$, $t \in [0, 1]$, whose Morse datum $F_\Gamma$ is $t$-independent. Let $\tilde{\Gamma}$ be a quilted disk type, for which forgetting the quilting yields the disk type $\Gamma$. One may define a perturbation morphism $\mathfrak{p}^{01}_{\tilde{\Gamma}}$ connecting $\mathfrak{p}^0_\Gamma$ and $\mathfrak{p}^1_\Gamma$ by setting the domain-dependent data to be

$$J^{01}_{\tilde{\Gamma}}(z) := J^{\delta\circ d(z)}_\Gamma(z), \quad F^{01}_{\tilde{\Gamma}}(z) := F_\Gamma(z) \tag{10.33}$$

for any $z \in \mathcal{S}_{\tilde{\Gamma}}$ where $d$ is the "distance from the quilting circle' function from (10.29), and $\delta : [-\infty, \infty] \to [0, 1]$ is a fixed increasing diffeomorphism from (10.30). Thus on any surface component of a $J^{01}_{\tilde{\Gamma}}$-holomorphic quilted disk, the map is $J^t_\Gamma$-holomorphic for some $t \in [0, 1]$.

(b) (A path of regular perturbations) In some special cases, a one-dimensional component of quilted disks is made up of a family of unquilted $\mathfrak{p}_t$-holomorphic disks for $t \in [0, 1]$. Such a special case arises when the perturbation $\mathfrak{p}^t_\Gamma = (J^t_\Gamma, F_\Gamma)$ is regular for a disk homology class $\beta \in H_2(X, L)$ and input/output tuple $\underline{x} = (x_0, \ldots, x_d)$ for all $t$. Then the zero dimensional component of the moduli space of perturbed holomorphic maps $\mathcal{M}_\beta(\underline{\mathfrak{p}}^t, \underline{x})_0$ is in bijection with $\mathcal{M}_\beta(\underline{\mathfrak{p}}^0, \underline{x})_0$ for any $t \in [0, 1]$. Furthermore, for a perturbation morphism as in (10.33), a path of moduli spaces $\cup_{t\in[0,1]} \mathcal{M}_\Gamma(\underline{\mathfrak{p}}^t, \underline{x})^{\le E_0}_0$ of $\underline{\mathfrak{p}}^t$-holomorphic disks gives a one-dimensional component $(u_t)_{t\in[0,1]}$ of quilted $\underline{\mathfrak{p}}^{01}$-holomorphic disks where

- if $\delta^{-1}(t) > 0$ (where $\delta : [-\infty, \infty] \to [0, 1]$ is a fixed increasing diffeomorphism), $u_t$ is of the form in Figure 10.9 (a), $u_t$ is constant on quilted components, $J_t$-holomorphic on the dark shaded disk, and boundary edges have length $\tau := \delta^{-1}(t)$;
- if $\delta^{-1}(t) < 0$, $u_t$ is of the form in Figure 10.9 (c), $u_t$ is constant on quilted components and $J_t$-holomorphic on the light shaded disk, and boundary edges have length $\tau := -\delta^{-1}(t)$;
- if $\delta^{-1}(t) = 0$, $u_t$ is of the form in Figure 10.9 (b), on the quilted components $u_t$ is $J_t$-holomorphic.

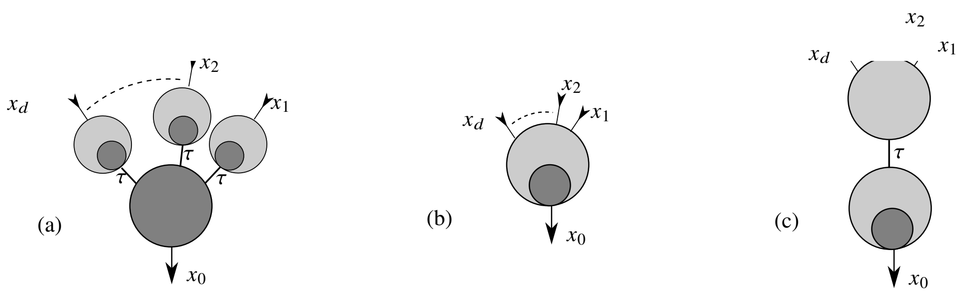



**Figure 10.9.** Types of quilted holomorphic disks occurring in the one-dimensional moduli space described in Remark 10.23 (b). In (a) and (c) the map is constant on the quilted component.

The analysis of the special case shows that in an idealized setting moduli spaces of quilted holomorphic disks interpolate between moduli spaces of unquilted holomorphic disks. But in general cases, one has to account for disk bubbling in the moduli spaces, leading to a variety of configurations occurring in the codimension one boundary of moduli spaces of quilted holomorphic disks, see Remark 10.25.

For any combinatorial type $\Gamma$ of quilted holomorphic disks we denote by $\overline{\mathcal{M}}^q_\Gamma(L)$ the compactified moduli space of equivalence classes of treed quilted holomorphic disks. The moduli space of quilted disks breaks into components depending on the limits along the root and leaf edges. Denote by $\mathcal{M}^q_\Gamma(L,\underline{x}) \subset \mathcal{M}^q_\Gamma(L)$ the moduli space of isomorphism classes of stable holomorphic quilted treed disks of type $\Gamma$ with boundary in $L$ and limits $\underline{x}$ along the root and leaf edges, where $\underline{x} = (x_0,\dots,x_{d(\circ)}) \in \widehat{\mathcal{I}}(L)$. A quilted holomorphic disk type $\Gamma$ is *rigid* if it satisfies all the conditions for a holomorphic map type to be rigid, as in Definition 8.47. For a rigid quilted holomorphic disk type $\Gamma$, the expected dimension of the moduli space $\mathcal{M}^q_\Gamma(L,\underline{x})$ is

$$i^q(\Gamma,\underline{x}) = i(\Gamma',\underline{x}) + 1, \tag{10.34}$$

where $i$ is the index for types of treed holomorphic disks (see (6.30)), and $\Gamma'$ is the unquilted type obtained by forgetting the quilting in $\Gamma$. The extra dimension for the moduli space of quilted disks arises from the quilting datum. In case $n$ quilted disk components, there are $n-1$ equations in the (Equal lengths condition), leading to an extra dimension of 1 compared to the unquilted case.

For a comeager subset of perturbation morphisms extending those chosen for unquilted disks, the uncrowded moduli spaces of expected dimension at most one are smooth and of expected dimension. For sequential compactness, it suffices to consider a sequence $u_\nu : C_\nu \to X$ of quilted treed disks of fixed combinatorial type $\Gamma = \Gamma_\nu$ for all $\nu$. Coherence of the perturbation morphism implies the existence of a stable limit $u : C \to X$ which we claim is adapted. If a sequence of markings $z_{i,\nu} \in C_\nu$ converges to $z_i \in C$, then, $u(z_i) \in D^{\delta\circ d(z_i)}$. Indeed, since the distance from the quilting circle $d(z_{i,\nu})$ converges to $d(z_i)$, the divisor $D^{\delta\circ d(z_i)}$ is the limit of the divisors $D^{\delta\circ d(z_{i,\nu})}$. For types of index at most one, each component of $u^{-1}(D^{\delta\circ d(z_i)})$ is a limit of a unique component of $u_\nu^{-1}(D^{\delta\circ d(z_{i,\nu})})$, otherwise the intersection degree would be more than one which is a codimension two condition. Therefore, every marking in $C$ is a transverse divisor intersection. There are no other divisor intersections because the intersection number with $D^{\delta\circ d}$ is preserved in the limit for topological reasons. The moduli space of quilted broken disks then has the same transversality and compactness property as in the unquilted case, by similar arguments.

Counts of quilted disks define $A_\infty$ morphisms. Given a regular, stabilized and coherent perturbation morphism $\underline{\mathfrak{p}}^{01}$ from $\underline{\mathfrak{p}}^0$ to $\underline{\mathfrak{p}}^1$, define an $A_\infty$ morphism $\phi = (\phi^d)_{d\geq 0} : CF(L,\underline{\mathfrak{p}}^0) \to CF(L,\underline{\mathfrak{p}}^1)$ as

$$\begin{aligned}\phi^d : CF(L;\underline{\mathfrak{p}}^0)^{\otimes d} &\to CF(L;\underline{\mathfrak{p}}^1)\\ (x_1,\dots,x_d) &\mapsto \sum_{x_0,u\in\mathcal{M}_\Gamma(L,D,x_0,\dots,x_d)_0} (-1)^{\heartsuit} w(u) x_0\end{aligned} \tag{10.35}$$

where the weight $w(u)$ is given by

$$w(u) = \epsilon([u])(d_\bullet(\Gamma)!)^{-1} q^{E([u])} \operatorname{Hol}_L([\partial u]) x_0 \tag{10.36}$$

the sum is over uncrowded strata $\Gamma$ of weighted treed quilted holomorphic disks whose boundary edges $e \in \operatorname{Edge}_{\circ,-}(\Gamma)$ have finite non-zero length $\ell(e) \in (0, \infty)$ and whose input and output labels are compatible with $(x_0, \dots, x_d)$ in terms of the (Label axiom) in Definition 10.8, and $\epsilon([u]) = \pm 1$ is the orientation sign.

*Remark* 10.24. (Lowest area terms) For any $x \in \operatorname{crit}(F^0) \cup \{x^{\blacktriangledown}, x^{\triangledown}\}$, the element $\phi_1(x)$ contains zero area terms coming from the count of a quilted treed disk with no interior marking, that is, a treed disk with only one disk that is quilted and mapped to a point. The domain is one of those in Figure 10.8. If $x$ is $x^{\blacktriangledown}$ resp. $x^{\triangledown}$ there is one such configuration whose output is weighted resp. forgettable. In the latter case, it will be the only term with a forgettable output.

*Remark* 10.25. (Codimension one boundary strata) The codimension one strata are of several possible types: either there is one (or a collection of) edge(s) $e$ of length $\ell(e)$ infinity, there is one (or a collection of) edge(s) $e$ of length $\ell(e)$ zero, or equivalently, boundary nodes, or there is an edge $e$ with zero or infinite weight $\rho(e)$. The case of an edge of zero or infinite weighting is equivalent to breaking off a constant trajectory, and so may be ignored. In the case of edges of infinite length(s), then either $\Gamma$ is

(a) (Breaking off an unquilted tree) a pair $\Gamma_1 \sqcup \Gamma_2$ consisting of a quilted tree $\Gamma_1$ and an unquilted tree $\Gamma_2$ as in the left side of Figure 10.6; necessarily the breaking must be a leaf of $\Gamma_1$; or

(b) (Breaking off quilted trees) a collection consisting of an unquilted tree $\Gamma_0$ containing the root and a collection $\Gamma_1, \dots, \Gamma_r$ of quilted trees attached to each of its $r$ leaves as in the right side of Figure 10.6. Such a stratum $\mathcal{M}_\Gamma$ is codimension one because of the (Equal lengths condition) which implies that if the length of any edge between $e_0$ to $e_i$ is infinite for some $i$ then the path from $e_0$ to $e_i$ for any $i$ has the same property.

In the case of a zero length(s), one obtains a fake boundary component with normal bundle $\mathbb{R}$, corresponding to either deforming the edge(s) to have non-zero length or deforming the node(s). This ends the Remark.

We can now prove the first part of Proposition 10.18.

*Proof of Proposition* 10.18. We have so far chosen a regular coherent perturbation morphism $\underline{\mathfrak{p}}^{01}$, and defined a collection $\phi = (\phi^d)_{d \geq 0}$ of maps in (10.35), which is a convergent unital $A_\infty$ morphism for the following reason: The true boundary strata of one-dimensional moduli spaces of quilted holomorphic disks are those described in Remark 10.25 and correspond to the terms in the axiom for $A_\infty$ morphisms (10.8). The signs are similar to those in [17] and omitted. The assertion on the strict units is a consequence of the existence of forgetful maps for infinite values of the weights. By assumption the terms involving $\phi_{d(\circ)}$ and $x^{\triangledown}$ as inputs involve counts of quilted treed disks using perturbation that are independent of the disk boundary incidence points of the first leaf marked $x^{\triangledown}$ asymptotic to $x_M \in X$. Since forgetting that semi-infinite edge gives a configuration of negative expected dimension, if non-constant, the only configurations contributing to these terms must be the constant maps. Hence

$$\phi^1(x^{\triangledown}) = x^{\triangledown}, \quad \phi^{d(\circ)}(\ldots, x^{\triangledown}, \ldots) = 0, n \geq 2.$$

In other words, the only regular quilted trajectories from the maximum, considered as $x^{\triangledown}$, being regular are the ones reaching the other maximum that do not have interior markings (i.e. non-constant disks). The convergence property follows from the fact that any treed quilted disk with no incoming edges must, by stability, have a disk component with interior markings. The $A_\infty$ morphism $\phi$ is an $A_\infty$ homotopy equivalence by Corollary 10.30. ∎

## 10.6 Homotopies

In this Section, we define homotopies between $A_\infty$ morphisms via counts of twice-quilted disks. The main result is that the $A_\infty$ morphism between Fukaya algebras constructed in the previous Section is a homotopy equivalence.

A *twice-quilted disk* $(C, Q_1, Q_2)$ is defined in the same way as once-quilted disks, but with two quilting circles $Q_1, Q_2 \subset C$ that are either equal $Q_1 = Q_2$ or with the second $Q_2 \subset \mathrm{int}(Q_1)$ contained inside the first, say with radii $\rho_1 < \rho_2$ satisfying certain balancing conditions. The combinatorial type $\Gamma$ of a nodal quilted disk comes with the data of subsets of the boundary vertex set

$$\mathrm{Vert}_1^q(\Gamma), \mathrm{Vert}_2^q(\Gamma) \subset \mathrm{Vert}_\circ(\Gamma), \quad \mathrm{Vert}_{12}^{qq}(\Gamma) = \mathrm{Vert}_1^q(\Gamma) \cap \mathrm{Vert}_2^q(\Gamma),$$

where $\mathrm{Vert}_k^q(\Gamma)$ corresponds to disk components which contain the quilting circle $Q_i$. The vertices in $\mathrm{Vert}_{12}^{qq}(\Gamma)$ are *twice-quilted vertices*. See Figure 10.10. For a twice-quilted vertex $v \in \mathrm{Vert}_{12}^{qq}(\Gamma)$, we denote by

$$\lambda_S(v) := \rho_2(v)/\rho_1(v) \qquad (10.37)$$

the ratio of the radii of the quilting circles.

Treed twice-quilted disks are defined by replacing disk nodes by treed segments, whose lengths satisfy two equal lengths conditions: The lengths of edges occurring in a path $P$ from the root vertex to a quilted disk vertex $v \in \mathrm{Vert}_2^q(\Gamma)$ containing the inner quilting circle $Q_2$ is independent of $v$: For any two vertices $v_1, v_2 \in \mathrm{Vert}_2^q(\Gamma)$ connected to the root vertex $v_0$ by non self-crossing paths $P(v_0, v_1), P(v_0, v_2)$ in $\Gamma$,

$$\text{(Equal lengths to inner circle)} \quad \sum_{e \in P(v_0, v_1)} \ell(e) = \sum_{e \in P(v_0, v_2)} \ell(e). \qquad (10.38)$$

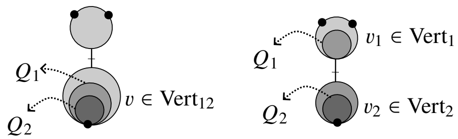



**Figure 10.10.** For $k = 1, 2$, components corresponding to vertices in $\mathrm{Vert}_k^q \subset \mathrm{Vert}_\circ$ contain the quilting circle $Q_i$. Components corresponding to vertices in $\mathrm{Vert}_{12}^{qq} \subset \mathrm{Vert}_\circ$ are twice-quilted.

Secondly, for a fixed disk vertex $v_2 \in \mathrm{Vert}_2^q(\Gamma)$ containing an inner quilting circle, and any two vertices $v_1, v_1' \in \mathrm{Vert}_1^q(\Gamma)$ containing an outer quilting circle and for which the paths $P(v_1, v_2)$, $P(v_1', v_2)$ do not containing the root vertex (unless $v_2$ is the root vertex),

$$\text{(Equal lengths from inner to outer circle)} \qquad \sum_{e \in P(v_2, v_1)} \ell(e) = \sum_{e \in P(v_2, v_1')} \ell(e). \quad (10.39)$$

Note that if $v_2$ is a twice-quilted vertex, then (10.39) is automatically satisfied since all path lengths occurring in (10.39) are zero. A twice-quilted disk is *stable* if it has at least two special points.

The treed disks with two quilting circles defined so far are called *unbalanced twice-quilted disks*. The moduli space of unbalanced treed twice quilted disks of type $\Gamma$ is denoted by $M_\Gamma^{qq,\mathrm{univ}}$. An unbalanced twice-quilted disk $C \in \mathcal{M}_\Gamma^{qq,\mathrm{univ}}$ is a *twice-quilted disk* if it satisfies the *balanced ratio of radii* condition, which is as follows:

- Either all quilted disks are twice-quilted, that is $\mathrm{Vert}_1^q(\Gamma) = \mathrm{Vert}_2^q(\Gamma) = \mathrm{Vert}_{12}^{qq}(\Gamma)$, and the ratio (10.37) of radii is required to be the same for all twice-quilted vertices $v \in \mathrm{Vert}_{12}^{qq}(\Gamma)$, that is,

$$\text{(Balanced ratios, surfaces)} \quad \lambda_S(v) = \lambda_S(v') \quad \forall v, v' \in \mathrm{Vert}_{12}^{qq}(\Gamma); \, or \quad (10.40)$$

- there are no twice-quilted disks and the following treed version of the balanced ratio condition holds: For a disk vertex $v_2 \in \mathrm{Vert}^2(\Gamma)$ containing the inner circle, and a vertex $v_1 \in \mathrm{Vert}^1(\Gamma)$ containing the outer quilting circle, and whose path to the root vertex contains $v_2$, denoting

$$\lambda_T(v_2) := \sum_{e \in P(v_1, v_2)} \ell(e), \quad (10.41)$$

  where $P(v_1, v_2)$ is the path in $\Gamma$ from $v_1$ to $v_2$, we require that

$$\text{(Balanced ratios, trees)} \quad \lambda_T \text{ is independent of } v_2. \quad (10.42)$$

Note that by (10.39), the quantity $\lambda_T(v_2)$ does not depend on the choice of $v_1$. The balanced ratio conditions (10.40), (10.42) on surfaces and trees can be stated in a unified way as follows. First, we fix an identification

$$\phi : ((\{1 \le \lambda_S \le \infty\} \sqcup \{0 \le \lambda_T \le \infty\})/(\lambda_S = \infty) \sim (\lambda_T = 0)) \xrightarrow{\simeq} [0, \infty].$$

Define a *generalized ratio of radii* as

$$\lambda_\Gamma : \mathcal{M}_\Gamma^{qq,\text{univ}} \to [0,\infty]^{\mathrm{Vert}_2^q(\Gamma)}, \quad \lambda_\Gamma(C) = (\lambda_\Gamma(C)_v)_{v \in \mathrm{Vert}_2^q(\Gamma)},$$

where, for any vertex $v \in \mathrm{Vert}_2^q(\Gamma)$ containing an inner quilting circle,

$$\lambda_\Gamma(C)_v := \phi\left( \begin{Bmatrix} \lambda_S(v), & v \text{ is twice-quilted,} \\ \lambda_T(v). & v \text{ is not twice-quilted} \end{Bmatrix} \right),$$

An unbalanced twice-quilted curve $C$ of type $\Gamma$ then satisfies the balanced ratio condition if $\lambda_\Gamma(C)$ maps to the diagonal $\Delta$, that is, $\lambda_\Gamma(C)_v$ is the same for all $v \in \mathrm{Vert}_1^q(\Gamma)$. The moduli space of twice quilted disks of type $\Gamma$ is denoted by

$$\mathcal{M}_\Gamma^{qq} := \lambda_\Gamma^{-1}(\Delta).$$

The moduli space of twice-quilted treed disks is a cell complex constructed in a similar way to the space of once-quilted treed disks. Denote the moduli space of twice-quilted disks with $d(\circ)$ boundary leaves and $d(\bullet)$ interior leaves by $\mathcal{M}_{d(\circ),d(\bullet)}^{qq}$. The moduli space admits a stratification by combinatorial type, that is,

$$\mathcal{M}_{d(\circ),d(\bullet)}^{qq} = \cup_\Gamma \mathcal{M}_\Gamma^{qq},$$

where $\Gamma$ ranges over all twice-quilted disk types with $d(\circ)$ non-root boundary leaves and $d(\bullet)$ interior leaves. The open strata in $\mathcal{M}_{d(\circ),d(\bullet)}^{qq}$ have dimension $d(\circ) + 2d(\bullet)$. A twice-quilted holomorphic disk type $\Gamma$ is *rigid* if it satisfies all the conditions for a holomorphic map type to be rigid, as in Definition 8.47.

For example, the moduli space $\mathcal{M}_{1,0}^{qq}$ of twice-quilted disks with a single non-root boundary marking is one-dimensional. It has two open strata interspersed by three strata of codimension one, whose types are shown in Figure 10.11.

Coherent domain-dependent perturbations for twice-quilted disks are defined in a similar way to quilted disks, see [17, Definition 5.9]. That is, given coherent perturbations $\underline{\mathfrak{p}}_0$, $\underline{\mathfrak{p}}_1$, $\underline{\mathfrak{p}}_2$ on $X$, and coherent perturbation morphisms $\underline{\mathfrak{p}}_{01}$, $\underline{\mathfrak{p}}_{12}$, perturbation data $\underline{\mathfrak{p}}_{012}$ for twice quilted disks are required to interpolate between these. In particular, if in a stratum, (Cutting edges) produces an unquilted disk type $\Gamma$ of shading $k$ resp. a once-quilted disk type $\Gamma$ of shading $k$ and $k+1$, then the domain-dependent perturbation on these strata is defined by the pullback of $(\mathfrak{p}_k)_\Gamma$ resp. $(\mathfrak{p}_{k,k+1})_\Gamma$.

The following is the main result of the Section.

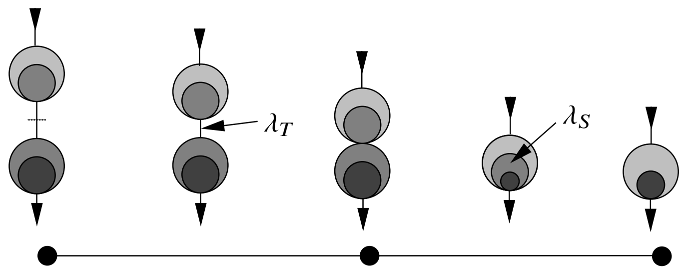



**Figure 10.11.** Moduli space of treed twice-quilted disks with one leaf

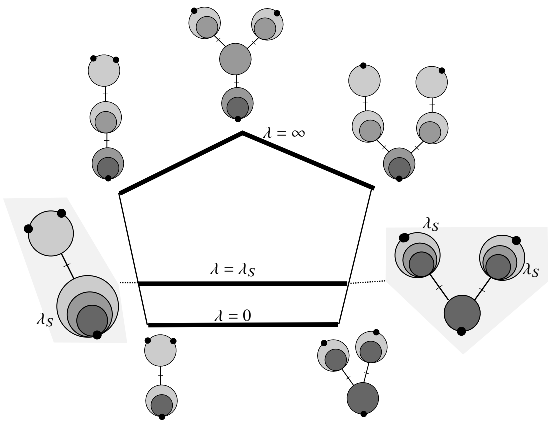



**Figure 10.12.** Moduli space $\mathcal{M}_{0,2}^{qq}$ of twice-quilted disks with two boundary inputs. The figure shows the configurations at the codimension two corners, and also the end-points of the moduli space $\mathcal{M}_{0,2}^{qq,\lambda_S}$ with fixed ratio of radii.

**Theorem 10.26.** *($A_\infty$ homotopies via twice-quilted disks) Given coherent perturbations $\underline{\mathfrak{p}}_{01}, \underline{\mathfrak{p}}_{12}, \underline{\mathfrak{p}}_{02}$ defining morphisms*

$$\phi_{ij} : CF(L, \underline{\mathfrak{p}}_i) \to CF(L, \underline{\mathfrak{p}}_j), \quad 0 \le i < j \le 2$$

*and coherent perturbation data $\underline{\mathfrak{p}}_{012}$ for twice-quilted disks, there is a convergent $A_\infty$ homotopy*

$$\mathcal{T}^{02} \in \operatorname{Hom}(\phi_{02}, \phi_{01} \circ \phi_{12}).$$

*Remark* 10.27. (Convergent $A_\infty$ homotopies) Let $A_0, A_1$ be $A_\infty$ algebras with Novikov coefficients, and let $\mathcal{F}_0, \mathcal{F}_1 : A_0 \to A_1$ be $A_\infty$ morphisms An $A_\infty$ homotopy $\mathcal{T} \in \operatorname{Hom}(\mathcal{F}_0, \mathcal{F}_1)$ from $\mathcal{F}_0$ to $\mathcal{F}_1$ is said to be *convergent* if $\mathcal{T}^0$ has positive $q$-valuation. For a convergent $A_\infty$ algebra $A$, if a convergent morphism $\mathcal{F} : A \to A$ is convergent homotopy equivalent to the identity morphism, then the map on the space of Maurer-Cartan solutions $MC(\mathcal{F}) : MC(A) \to MC(A)$, and the map on cohomology

$H(\mathcal{F}) : H(A, b) \to H(A, \mathcal{F}(b))$ for any $b \in MC(A)$ are isomorphisms modulo gauge transformations, see [17, Lemma 5.2, 5.3].

*Proof of Theorem* 10.26. We first introduce the necessary moduli spaces of holomorphic twice-quilted disks. For a generic perturbation $\underline{\mathfrak{p}}_{012}$ that extends perturbation data $\underline{\mathfrak{p}}_i$, $i = 0, 1, 2$, and perturbation morphisms $\underline{\mathfrak{p}}_{01}, \underline{\mathfrak{p}}_{12}, \underline{\mathfrak{p}}_{02}$, and $\lambda$ lying in a comeager subset $\mathbb{R}^{reg}_{\geq 0} \subset [0, \infty]$, for any type $\Gamma$ of twice-quilted disks, and any collection $\underline{x}$ of labels on the leaves,

- the moduli space $\mathcal{M}^{qq,\lambda}_{\Gamma}(L, \underline{\mathfrak{p}}_{012}, \underline{x})_d$ of twice quilted disks of type $\Gamma$ with ratio of radii is $\lambda$ and expected dimension $d \leq 1$, and
- the moduli space $\mathcal{M}^{qq}_{\Gamma}(L, \underline{\mathfrak{p}}_{012}, \underline{x})_d$ of twice quilted disks with the ratio of radii unprescribed with expected dimension $d \leq 0$

are transversally cut out. For any $\lambda \in \mathbb{R}^{reg}_{\geq 0}$ as above, counts of rigid twice-quilted holomorphic disks with ratio of radii $\lambda$ and index zero defines an $A_\infty$ morphism

$$\phi^{\lambda}_{02} = ((\phi^{\lambda}_{02})^d)_d, \qquad (\phi^{\lambda}_{02})^d : CF(L, \underline{\mathfrak{p}}_0)^{\otimes d} \to CF(L, \underline{\mathfrak{p}}_2).$$

For $\lambda = 0$, the $A_\infty$ morphism $\phi^0_{02}$ corresponds to $\phi_{02}$, and for $\lambda = \infty$, $\phi^{\infty}_{02}$ corresponds to $\phi_{12} \circ \phi_{01}$.

The homotopy between $\phi^0_{02}$ and $\phi^{\infty}_{02}$ is defined by combining homotopies constructed from small variations of ratio. We aim to define a pre-natural transformation $\mathcal{T}^{02} = ((\mathcal{T}^{02})^d)_{d \geq 0} \in \mathrm{Hom}(\phi^0_{02}, \phi^{\infty}_{02})$ which will later be shown to be a homotopy. Given an energy bound $E > 0$, we first define $\mathcal{T}^{02}$ modulo terms involving $q^{\geq E}$. We may divide $[0, \infty]$ into a finite number of intervals

$$[\lambda_i, \lambda_{i+1}], \quad i = 1, \ldots, k, \tag{10.43}$$

so that there is at most one rigid twice-quilted disk with ratio $\lambda$ and symplectic area $\leq E$ (which, we recall from (5.2), is proportional to the number of interior markings $d_\bullet$) in each such interval $[\lambda_i, \lambda_{i+1}]$. We define $\mathcal{T}^{\lambda_i, \lambda_{i+1}, \leq E}_{02}$ by counting twice-quilted disks with ratio in the specified interval,

$$\begin{aligned} (\mathcal{T}^{\lambda_i, \lambda_{i+1}, \leq E}_{02})^{d(\circ)} &: CF(L, \underline{\mathfrak{p}}_0)^{\otimes d(\circ)} \to CF(L, \underline{\mathfrak{p}}_2) \\ &(x_1, \ldots, x_{d(\circ)}) \mapsto \sum_{x_0, u \in \mathcal{M}^{qq,\lambda}_{\Gamma}(\underline{\mathfrak{p}}, L, \underline{x})_0} w(u) x_0, \end{aligned} \tag{10.44}$$

where $\underline{x} = (x_0, x_1, \ldots, x_{d(\circ)})$ and the sum is over rigid combinatorial types $\Gamma$ of twice-quilted disks with symplectic area $\leq E$. We will prove in Lemma 10.28 below that for any $d(\circ) \geq 0$ and any tuple of leaf labels $(x_1, \ldots, x_{d(\circ)}) \in \hat{\mathcal{I}}(L)^{d(\circ)}$,

$$(\phi^{\lambda_i}_{02} - \phi^{\lambda_{i+1}}_{02})^{d(\circ)}(x_1, \ldots, x_{d(\circ)}) = (m^1_Q \mathcal{T}^{\lambda_i, \lambda_{i+1}, \leq E}_{02})^{d(\circ)}(x_1, \ldots, x_{d(\circ)}) \mod q^E.$$

Composition using (10.14) produces a homotopy

$$\mathcal{T}_{02}^{\leq E} := \mathcal{T}_{02}^{\lambda_{k-1},\lambda_k,\leq E} \circ (\ldots \circ (\mathcal{T}_{02}^{\lambda_1,\lambda_2,\leq E} \circ \mathcal{T}_{02}^{\lambda_0,\lambda_1,\leq E}))$$

between $\phi_{12} \circ \phi_{01}$ and $\phi_{02}$ modulo terms involving powers $q^{\geq E}$, that is,

$$\phi_{02} - (\phi_{12} \circ \phi_{01}) = \mathcal{T}_{02}^{\leq E} \mod q^E.$$

Taking the limit $E \to \infty$ defines a homotopy

$$\mathcal{T}_{02} := \lim_{E\to\infty} \mathcal{T}_{02}^{\leq E}$$

between $\phi_{02}$ and $\phi_{12} \circ \phi_{01}$. The limit exists because for any $E_1 > E$, $\mathcal{T}_{02}^{\leq E} = \mathcal{T}_{02}^{\leq E_1} \mod q^E$, and therefore terms in $\mathcal{T}_{02}$ with valuation $< E$ are given by $\mathcal{T}_{02}^{\leq E}$. Convergence, that is, that $\mathcal{T}_{02}^0(1)$ has coefficients with positive $q$-valuation, holds since any treed disk with no incoming edges must, by stability, have a disk component with interior markings, and therefore has positive area. ■

**Lemma 10.28.** *For an area bound $E > 0$, let the interval $[\lambda_i, \lambda_{i+1}]$ be as in (10.43). Then,*

$$\phi_{02}^{\lambda_i,\leq E} - \phi_{02}^{\lambda_{i+1},\leq E} = m_Q^1(\mathcal{T}_{02}^{\lambda_i,\lambda_{i+1},\leq E}) \mod q^E. \quad (10.45)$$

Configurations consisting of multiple twice-quilted disks attached to an unquilted (dark shaded) disk by broken edges are not transversally cut out, and they occur in the boundary of one-dimensional moduli spaces. Indeed, our current definition of holomorphic twice quilted disks require the ratios of radii (the surface version (10.40) or the treed version (10.42)) to be equal for all twice quilted components. This requirement may be viewed as a fiber product, and the resulting moduli space is not transversally cut out.

In order to obtain transversality, the fiber products (10.40), (10.42) involved in the definition of the twice-quilted disks must be perturbed, using *delay functions*, as in Seidel [76] and Ma'u-Wehrheim-Woodward [52]. In a twice-quilted disk type, denote by

$$\mathrm{Edge}^2_{\circ,-}(\Gamma) \subset \mathrm{Edge}_{\circ,-}(\Gamma)$$

the subset of boundary edges that lie below the inner quilting circle. In terms of Figure 10.11, edges in $\mathrm{Edge}^2_{\circ,-}(\Gamma)$ connect vertices with the darkest shading. A *delay function* is a collection of maps

$$\tau_{d(\circ),d(\bullet)} = (\tau_\Gamma)_\Gamma, \quad \tau_\Gamma = (\tau_{\Gamma,e})_{e\in\mathrm{Edge}^2_{\circ,-}(\Gamma)} : \overline{\mathcal{M}}_\Gamma^{qq,\mathrm{univ}} \to \mathbb{R}^{\mathrm{Edge}^2_{\circ,-}(\Gamma)},$$

where $\Gamma$ ranges over all rigid twice-quilted disk types with $d(\bullet)$ interior leaves and $d(\circ)$ non-root boundary leaves, and $\tau_\Gamma$ satisfies the following:

(a) (Locality) The map $\tau_\Gamma$ depends only on the edge lengths of edges in $\mathrm{Edge}^2_{\circ,-}(\Gamma)$,

(b) (Zero edges) If for a curve $C \in \overline{\mathcal{M}}_\Gamma^{qq,\mathrm{univ}}$ the edge length of $e \in \mathrm{Edge}^2_{\circ,-}(\Gamma)$ is zero, then $\tau_{\Gamma,e}(C) = 0$.

(c) (Continuity) If $\Gamma'$ is a stratum containing an edge $e$ of length zero, and $\Gamma_0$ and $\Gamma_1$ are open stratum in $\mathcal{M}^{qq,\mathrm{univ}}_{d(\circ),d(\bullet)}$ obtained by collapsing the edge $e$ and making $\ell(e)$ non-zero respectively, then,

$$\tau_{\Gamma_0}|\mathcal{M}_\Gamma^{qq,\mathrm{univ}} = \tau_{\Gamma_1}|\mathcal{M}_\Gamma^{qq,\mathrm{univ}}.$$

(d) (Infinite edges) For any rigid twice quilted disk type $\Gamma'$ which has $\leq d(\circ)$ boundary leaves and $\leq d(\bullet)$ interior leaves, with at least one of the inequalities being strict, there exist delay maps

$$\tau_{\Gamma,\Gamma'} = (\tau_{\Gamma,\Gamma',e})_e : \mathcal{M}_{\Gamma'}^{qq,\mathrm{univ}} \to \mathbb{R}^{\mathrm{Edge}^2_{\circ,-}(\Gamma')}$$

such that the following holds: If $\mathcal{M}_\Gamma^{qq,\mathrm{univ}}$ is a stratum with a collection of edges of infinite length, that separate $\Gamma$ into an unquilted type $\Gamma_0$ with the darkest shading, and a set $\{\Gamma_1, \ldots, \Gamma_k\}$ of twice-quilted disk types connected to $\Gamma_0$ by edges $e_1, \ldots, e_k$, so that $\mathcal{M}_\Gamma^{qq,\mathrm{univ}}$ is a product

$$\mathcal{M}_\Gamma^{qq,\mathrm{univ}} \xrightarrow{(\pi_0,\ldots,\pi_k)} \mathcal{M}_{\Gamma_0} \times \prod_{i=1}^k \mathcal{M}_{\Gamma_i}^{qq,\mathrm{univ}},$$

then for any edge $e \in \mathrm{Edge}^2_{\circ,-}(\Gamma_i)$, $i = 1, \ldots, k$, or $e \in \mathrm{Edge}_\circ(\Gamma_0)$ and $i = 0$,

$$\tau_{\Gamma,\Gamma_i,e} \circ \pi_i = \tau_{\Gamma,e}. \tag{10.46}$$

Furthermore, for any $i = 1, \ldots, k$,

$$\tau_{\Gamma,e_i} = \text{constant} \quad \text{on } \overline{\mathcal{M}}_\Gamma^{qq,\mathrm{univ}}. \tag{10.47}$$

(e) (Increasing condition) Let $\gamma_1, \gamma_2$ be paths in $\Gamma$ starting from the root vertex $v_0$ and such that neither of them is contained in the other. If $\gamma_1 < \gamma_2$ (in the sense that if, for $i = 1, 2$, $\gamma_i$ is contained in the path from the root vertex $v_0$ to a non-root leaf $e_i$, then $e_1$ occurs before $e_2$), then

$$\sum_{e \in \gamma_1} \tau_{\Gamma,e} < \sum_{e \in \gamma_2} \tau_{\Gamma,e}. \tag{10.48}$$

We now define *delayed twice-quilted disks* where the fiber product is replaced by a perturbed version. Define

$$\lambda_{\tau_\Gamma} : \mathcal{M}_\Gamma^{qq,\mathrm{univ}} \to [0,\infty]^{\mathrm{Vert}_{12}(\Gamma)}, \quad \lambda_{\tau_\Gamma}(C)_v := (\lambda_\Gamma(C))_v + \sum_{e \in P(v_0,v)} (\tau_\Gamma(C))_e,$$

where $P(v_0, v)$ is the path in $\Gamma$ from the root vertex $v_0$ to $v$. An unbalanced twice-quilted disk $C$ is a *$\tau$-delayed twice-quilted disk* if $\lambda_{\tau_\Gamma}(C)_v$ is equal for all vertices $v \in \mathrm{Vert}^{12}(C)$ containing an inner quilting circle $Q_2$, and thus, the moduli space of $\tau$-delayed twice-quilted disks of type $\Gamma$ is

$$\mathcal{M}^{qq,\lambda_0}_{\Gamma,\tau} = \lambda^{-1}_{\tau_\Gamma}(\Delta),$$

equipped with a *delayed ratio of radius* map

$$\psi : \mathcal{M}^{qq,\lambda_0}_{\Gamma,\tau} \to \mathbb{R}, \quad C \mapsto \lambda_{\tau_\Gamma}(v), \tag{10.49}$$

where $v \in \mathrm{Vert}^{12}(\Gamma)$ can be any disk vertex containing an inner quilting circle. Delayed twice-quilted disks which have specific values of the delayed ratio of radii are denoted by

$$\mathcal{M}^{qq,\lambda_0}_{\Gamma,\tau} := \psi^{-1}(\lambda), \quad \mathcal{M}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma,\tau} := \psi^{-1}([\lambda_i, \lambda_{i+1}]).$$

*Proof of Lemma* 10.28. To define delayed holomorphic twice-quilted disks, we choose perturbations close to the ones used to define ordinary holomorphic twice-quilted disks. Given perturbation data $\underline{\mathfrak{p}}_{012}$ for holomorphic twice quilted disks, we take the delay functions to be small enough and define coherent perturbations

$$\underline{\mathfrak{p}}^{\tau}_{012} = (\mathfrak{p}^{\tau}_{012,\Gamma})_\Gamma$$

so that the moduli spaces

$$\mathcal{M}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma,\tau}(L, \underline{\mathfrak{p}}_{012})_d, \quad \mathcal{M}^{qq,\lambda_i}_{\Gamma,\tau}(L, \underline{\mathfrak{p}}_{012})_d, \quad \mathcal{M}^{qq,\lambda_{i+1}}_{\Gamma,\tau}(L, \underline{\mathfrak{p}}_{012})_d$$

with expected dimension $d \leq 1$ are transversally cut out, and there are cobordisms of zero-dimensional moduli spaces

$$\mathcal{M}^{qq,\lambda_i}_{\Gamma}(L, \underline{\mathfrak{p}}_{012})_0 \sim \mathcal{M}^{qq,\lambda_i}_{\Gamma,\tau}(L, \underline{\mathfrak{p}}^{\tau}_{012})_0, \quad \mathcal{M}^{qq,\lambda_{i+1}}_{\Gamma}(L, \underline{\mathfrak{p}}_{012})_0 \sim \mathcal{M}^{qq,\lambda_{i+1}}_{\Gamma,\tau}(L, \underline{\mathfrak{p}}^{\tau}_{012})_0, \tag{10.50}$$

and

$$\mathcal{M}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma'}(L, \underline{\mathfrak{p}}_{012})_0 \sim \mathcal{M}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma',\tau}(L, \underline{\mathfrak{p}}^{\tau}_{012})_0 \tag{10.51}$$

for all types $\Gamma' \leq \Gamma$.

We finally prove the identity required by the Lemma. For any $d(\circ)$, we consider input labels $(x_1, \ldots, x_{d(\circ)})$ and output label $x_0$, and we aim to prove

$$(\phi^{\lambda_i,\leq E}_{02} - \phi^{\lambda_{i+1},\leq E}_{02})^{d(\circ)}(x_1, \ldots, x_{d(\circ)}) = (m^1_Q(\mathcal{T}^{\lambda_i,\lambda_{i+1}}_{02}))^{\leq E, d(\circ)}(x_1, \ldots, x_{d(\circ)}). \tag{10.52}$$

We will show that (10.52) is obtained by summing the contributions from boundaries of one-dimensional moduli spaces $\overline{\mathcal{M}}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma,\tau}(L, \underline{\mathfrak{p}}^{\tau}_{012}, \underline{x})_1$ where $\Gamma$ ranges over rigid types of twice-quilted holomorphic disks with $d(\circ)$ boundary inputs and $\leq \frac{E}{k}$ interior leaves as follows: A configuration occurring in the boundary of

$$\overline{\mathcal{M}}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma,\tau}(L,\underline{\mathfrak{p}}^{\tau}_{012},\underline{x})_1$$

is

(a) either a twice-quilted disk in $\mathcal{M}^{qq,\lambda_i}_{\Gamma,\tau}(L,\underline{x})_0$ or $\mathcal{M}^{qq,\lambda_{i+1}}_{\Gamma,\tau}(L,\underline{x})_0$; or
(b) it has an unquilted disk with the lightest shading breaking off (as in (A), (B) in Figure 10.13); or
(c) it has a collection of twice-quilted disks breaking off (as in (C), (D) in Figure 10.13).

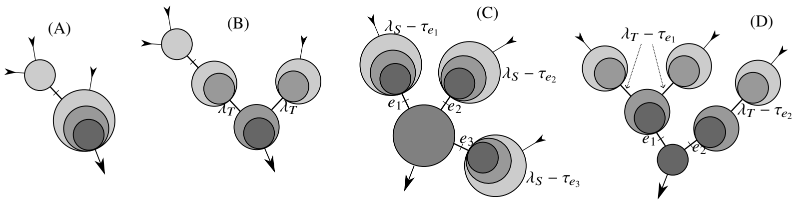



**Figure 10.13.** Configurations occurring in codimension one boundaries of moduli spaces of delayed twice-quilted disks with a fixed (generalized) ratio of radii. In (A) and (B), an unquilted disk breaks off outside the outer quilting circle. In (C) and (D), a collection of twice quilted disks break off.

A count of configurations of type (a), by the cobordism in (10.50), is equal to the left hand side of (10.52). A configuration $u = (u_0, u_1)$ of type (b), consists of an unquilted $\underline{\mathfrak{p}}_0$-holomorphic disk $u_0$ and a twice-quilted disk $u_1$ whose ratio of radius $\lambda$ lies in $(\lambda_i, \lambda_{i+1})$, and thus, $u_1$ lies in $\mathcal{M}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma',\tau}(L,\underline{x})_0$ for some type $\Gamma' \leq \Gamma$. By the cobordism (10.51), the count of configurations of type (b) is equal to the sum of terms of the form

$$\mathcal{T}^{\lambda_i,\lambda_{i+1},\leq E}_{02}(\ldots, m^e_{CF(L,\underline{\mathfrak{p}}_0)}(\ldots),\ldots),$$

occurring in $(m^1_Q \mathcal{T}^{\lambda_i,\lambda_{i+1}}_{02})^{\leq E,d(\circ)}$, which is the right hand side of (10.52). Finally, a configuration $u = (u_0,\ldots,u_r)$ of type (c) consists of an unquilted $\underline{\mathfrak{p}}_2$-holomorphic disk, and a collection $u_1,\ldots,u_r$ of twice-quilted disks, with $u_i$ $(i \geq 1)$ connected to $u_0$ by a broken edge $e_i$. Let

$$\Lambda_0 := \psi(u)$$

be the delayed ratio of radius (defined in (10.49)). For degree reasons, because of the fiber product with the diagonal, exactly one of the twice-quilted disks in the set $\{u_1,\ldots,u_r\}$, say the $k$-th, lies in the moduli space of expected dimension zero, while the rest have index one. The interval $[\lambda_i, \lambda_{i+1}]$ was chosen so that there is at most one twice-quilted disk in the zero-dimensional moduli space $\mathcal{M}^{qq,[\lambda_i,\lambda_{i+1}],\leq E}(L)_0$ with at most $d(\circ)$ inputs. That is, $u_k$ is the only element in this set, and its $\tau$-delayed ratio is $\psi(u_k) = \Lambda_0 + \tau_{\Gamma,e_k}(C_u)$, where $C_u$ is the domain curve of $u$. For any $j \neq k$,

$$\psi(u_j) = \Lambda_0 + \tau_{\Gamma,e_j}(C_u),$$

$u_j$ has index 1, and lies in the one-dimensional moduli space

$$\overline{\mathcal{M}}(u_j) := \mathcal{M}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma_j,\tau}(L, \underline{\mathfrak{p}}^{\tau}_{012}, \underline{x}_j)_1$$

for some input labels $\underline{x}_j$. We then have the following Claim.

*Claim* 10.29. For $j < k$, $\overline{\mathcal{M}}(u_j)$ does not have end-points in $\psi^{-1}(\lambda_i, \Lambda_0 + \tau_{\Gamma,e_j}(C_u))$ and for $j < k$, $\overline{\mathcal{M}}(u_j)$ does not have end-points in $\psi^{-1}(\Lambda_0 + \tau_{\Gamma,e_j}(C_u), \lambda_{i+1})$.

Assuming Claim 10.29, we finish the proof of the Lemma. The Claim implies that for $j < k$, there is a cobordism

$$\psi^{-1}(\Lambda_0 + \tau_{\Gamma,e_j}(C_u)) \cap \mathcal{M}(u_j)_1 \sim \mathcal{M}^{qq,\lambda_i}_{\Gamma_j,\tau}(L, \underline{\mathfrak{p}}^{\tau}_{012}, \underline{x}_j)_0, \tag{10.53}$$

since the disjoint union of both the spaces in (10.53) form the boundary of the one-dimensional moduli space $\overline{\mathcal{M}}(u_j)_1 \cap \{\psi \geq (\Lambda_0 + \tau_{\Gamma,e_j}(C_u))\}$. Similarly, for $j > k$, there is a cobordism

$$\psi^{-1}(\Lambda_0 + \tau_{\Gamma,e_j}(C_u)) \cap \mathcal{M}(u_j)_1 \sim \mathcal{M}^{qq,\lambda_{i+1}}_{\Gamma_j,\tau}(L, \underline{\mathfrak{p}}^{\tau}_{012})_0. \tag{10.54}$$

In the case when $j < k$ resp. $j > k$, the cobordism (10.53) resp. (10.54) combined with the cobordism in (10.50) implies that the signed count of maps $u_j$, with inputs $\underline{x}^{in}_j$ and delayed ratio of radius given by $\psi(u_j) = \Lambda_0 + \tau_{\Gamma,e_j}(C_u)$, is equal to

$$\phi^{\lambda_i}_{02}(\underline{x}^{in}_j) \quad \text{resp.} \quad \phi^{\lambda_{i+1}}_{02}(\underline{x}^{in}_j).$$

We conclude that the count of configurations $(u_0, \ldots, u_r)$ occurring in the boundary of $\overline{\mathcal{M}}^{qq,[\lambda_i,\lambda_{i+1}]}_{\Gamma,\tau}(L, \underline{\mathfrak{p}}^{\tau}_{012}, \underline{x})_1$ that are of type (c), and for which the $k$-th twice-quilted disk $u_k$ has index 0 is equal to the count of terms

$$m^r_{CF(L,\underline{\mathfrak{p}}_2)}(\underbrace{\phi^{\lambda_i}_{02}(\ldots), \ldots, \phi^{\lambda_i}_{02}(\ldots)}_{k-1}, \mathcal{T}^{02}(\ldots), \underbrace{\phi^{\lambda_{i+1}}_{02}(\ldots), \ldots, \phi^{\lambda_{i+1}}_{02}(\ldots)}_{r-k}) \mod q^E,$$

occurring in $(m^1_Q \mathcal{T}^{\lambda_i,\lambda_{i+1}})^{\leq E, d(\circ)}$, which is the right hand side of (10.52), finishing the proof of the Lemma.

It remains to prove Claim 10.29. We consider the case $j < k$, the proof of the other case, $j > k$, being similar. A map $v$ occurring in the boundary of the moduli space $\overline{\mathcal{M}}(u_j)_1$ in $\psi^{-1}(\lambda_i, \Lambda_0 + \tau_{\Gamma,e_j}(C_u))$ is of one of the two forms:

- Either $v$ has a collection of twice-quilted disks breaking off, that is, $v = (v_0, \dots, v_{r'})$, where $v_0$ is an unquilted disk with darkest shading, the others are $\tau$-delayed twice-quilted disks with one of them, say $v_\ell$, having index 0. Since $u_k$ is the only delayed twice-quilted disk of index 0 in $\psi^{-1}([\lambda_i, \lambda_{i+1}])$, we have that $v_\ell$ is the same as $u_k$. In particular, $\psi(v_\ell) = \Lambda_0 + \tau_{\Gamma,e_k}(C_u)$. Suppose the root vertex of $v_k$ is connected to the root vertex of $v$ by a path $\gamma$. By our assumption that $\psi(v) \in (\lambda_i, \Lambda_0 + \tau_{\Gamma,e_j})$, we have

$$\Lambda_0 + \tau_{\Gamma,e_j}(C_u) > \psi(v) = \Lambda_0 + \tau_{\Gamma,e_k}(C) - \sum_{e \in \gamma} \tau_{\Gamma,e}(C), \tag{10.55}$$

where the curve $C$ is obtained by replacing the subdomain of $u_j$ in $C_u$ with the domain of $v$. Note that (10.46) implies $\tau_{\Gamma,e_k}(C) = \tau_{\Gamma,e_k}(C_u)$, and (10.47) implies $\tau_{\Gamma,e_j}(C) = \tau_{\Gamma,e_j}(C_u)$. Therefore, (10.55) violates the increasing condition (10.48) at the curve $C$.

- Otherwise, $v$ has an unquilted disk bubbling off, that is $v = (v_0, v_1)$ where $v_0$ is twice-quilted and $v_1$ is unquilted. Since $u_k$ is the only delayed twice-quilted disk of index 0 in $\psi^{-1}([\lambda_i, \lambda_{i+1}])$, we have $\psi(u_k) = \psi(v_1) = \Lambda_0 + \tau_{\Gamma,e_k}(C_u)$, which can not lie in the interval $(\lambda_i, \Lambda_0 + \tau_{\Gamma,e_j}(C_u))$, since the increasing condition (10.48) implies that $\tau_{\Gamma,e_j}(C_u) < \tau_{\Gamma,e_k}(C_u)$.

This proves Claim 10.29, and hence, also proves the Lemma. ∎

**Corollary 10.30.** *For any two coherent convergent collections of perturbation data* $\underline{\mathfrak{p}}_0, \underline{\mathfrak{p}}_1$ *the Fukaya algebras* $CF(L, \underline{\mathfrak{p}}_0)$ *and* $CF(L, \underline{\mathfrak{p}}_1)$ *are homotopy equivalent via unital convergent* $A_\infty$ *morphisms*

$$\phi : CF(L, \underline{\mathfrak{p}}_0) \to CF(L, \underline{\mathfrak{p}}_1), \quad \psi : CF(L, \underline{\mathfrak{p}}_1) \to CF(L, \underline{\mathfrak{p}}_0)$$

*and unital convergent homotopies* $h, g$ *satisfying*

$$\phi \circ \psi - \mathrm{Id} = m_Q^1(h), \quad \psi \circ \phi - \mathrm{Id} = m_Q^1(g).$$

*Proof.* The $A_\infty$ morphisms $\phi, \psi$ are as constructed in the proof of Proposition 10.18, and the morphisms are unital and convergent. We apply Theorem 10.26 taking $\mathfrak{p}_2 = \mathfrak{p}_0$, $\phi_{01} = \phi$, $\phi_{12} = \psi$, $\phi_{02} = \mathrm{Id}$, and obtain a unital convergent homotopy $h$. The homotopy $g$ is obtained by switching the role of $\mathfrak{p}_0$ and $\mathfrak{p}_1$, ∎

## 10.7 Homotopy equivalence: unbroken to broken

In this section, we show that the Fukaya algebra of the broken manifold $\mathfrak{X}$ is homotopy equivalent to the unbroken Fukaya algebra on the manifold $X$. A rough outline of the proof is as follows. An $A_\infty$ morphism from an unbroken Fukaya algebra $CF(L, \underline{\mathfrak{p}}^\nu)$, where $\underline{\mathfrak{p}}^\nu$ is a perturbation system on the neck-stretched manifold $X^\nu$, to a broken one $CF^{\text{brok}}(L, \underline{\mathfrak{p}}^\infty)$ is defined as a limit of $A_\infty$ morphisms $\phi_n : CF(L, \underline{\mathfrak{p}}^\nu) \to CF(L, \underline{\mathfrak{p}}^{\nu+n})$ as $n \to \infty$. The $A_\infty$ morphism $\phi_n$ is defined via a count of quilted holomorphic disks in $(X, L)$. The existence of the limit relies on the bijection between moduli spaces of broken and unbroken disks obtained from the compactness and gluing theorems (Theorem 8.2 and Theorem 9.1).

We use a breaking perturbation datum on neck-stretched manifolds, so that the bijection between moduli spaces of broken and unbroken disks holds. We recall from Definition 8.1 that given a perturbation datum $\underline{\mathfrak{p}}^\infty$ on the broken manifold $\mathfrak{X}$, by gluing on the neck we obtain a perturbation datum $\rho_\nu(\underline{\mathfrak{p}}^\infty)$ on any $X^\nu$. By Proposition 5.23 there is a stabilizing pair $(\mathfrak{J}, \mathfrak{D})$ consisting of a cylindrical almost complex structure $\mathfrak{J}$ on $\mathfrak{X}$ and a $\mathfrak{J}$-holomorphic cylindrical stabilizing divisor $\mathfrak{D}$, for which the glued family $(J^\nu, D^\nu)$ is a stabilizing pair on the neck-stretched manifold $X^\nu$ for all $\nu$. For the perturbation datum $\underline{\mathfrak{p}}^\infty$ we use $\mathfrak{D}$ as the stabilizing divisor and $\mathfrak{J}$ as the background almost complex structure.

**Proposition 10.31.** *Let $\underline{\mathfrak{p}}^\infty$ be a regular perturbation datum on $\mathfrak{X}$. For any $E_0 > 0$, there exists $\nu_0(E_0)$ such that the following holds.*

(a) *There is a bijection between the zero-dimensional moduli spaces of rigid broken maps and rigid unbroken maps:*

$$\mathcal{M}^{\text{brok}}(\mathfrak{X}, L, \underline{\mathfrak{p}}^\infty)_0^{<E_0} \simeq \mathcal{M}(X^\nu, L, \rho_\nu(\underline{\mathfrak{p}}^\infty))_0^{<E_0}, \quad \nu \geq \nu_0(E_0).$$

*Furthermore, the moduli spaces of rigid quilted and twice-quilted disks relating $X^\nu$ and $X^{\nu+1}$ with energy at most $E_0$ are empty for $\nu \geq \nu_0(E_0)$.*

(b) *For any $\nu \in \mathbb{Z}_{>0}$, there exists a regular perturbation datum $\underline{\mathfrak{p}}^\nu$, and a perturbation morphism $\underline{\mathfrak{p}}^{\nu,\nu+1}$ extending $\underline{\mathfrak{p}}^\nu$ and $\underline{\mathfrak{p}}^{\nu+1}$ such that for all $E_0 > 0$ and for all $\nu \geq \nu_0(E_0)$, the (possibly curved) $A_\infty$ homotopy equivalence induced by $\underline{\mathfrak{p}}^{\nu,\nu+1}$*

$$\phi_\nu^{\nu+1} : CF(L, \underline{\mathfrak{p}}^\nu) \to CF(L, \underline{\mathfrak{p}}^{\nu+1})$$

*is the identity map modulo $q^{E_0}$. Similarly, there is a (possibly curved) $A_\infty$ homotopy equivalence*

$$\psi_{\nu+1}^\nu : CF(L, \underline{\mathfrak{p}}^{\nu+1}) \to CF(L, \underline{\mathfrak{p}}^\nu)$$

*that is the identity map modulo $q^{E_0}$. Furthermore, the homotopy equivalence between $\phi_\nu^{\nu+1} \circ \psi_{\nu+1}^\nu$ and the identity, and $\psi_{\nu+1}^\nu \circ \phi_\nu^{\nu+1}$ and the identity may be taken to be zero modulo $q^{E_0}$.*

*Proof.* The proof of bijection of moduli spaces is a consequence of the compactness and gluing theorems – Theorem 8.2 and Theorem 9.1. By the gluing Theorem 9.1, any regular $\underline{\mathfrak{p}}^\infty$-disk $u : C \to \mathfrak{X}$ can be glued to yield regular disks $u_\nu : C_\nu \to X^\nu$. Conversely, any sequence $(u_\nu)_\nu$ of maps with area $\leq E_0$ converges to a broken disk $u_\infty$. By surjectivity of gluing in Theorem 9.1, for large enough $\nu$, $u_\nu$ is contained in the image of the gluing map for $u_\infty$. Since the moduli space $\mathcal{M}(\mathfrak{X}, L)_0^{\leq E_0}$ has a finite number of points, the constant $\nu_0(E_0)$ can be chosen as the maximum obtained from gluing each of the broken maps.

To prove the statements about quilted disks and part (b), we construct regular perturbations and perturbation morphisms on neck-stretched manifolds. A regular perturbation datum is constructed by extending the breaking perturbation datum. For any $E_0 > 0$ and $\nu > \nu_0(E_0)$, the perturbation datum is defined as

$$\mathfrak{p}_\Gamma^\nu := \rho_\nu(\mathfrak{p}_\Gamma^\infty) = (J_\Gamma^\nu, F_\Gamma^\infty) \tag{10.56}$$

if $E(\Gamma) \leq E_0$. For the other strata, a regular perturbation $\mathfrak{p}_\Gamma^\nu$ can be chosen using the transversality result Theorem 6.28 for all integers $\nu$.

For strata with low enough area, perturbation morphisms are constructed using the breaking perturbation data. For $\nu \in \mathbb{Z}_+$, $\Gamma$ satisfying $\nu \geq \nu_0(E(\Gamma))$, the perturbation morphism is defined using the recipe in Remark 10.23 as

$$\mathfrak{p}_\Gamma^{\nu,\nu+1} = (J_\Gamma^{\nu,\nu+1}, F_{\Gamma'}^\infty)$$

where $F_{\Gamma'}^\infty$ is part of the perturbation data $\underline{\mathfrak{p}}^\infty = (\underline{J}^\infty, \underline{F}^\infty)$ for broken disks, $\Gamma'$ is the type of broken disk obtained by forgetting the quilting, and for any $z \in \mathcal{S}_\Gamma$, $J_\Gamma^{\nu,\nu+1}$ is the neck-stretched domain-dependent almost complex structure from (10.56). The perturbation morphisms are extended to higher strata so that they are coherent and regular.

We claim that the quilted and twice-quilted moduli spaces satisfying an energy bound $A(u) < E_0$ are empty for sufficiently large $\nu > \nu(E_0)$. First, we point out that this count includes quilted holomorphic disks with a single input and output with the same label (as in Figure 10.8) and a constant map on the surface and tree components, which gives an identity term in $\phi_1^\nu$. We claim that for any $E_0$ there is a constant $\nu_0(E_0)$ such that for $\nu \geq \nu_0(E_0)$ there are no other $\underline{\mathfrak{p}}^{\nu,\nu+1}$-holomorphic quilted disks with area $\leq E_0$. Suppose, by way of contradiction, that there is a sequence $u_\nu$ of quilted $\underline{\mathfrak{p}}^{\nu,\nu+1}$-disks with bounded area. Any surface component of $u_\nu$ is $J^{\nu'}$-holomorphic for some $\nu' \in [\nu, \nu+1]$. Therefore, after passing to a subsequence, the surface part of the map $u_\nu$ converge to a limit $J^\infty$-holomorphic broken map $u$. For large enough $\nu$, the Maslov index $I(u_\nu)$ is $\nu$-independent, and therefore the number of disk inputs is bounded. After passing to a subsequence, we can assume that the disk input/output tuple $\underline{x}$ and the type $\Gamma$ is the same for all the quilted disks $u_\nu$. Since the $A_\infty$ morphism counts quilted holomorphic disks with index 0, we may assume that the index of the quilted disks is zero: $i^q(\Gamma, \underline{x}) = 0$. The sequence $u_\nu$ converges to a $\underline{\mathfrak{p}}^\infty$-unquilted broken disk $u$ of type $\Gamma_\tau$ and input/output labels given by $\underline{x}$. Here $\Gamma_\tau$ is the type of a broken treed holomorphic map for which collapsing the tropical edges yields a type $\Gamma'$ of a treed holomorphic map, which is the same as the type obtained by forgetting the quilting in $\Gamma$. The unquilted map $u$ is stable in all cases except the one contributing to the identity term, that is, when there is a single input and output, and the map is constant on the surface and tree part of the domain. In all other cases, the index of the unquilted disk is one less (see (10.34)), and we have $i(\Gamma', \underline{x}) = -1$. Since collapsing tropical edges does not affect the dimension, we have $i(\Gamma_\tau, \underline{x}) = -1$. The existence of such a disk $u$ is a contradiction, since $\underline{\mathfrak{p}}^\infty$ is a regular perturbation. The argument for twice-quilted disks is similar.

Finally, the $A_\infty$ morphisms $\phi_\nu^{\nu+1}, \psi_{\nu+1}^\nu$ are defined by counting $\underline{\mathfrak{p}}^{\nu,\nu+1}$-holomorphic quilted holomorphic disks. The previous paragraph shows that if $\nu \geq \nu_0(E_0)$,

$$\phi_\nu^{\nu+1} = \mathrm{Id} \mod q^{E_0}, \forall \nu > \nu_0(E_0)$$

and similarly for $\psi_{\nu+1}^\nu$. ∎

**Corollary 10.32.** *Let $CF(\underline{\mathfrak{p}}^\nu, L)$ and $CF_{\mathrm{brok}}(\underline{\mathfrak{p}}^\infty, L)$ be the unbroken and broken Fukaya algebras defined by the perturbation data $\underline{\mathfrak{p}}^\nu$, $\underline{\mathfrak{p}}^\infty$ from Proposition 10.31. The structure maps $m_\nu^d$ defining $CF(\underline{\mathfrak{p}}^\nu, L)$ converge as $\nu \to \infty$ to the structure maps $m_{\mathrm{brok}}^d$ defining $CF_{\mathrm{brok}}(\underline{\mathfrak{p}}^\infty, L)$, in the sense that for any $E_0$ there exists $\nu(E_0)$ so that $m_\nu^d$ agrees with $m_{\mathrm{brok}}^d$ up to terms divisible by $q^{E_0}$ for $\nu > \nu(E_0)$.*

*Completion of the second proof of Theorem 10.4.* It suffices to prove that for any $E_0 > 0$ that homotopy-associativity holds up to order $q^{E_0}$. This follows from Corollary 10.32 and Theorem 10.2, since the broken and unbroken composition maps are equal to up arbitrary order, for sufficiently large neck length. ∎

**Proposition 10.33.** *For any $\nu_0$, let $\underline{\mathfrak{p}}_{\nu_0}$ be the perturbation defined in Proposition 10.31. Then there are convergent strictly unital $A_\infty$ morphisms*

$$\phi : CF(L, \underline{\mathfrak{p}}^{\nu_0}) \to CF_{\mathrm{brok}}(L, \underline{\mathfrak{p}}^\infty), \quad \psi : CF_{\mathrm{brok}}(\underline{\mathfrak{p}}^\infty) \to CF(L, \underline{\mathfrak{p}}^{\nu_0})$$

*such that $\psi \circ \phi$ and $\phi \circ \psi$ are $A_\infty$ homotopy equivalent by a convergent $A_\infty$ homotopy.*

*Proof.* The $A_\infty$ morphisms $\phi$ and $\psi$ required by the Proposition are defined to be the limits of the compositions of the $A_\infty$ morphisms from Proposition 10.31 (b), namely,

$$\phi_\nu^{\nu+1} : CF(L, \underline{\mathfrak{p}}^\nu) \to CF(L, \underline{\mathfrak{p}}^{\nu+1}), \quad \psi_{\nu+1}^\nu : CF(L, \underline{\mathfrak{p}}^{\nu+1}) \to CF(L, \underline{\mathfrak{p}}^\nu).$$

For any $E_0$, if $\nu \geq \nu'(E_0)$, the $A_\infty$ morphisms $\psi_\nu^{\nu+1}$ and $\psi_{\nu+1}^\nu$ are equal to Id modulo $q^{E_0}$. Therefore, there exist limits of the successive compositions: Let

$$\phi_n := \phi_{\nu_0+n-1}^{\nu_0+n} \circ \ldots \circ \phi_{\nu_0+1}^{\nu_0+2} \circ \phi_{\nu_0}^{\nu_0+1} : CF(L, \underline{\mathfrak{p}}^{\nu_0}) \to CF(L, \underline{\mathfrak{p}}^{\nu_0+n})$$

for any choice of $\nu_0$. The limit

$$\phi = \lim_{n\to\infty} \phi_n : CF(L, \underline{\mathfrak{p}}^{\nu_0}) \to \lim_{n\to\infty} CF(L, \underline{\mathfrak{p}}^{\nu_0+n}) = CF_{\text{brok}}(L, \underline{\mathfrak{p}}^{\infty})$$

exists, and similarly, for

$$\psi_n := \psi^{\nu_0}_{\nu_0+1} \circ \ldots \circ \psi^{\nu_0+n-1}_{\nu_0+n} : CF(L, \underline{\mathfrak{p}}^{\nu_0+n}) \to CF(L, \underline{\mathfrak{p}}^{\nu_0}),$$

the limit

$$\psi = \lim_{n\to\infty} \psi_n : CF_{\text{brok}}(L, \underline{\mathfrak{p}}^{\infty}) \to CF(L, \underline{\mathfrak{p}}^{\nu_0})$$

exists. Since the composition of strictly unital morphisms is strictly unital, the composition $\psi$ is strictly unital mod terms divisible by $q^E$ for any $E$, hence strictly unital. The limit map $\psi$ resp. $\phi$ is an $A_\infty$ morphism whose domain resp. target is $CF_{\text{brok}}(L, \underline{\mathfrak{p}}^{\infty})$ because the composition maps in $CF(L, \underline{\mathfrak{p}}^{\nu})$ converge to $CF_{\text{brok}}(L, \underline{\mathfrak{p}}^{\infty})$. The composition maps of $CF(L, \underline{\mathfrak{p}}^{\nu})$ converge to those of $CF_{\text{brok}}(L, \underline{\mathfrak{p}}^{\infty})$ by the bijection of the moduli spaces of disks in Proposition 10.31 (a). Indeed, the bijection preserves the orientation $\epsilon(u)$, area $A(u)$, and holonomy $\text{Hol}_L([\partial u])$ since gluing preserves these quantities.

We claim that $\phi$ and $\psi$ are homotopy equivalences. Let $h_n, g_n$ denote the homotopies satisfying

$$\phi_n \circ \psi_n - \text{Id} = m^1_Q(h_n), \quad \psi_n \circ \phi_n - \text{Id} = m^1_Q(g_n),$$

from the homotopies relating $\phi_n \circ \psi_n$ and $\psi_n \circ \phi_n$ to the identities, see Corollary 10.30. Given $E \geq 0$, there is a constant $n_0(E)$ such that for any $n_1, n_2 \geq n_0(E)$, the perturbation systems underlying the $A_\infty$ morphisms $\psi_{n_1}$ and $\psi_{n_2}$ resp. $\phi_{n_1}$ and $\psi_{n_2}$ are the same for domain types $\Gamma$ with at most $\frac{E}{k}$ interior markings, and

$$\phi_{n_1} = \phi_{n_2} \mod q^E, \quad \psi_{n_1} = \psi_{n_2} \mod q^E.$$

Then, the perturbations underlying the homotopies $h_{n_1}$ and $h_{n_2}$ resp. $g_{n_1}$ and $g_{n_2}$ can also be chosen so that they are equal on domain types with at most $\frac{E}{k}$ interior markings, and

$$h_{n_1} = h_{n_2} \mod q^E, \quad g_{n_1} = g_{n_2} \mod q^E.$$

Therefore, the limits $h := \lim_{n\to\infty} h_n$, $g := \lim_{n\to\infty} g_n$ are well-defined. Since the $A_\infty$ algebra $CF_{\text{brok}}(L, \underline{\mathfrak{p}}^{\infty})$ is a limit $\lim_{\nu\to\infty} CF(L, \underline{\mathfrak{p}}^{\nu})$ (in the sense that composition maps converge), and the $A_\infty$ morphism $\psi_n$ resp. $\phi_n$ limits to $\psi$ resp. $\phi$, we conclude

$$\phi \circ \psi - \text{Id} = m^1_Q(h), \quad \psi \circ \phi - \text{Id} = m^1_Q(g).$$

The $A_\infty$ homotopies $h, g$ are convergent since each of them is the limit of convergent homotopies. ∎

## Chapter 11

# Split perturbations and potentials for semi-Fano manifolds

So far, we have used perturbations on broken manifolds that are gluable on the neck regions. This requirement was needed for proving the $A_\infty$ homotopy equivalence between broken and unbroken Fukaya algebras. In this section, we allow perturbations of almost complex structures that are independently defined in different components of the broken manifold. We rigorously define this new kind of perturbation, called *split perturbation*, and show that the broken Fukaya algebras it produces are homotopy equivalent to the ones we have defined so far. Split perturbations were used in Chapter 2 to compute the disk potentials in the cubic surface and the second Hirzebruch surface. To complete the circle of ideas, in this concluding Chapter, we also prove a technical result used in the computations of disk potentials in semi-Fano toric manifolds in Chapter 2, namely that, the multiple cut of a semi-Fano toric manifold, under some hypothesis, has the same disk potential as the original smooth manifold.

## 11.1 Split perturbations

A split perturbation depends not just on the domain curve but also on an accompanying tropical structure on it, recorded by the *base tropical graph*. The domain type for maps for split perturbations, denoted by $\tilde{\Gamma}$, may have additional vertices than those in the base tropical graph $\Gamma$. However, for generic perturbations, for maps of index $\leq 1$, the domain type $\tilde{\Gamma}$ is equal to the base tropical graph $\Gamma$.

**Definition 11.1.** (Based curves)A *curve with base type* consists of

(a) a stable treed disk $C$ of type $\tilde{\Gamma}$,

(b) a *base tropical graph* $\Gamma$,

(c) and an edge collapse morphism $\kappa : \tilde{\Gamma} \to \Gamma$ that necessarily collapses all disk edges $e \in \mathrm{Edge}_\circ(\tilde{\Gamma})$ and treed segments $T_e$, $e \in \mathrm{Edge}_\circ(\tilde{\Gamma})$, and possibly some interior edges. (The map $\kappa$ is an edge collapse of graphs, and not of tropical graphs, because $\tilde{\Gamma}$ does not have a tropical structure.)

The *type* of such a curve consists of the datum $\tilde{\Gamma} \xrightarrow{\kappa} \Gamma$. The moduli space of stable treed curves of type $\tilde{\Gamma}$ with base type $\Gamma$ is denoted by $\mathcal{M}_{\tilde{\Gamma},\Gamma}$, or simply $\mathcal{M}_{\tilde{\Gamma}}$ when the context allows it. The topology on the space of based curves is the standard topology on the space of stable curves with the additional axiom that the base tropical graph is preserved in the limit. This ends the Definition.

**Definition 11.2.** (Based graph morphisms) The following morphisms are defined on types of curves with a base :

(a) A based curve type $\tilde{\Gamma}_1 \xrightarrow{\kappa_1} \Gamma$ is obtained by *(Collapsing edges)* resp. *(Making an edge length/weight finite/non-zero)* in the based curve type $\tilde{\Gamma}_0 \xrightarrow{\kappa_0} \Gamma$ if there is a (Collapsing edges) resp. (Making an edge length/weight finite/non-zero) morphism $\tilde{\Gamma}_0 \xrightarrow{\tilde{\kappa}} \tilde{\Gamma}_1$ of treed curve types (see Definition 6.5) and $\tilde{\kappa} \circ \kappa_0 = \kappa_1$. Note that the (Collapsing edges) morphism does not collapse an edge $e$ in $\tilde{\Gamma}_0$ that is present in the base graph $\Gamma$.

(b) A type with base $\tilde{\Gamma} \xrightarrow{\kappa} \Gamma$ is obtained by *(Cutting an edge)* in $\tilde{\Gamma}' \xrightarrow{\kappa'} \Gamma'$ if
- $\tilde{\Gamma}$ is obtained by cutting an $e = (v_+, v_-) \in \mathrm{Edge}_{\circ,-}(\tilde{\Gamma}')$ containing a breaking,
- $\Gamma'$ is obtained by identifying the vertices $\kappa(v_+)$, $\kappa(v_-)$ in $\Gamma$.

**Definition 11.3.** (Split perturbation datum) Let $\Gamma$ be a tropical graph. A *coherent split perturbation datum* is a collection

$$\underline{\mathfrak{p}} = (\mathfrak{p}_{(\tilde{\Gamma},\Gamma)})_{(\tilde{\Gamma},\Gamma)},$$

consisting of a perturbation datum

$$\mathfrak{p}_{(\tilde{\Gamma},\Gamma)} = (J_{(\tilde{\Gamma},\Gamma)}, F_{(\tilde{\Gamma},\Gamma)}), \quad J_{\tilde{\Gamma},\Gamma} : \mathcal{S}_{\tilde{\Gamma}} \to \mathcal{J}^{\mathrm{cyl}}(\mathfrak{X}), \quad F_{\tilde{\Gamma},\Gamma} : \mathcal{T}_{\tilde{\Gamma}} \to C^\infty(L, \mathbb{R})$$

for each based curve type $(\tilde{\Gamma}, \Gamma)$, that is coherent under the based graph morphisms of (Collapsing edges), (Making an edge length or weight finite/non-zero) and (Cutting edges) from Definition 11.2, and the (Locality axiom) from Definition 6.5.

*Remark* 11.4. An important difference between ordinary perturbations (from Definition 6.5) on $\mathfrak{X}$ and the split perturbations in the above Definition 11.3 is that the split perturbation datum $\mathfrak{p}_{\tilde{\Gamma},\Gamma}$ is not required to be coherent under collapsing edges that are present in the base tropical graph $\Gamma$. Indeed, for a based graph $\tilde{\Gamma} \xrightarrow{\kappa} \Gamma$, the (Collapsing edges) morphism is only applicable on edges $e \in \mathrm{Edge}(\tilde{\Gamma}) \setminus \mathrm{Edge}(\Gamma)$.

**Definition 11.5.** (Isomorphism of tropical graphs) For a based curve type $(\tilde{\Gamma}, \Gamma)$, the *root vertex* $v_\circ$ of the base tropical graph is the vertex $v_\circ \in \mathrm{Vert}(\Gamma)$ containing the disk components of $\tilde{\Gamma}$, that is, $\kappa(v) = v_\circ$ for all $v \in \mathrm{Vert}_\circ(\tilde{\Gamma})$. Two tropical graphs $\Gamma_0, \Gamma_1$ are *isomorphic* if there is an isomorphism of graphs $\phi : \Gamma_0 \to \Gamma_1$ that preserves polytope assignments on vertices, edge slopes, and the root vertex.

*Remark* 11.6. (Background almost complex structures for split perturbations) For a split perturbation datum $\mathfrak{p}_{\tilde{\Gamma},\Gamma}$, the stabilizing divisor and background almost complex structure can be chosen to be different for curve components corresponding to different vertices of $\Gamma$. A stabilizing pair underlying a coherent split perturbation datum consists of a collection of stabilizing pairs $(D_v, J_v)$ on $\overline{X}_{P(v)}$ for all vertices $v \in \mathrm{Vert}(\Gamma)$ such that for any automorphism $\phi : \Gamma \to \Gamma$ of the tropical graph (as in Definition 11.5), $(D_{\phi(v)}, J_{\phi(v)}) = (D_v, J_v)$.

*Remark* 11.7. (Cylindrical structures for split perturbations) The cylindrical almost complex structures used to define based perturbations need not be gluable. Therefore, we may choose the cylindrical almost complex structure $J_P$ on $X_P$ so that there is a taming embedding of $(X_P, J_P)$ into the symplectic broken manifold $(\overline{X}_P, \omega_{X_P})$.

**Definition 11.8.** (Broken maps with base) Let $\underline{\mathfrak{p}}$ be a coherent split perturbation datum.

(a) A $\underline{\mathfrak{p}}$-*holomorphic broken map with base* consists of

- a based type $\kappa : \tilde{\Gamma} \to \Gamma$,
- and a broken map $u$ whose domain is of type $\tilde{\Gamma}$ and $u$ is $\mathfrak{p}_{\tilde{\Gamma},\Gamma}$-holomorphic in the ordinary sense (as in Definition 6.12). Furthermore, if $\tilde{\Gamma}_{\mathrm{tr}}$ is the tropical graph of $u$ then $\kappa$ factors through $\tilde{\Gamma}_{\mathrm{tr}}$ as

$$\kappa = \kappa_1 \circ \mathrm{tr}, \quad \tilde{\Gamma} \xrightarrow{\mathrm{tr}} \tilde{\Gamma}_{\mathrm{tr}} \xrightarrow{\kappa_1} \Gamma,$$

where $\kappa_1$ is a tropical edge collapse.

(b) (Type) The *type* of an broken map with base consists of the tropical graph $\tilde{\Gamma}$, the tropical edge collapse map $\tilde{\Gamma} \to \Gamma$, and the homology and tangency data for the map $u$ (as in the type of a broken map, see Definition 6.14). Whenever it is possible, the base tropical type $\Gamma$ is suppressed in the notation.

(c) (Tropical symmetry) For a type $\tilde{\Gamma}$ of a broken map with base, the tropical symmetry group $T_{\mathrm{trop}}(\tilde{\Gamma})$ is the symmetry group obtained by viewing $\tilde{\Gamma}$ as a type of broken map (by Definition 4.35) and forgetting the base type $\Gamma$.

(d) (Rigidity) The type $\tilde{\Gamma} \to \Gamma$ of a broken map with base is *rigid* if $\tilde{\Gamma}$ is rigid as a type of broken map. Thus for a rigid type, the base tropical graph $\Gamma$ is rigid, the tropical graph $\tilde{\Gamma}_{\mathrm{tr}}$ of the map is equal to $\Gamma$, and the morphism $\tilde{\Gamma} \to \Gamma$ only collapses treed components and boundary nodes.

Split perturbations can be used to define broken Fukaya algebras. Given a coherent split perturbation datum $\underline{\mathfrak{p}}$ on $\mathfrak{X}$, define the Fukaya algebra as the graded vector space

$$CF_{\mathrm{brok}}(L, \underline{\mathfrak{p}}) := CF^{\mathrm{geom}}(L) \oplus \Lambda_{\geq 0} x^{\blacktriangledown}[1] \oplus \Lambda_{\geq 0} x^{\triangledown}$$

with composition maps

$$m^d_{\mathrm{brok}}(x_1, \ldots, x_d) = \sum_{x_0, u \in \mathcal{M}_{\tilde{\Gamma},\Gamma}(\underline{\mathfrak{p}}, \underline{\eta}, \underline{x})_0} w_s(u) x_0, \tag{11.1}$$

where $u$ ranges over all rigid types $(\tilde{\Gamma}, \Gamma)$ of broken maps with base that have $d$ inputs (see Definition 11.8 (d) for rigidity), and

$$w_s(u) := (-1)^{\heartsuit}(d_\bullet(\Gamma)!)^{-1} \operatorname{Hol}_L([\partial u])\epsilon(u)q^{A(u)}, \tag{11.2}$$

where the symbols in (11.2) are as in (10.16), and in particular the orientation is computed in the same way as for broken map without base, see Remark 6.29. As in the case of broken maps (without base), for a one-dimensional component of the moduli space, the configurations with a boundary edge of length zero constitute a fake boundary, whereas those with a broken boundary edge constitute the true boundary of the moduli space. Setting the counts of the maps occurring in the true boundary to zero yields the $A_\infty$ associativity relations, and we conclude that $CF(L, \underline{\mathfrak{p}})$ is an $A_\infty$ algebra.

Broken Fukaya algebras defined by counts of broken maps with base are independent of the choice of perturbation up to homotopy equivalence. The proof is similar to the proof of independence of perturbations for unbroken Fukaya algebras shown in Proposition 10.18. Assuming that the stabilizing divisors have the same degree, the proof is by constructing an $A_\infty$ morphism by counts of quilted broken holomorphic disks with base. These are holomorphic disks with base, with the additional datum of a quilting circle on disks. Split perturbations differ in the sense that they are not coherent under collapsing the edges in the base tropical graph. The extension is straightforward because quilting phenomena take place on disk components, and the base tropical graph collapses disk components. We obtain the following result, which is an extension of Proposition 10.18.

**Proposition 11.9.** *(Independence of perturbations, split version) Suppose $\underline{\mathfrak{p}}^0$, $\underline{\mathfrak{p}}^1$ are regular split perturbation data that are defined using stabilizing pairs $(J^0, D^0)$ and $(J^1, D^1)$, which are connected by a path of stabilizing pairs $\{(J^t, D^t)\}_{t\in[0,1]}$. There exists a coherent perturbation datum $\underline{\mathfrak{p}}^{01}$ which induces a convergent unital $A_\infty$ morphism (as in (10.9))*

$$\phi_{01} : CF_{\text{brok}}(L, \underline{\mathfrak{p}}^0) \to CF_{\text{brok}}(L, \underline{\mathfrak{p}}^1)$$

*which is a homotopy equivalence.*

In fact, Fukaya algebras defined using ordinary perturbation data are $A_\infty$ homotopy equivalent to those defined using split perturbation data:

**Proposition 11.10.** *Let $\underline{\mathfrak{p}}$ be a regular coherent perturbation datum for the broken manifold $(\mathfrak{X}, L)$ and let $\underline{\mathfrak{p}}' = (\underline{\mathfrak{p}}'_\Gamma)_\Gamma$ be a collection of regular coherent split perturbation data for all rigid tropical graphs $\Gamma$. Then the broken Fukaya algebras $CF_{\text{brok}}(L, \underline{\mathfrak{p}})$, $CF_{\text{brok}}(L, \underline{\mathfrak{p}}')$ are homotopy equivalent.*

*Proof.* The result follows from the homotopy equivalence of any two broken Fukaya algebras defined by split perturbations, and the fact that a non-split perturbation may be regarded as a split perturbation in the following way: Given a perturbation $\underline{\mathfrak{p}}$ without base, we define a split perturbation $\underline{\mathfrak{p}}'$ by forgetting the base tropical graph. The resulting curve counts defining the composition maps are the same for $\underline{\mathfrak{p}}$ and $\underline{\mathfrak{p}}'$. Indeed, a $\underline{\mathfrak{p}}$-holomorphic map of type $\Gamma$ corresponds to a unique $\underline{\mathfrak{p}}'$-holomorphic map of type $\Gamma \xrightarrow{\text{tr}} \Gamma_{\text{tr}}$ where tr is the tropicalization map; the uniqueness follows from the fact that since $\Gamma$ is rigid, there is no other tropical graph that can be obtained by edge collapses. ∎

## 11.2 Application: Disk potentials for semi-Fano toric surfaces

Recall that in Section 2.2 we counted disks in a cubic surface by deforming the symplectic form on the cubic surface to that of a *semi-Fano* toric surface. A complex manifold $X$ is semi-Fano if the first Chern number $c_1(TX|S)$ is non-negative on any holomorphic curve $S \subset X$. In this section, we show that the disk potential of a toric Lagrangian in a semi-Fano toric surface is well-defined if we use a perturbation datum for which the almost complex structure is close enough to a divisor-preserving almost complex structure – these are almost complex structures for which the torus-invariant divisors are holomorphic. We also show that under some homological hypotheses, the potential is preserved by a multiple cut on a semi-Fano toric surface. These results were used in Sections 2.1 and 2.2 to obtain the disk potential by counting broken disks.

**Definition 11.11.** Let $X$ be a toric symplectic manifold. A tamed almost complex structure $J$ on $X$ is called *divisor-preserving* if all toric divisors $Y$ of $X$ are $J$-holomorphic, that is, $J(TY) = TY$.

**Definition 11.12.** (Potential for a non-regular almost complex structure) Suppose $J_0$ is a compatible almost complex structure on a symplectic manifold $(M, \omega)$ and $L \subset M$ be a Lagrangian with a brane structure. Let $\underline{F} = (F_\Gamma)_\Gamma$ be a coherent regular domain-dependent Morse function on $L$. The manifold $(X, J_0, L)$ with Morse datum $\underline{F}$ has potential

$$W_0(J_0) \in \Lambda_{>0} \tag{11.3}$$

if the following holds: Given an area level $E_0$, there exists $\epsilon > 0$ such that for any coherent regular perturbation $\underline{\mathfrak{p}} = (\underline{J}, \underline{F})$ such that $\underline{J}$ maps to an $\epsilon$-neighborhood of $J_0$,

$$m^0_{CF(L,\underline{\mathfrak{p}})}(1) = W_0 1^{\blacktriangledown} \mod q^{E_0}.$$

**Proposition 11.13.** *(Potential on semi-Fano toric manifolds) Let $X$ be a semi-Fano symplectic toric manifold and $L \subset X$ is a toric Lagrangian. Let $J_0$ be a divisor-preserving almost complex structure, and let $\underline{F}$ be a generic coherent domain-dependent Morse function on $L$. The potential for the manifold $(X, J_0, L)$ is well-defined in the sense of Definition 11.12.*

*Proof.* As a first step, we define the element $m^0_{J_0}(1) \in \Lambda_{\geq 0}\langle \mathrm{Crit}(F)\rangle$ for the divisor-preserving almost complex structure $J_0$.

*Claim.* There is an element $m^0_{J_0}(1) \in \Lambda_{\geq 0}\langle \mathrm{Crit}(F)\rangle$ such that for any area level $E_0$ there exists $\epsilon > 0$ such that for any coherent regular perturbation $\underline{\mathfrak{p}}$ that is $\epsilon$-close to $J_0$,

$$m^0_{CF(L,\underline{\mathfrak{p}})}(1) = m^0_{J_0}(1) \mod q^{E_0}.$$

To prove the Claim, let us assume the opposite, that is, there is a sequence of coherent regular perturbations $\underline{\mathfrak{p}}^\nu$ that converge uniformly to the constant $J_0$, and such that

$$m^0_{CF(L,\underline{\mathfrak{p}}^\nu)}(1) \neq m^0_{CF(L,\underline{\mathfrak{p}}^{\nu+1})}(1) \mod q^{E_0}$$

for all $\nu$. Choose a regular perturbation morphism

$$\underline{\mathfrak{p}}^{\nu,\nu+1} = (\underline{J}^{\nu,\nu+1}, \underline{F})$$

with end points $\underline{\mathfrak{p}}^\nu$, $\underline{\mathfrak{p}}^{\nu+1}$, such that the sequence $\underline{J}^{\nu,\nu+1}$ converges uniformly to the constant $J_0$ as $\nu \to \infty$. Since the potentials are unequal for $\underline{\mathfrak{p}}^\nu$ and $\underline{\mathfrak{p}}^{\nu+1}$, there is a $\underline{\mathfrak{p}}^{\nu,\nu+1}$-quilted disk $u_\nu$ with no inputs, having index zero, and area at most $E_0$, which contains a non-constant quilted surface component $S_v$. Indeed, the boundary of the one-dimensional part of the moduli space of quilted disks with no inputs consists of quilted disks with a broken edge $e$ (that is, $\ell(e) = \infty$), and the only configurations where the map is constant on the quilted disk are those counted in $m^0_{CF(L,P^\nu)}(1)$ and $m^0_{CF(L,P^{\nu+1})}(1)$.

We now rule out the appearance of quilted disks by a dimension argument. The quilted disk $u_\nu$ from the previous paragraph is of one of the forms shown in Figure 11.1. A subsequence of the sequence $u_\nu$ converges to a $J_0$-holomorphic treed disk $u_\infty$ of index $-1$. The limit $u_\infty$ also has a broken edge, and let $u^i_\infty$, $i = 0, 1, \cdots$, be the maps obtained by cutting the broken edges in $u_\infty$. One of these maps, say $u^j_\infty$, has index $-1$ and the map on the surface part is non-constant. The surface component in $u^j_\infty$ is $J_0$-holomorphic, and so, it consists of

- disks that have positive isolated intersections with the toric divisors, and so, have Maslov index at least 2;
- and spheres that have $c_1 \geq 0$ by the semi-Fano condition.

Consequently, if $u^j_\infty$ does not have inputs, the index $i(u^j_\infty) \geq 0$, which rules out configurations of the form (b) in Figure 11.1 for large enough $\nu$. Suppose an input leaf of $u^j_\infty$ is the output of a treed disk $u^k_\infty$ as in Figure 11.1 (a). By the same reasoning used for $u^j_\infty$ the surface part of $u^k_\infty$ has Maslov index at least two, and so, the output is necessarily the maximum point $x^\blacktriangledown$. Therefore, in Figure 11.1 (a), any input to the $(-1)$-index disk $u^j_\infty$ is $x^\blacktriangledown$, and consequently, the disk $u^j_\infty$ does not exist for degree reasons since the

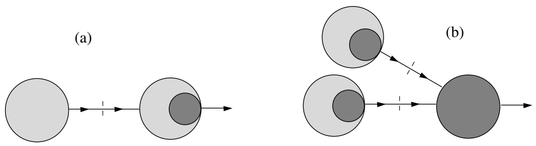



**Figure 11.1.** Codimension one strata in a moduli space of quilted disks with no inputs.

surface component has Maslov index at least 2. Therefore the configuration (a) is also ruled out for large $\nu$. Thus the Claim is proved.

Finally, we show that $m^0_{J_0}(1)$ is a multiple of $x^\blacktriangledown$: For any sequence of regular perturbations $\underline{\mathfrak{p}}^\nu = (\underline{J}^\nu, \underline{F})$ that uniformly converge to $(J_0, \underline{F})$, and a sequence of treed disks $\underline{\mathfrak{p}}_\nu$ disks $u_\nu$ with no inputs, of index zero, and uniformly bounded area, a subsequence converges to a $(J_0, \underline{F})$-disk $u_\infty$ of index 0. By the semi-Fano condition the Maslov index of $u_\infty$ is at least 2, and since there are no inputs, a dimension count dictates that $x^\blacktriangledown$ is the only possible output. ∎

**Proposition 11.14.** *For a semi-Fano toric symplectic manifold, the potential $W(J_0)$ does not depend on the choice of divisor-preserving almost complex structure $J_0$.*

*Outline of proof.* Assume the opposite, so that there is a path $\{J_t\}_{t\in[0,1]}$ of divisor-preserving almost complex structures on which the terms in the potential with area level $\le E_0$ are not $t$-independent. Then, there is a sequence of quilted disks with area $\le E_0$ that converge to a $J_t$-holomorphic disk of index $-1$ for some $t \in [0, 1]$. The existence of such a disk is ruled out as in the proof of Proposition 11.13. ∎

The following Proposition gives a sufficient condition under which the potential is not altered by multiple cutting of a semi-Fano toric surface.

**Proposition 11.15.** *(When breaking does not alter the potential) Let $X$ be a semi-Fano toric surface, $\mathcal{P}$ a toric multiple cut, and $L \subset \overline{X}_{P_0}$ a toric Lagrangian in a component $\overline{X}_{P_0}$ of the broken manifold $\mathfrak{X}_{\mathcal{P}}$. Let $\mathfrak{J}_0$ be a broken divisor-preserving tamed almost complex structure on $\mathfrak{X}$. Suppose for any broken $\mathfrak{J}_0$-holomorphic disk $u$ in $\mathfrak{X}$, $I(u_{\text{glue}}) \ge 2$. Then the potential for $(\mathfrak{X}, \mathfrak{J}_0, L)$ is well-defined and is the same as the toric potential for $X$ equipped with a divisor preserving almost complex structure.*

*Proof.* We argue by contradiction, using Gromov compactness. Let $J_0^\nu$ be the divisor preserving almost complex structure obtained from $\mathfrak{J}_0$ by gluing the cylindrical ends of $\mathfrak{X}$. Following the proof of Proposition 11.13, if the Proposition were not true, there would exist

- a sequence of regular perturbations $\underline{\mathfrak{p}}_\nu$ on $X^\nu$ that converge to the constant broken almost complex structure $J_0$,

- perturbation morphisms $\underline{\mathfrak{p}}^{\nu,\nu+1}$ that also converge to $J_0$,
- a sequence $u_\nu$ of $\underline{\mathfrak{p}}^{\nu,\nu+1}$-quilted disks of index zero, uniformly bounded area, each having a broken edge $e$ (that is, $\ell(e) = \infty$), and a quilted surface component on which the map is non-constant.

Consequently, after passing to a subsequence, the sequence $\{u_\nu\}_\nu$ converges to a $\mathfrak{J}_0$-holomorphic broken treed disk $u_\infty$ of index $-1$. Indeed, this is the only case to consider, since in the proof of Proposition 11.13 we have already ruled out quilted disks of the above form that converge to a $J_0^\nu$-holomorphic disk for a finite $\nu$. The quilted disks in the sequence $\nu$ are of one of the two forms shown in Figure 11.1. Copying the arguments from the proof of Proposition 11.13, each of the cases is ruled out using the fact that for a broken $\mathfrak{J}_0$-holomorphic disk $u$ can not have index $i^{\mathrm{brok}}(u) = -1$ as follows: By Proposition 6.22 (b), the glued type has the same index, that is, $i^{\mathrm{brok}}(u) = i(u_{\mathrm{glue}}) = -1$, which is not possible, since by the hypothesis, the Maslov index $I(u_{\mathrm{glue}})$ is at least two. ∎

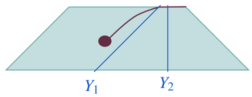


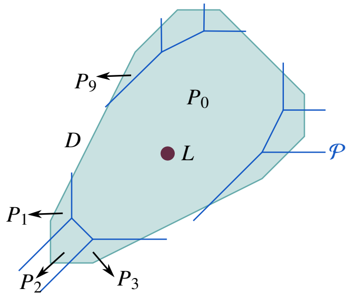

**Figure 11.2.** A broken disk with Maslov index 0 in the second Hirzebruch surface with the multiple cut $\mathcal{P}_1$ from Example 2.4.

*Remark* 11.16. (A counter-example) In Example 2.4 we saw that there is a multiple cut $\mathcal{P}_2$ on the second Hirzebruch surface $X := F_2$ for which the potential of the broken manifold $\mathfrak{X}_{\mathcal{P}_2}$ differs from that of the unbroken one $X$. The broken manifold $\mathfrak{X}_{\mathcal{P}_2}$ does not satisfy the hypothesis of Proposition 11.15 because there is a broken disk $u$ (shown in Figure 11.2) whose glued Maslov index $I(u_{\mathrm{glue}})$ is 0. The map $u$ contains a disk of Maslov index 4 and two spheres with self-intersection $-1$.

In the following result, we verify that the multiple cut $\mathcal{P}$ on the cubic surface given in Figure 11.3 satisfies the hypothesis of Proposition 11.15, and use it to conclude that the multiple cut preserves the potential of the toric Lagrangian in the cubic surface.

**Proposition 11.17.** *In the multiply cut cubic surface $\mathfrak{X}$ as in Figure 11.3 equipped with a divisor-preserving domain-dependent almost complex structure $\mathfrak{J}_0$ and a toric Lagrangian $L \subset \overline{X}_{P_0}$, for any $\mathfrak{J}_0$-holomorphic broken disk u, the glued Maslov index $I(u_{\mathrm{glue}})$ is at least 2. Consequently, the multiple cut of Figure 11.3 preserves the disk potential of the toric Lagrangian L.*

*Proof.* The proof is combinatorial and proceeds by analyzing various cases. We first consider disks whose tropical nodes do not map to orbifold singularities, and prove



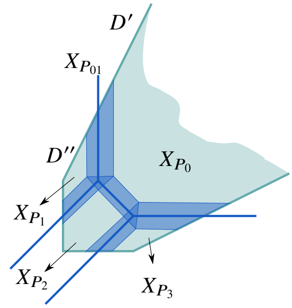




**Figure 11.3.** Left: Multiple cut $\mathcal{P}$ on the cubic surface $X$. Right: Components of the broken cubic surface $\mathfrak{X}$.

that their glued Maslov index is at least 2. For a rigid broken map $u : C \to \mathfrak{X}$ of type $\Gamma$, the Maslov index of the glued type $\Gamma_{\text{glue}}$ is

$$I(\Gamma_{\text{glue}}) = \sum_{v \in \text{Vert}(\Gamma)} \overline{I}(u_v), \quad \overline{I}(u_v) := I(u_v) - \sum_{e \in \text{Edge}_{\text{trop}}(\Gamma): e \ni v} 2\mu_{e,v}, \tag{11.4}$$

where $\mu_{e,v}$ is the sum of the intersection multiplicities with all the relative divisors at the lift of the node $w_e$ on $C_v$, and $I(u_v)$ is the Maslov index of the disk/sphere $u_v$. Indeed, (11.4) follows from (6.32) by observing that $\overline{I}(u_v) = I_{\text{adj}}(u_v) - 2|\text{Edge}_{\text{trop}}(\Gamma_v)|$ where $\text{Edge}_{\text{trop}}(\Gamma_v)$ is the set of tropical edges incident on $v$ (see also the related Remark 6.23). Via a case-by-case analysis of the components of the map, we will show that $I(\Gamma_{\text{glue}}) \geq 2$. In the sequel, we say that $v \in \text{Vert}(\Gamma)$ is a *descendent* of $w \in \text{Vert}(\Gamma)$ if there is an edge between $v$ and $w$, and $v$ is further from the root than $w$.

Case 1: If the map $u_v$ does not map to a toric divisor of $\overline{\mathfrak{X}}_{P(v)}$ then $\overline{I}(u_v) \geq 0$. Indeed, in this case $\overline{I}(u_v)$ is equal to the number of intersections of the map $u_v$ with non-boundary toric divisors counted with multiplicity, which is a non-negative integer. An inspection of the multiply cut cubic surface $\mathfrak{X}$ tells us that this case covers all maps $u_v$ where $\text{codim}(P(v)) > 0$.

Case 2: Suppose $P(v)$ is top-dimensional, $P(v) \neq P_0$, and $u_v$ maps to a non-boundary toric divisor $Y \subset \overline{X}_{P(v)}$. Then $Y$ is a $(-1)$-curve and one of the ends of $Y$ intersects a non-boundary toric divisor $Y_1 \subset \overline{X}_{P(v)}$. Therefore, $\overline{I}(u_v) = 0$.

Case 3: Suppose $P(v) = P_0$ and $u_v$ maps to a long divisor, say $D_1'$. In this case $\overline{I}(u_v)$ is negative, but we show that its contribution is cancelled by positive contributions from some descendent vertices of $v$. For the moment, we assume $u_v$ is a simple cover of $D_1'$. Then $v$ has a descendent $v_1$ such that

$$\text{(Case A) } P(v_1) = P_1 \text{ or } P_{01} \qquad \text{or} \qquad \text{(Case B) } P(v_1) = P_4 \text{ or } P_{04}.$$

Since Cases (A) and (B) are symmetric, we only consider (A). If $P(v_1) = P_1$, $u_{v_1}$ intersects either $D_1''$ or $E_1'$, both of which are non-relative toric divisors of $\overline{X}_{P_1}$; and if $P(v_1) = P_{01}$, $u_{v_1}$ has an intersection with the thickening of $D_1' \cap D_1''$, which is a non-boundary toric divisor of $\overline{\mathfrak{X}}_{P_{01}}$. In both cases $I(u_{v_1}) = 2$, which cancels the negative contribution $I(u_v) = -2$. In case $u_v$ is a $k$-cover of $D_1'$, then a similar cancellation argument applies, using descendent vertices of $v$.

So far, we have shown that the sum of $\overline{I}(u_v)$ over all vertices is non-negative. We now show that the sum is positive. Consider a disk component $v \in \mathrm{Vert}_\circ(\Gamma)$ on which the map is non-constant. The disk $u_v$ either intersects a long divisor, in which case $\overline{I}(u_v) \geq 2$; otherwise it intersects the relative divisor, say $\overline{X}_{P_{01}}$. In the latter case, there is a descendent vertex $v_1$ of $v$ for which $\overline{I}(u_{v_1}) \geq 2$ using the same argument as in Case 3. This finishes the proof in the case when the tropical nodes of the broken disk $u$ do not map to orbifold singularities.

In the case when a tropical node on a component $u_v$ maps to an orbifold singularity in $\overline{\mathfrak{X}}_{P(v)}$, we consider a toric resolution $\widetilde{\mathfrak{X}}_v$ of $\overline{\mathfrak{X}}_{P(v)}$ and a lift $\tilde{u}_v : C_v \to \widetilde{\mathfrak{X}}_v$. The formula (11.4) is applicable if the Maslov index $I(u_v)$ and the sum of intersection multiplicities $\mu_{e,v}$ are replaced by the corresponding quantities for the lift $\tilde{u}_v$. The other arguments in the proof now carry over. In fact, the components $u_v$ mapping to orbifold points are covered by Case 1, since in the multiply cut cubic surface, toric divisors in $X_{P(v)}$ do not contain orbifold singularities in their closure.

We have shown that in the multiply cut cubic surface $\mathfrak{X}$, the glued Maslov index of any broken disk is at least 2. Therefore, we may apply Proposition 11.15 on $\mathfrak{X}$ and conclude that the potential of the toric Lagrangian $L$ in the cubic surface is preserved under the multiple cut $\mathcal{P}$. This finishes the proof of Proposition 11.17. ∎

*Remark* 11.18. A curious reader may wonder whether the potential has the same terms if the Lagrangian is in some other top-dimensional piece of the multiply cut cubic surface. The potential has the same terms as the unbroken case when $L$ is a toric Lagrangian in any top-dimensional piece $\overline{X}_{P_i}, i = 0, \ldots, 9$ in the broken manifold. Indeed, the proof of Proposition 11.17 can be replicated, and there are no disks of Maslov index less than 2 in each of the cases. However, if $L \subset \overline{X}_{P_i}, i \neq 0$, the disks can not be counted in a straightforward way, since some of the broken disks contain components that are multiple covers of $(-1)$-spheres. For example, for the disk $u$ in Figure 11.4 the gluing $u_{\text{glue}}$ has homology class $\delta_{E_5}$. The component $u_{v_3}$ in $u$ is homologous to a double cover $2[D_1']$; the count of such a configuration is $-\frac{1}{2}$ by the Graber-Pandharipande multiple cover formula [12] (and for which we provide an alternate proof in [82] without using virtual localization). Therefore, after accounting for the multiplicity of the edge $e$, the map $u$ contributes $-1$ to the count. There is another broken disk $u'$ in the same homology class, where the sphere $u_{v_3}$ is replaced by two spheres $u_{v_3}, u_{v_3'}$, each of homology class $[D_1']$ (see Figure 11.5); there are two broken maps of this type obtained by interchanging the sub-trees attached at the vertices $v_3$, $v_3'$, adding $+2$ to the curve count.

Together, the two types of maps give a contribution of +1, which is the contribution of a disk of class $\delta_{E_5}$ in the unbroken case.

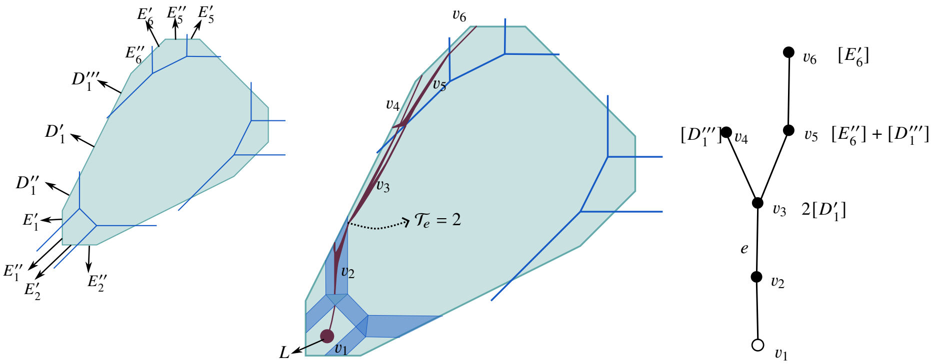



**Figure 11.4.** A broken map $u$ whose gluing is in the class $\delta_{E_5}$, and which contains a multiple cover. The node $(v_1, v_2)$ (whose $v_1$-end maps to an orbifold singularity) is similar to the node in Example 6.24 (b).

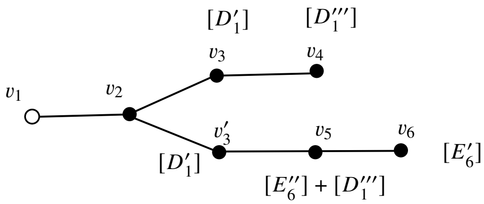



**Figure 11.5.** The graph of a broken map $u'$ whose gluing is in the class $\delta_{E_5}$, but which does not have any component that is homologous to a multiple cover.

# References


[1] Mohammed Abouzaid. Framed bordism and Lagrangian embeddings of exotic spheres. *Ann. of Math. (2)*, 175(1):71–185, 2012.

[2] Dan Abramovich, Qile Chen, Mark Gross, and Bernd Siebert. Decomposition of degenerate Gromov-Witten invariants. *Compos. Math.*, 156(10):2020–2075, 2020.

[3] Byung Hee An, Yunhyung Cho, and Jang Soo Kim. On the $f$-vectors of Gelfand-Cetlin polytopes. *European J. Combin.*, 67:61–77, 2018.

[4] D. Auroux. A remark about Donaldson's construction of symplectic submanifolds. *J. Symplectic Geom.*, 1(3):647–658, 2002.

[5] Denis Auroux, Damien Gayet, and Jean-Paul Mohsen. Symplectic hypersurfaces in the complement of an isotropic submanifold. *Math. Ann.*, 321(4):739–754, 2001.

[6] Sam Bardwell-Evans, Man-Wai Mandy Cheung, Hansol Hong, and Yu-Shen Lin. Scattering diagrams from holomorphic discs in log calabi-yau surfaces. *arXiv eprint*, 2110.15234, 2024.

[7] Lawrence Jack Barrott. Explicit equations for mirror families to log calabi-yau surfaces. *arXiv eprint*, 1810.08356, 2018.

[8] Victor V. Batyrev, Ionu¸t Ciocan-Fontanine, Bumsig Kim, and Duco van Straten. Mirror symmetry and toric degenerations of partial flag manifolds. *Acta Math.*, 184(1):1–39, 2000.

[9] Paul Biran and Octav Cornea. Quantum structures for lagrangian submanifolds. *arXiv eprint*, 0708.4221, 2007.

[10] Nathaniel Bottman and Alexei Oblomkov. A compactification of the moduli space of marked vertical lines in $\mathbb{C}^2$. *arXiv eprint*, 1910.02037, 2019.

[11] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in symplectic field theory. *Geom. Topol.*, 7:799–888, 2003.

[12] Jim Bryan and Rahul Pandharipande. The local Gromov-Witten theory of curves. *J. Amer. Math. Soc.*, 21(1):101–136, 2008. With an appendix by Bryan, C. Faber, A. Okounkov and Pandharipande.

[13] Marco Castronovo. *Fukaya category of Grassmannians: rectangles*, volume 372. 2020.

[14] Kwokwai Chan, Cheol-Hyun Cho, Siu-Cheong Lau, and Hsian-Hua Tseng. Gross fibrations, SYZ mirror symmetry, and open Gromov-Witten invariants for toric Calabi-Yau orbifolds. *J. Differential Geom.*, 103(2):207–288, 2016.

[15] Kwokwai Chan and Siu-Cheong Lau. Open Gromov-Witten Invariants and Superpotentials for Semi-Fano Toric Surfaces. *International Mathematics Research Notices*, 2014(14):3759–3789, 03 2013.

[16] Fran¸cois Charest and Chris Woodward. Floer trajectories and stabilizing divisors. *J. Fixed Point Theory Appl.*, 19(2):1165–1236, 2017.

[17] François Charest and Chris T. Woodward. Floer theory and flips. *arXiv eprint*, 1508.01573, 2022.

[18] Qile Chen. Stable logarithmic maps to Deligne-Faltings pairs I. *Ann. of Math. (2)*, 180(2):455–521, 2014.

[19] Weimin Chen and Yongbin Ruan. Orbifold Gromov-Witten theory. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 25–85. Amer. Math. Soc., Providence, RI, 2002.

[20] Weimin Chen and Yongbin Ruan. A new cohomology theory of orbifold. *Comm. Math. Phys.*, 248(1):1–31, 2004.

[21] Cheol-Hyun Cho. Holomorphic discs, spin structures, and Floer cohomology of the Clifford torus. *Int. Math. Res. Not.*, (35):1803–1843, 2004.

[22] Cheol-Hyun Cho and Yong-Geun Oh. Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds. *Asian J. Math.*, 10(4):773–814, 2006.

[23] Yunhyung Cho and Yoosik Kim. Monotone Lagrangians in flag varieties. *Int. Math. Res. Not. IMRN*, (18):13892–13945, 2021.

[24] Yunhyung Cho, Yoosik Kim, and Yong-Geun Oh. Lagrangian fibers of Gelfand-Cetlin systems. *Adv. Math.*, 372:107304, 57, 2020.

[25] K. Cieliebak and K. Mohnke. Compactness for punctured holomorphic curves. volume 3, pages 589–654. 2005. Conference on Symplectic Topology.

[26] Kai Cieliebak and Klaus Mohnke. Symplectic hypersurfaces and transversality in Gromov-Witten theory. *J. Symplectic Geom.*, 5(3):281–356, 2007.

[27] Octav Cornea and Fran¸cois Lalonde. Cluster homology: an overview of the construction and results. *Electron. Res. Announc. Amer. Math. Soc.*, 12:1–12, 2006.

[28] Thomas Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France*, 116(3):315–339, 1988.

[29] S. K. Donaldson. Symplectic submanifolds and almost-complex geometry. *J. Differential Geom.*, 44(4):666–705, 1996.

[30] J. J. Duistermaat. On global action-angle coordinates. *Comm. Pure Appl. Math.*, 33(6):687–706, 1980.

[31] Jonny Evans. *Lectures on Lagrangian torus fibrations*, volume 105 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2023.

[32] Mohammad Farajzadeh-Tehrani. Deformation theory of log pseudo-holomorphic curves and logarithmic ruan-tian perturbations. *arXiv eprint*, 1910.05201, 2019.

[33] Mohammad Farajzadeh-Tehrani. Limits of stable maps in a semi-stable degeneration. *Geom. Dedicata*, 216(6):Paper No. 66, 42, 2022.

[34] Mohammad Farajzadeh-Tehrani. Pseudoholomorphic curves relative to a normal crossings symplectic divisor: compactification. *Geom. Topol.*, 26(3):989–1075, 2022.

[35] Andreas Floer. The unregularized gradient flow of the symplectic action. *Comm. Pure Appl. Math.*, 41(6):775–813, 1988.

[36] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46.1 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[37] Fabio Gironella, Vicente Muñoz, and Zhengyi Zhou. Asymptotically holomorphic theory for symplectic orbifolds. *Math. Z.*, 304(1):Paper No. 21, 33, 2023.

[38] Alexander Givental. Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture. In *Topics in singularity theory*, volume 180 of *Amer. Math. Soc. Transl. Ser. 2*, pages 103–115. Amer. Math. Soc., Providence, RI, 1997.

[39] Mark Gross. *Tropical geometry and mirror symmetry*, volume 114 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2011.

[40] Mark Gross and Bernd Siebert. From real affine geometry to complex geometry. *Ann. of Math. (2)*, 174(3):1301–1428, 2011.

[41] Mark Gross and Bernd Siebert. An invitation to toric degenerations. In *Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics*, volume 16 of *Surv. Differ. Geom.*, pages 43–78. Int. Press, Somerville, MA, 2011.

[42] V Guillemin and S Sternberg. The gelfand-cetlin system and quantization of the complex flag manifolds. *Journal of Functional Analysis*, 52(1):106–128, 1983.

[43] Helmut Hofer. Pseudoholomorphic curves in symplectizations with applications to the weinstein conjecture in dimension three. *Invent Math*, (114):515–563, 1993.

[44] Eleny-Nicoleta Ionel. GW invariants relative to normal crossing divisors. *Adv. Math.*, 281:40–141, 2015.

[45] Eleny-Nicoleta Ionel and Thomas H. Parker. Relative Gromov-Witten invariants. *Ann. of Math. (2)*, 157(1):45–96, 2003.

[46] M. Kontsevich and Y. Soibelman. Notes on $A_\infty$-algebras, $A_\infty$-categories and non-commutative geometry. In *Homological mirror symmetry*, volume 757 of *Lecture Notes in Phys.*, pages 153–219. Springer, Berlin, 2009.

[47] Maxim Kontsevich and Yan Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 203–263. World Sci. Publ., River Edge, NJ, 2001.

[48] Eugene Lerman. Symplectic cuts. *Math. Res. Lett.*, 2(3):247–258, 1995.

[49] Naichung Conan Leung and Margaret Symington. Almost toric symplectic four-manifolds. *J. Symplectic Geom.*, 8(2):143–187, 2010.

[50] Jun Li. A degeneration formula of GW-invariants. *J. Differential Geom.*, 60(2):199–293, 2002.

[51] Robert B. Lockhart and Robert C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):409–447, 1985.

[52] S. Ma'u, K. Wehrheim, and C. Woodward. $A_\infty$ functors for Lagrangian correspondences. *Selecta Math. (N.S.)*, 24(3):1913–2002, 2018.

[53] S. Ma'u and C. Woodward. Geometric realizations of the multiplihedra. *Compos. Math.*, 146(4):1002–1028, 2010.

[54] Dusa McDuff. From symplectic deformation to isotopy. In *Topics in symplectic 4-manifolds (Irvine, CA, 1996)*, volume I of *First Int. Press Lect. Ser.*, pages 85–99. Int. Press, Cambridge, MA, 1998.

[55] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, second edition, 2012.

[56] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, third edition, 2017.

[57] Eckhard Meinrenken. Symplectic surgery and the $Spin^c$-Dirac operator. *Adv. Math.*, 134(2):240–277, 1998.

[58] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18(2):313–377, 2005.

[59] John Morales, Sergey Galkin, and Alexandr Usnich. Mutations of potentials, 2011.

[60] David Mumford. *Algebraic geometry. I*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Complex projective varieties, Reprint of the 1976 edition.

[61] Takeo Nishinou, Yuichi Nohara, and Kazushi Ueda. Toric degenerations of Gelfand-Cetlin systems and potential functions. *Adv. Math.*, 224(2):648–706, 2010.

[62] Yuichi Nohara and Kazushi Ueda. Toric degenerations of integrable systems on Grassmannians and polygon spaces. *Nagoya Math. J.*, 214:125–168, 2014.

[63] Brett Parker. Exploded manifolds. *Adv. Math.*, 229(6):3256–3319, 2012.

[64] Brett Parker. Log geometry and exploded manifolds. *Abh. Math. Semin. Univ. Hambg.*, 82(1):43–81, 2012.

[65] Brett Parker. Holomorphic curves in exploded manifolds: compactness. *Adv. Math.*, 283:377–457, 2015.

[66] Brett Parker. Tropical enumeration of curves in blowups of the projective plane. *arXiv eprint*, 1411.5722, 2017.

[67] Brett Parker. Tropical gluing formulae for gromov-witten invariants. *arXiv eprint*, 1703.05433, 2017.

[68] Brett Parker. De Rham theory of exploded manifolds. *Geom. Topol.*, 22(1):1–54, 2018.

[69] Brett Parker. Holomorphic curves in exploded manifolds: Kuranishi structure. *arXiv eprint*, 1301.4748, 2019.

[70] Brett Parker. Holomorphic curves in exploded manifolds: regularity. *Geom. Topol.*, 23(4):1621–1690, 2019.

[71] Brett Parker. Notes on exploded manifolds and a tropical gluing formula for Gromov-Witten invariants. In *Gromov-Witten theory, gauge theory and dualities*, volume 48 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, page 17. Austral. Nat. Univ., Canberra, 2019.

[72] James Pascaleff and Dmitry Tonkonog. The wall-crossing formula and Lagrangian mutations. *Adv. Math.*, 361:106850, 67, 2020.

[73] Matthias Schwarz. *Morse homology*, volume 111 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1993.

[74] Paul Seidel. Graded Lagrangian submanifolds. *Bull. Soc. Math. France*, 128(1):103–149, 2000.

[75] Paul Seidel. $A_\infty$-subalgebras and natural transformations. *Homology Homotopy Appl.*, 10(2):83–114, 2008.

[76] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[77] Paul Seidel. Homological mirror symmetry for the genus two curve. *J. Algebraic Geom.*, 20(4):727–769, 2011.

[78] Nick Sheridan. On the Fukaya category of a Fano hypersurface in projective space. *Publ. Math. Inst. Hautes Études Sci.*, 124:165–317, 2016.

[79] Bernd Siebert. Gromov-witten invariants of general symplectic manifolds. *arXiv eprint*, dg-ga/9608005, 1998.

[80] James Stasheff. *$H$-spaces from a homotopy point of view*, volume Vol. 161 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.

[81] Margaret Symington. Four dimensions from two in symplectic topology. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 153–208. Amer. Math. Soc., Providence, RI, 2003.

[82] Sushmita Venugopalan and Chris Woodward. Disk potentials for almost toric manifolds. In preparation.

[83] Sushmita Venugopalan and Chris Woodward. Split fukaya algebras. In preparation.

[84] Renato Vianna. Infinitely many monotone Lagrangian tori in del Pezzo surfaces. *Selecta Math. (N.S.)*, 23(3):1955–1996, 2017.

[85] Fuzhen Zhang. *Matrix theory*. Universitext. Springer, New York, second edition, 2011. Basic results and techniques.

[86] Aleksey Zinger. Notes on j-holomorphic maps. *arXiv eprint*, 1706.00331, 2017.

# Index